Zeitschrift: Helvetica Physica Acta
Band: 57 (1984)
Heft: 1

Artikel: Exact decay of correlations for infinite range continuous systems
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DOI: https://doi.org/10.5169/seals-115497

Helvetica Physica Acta, Vol. **57** (1984) 63–85
0018–0238/84/010063–23$1.50 + 0.20/0

Exact decay of correlations for infinite range continuous systems

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(25. IX. 1983)

Abstract. We give the exact asymptotic form at low activity of the correlations $\rho(x_1 \cdots x_n)$ of a classical fluid of particles interacting by two body potentials Φ with integrable power law decay.

These low activity results are extended to the whole domain of activities and temperatures where the state is unique and the (truncated) correlations have a power law decay. For homogeneous systems, this yields in particular a rigorous proof of the formula

$$\rho(x_1x_2) \sim \rho^2(1-\beta\Phi(x_1-x_2)(\beta^{-1}\rho\chi_T)^2), \qquad |x_1-x_2| \to \infty$$

with χ_T the compressibility. Moreover it is shown that for all activities and temperatures, the decay of the correlations cannot be faster than that of the potential when $\chi_T \neq 0$.

0. Introduction

The study of the asymptotic form of the correlation $\rho(x, y)$ is an old problem in the theory of fluids. It is of interest since measurements of $\rho(x, y)$ can yield informations on the pair potential between atoms or molecules.

In this paper we consider a classical system of particles in $\mathbb{R}^\nu$, with pair potential Φ such that $\Phi(x) \sim d\,|x|^{-\gamma}$, $\gamma > \nu$, as $|x| \to \infty$. Assuming that the direct correlation $c(x)$ behaves as $-\beta\Phi(x)$, $|x| \to \infty$, Enderby *et al.* obtained in 1965 the relation $|x-y|^\gamma \rho^T(x, y) \sim -d\beta\rho^2(\beta^{-1}\rho\chi_T)^2$ as $|x-y| \to \infty$ [1a], where $\rho^T(x, y) = \rho(x, y) - \rho(x)\rho(y)$ and χ_T is the compressibility. On the other hand a low activity study of $\rho(x, y)$ led Groeneveld in 1967 to conjecture that $|x-y|^\gamma\, \rho^T(x, y)$ should have a limit as $|x-y| \to \infty$ [1b]. An argument for the asymptotic form of $c(x)$ can be found in Verlet 1968 [2a] and a formal proof of these properties was given by Stell in 1977 [2b].

Since these early works, a large number of authors have studied the decay of the correlation functions for systems with long range interactions. Almost all these investigations were concerned with lattice systems and the results were restricted

*) Partially supported by CNR-GNFM and Ministero Pubblica Istruzione.
†) Partially supported by the Swiss National Foundation for Scientific Research.

to high temperature. Typically their analysis have shown that, at high temperature, the correlation functions have a power law decay (if the potential has a power law decay) and bounds on this power have been obtained. Furthermore, it has been shown that the correlations decay in the same weighted summability sense as the potential. For a survey of the literature up to 1979, we refer to the article of L. Gross [3]; among the more recent works, we mention those of R. Israel and C. Nappi [4] and C. Cammarota [5]. Results in the whole domain of uniqueness for the equilibrium state have been obtained by H. Künsch [6], while J. Imbrie obtained bounds on the decay in the low temperature domain for 1-dimensional spin systems with interaction $J(x, y)=|x-y|^{-2}$ [7]. For the case of continuous systems we recall the results of M. Duneau and B. Souillard [8]; more recently, assuming some regularity property of the correlation functions, it was shown that their decay cannot be faster than that of the potential for all values of temperature $T>0$ [9, 10, 11].

This last result should be compared to the analogous property for ferromagnetic spin $\frac{1}{2}$ lattice systems. Indeed in this latter case the Griffith inequality yields [25]

$$\langle\sigma_0\sigma_x\rangle \geq \tanh \beta J_{0x} \geq \beta J_{0x}, \qquad |x|\to\infty.$$

In this article, we first give a rigorous derivation of Enderby's result which is valid at low activity (Section 2). Our result is in fact more general since we obtain for any $(n+m)$-point functions the exact value of $\lambda^\gamma(\rho(x_1\cdots x_n y_1+\lambda\hat{u}\cdots y_m+\lambda\hat{u})-\rho(x_1\cdots x_n)\rho(y_1+\lambda\hat{u}\cdots y_m+\lambda\hat{u}))$ in the limit $\lambda\to\infty$, (Proposition 1 and Corollary). This result is derived using the low activity expansion: we show that the limit $\lambda\to\infty$ can be permuted with the sum and we evaluate the limit for each term of the series.

In Section 3, we discuss the decay property for arbitrary values of activity. We prove under reasonable hypothesis that the clustering cannot be faster than the decay of the potential if the system is compressible, i.e. if $\chi_T\neq 0$. Moreover, in the domain of uniqueness of the equilibrium state, it is shown that if

$$\lambda^\alpha(\rho(x_1\cdots x_n y_1+\lambda\hat{u}\cdots y_m+\lambda\hat{u})-\rho(x_1\cdots x_n)\rho(y_1+\lambda\hat{u}\cdots y_m+\lambda\hat{u}))$$

has a non zero limit when $\lambda\to\infty$, then necessarily $\alpha=\gamma$ (=power of the potential) and the asymptotic form is the same as in the low activity domain. These properties are obtained by an asymptotic analysis of the Kirkwood-Salsburg equation. We should stress that all our results are derived without assuming translation invariance of the state, but under the condition that the potential is integrable i.e. $\gamma>\nu$; furthermore it yields a rigorous proof that under these conditions $c(x)\sim-\beta\Phi(x)$ as $|x|\to\infty$. It is interesting to compare these results with the decay property of Coulomb systems for which $\gamma=\nu-2$ and this is done at the end of Section 3 for different situations. In the case of exponential interactions, or finite range and hard cores interactions, our analysis does not apply. In the latter case, it is well known that the correlations have an oscillatory behaviour at infinity [2b, 24].

1. Definitions

We consider an infinite system of classical particles in thermal equilibrium defined by (T, z) ($T=$ temperature, $z=$ activity) and moving in some domain $\mathscr{D}$ o

$\mathbb{R}^\nu$. The domain $\mathcal{D}$ may be the whole space $\mathbb{R}^\nu$, or some region of $\mathbb{R}^\nu$ bounded by hard walls. We require that $\mathcal{D}$ extends to infinity at least in some direction $\hat{u}$ and that the limiting ensemble $\lim_{\lambda\to\infty}(\mathcal{D}-\lambda\hat{u})=\tilde{\mathcal{D}}$ exists.

The particles interact by means of a symmetric, translation invariant, pair potential $\Phi(x_1-x_2)=\Phi(x_2-x_1)$ having the following properties:[1])

$$\sum_{1\leq i<j\leq n}\Phi(x_i-x_j)\geq -nB \quad \text{for all } B\geq 0 \qquad (x_1,\ldots,x_n)\subset\mathbb{R}^\nu \tag{1a}$$

$$\lim_{\lambda\to\infty}\lambda^\gamma\Phi(\lambda\hat{x})=d(\hat{x}), \quad \text{uniformly with respect to } \hat{x} \tag{1b}$$

where $d(\hat{x})$ is continuous and not identically zero on the unit sphere $|\hat{x}|=1$; $\gamma>\nu$.

Notice that (1a) and (1b) imply that $|e^{-\beta\Phi(x)}-1|$ is everywhere bounded and for all β

$$|e^{-\beta\Phi(x)}-1|\leq\frac{C(\beta)}{|x|^\gamma+1} \tag{2a}$$

$$\int dx\,|e^{-\beta\Phi(x)}-1|=b(\beta)<\infty \tag{2b}$$

Let $X, Y, \ldots,$ denote finite sets of points in $\mathbb{R}^\nu$ and $|X|$, $|Y|$ their cardinality. We write $Y^{(n)}=(y_1,\ldots,y_n)$ when it is necessary to specify that Y has n points and

$$\int_{\mathcal{D}} dY^{(n)}=\int_{\mathcal{D}} dy_1\cdots\int_{\mathcal{D}} dy_n$$

2. Clustering properties at low activities

In this section we prove that, at low activities, the asymptotic behaviour of the truncated correlation functions is the same as that of the potential.

Let G_X^n be the set of fully connected graphs with $|X|+n$ vertices $XY^{(n)}$, with X fixed and y_i arbitrary in $\mathcal{D}$. We denote by g a graph in G_X^n, and by l a line of the graph g. Any pair of vertices in g is linked by a chain, i.e. a sequence of consecutive lines.

To each g we associate the product of Mayer functions:

$$F_g(XY^{(n)})=\prod_{l\in g}(e^{-\beta\Phi(l)}-1) \tag{3}$$

$F_g(x)=1$ when g consists of a single vertex x.

In this section, the truncated correlation functions are defined by the series

$$\rho^T(X)=\sum_{n=0}^\infty\frac{z^{|X|+n}}{n!}I_n(X) \tag{4}$$

with

$$\begin{aligned} I_n(X)&=\sum_{g\in G_X^n}\int_{\mathcal{D}} dY^{(n)}F_g(XY^{(n)}) \qquad n>0\\ I_0(X)&=\sum_{g\in G_X^0}F_g(X)\end{aligned} \tag{5}$$

[1]) We do not assume rotation invariance of $\Phi(x)$.

It is well known that the series (4) converges absolutely for $z < z_0$. In fact, one can show that

$$I_n(X) \leq (|X| + n - 1)!\, A^{|X|-1} B^n$$

where A and B are functions of β only; these functions depend on the techniques used for the estimation of (5) (see e.g. [12], [13]). Of course z_0 goes to infinity as β goes to zero.

We denote by X^λ, or $X + \lambda\hat{u}$, the translate of X by $\lambda\hat{u}$, i.e.

$$X^\lambda = X + \lambda\hat{u} = (x_1 + \lambda\hat{u}, \ldots, x_m + \lambda\hat{u})$$

Our main result is given in Proposition 1:

Proposition 1. *If the potential satisfies the conditions* (1a, b), *then for any* $X_1 \neq \emptyset$, $X_2 \neq \emptyset$, *and* $0 \leq z < z_0$,

$$\lim_{\lambda\to\infty} \lambda^\gamma \rho^T(X_1, X_2 + \lambda\hat{u}) = -\beta\, d(\hat{u}) H^T(X_1) \tilde{H}^T(X_2)$$

where

$$H^T(X) = |X|\, \rho^T(X) + \int_{\mathscr{D}} dy \rho^T(Xy) = z \frac{d}{dz} \rho^T(X) \tag{6}$$

and $\tilde{H}^T$ *is the same quantity as* H^T *but defined with respect to* $\tilde{\mathscr{D}}$; *let us note that*

$$\tilde{H}^T(X) = \lim_{\lambda\to\infty} H^T(X + \lambda\hat{u}) \quad \text{and} \quad \tilde{\rho}^T(X) = \lim_{\lambda\to\infty} \rho^T(X + \lambda\hat{u})$$

Corollary. *Under the conditions of Proposition* 1

$$\lim_{\lambda\to\infty} \lambda^\gamma [\rho(X_1, X_2 + \lambda\hat{u}) - \rho(X_1)\rho(X_2 + \lambda\hat{u})] = -\beta\, d(\hat{u}) H(X_1) \tilde{H}(X_2)$$

where

$$H(X) = |X|\, \rho(X) + \int_{\mathscr{D}} dy [\rho(Xy) - \rho(X)\rho(y)] \tag{6'}$$

To establish the proposition we first compute explicitly in Lemma 1 the limit as $\lambda \to \infty$ of the integrals $\lambda^\gamma I_n(X_1, X_2^\lambda)$ occurring in the development (4), i.e. in

$$\lambda^\gamma \rho^T(X_1, X_2^\lambda) = \sum_{n=0}^{\infty} \frac{z^{|X_1|+|X_2|+n}}{n!} \lambda^\gamma I_n(X_1, X_2^\lambda) \tag{7}$$

Then we give in Lemma 2 a bound on the nth order term of the series (7) uniform with respect to λ which enables us to permute the limit $\lambda \to \infty$ and the summation in (7).

Lemma 1. *If the potential satisfies the conditions* (1a, b), *then*

$$\lim_{\lambda\to\infty} \lambda^\gamma I_n(X_1, X_2 + \lambda\hat{u})$$

$$= -\beta\, d(\hat{u}) \sum_{\substack{p=0 \\ p+q=n}}^{n} \frac{n!}{p!\, q!} (p + |X_1|)(q + |X_2|) I_p(X_1) \tilde{I}_q(X_2) \tag{8}$$

Lemma 2. *Under the same conditions on the potential, there exists $C_n > 0$ independent of λ such that*

$$\frac{1}{n!}|\lambda^{\gamma} I_n(X_1, X_2+\lambda\hat{u})| \leq C_n \quad \text{with} \quad \sum_{n=0}^{\infty} z^n C_n < \infty$$

for $0 \leq z < z_0$.

The result of the proposition is then immediate since Lemmas 1 and 2 imply that:

$$\begin{aligned}
\lim_{\lambda\to\infty} \lambda^{\gamma}\rho^T(X_1, X_2+\lambda\hat{u}) &= \sum_{n=0}^{\infty} \frac{z^{|X_1|+|X_2|+n}}{n!} \lim_{\lambda\to\infty} \lambda^{\gamma} I_n(X_1, X_2+\lambda\hat{u}) \\
&= -\beta\, d(\hat{u}) \left(\sum_{p=0}^{\infty} \frac{z^{|X_1|+p}}{p!} (p+|X_1|) I_p(X_1) \right) \\
&\quad \times \left(\sum_{q=0}^{\infty} \frac{z^{|X_2|+q}}{q!} (q+|X_2|) \tilde{I}_q(X_2) \right) \\
&= -\beta\, d(\hat{u}) H^T(X_1) \tilde{H}^T(X_2)
\end{aligned} \tag{9}$$

This last expression follows from the definitions (4) and (5) which give

$$\begin{aligned}
I_{n+1}(X) &= \sum_{g\in G_X^{n+1}} \int_{\mathscr{D}} dy \int_{\mathscr{D}} dY^{(n)} F_g(XyY^{(n)}) \\
&= \int_{\mathscr{D}} dy \sum_{g\in G_{Xy}^n} \int_{\mathscr{D}} dY^{(n)} F_g(XyY^{(n)})
\end{aligned}$$

i.e.

$$I_{n+1}(X) = \int_{\mathscr{D}} dy I_n(Xy) \tag{10}$$

and thus

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{z^{|X|+n}}{n!} n I_n(X) &= \sum_{n=0}^{\infty} \frac{z^{|X|+n+1}}{n!} \int_{\mathscr{D}} dy I_n(Xy) \\
&= \int_{\mathscr{D}} dy \rho^T(Xy)
\end{aligned} \tag{11}$$

where all sums are absolutely convergent for $0 \leq z < z_0$. The rest of this section is devoted to the proof of Lemma 1. The proof of Lemma 2 and of the corollary can be found in Appendix A.

Proof of Lemma 1

We divide the domain $\mathscr{D}$ into three disjoint regions, $\mathscr{D} = \mathscr{D}_1^{(\lambda)} \cup \mathscr{D}_2^{(\lambda)} \cup \mathscr{D}_3^{(\lambda)}$ with $\mathscr{D}_1^{(\lambda)} = \{x \in \mathscr{D} \mid |x| \leq \lambda/4\}$ around X_1 and $\mathscr{D}_2^{(\lambda)} = \{x \in \mathscr{D} \mid |x-\lambda\hat{u}| \leq \lambda/4\}$ around X_2^{λ}. We then have:

$$\int_{\mathscr{D}} dY^{(n)} = \sum_{\alpha_1,\ldots,\alpha_n=1}^{3} \int_{\mathscr{D}_{\alpha_1}^{(\lambda)}} dy_1 \cdots \int_{\mathscr{D}_{\alpha_n}^{(\lambda)}} dy_n$$

Furthermore for any subset $I \subset \{1, 2, \ldots, n\}$, we denote $Y_I = \{y_i \in Y^{(n)} \mid i \in I\}$

and $\chi_\alpha^{(\lambda)}(Y_I)=\prod_{i\in I}\chi_\alpha^{(\lambda)}(y_i)$ where $\chi_\alpha^{(\lambda)}$ is the characteristic function of $\mathscr{D}_\alpha^{(\lambda)}$, $\alpha = 1, 2, 3$.

The idea of the proof is that only the graphs in the definition (5) of $I_n(X_1, X_2+\lambda\hat{u})$ with just one line connecting $\mathscr{D}_1^{(\lambda)}$ to $\mathscr{D}_2^{(\lambda)}$ and no vertices in $\mathscr{D}_3^{(\lambda)}$ will give a non-zero contribution to $\lambda^\gamma I_n(X_1, X_2+\lambda\hat{u})$ in the limit $\lambda\to\infty$. Indeed, in this case only, all the Mayer functions are non-vanishing except for the one connecting $\mathscr{D}_1^{(\lambda)}$ with $\mathscr{D}_2^{(\lambda)}$ which is of the order $1/\lambda^\gamma$.

We thus decompose $I_n(X_1, X_2^\lambda)$ as the sum of two contributions

$$I_n(X_1, X_2^\lambda) = I'_n(X_1, X_2^\lambda) + I''_n(X_1, X_2^\lambda) \tag{12}$$

with

$$I'_n(X_1, X_2^\lambda) = \sum_{I\subset\{1\cdots n\}} \sum_{g\in G^n_{X_1,X_2^\lambda}} \int_{\mathscr{D}_1^{(\lambda)}} dY_I \int_{\mathscr{D}_2^{(\lambda)}} dY_J F_g(X_1X_2^\lambda Y) \tag{13}$$

where $Y_J = Y/Y_I$. In (13) the integration variables $Y_I=\{y_i \mid i\in I\}$ remain close to X_1 and the integration variables Y_J close to X_2^λ.

On the other hand $I''_n(X_1, X_2^\lambda)$ is characterized by the fact that there is at least one variable y_r in $\mathscr{D}_3^{(\lambda)}$, $r=1\cdots n$.

In the following subsections (i) and (ii) we treat separately the contributions I'_n and I''_n, showing that the first one gives the result of Lemma 1.

i) From (13) we have

$$\lambda^\gamma I'_n(X_1, X_2^\lambda) = \sum_{I\subset\{1\cdots n\}} \sum_{g\in G^n_{X_1,X_2^\lambda}} \times \int_{\mathbb{R}^\nu} dY_I \int_{\mathbb{R}^\nu} dY_J \chi_1^{(\lambda)}(Y_I)\chi_2^{(\lambda)}(Y_J^\lambda)\lambda^\gamma F_g(X_1Y_IX_2^\lambda Y_J^\lambda) \tag{14}$$

$$Y_J^\lambda = Y_J + \lambda\hat{u}$$

For a given $I\subset\{1,\ldots,n\}$ we consider the set $\mathscr{L}_{I,J}$ of lines linking one vertex in X_1Y_I to another vertex in $X_2^\lambda Y_J^\lambda$; clearly $|\mathscr{L}_{I,J}| = (|X_1|+|Y_I|)(|X_2|+|Y_J|)$. Any l in $\mathscr{L}_{I,J}$ is of the form $l=v-w-\lambda\hat{u}$ with $v\in X_1Y_I$, $w\in X_2Y_J$, and since $y_j\in\mathscr{D}_2^{(\lambda)}-\lambda\hat{u}$ we have $|v|\leq\lambda/4$, $|w|\leq\lambda/4$ as soon as $\lambda>\lambda_0$; therefore

$$|l|\geq\lambda-|v-w|\geq\frac{\lambda}{2}\geq|v-w|$$

and thus from (1b) and (2a)

$$\lim_{\lambda\to\infty}\lambda^\gamma[e^{-\beta\Phi(v-w-\lambda\hat{u})}-1] = -\beta\, d(\hat{u}) \tag{16}$$

$$|e^{-\beta\Phi(v-w-\lambda\hat{u})}-1|\leq\frac{C}{\left(\frac{\lambda}{2}\right)^\gamma+1}\leq\frac{C}{|v-w|^\gamma+1} \tag{17}$$

and

$$\lambda^\gamma\,|e^{-\beta\Phi(v-w-\lambda u)}-1|\leq M \tag{18}$$

For each $I\subset\{1,2,\ldots,n\}$ in the sum (14), we distinguish the contribution of the two following classes of graphs:

(a) g has only one line in $\mathscr{L}_{I,J}$
(b) g has at least two lines in $\mathscr{L}_{I,J}$

and we decompose I'_n accordingly, i.e.

$$I'_n(X_1, X_2^\lambda) = I_n'^{(a)}(X_1, X_2^\lambda) + I_n'^{(b)}(X_1, X_2^\lambda) \tag{20}$$

The class (a) is the set of graphs of the type $g = g_1 \cup g_2 \cup l$ with $g_1 \in G_{X_1}^{|I|}$, $g_2 \in G_{X_2^\lambda}^{|J|}$ and $l \in \mathscr{L}_{I,J}$.

For such graphs one has by (18)

$$\lambda^\gamma |F_g(X_1, X_2^\lambda, Y_I, Y_J^\lambda)| = \lambda^\gamma |F_{g_1}(X_1 Y_I)F_{g_2}(X_2 Y_J)[e^{-\beta\Phi(l)} - 1]|$$
$$\leq M |F_{g_1}(X_1 Y_I)F_{g_2}(X_2 Y_J)| \tag{21}$$

showing that the integrand in (14) is uniformly bounded by an integrable function of $Y_I Y_J$. Moreover, by (16), this integrand converges pointwise to

$$-\beta\, d(\hat{u})F_{g_1}(X_1 Y_I)F_{g_2}(X_2 Y_J)\chi_{\mathscr{D}}(Y_I)\chi_{\tilde{\mathscr{D}}}(Y_J)$$

Therefore by dominated convergence

$$\lim_{\lambda\to\infty} \lambda^\gamma I_n'^{(a)}(X_1, X_2^\lambda)$$
$$= -\beta d(\hat{u}) \sum_{I\subset\{1,\ldots,n\}} (|I| + |X_1|)(|J| + |X_2|)$$
$$\times \left(\sum_{g_1\in G_{X_1}^{|I|}} \int_{\mathscr{D}} dY_I F_{g_1}(X_1 Y_I)\right)\left(\sum_{g_2\in G_{X_2}^{|J|}} \int_{\tilde{\mathscr{D}}} dY_J F_{g_2}(X_2 Y_J)\right) \tag{22}$$

which is identical to the result of Lemma 1 when we note that the terms in the sum (22) depend only on $|I|$ and $|J|$.

We show now that the term $\lambda^\gamma I_n'^{(b)}(X_1, X_2^\lambda)$ converges to zero as $\lambda\to\infty$. For each graph of the class (b), we select a pair of points (x_1, x_2), $x_1 \in X_1$, $x_2 \in X_2$ and a chain linking x_1 to $x_2 + \lambda\hat{u}$. This chain has certainly a line l in $\mathscr{L}_{I,J}$ and therefore by (18)

$$\lambda^\gamma |F_g(X_1, X_2^\lambda, Y_I, Y_J^\lambda)| \leq M |F_{g/l}(X_1, X_2^\lambda, Y_I, Y_J^\lambda)| \tag{23}$$

The graph $\bar{g} = g/l$ is necessarily of one of the following type:

Γ_1: $\bar{g}$ belongs to $G^n_{X_1 X_2^\lambda}$

Γ_2: $\bar{g}$ is the union of two disjoint connected graphs g_1, g_2 $g_1 \in G^p_X$, $g_2 \in G^q_{X'}$ having vertices $XY^{(p)}$, $X'Y^{(q)}$, with $X \neq \varnothing$, $X' \neq \varnothing$, $X \cup X' = X_1 \cup X_2^\lambda$, $X \cap X' = \varnothing$, $p + q = n$

Since g is of the class (b), g/l has still a line in $\mathscr{L}_{I,J}$, and by (17), its corresponding Mayer factor vanishes as $\lambda\to\infty$. Thus, by (23), $F_g(X_1, X_2^\lambda, Y_I, Y_J^\lambda)$ tends pointwise to zero as $\lambda\to\infty$. Moreover, $F_{g/l}(X_1 X_2^\lambda Y_I Y_J^\lambda)$ depends on λ only through the Mayer factors involving lines in $\mathscr{L}_{I,J}$. Each of them can be majorized uniformly with respect to λ by the integrable function (17). Hence, $F_{g/l}(X_1 X_2^\lambda Y_I Y_J^\lambda)$ is majorized uniformly with respect to λ by a product of integrable factors associated to the lines of g/l. Because g/l is either connected, or the union of connected graphs with roots in $X_1 \cup X_2^\lambda$, this function is jointly integrable in $Y_I Y_J$, and $\lim_{\lambda\to\infty} \lambda^\gamma I_n'^{(b)}(X_1 X_2^\lambda) = 0$ follows by dominated convergence.

ii) We treat the second contribution in (12). Let

$$\mathscr{D}^{(\lambda)}_{[\alpha]} = \mathscr{D}^{(\lambda)}_{\alpha_1} \cup \cdots \cup \mathscr{D}^{(\lambda)}_{\alpha_n} \qquad [\alpha] = (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i = 1, 2, 3;$$

$$I''_n(X_1, X_2^\lambda) = \sum_{[\alpha]}{}'' \sum_{g\in G^n_{X_1,X_2^\lambda}} \int_{\mathscr{D}^{(\lambda)}_{[\alpha]}} dY^{(n)} F_g(X_1, X_2^\lambda, Y^{(n)}) \tag{24}$$

where $\sum''$ is the sum over those $[\alpha]$ such that there is at least some $\alpha_r \in [\alpha]$ with $\alpha_r = 3$.

Let us choose once for all a pair (x_1, x_2), $x_1 \in X_1$, $x_2 \in X_2$; for a given graph g there exists $(z_1, \ldots, z_p) \subset X_1 \cup X_2^\lambda \cup Y$ such that $(x_1 = z_1, z_2, \ldots, z_{p-1}, x_2 + \lambda\hat{u} = z_p)$ is a chain between x_1 and $x_2 + \lambda\hat{u}$. Since $\lambda - |x_1 - x_2| \leq \sum_{k=1}^{p-1} |l_k|$ for $\lambda > \lambda_0$, there exists some k, $0 \leq k \leq p-1$ such that

$$|l_k| = |z_{k+1} - z_k| \geq \frac{\lambda - |x_1 - x_2|}{p-1} \geq \frac{\lambda - |x_1 - x_2|}{n + |X_1| + |X_2| - 1} \tag{25}$$

and with (2a)

$$|\lambda^\gamma (e^{-\beta\phi(l_k)} - 1)| \leq \lambda^\gamma \frac{C}{\left(\dfrac{\lambda - |x_1 - x_2|}{n + |X_1| + |X_2| - 1}\right)^\gamma + 1} \leq C_0 (n + |X_1| + |X_2| - 1)^\gamma \tag{26}$$

with C_0 independent of x_1, x_2 and λ.

We write $(\mathbb{R}^\nu)^n = \bigcup_{k=1}^{p-1} \mathcal{P}_k$ as an union of (non-disjoint) domains $\mathcal{P}_k$ defined by

$$\mathcal{P}_k = \left\{ Y^{(n)} \,\middle|\, |l_k| = |z_{k+1} - z_k| \geq \frac{\lambda - |x_1 - x_2|}{n + |X_1| + |X_2| - 1} \right\} \tag{27}$$

Clearly (25) and (26) apply on $\mathcal{P}_k$ and hence for the given graph g:

$$\begin{aligned} \lambda^\gamma \left| \int_{\mathcal{D}_{[\alpha]}^{(\lambda)}} dY^{(n)} F_g(X_1, X_2^\lambda, Y) \right| &\leq \lambda^\gamma \sum_{k=1}^{p-1} \int_{\mathcal{D}_{[\alpha]}^{(\lambda)} \cap \mathcal{P}_k} dY^{(n)} |F_g(X_1 X_2^\lambda Y^{(n)})| \\ &\leq C_0 (n + |X_1| + |X_2| - 1)^\gamma \sum_{k=1}^{p-1} \int_{\mathcal{D}_{[\alpha]}^{(\lambda)}} dY^{(n)} |F_{g/l_k}(X_1 X_2^\lambda Y^{(n)})| \end{aligned} \tag{28}$$

It remains to show that each term of the sum (28) tends to zero as $\lambda \to \infty$. Notice that for each k, g/l_k is of the form Γ_1 or Γ_2; thus in both cases g/l_k has a subgraph $\tilde{g}$ with the following properties:

— $\tilde{g} = \bigcup_s t_s$ is the union of disjoint trees t_s.

— the t_s have vertices of the form (v_s, y_s) with $v_s \in X_1$ or $v_s \in X_2^\lambda$, $Y_s \subseteq Y^{(n)}$ and $\bigcup_s Y_s = Y^{(n)}$.

Then, we have

$$|F_{g/l_k}(X_1 X_2^\lambda Y^{(n)})| \leq A \prod_s |F_{ts}(v_s Y_s)| \tag{29}$$

where A depends only on g/l_k, and with (2b)

$$\int dY_s \, |F_{ts}(v_s Y_s)| = b^{|Y_s|} \tag{30}$$

We know that there is at least one integration variable y_r in (28) which lies in $\mathcal{D}_3^{(\lambda)}$. Call t_1 the tree to which y_r belongs.

Then with (29) and (30) we get for the graph g

$$\int_{\mathcal{D}_{[\alpha]}^{(\lambda)}} dY^{(n)} |F_{g/l_k}(X_1 X_2^\lambda Y^{(n)})| \leq A b^{n - |Y_1|} \int dY_1 \chi_3^{(\lambda)}(y_r) \, |F_{t_1}(v_1 Y_1)| \tag{31}$$

Let $(v_1 - y_{j_1}, y_{j_1} - y_{j_2}, \ldots, y_{j_q} - y_r)$ be the chain which connects v_1 to y_r in t_1 and perform all other integrations in t_1. We find after a change of variables

$$\int dY_1 \chi_3^{(\lambda)}(y_r)\, |F_{t_1}(v_1 Y_1)|$$

$$\leq b^{|Y_1|-q-1} \int dy_{j_1} \cdots dy_{j_q}\, dy_r \chi_3^{(\lambda)}(y_r)$$

$$\times |e^{-\beta\phi(y_r - y_{j_q})} - 1| \cdots |e^{-\beta\phi(y_{j_1} - y_{j_2})} - 1|\, |e^{-\beta\phi(v_1 - y_{j_1})} - 1| =$$

$$= b^{|Y_1|-q-1} \int dy_1 \cdots dy_q \left[\left[\int dy \chi_3^{(\lambda)}(y + v_1)\, |e^{-\beta\phi(y - y_q)} - 1|\right]\right.$$

$$\times |e^{-\beta\phi(y_1 - y_2)} - 1|\, |e^{-\beta\phi(y_1)} - 1| \tag{32}$$

Since either $v_1 = x_1 \in X_1$ or $v_1 = x_2 + \lambda\hat{u}$, $x_2 \in X_2$, we have $|y + v_1| \leq |y| + |x_1|$ or $|y + v_1 - \lambda\hat{u}| = |y + x_2|$ and in both cases $\lim_{\lambda\to\infty} \chi_3^{(1)}(y + v_1) = 0$. Therefore $\int dy \chi_3^{(\lambda)}(y - v_1)\, |e^{-\beta\phi(y - y_q)} - 1|$ also vanishes as $\lambda \to \infty$ and is bounded by b independently of y_q. Then the expression (32) vanishes as $\lambda \to \infty$ by dominated convergence, and this concludes the proof of Lemma 1.

3. Kirkwood–Salsburg equation and clustering property

In this section, we consider equilibrium states defined by means of the Kirkwood–Salsburg (K–S) equation. Assuming that the potential satisfies $\lim_{\lambda\to\infty} \lambda^\gamma \phi(\lambda\hat{u}) = d(\hat{u})$ with $\gamma > \nu$, we show that the correlation functions cannot have a clustering which is faster than the potential; furthermore, if $\lambda^\alpha \rho^T(X_1, X_2 + \lambda\hat{u})$ has a limit when $\lambda \to \infty$ with $\alpha < \gamma$ then this limit is necessarily zero whenever the state is unique at a given (T, z), i.e. the clustering cannot be slower than the potential. Therefore, if the correlation functions have a clustering with a power law, then necessarily the decay of the correlations must be the same as the decay of the potential. Finally if $\lambda^\gamma \rho^T(X_1, X_2 + \lambda\hat{u})$ has a limit when $\lambda \to \infty$, then

$$\lim_{\lambda\to\infty} \lambda^\gamma \rho^T(X_1, X_2 + \lambda\hat{u}) = -\beta\, d(\hat{u}) H^T(X_1) \tilde{H}^T(X_2)$$

where

$H^T(X_1)$ and $\tilde{H}^T(X_2)$ are given by equation (6)

Our starting point is the K–S equation [12]

$$\rho(xX) = z e^{-\beta W(x;X)} \int_{\mathcal{D}} dY K(x; Y) \rho(XY) \tag{33}$$

where:

$$\rho(\Phi) = 1$$

$$W(x; \Phi) = 0 \qquad W(x; X) = \sum_{x_i \in X} \Phi(x - x_i)$$

$$K(x; \Phi) = 1$$

$$K(x; y) = e^{-\beta\Phi(x-y)} - 1 \qquad K(x; Y) = \prod_{y_i \in Y} K(x; y_i)$$

and

$$\int_{\mathcal{D}} dY = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dy_1 \cdots dy_n$$

from which we obtain:

$$\rho(xX_1X_2) - \rho(X_1)\rho(xX_2) = ze^{-\beta W(x;X_1X_2)} \int_{\mathcal{D}} dYK(x;Y) \times [\rho(X_1X_2Y) - \rho(X_1)\rho(X_2Y)]$$

$$+ \rho(X_1) \cdot ze^{-\beta W(x;X_2)}[e^{-\beta W(x;X_1)} - 1] \int_{\mathcal{D}} dYK(x;Y)\rho(X_2Y) \quad (34)$$

Our main result is given by the following proposition:

Proposition 2. *If the two-body potential $\phi(x)$ satisfies the condition (1a, b) and if ρ, solution of the K.S.-equation, satisfies the following conditions:*

1) $\lim_{\lambda\to\infty} \rho(X+\lambda\hat{u}) = \tilde{\rho}(X)$ *for some $\hat{u}$ such that* $\lim_{\lambda\to\infty} \mathcal{D} - \lambda\hat{u} = \tilde{\mathcal{D}}$

2) $|\rho(X)| \leq \xi^{|X|}$ *for some* $\xi > 0$

3) $|\rho(X_1, X_2+\lambda\hat{u}) - \rho(X_1)\rho(X_2+\lambda\hat{u})| \leq \dfrac{\xi^{|X_1|+|X_2|}}{d(X_1;X_2)^\alpha + 1}$

where

$$d(X_1; X_2) = \min_{\substack{x_1\in X_1\\ x_2\in X_2}} |x_1 - x_2|$$

then

a) if the fluid is compressible, i.e. $\chi_T > 0$, the decay of the correlation functions cannot be faster than that of the potential

b) if the K–S equation has a unique solution, and if

4) $\lambda^\alpha[\rho(X_1, X_2+\lambda\hat{u}) - \rho(X_1)\rho(X_2+\lambda\hat{u})]$ has a limit when $\lambda\to\infty$ for some α, $\nu < \alpha \leq \gamma$,

then

$$\lim_{\lambda\to\infty} \lambda^\alpha[\rho(X_1, X_2+\lambda\hat{u}) - \rho(X_1)\rho(X_2+\lambda\hat{u})] = -\beta\, d(\hat{u})\delta_{\alpha,\gamma} H(X_1)\tilde{H}(X_2)$$

where $H(X_1)$ and $\tilde{H}(X_2)$ are given by equation (6′) and

$$\delta_{\alpha,\gamma} = \begin{cases} 1 & \alpha = \gamma \\ 0 & \alpha \neq \gamma \end{cases}$$

Remarks

1) It is known that for superstable potentials condition 2) is always satisfied [14]. Moreover, following the proof of lemma 2 (Appendix A) and using the

corresponding graphical structure, it is easy to see that

$$\lambda^{\gamma}\,|\rho(X_1, X_2+\lambda\hat{u})-\rho(X_1)\rho(X_2+\lambda\hat{u})|$$
$$\leq C_0\sum_n z^{n+|X_1|+|X_2|}(n+|X_1|+|X_2|-1)^{\gamma+1}(n+|X_1|)(n+|X_2|)\tilde{a}_n(X_1, X_2+\lambda\hat{u})$$

where $\tilde{a}_n(X)$ are the coefficient of the activity expansion

$$\rho(X)=\sum_n z^{n+|X|}\tilde{a}_n(X)$$

Therefore using the inequality (see [12], equation (4.27))

$$|\tilde{a}_n(X)|\leq A^{|X|-1}B^n$$

condition 3) is satisfied with $\alpha\leq\gamma$ in the domain of convergence of the activity expansion.

2) For superstable potentials, it can be shown that χ_T is strictly positif if the state is invariant under translations extending the arguments of Refs. [11, 15] from infinite spin systems to particle systems. We thus recover the result of the prop. 9 of [9] without the regularity assumption which was needed in this previous work.

3) We have taken $\alpha>\nu$ which corresponds to the *fluid phase* (outside the phase transition points) where one usually assumes that the correlation functions ρ^T are integrable.

To establish Proposition 2, we study $\lambda^{\alpha}[\rho(X_1, x^{\lambda}, X_2^{\lambda})-\rho(X_1)\rho(x^{\lambda}, X_2^{\lambda})]$ in the limit $\lambda\to\infty$ using equation (34).

Lemma 3. *If the conditions of* 1)–4) *of Proposition* 2 *are satisefied,*

1) $$\lim_{\lambda\to\infty}\lambda^{\alpha}\int_{\mathcal{D}} dY^{(n)}K(x^{\lambda}; Y)[\rho(X_1, X_2^{\lambda}, Y)-\rho(X_1)\rho(X_2^{\lambda}Y)]$$
$$=-\beta\int_{\mathcal{D}} dY^{(n)}K(x; Y)g_{\hat{u}}(X_1; X_2Y)$$
$$-\delta_{\alpha,\gamma}\beta\, d(\hat{u})n\int_{\mathcal{D}} dY[\rho(X_1y)$$
$$-\rho(X_1)\rho(y)]\int_{\hat{\mathcal{D}}} d\bar{Y}^{(n-1)}K(x; \bar{Y})\bar{\rho}(X_2, \bar{Y}) \quad (35)$$

where $g_{\hat{u}}$ *is defined by*

$$\lim_{\lambda\to\infty}\lambda^{\alpha}[\rho(X_1, X_2^{\lambda})-\rho(X_1)\rho(X_2^{\lambda})]=-\beta g_u(X_1; X_2) \quad (35')$$

2) $$\lambda^{\alpha}\int_{\mathcal{D}} dY^{(n)}\,|K(x^{\lambda}; Y)|\,|\rho(X_1, X_2^{\lambda}, Y)-\rho(X_1)\rho(X_2^{\lambda}, Y)|$$
$$\leq \xi^{|X_1|+|X_2|+n}n^{\gamma}C_1^n\,|X_1| \quad (36)$$

Note that

$$g_{\hat{u}}(X_1; \Phi)=g_u(\Phi; X_2)=0$$

As we shall see in the proof of this lemma, the conditions (1) and (4) are the only conditions required to obtain the limit (35); the conditions (2) and (3) are necessary to obtain the bounds (36) which are necessary to permute the limit $\lambda \to \infty$ and the sum over n in (34).

Proof. We consider only the case $\alpha = \gamma$; the result for $\alpha < \gamma$ follows immediately using the fact that if $\alpha < \gamma$, then

$$\lim_{\lambda \to \infty} \lambda^{\alpha} \Phi(\lambda \hat{u}) = 0$$

The idea of the proof is the following:

Let $\mathcal{B}^{\lambda}$ be the ball centered at the origin with radius $\lambda/8$; the K–S operator $K(x+\lambda\hat{u}; Y)$ can be non-zero in the limit $\lambda \to \infty$ only if all $y_i \in Y$ are outside $\mathcal{B}^{\lambda}$ and in this case $[\rho(X_1, X_2+\lambda\hat{u}, Y) - \rho(X_1)\rho(X_2+\lambda\hat{u}, Y)]$ will be of the order $1/\lambda^{\gamma} g_{\hat{u}}(X_1; X_2 Y)$; the integral over $[\mathcal{B}^{\lambda}]^c$ will thus give the first term of (35). If on the other hand one variable $y_i \in Y$ is in $\mathcal{B}^{\lambda}$ and $Y_i = Y/y_i \subset [\mathcal{B}^{\lambda}]^c$ then $K(x+\lambda\hat{u}; Y) = K(x+\lambda\hat{u}; y_i) \cdot K(x+\lambda\hat{u}; Y_i)$ will be of the order $-\beta\lambda^{-\gamma} d(\hat{u}) K(x+\lambda\hat{u}; Y_i)$ and $[\rho(X_1, X_2+\lambda\hat{u}, Y) - \rho(X_1)\rho(X_2+\lambda\hat{u}, Y)]$ will be of the order $[\rho(X_1 y_i) - \rho(X_1)\rho(y_i)] \cdot \rho(X_2+\lambda\hat{u}, Y_i)$; therefore, the integral over this domain will give the second term of (35) with the factor n coming from $i = 1, \ldots, n$. Finally for $Y^{(k)} \subset Y$, $k \geq 2$, in $\mathcal{B}^{\lambda}$ and $Y/Y^{(k)}$ outside the ball $\mathcal{B}^{\lambda}$, then $K(x+\lambda\hat{u}; Y) = K(x+\lambda\hat{u}; Y^{(k)}) \cdot K(x+\lambda\hat{u}; Y/Y^{(k)})$ will be of the order $(\lambda^{-\gamma})^k \cdot K(x+\lambda\hat{u}; Y/Y^{(k)})$ and these terms will not contribute. The technical details of the proof are given in Appendix B.

Lemma 4. Under the conditions of Propositions 2, the functions $g_{\hat{u}}(X_1; X_2)$ satisfy the following identity for $\alpha \leq \gamma$:

$$g_{\hat{u}}(X_1; xX_2) = \delta_{\alpha;\gamma}\, d(\hat{u}) H(X_1) \tilde{\rho}(xX_2) + z e^{-\beta W(x; X_2)} \int_{\tilde{\mathcal{D}}} dY K(x; Y) g_u(X_1; X_2 Y) \tag{37}$$

where

$$H(X_1) = |X_1|\, \rho(X_1) + \int_{\mathcal{D}} dy [\rho(yX_1) - \rho(y)\rho(X_1)] \tag{38}$$

Proof. Using (35′), Lemma 3 and the assumption $|\rho(X)| < \xi^{|X|}$, we have

$$\begin{aligned} \lim_{\lambda \to \infty} \lambda^{\alpha} \Big\{ & z e^{-\beta W(x+\lambda\hat{u}; X_1 X_2^{\lambda})} \int_{\mathcal{D}} dY K(x+\lambda\hat{u}; Y) [\rho(X_1 X_2^{\lambda} Y) - \rho(X_1)\rho(X_2^{\lambda} Y)] \\ &= -\beta z e^{-\beta W(x; X_2)} \int_{\tilde{\mathcal{D}}} dY K(x; Y) g_{\hat{u}}(X_1; X_2 Y) \\ &\quad - z\delta_{\alpha;\gamma} e^{-\beta W(x; X_2)} \beta\, d(\hat{u}) \int_{\mathcal{D}} dy [\rho(yX_1) - \rho(y)\rho(X_1)] \int_{\tilde{\mathcal{D}}} dY K(x; Y) \tilde{\rho}(X_2 Y) \end{aligned} \tag{39}$$

$$\begin{aligned} &= -\beta z e^{-\beta W(x; X_2)} \int_{\tilde{\mathcal{D}}} dY K(x; Y) g_{\hat{u}}(X_1; X_2 Y) \\ &\quad - \delta_{\alpha;\gamma} \beta\, d(\hat{u}) \int_{\mathcal{D}} dy [\rho(yX_1) - \rho(y)\rho(X_1)] \tilde{\rho}(xX_2) \end{aligned} \tag{40}$$

Equation (39) is justified because of the bounds (36) which implies that

$$\lambda^\alpha \int_{\mathscr{D}} dY\, |K(x+\lambda\hat{u};\, Y)|\, |\rho[X_1 X_2^\lambda Y) - \rho(X_1)\rho(X_2^\lambda Y)| \leq$$

$$\leq |X_1|\, \xi^{|X_1|+|X_2|} \sum_n \frac{1}{n!}\, n^\gamma (\xi C_1)^n < \infty$$

We can thus permute the limit $\lambda \to \infty$ with the sum in $\int dy = \sum_n 1/n! \int dY^{(n)}$ and apply Lemma 3 to each term. Equation (40) follows then from (39) using the K–S equation (33).

We thus obtain from equation (34)

$$\lim_{\lambda\to\infty} \lambda^\alpha [\rho(x+\lambda\hat{u}, X_1, X_2^\lambda) - \rho(X_1)\rho(x+\lambda u, X_2^\lambda)] = -\beta g_{\hat{u}}(X_1;\, xX_2)$$

$$= -\beta z e^{-\beta W(x;X_2)} \int_{\tilde{\mathscr{D}}} dY K(x;\, Y) g_{\hat{u}}(X_1;\, X_2 Y)$$

$$-\delta_{\alpha;\gamma} \beta d(\hat{u}) \int_{\mathscr{D}} dy[\rho(yX_1) - \rho(y)\rho(X_1)]\tilde{\rho}(xX_2)$$

$$+ \rho(X_1) z e^{-\beta W(x;X_2)} [-\delta_{\alpha;\gamma} \beta\, d(\hat{u})\, |X_1|] \int_{\tilde{\mathscr{D}}} dY K(x;\, Y)\tilde{\rho}(X_2 Y) \tag{41}$$

The result of Lemma 4 follows then from the fact that by equation (33) the last term in equation (41) is

$$-\delta_{\alpha;\gamma} \beta\, d(\hat{u})\, |X_1|\, \rho(X_1)\tilde{\rho}(xX_2)$$

Our next goal is to show that (37) has always the solution

$$g_{\hat{u}}^0(X_1;\, X_2) = d(\hat{u}) H(X_1) \tilde{H}(X_2) \tag{42}$$

and furthermore this solution is unique whenever the K–S equation has a unique solution, i.e. whenever the equilibrium state is unique.

Lemma 5. *If the K–S equation has a unique solution, then* (37) *has at most one solution. Furthermore, this solution is zero if* $\alpha < \gamma$.

Proof. Let $\mathscr{K}$ be the K–S operator defined on the usual space $\mathscr{E} = \{f = (f_0, f_1(x), f_2(x_1, x_2), \ldots)\}$ [12] defined by:

$$(\mathscr{K}f)_0 = 0$$

$$(\mathscr{K}f)(xX) = z e^{-\beta W(x;X)} \int dY K(x;\, Y) f(XY) \tag{43}$$

Then the K–S equation is simply

$$\rho = \delta_{|X|,0} + \mathscr{K}\rho \tag{44}$$

while equation (46) with $G(X) = g_{\hat{u}}(X_1;\, X)$ becomes

$$G_0 = 0$$

$$G(xX) = \delta_{\alpha;\gamma}\, d(\hat{u}) H(X_1) \tilde{\rho}(xX) + (\mathscr{K}G)(xX) \tag{45}$$

Since the K–S equation has a unique solution if and only if the operator $\mathcal{K}$ does not have the eigenvalue 1, we see that under the same condition, there is a unique G satisfying (45) and this solution is zero if $\alpha<\gamma$.

Lemma 6. *If* $\alpha=\gamma$ the function $g^0_{\hat{u}}(X_1; X_2)=d(\hat{u})H(X_1)\tilde{H}(X_2)$ is always solution of equation (37).

Proof. Let us first compute:

$$I=ze^{-\beta W(x;X)}\int_{\tilde{\mathcal{D}}} dYK(x;\,Y)\tilde{H}(XY).$$

$$I=ze^{-\beta W(x;X)}\int_{\tilde{\mathcal{D}}} dYK(x;\,Y)$$
$$\times\left\{(|X|+|Y|)\tilde{\rho}(XY)+\int_{\tilde{\mathcal{D}}} dy[\tilde{\rho}(yXY)-\tilde{\rho}(y)\tilde{\rho}(XY)]\right\}$$

Using (33)

$$I=|X|\,\tilde{\rho}(xX)+ze^{-\beta W(x;X)}\int_{\tilde{\mathcal{D}}} dyK(x;\,y)\int_{\tilde{\mathcal{D}}} dYK(x;\,Y)\tilde{\rho}(XyY)$$
$$+ze^{-\beta W(x;X)}\int_{\tilde{\mathcal{D}}} dy\int_{\tilde{\mathcal{D}}} dYK(x;\,Y)[\tilde{\rho}(yXY)-\tilde{\rho}(y)\tilde{\rho}(XY)]$$
$$=|X|\,\tilde{\rho}(xX)+\int_{\tilde{\mathcal{D}}} dye^{\beta\Phi(x-y)}\tilde{\rho}(xyX)$$
$$+\int_{\tilde{\mathcal{D}}} dy[e^{\beta\Phi(x-y)}\tilde{\rho}(xyX)-\tilde{\rho}(y)\tilde{\rho}(xX)]$$
$$=|X|\,\tilde{\rho}(xX)+\int_{\tilde{\mathcal{D}}} dy[\tilde{\rho}(xyX)-\tilde{\rho}(y)\tilde{\rho}(xX)]$$

we thus have:

$$d(\hat{u})\tilde{\rho}(xX_2)H(X_1)+ze^{-\beta W(x;X_2)}\,d(\hat{u})\int_{\tilde{\mathcal{D}}} dYK(x;\,Y)H(X_1)\tilde{H}(X_2Y)$$
$$=d(\hat{u})H(X_1)\left\{\tilde{\rho}(xX_2)+|X_2|\,\tilde{\rho}(xX_2)+\int_{\tilde{\mathcal{D}}} dy[\tilde{\rho}(yxX_2)-\tilde{\rho}(y)\rho(xX_2)]\right\}$$
$$=d(\hat{u})H(X_1)\tilde{H}(xX_2)=g^0_{\hat{u}}(X_1;\,xX_2)$$

which concludes the proof.

Proof of Proposition 2

a) If the clustering is faster than the decay of the potential, i.e.

$$\lim_{\lambda\to\infty}\lambda^{\gamma}[\rho(X_1,X_2^{\lambda})-\rho(X_1)\rho(X_2^{\lambda})]=0$$

then Lemmas 3 and 4 apply with $\alpha = \gamma$ and $g_{\hat{u}} = 0$. Hence equation (37) implies $H(X) = 0$, i.e. the following sum rule

$$|X|\, \rho(X) + \int_{\mathcal{D}} dy[\rho(yX) - \rho(y)\rho(X)] = 0 \tag{46}$$

and in particular that for all x in $\mathcal{D}$

$$\rho(x) + \int_{\mathcal{D}} dy \rho^T(x, y) = 0 \tag{47}$$

Thus the compressibility χ_T defined by

$$\begin{cases} \chi_T = \dfrac{\beta}{\rho^2} \lim\limits_{\Lambda \to \mathcal{D}} \dfrac{1}{|\Lambda|} \displaystyle\int_\Lambda dx \left[\rho(x) + \int_{\mathcal{D}} dy \rho^T(xy)\right] \\ \rho = \lim\limits_{\Lambda \to \infty} \dfrac{1}{|\Lambda|} \displaystyle\int_\Lambda dx \rho(x) \end{cases}$$

is identically zero, which contradicts the assumption of compressible fluids.

b) follows immediately from Lemmas 5 and 6.

Under the conditions of Proposition 2, we find that the direct correlation function $c(x)$ defined by the Ornstein–Zernike relation

$$\rho^T(x) = \rho^2 c(x) + \rho \int c(x-y)\rho^T(y)\, dy \tag{48}$$

has the well known asymptotic behaviour $c(x) \sim -\beta\Phi(x)$ as $|x| \to \infty$.

Precisely we have

Proposition 3. *In an homogeneous state and under the conditions of the Prop. 2*

a) *The decay of $c(x)$ cannot be faster than that of the potential*

b) *If $\lambda^\alpha c(x + \lambda\hat{u})$ has a limit as $\lambda \to \infty$ for some α, $\nu < \alpha \leq \gamma$, then*

$$\lim_{\lambda \to \infty} \lambda^\alpha c(x + \lambda\hat{u}) = -\delta_{\alpha\gamma}\beta\, d(\hat{u})$$

Proof. Since $c(x)$ is integrable, (48) implies

$$H\left(1 - \rho \int c(x)\, dx\right) = \rho > 0 \tag{49}$$

with

$$H = \rho + \int dy \rho^T(y)$$

Hence $H \neq 0$ and $\rho \int c(x)\, dx \neq 1$.

a) Multiplying (48) by λ^γ, taking the limit $\lambda \to \infty$ and using

$$\lim_{\lambda \to \infty} \lambda^\gamma \rho^T(x + \lambda\hat{u}) = -\beta\, d(\hat{u}) H^2$$

gives

$$-\beta\, d(\hat{u})H^2 = -\beta\, d(\hat{u})H^2\rho \int c(x)\, dx$$

This implies either $H = 0$ or $\rho\int c(x)\, dx = 1$, which contradicts (49).

b) If $\alpha < \gamma$, multiplying (48) by λ^α and taking the limit yields, with $c(\hat{u}) = \lim_{\lambda\to\infty} \lambda^\alpha c(x+\lambda\hat{u})$

$$0 = \rho^2 c(\hat{u}) + \rho c(\hat{u}) \int dy \rho^T(y) = \rho c(\hat{u}) H$$

and hence $c(\hat{u}) = 0$.

Taking now $\alpha = \gamma$, the same limit gives

$$-\beta\, d(\hat{u})H^2 = \rho^2 c(\hat{u}) + \rho c(\hat{u}) \int dy \rho^T(y) - \beta\, d(\hat{u})H^2\rho \int c(x)\, dx$$

and thus, with (49) $c(\hat{u}) = -\beta\, d(\hat{u})$.

Concluding remarks

1) It is of interest to compare the situation of Proposition 2 with known results on charged fluids. Whereas the sum rule (46) cannot be valid for compressible neutral fluids with integrable potentials, they are true identities for the one component Coulomb system (Jellium). In the latter case, they express the typical screening properties of the system's charges in the fluid. This is in close relation with the fact that the square of the particle number fluctuations are extensive in a neutral fluid (outside of critical points), but the charge fluctuations are always abnormal [9, 19, 20]. Sum rules analogous to (46) (with summations on charges) hold in several component Coulomb systems ($\gamma = \nu - 2$) and in general for long range potentials $\upsilon - 2 \leqslant \gamma \leqslant \nu - 1$ whenever the state has an integrable clustering [9, 16, 20].

2) In an equilibrium state of a semi infinite system bounded by a plane wall, i.e. $\mathcal{D} = \{x \in \mathbb{R}^\nu;\, x^1 \geqslant 0\}$, and translation invariant parallel to the wall, we have

$$\lim_{\lambda\to\infty} \lambda^\gamma \rho^T(x_1, x_2 + \lambda\hat{u}) = -d(\hat{u})\rho^2\chi_T \times \left[\rho(x_1) + \int_{\mathcal{D}} dy \rho^T(x_1, y)\right]$$

when $\hat{u}$ is perpendicular to the wall and

$$\lim_{\lambda\to\infty} \lambda^\gamma \rho^T(x_1, x_2 + \lambda\hat{u}) = -\beta\, d(\hat{u})\left(\rho(x_1) + \int_{\mathcal{D}} dy \rho^T(x_1, y)\right) \cdot \left(\rho(x_2) + \int_{\mathcal{D}} dy \rho^T(x_2, y)\right)$$

when $\hat{u}$ is parallel to the wall, showing that the decay parallel to the wall is the same as the decay in the bulk. This result has to be compared to the following situations:

(i) the case of an inhomogeneous state in $\mathbb{R}^\nu$ with a planar interface, where the decay parallel to this interface cannot be integrable [17, 21, 22].

(ii) the case of a semi infinite Coulomb system where the decay parallel to the wall is known to be weaker ($|x|^{-\nu}$) then the decay in the bulk even at low density [18, 23].

Appendix A

Proof of Lemma 2

Let us consider a graph $g \in G^n_{X_1X_2^\lambda}$. By the same argument used in the derivation of equation (28) (the domain of integration played there no role), we can write

$$\lambda^\gamma \int_{\mathcal{D}} dY^{(n)} |F_g(X_1X_2^\lambda Y)| \le c_0(n+|X_1|+|X_2|-1)^\gamma \sum_{k=1}^{p-1} \int_{\mathcal{D}} dY^{(n)} F_{g/l_k}(X_1X_2^\lambda Y) \tag{A1}$$

with g/l_k belonging to the class Γ_1, or Γ_2. Thus:

$$\lambda^\gamma |I_n(X_1, X_2^\lambda)| \le c_0(n+|X_1|+|X_2|-1)^\gamma \sum_{g\in G^n_{X_1,X_2^\lambda}} \sum_{k=1}^{p-1} \int_{\mathcal{D}} dY^{(n)} F_{g/l_k}(X_1X_2^\lambda Y) \tag{A2}$$

If $g/l_k \in \Gamma_1$, $\int dY^{(n)} F_{g/l_k}(X_1X_2^\lambda Y)$ occurs exactly once in the development of $\rho^T(X_1, X_2^\lambda)$. If $g/l_k \in \Gamma_2$:

$$\int_{\mathcal{D}} dY^{(n)} F_{g/l_k}(X_1X_2^\lambda Y) = \int_{\mathcal{D}} dY^{(p)} F_{g_1}(X'Y^{(p)}) \int_{\mathcal{D}} dY^{(q)} F_{g_2}(X''Y^{(q)})$$

$g_1 \in G^p_{X'}$, $g_2 \in G^q_{X''}$, $p+q=n$, occurs exactly once in the expansion of $\rho^T(X')\rho^T(X'')$ for some $X' \ne \varnothing$, $X'' \ne \varnothing$ with $X' \cup X'' = X_1 \cup X_2^\lambda$

However, each of these contributions can occur several times in (A2). If $g/l \in \Gamma_2$, one has to take in account the different ways of dividing $Y^{(n)}$ in two sets $Y^{(p)}$ and $Y^{(q)}$ with $p+q=n$. Furthermore one has to consider that, by removing a line l in a chain between X_1 and X_2^λ in a graph $g \in G^n_{X_1X_2^\lambda}$, one gets the same graph g/l a number of times which is bounded by $(n+|X_1|)(n+|X_2|)$. Thus,

$$\lambda^\gamma \frac{1}{n!} |I_n(X_1X_2^\lambda)| \le M(n+|X_1|+|X_2|-1)^{\gamma+1}(n+|X_1|)(n+|X_2|)a_n(X_1X_2^\lambda) \tag{A3}$$

where $\{a_n(X_1X_2^\lambda)\}$, $n \ge 0$, are the coefficients of the low activity expansion of

$$\rho^T(X_1, X_2^\lambda) + \sum_{\substack{X'\cup X''=X_1X_2^\lambda \\ X'\cap X''=\varnothing \\ X'\ne\varnothing X''\ne\varnothing}} \rho^T(X')\rho^T(X'') = \sum_{n=0}^{\infty} z^n a_n(X_1X_2^\lambda) \tag{A4}$$

The convergence of the Mayer series implies that $|a_n(X_1X_2^\lambda)| \le b_n$ with b_n independent of λ and $\sum_{n=0}^\infty z^n b_n < \infty$, for $0 \le z < z_0(\beta)$. This observation and (A3) immediately imply Lemma 2.

Proof of the Corollary of Proposition 1

Since

$$\lambda^{\gamma}[\rho(X_1, X_2^{\lambda}) - \rho(X_1)\rho(X_2^{\lambda})] = \sum_{\substack{\emptyset \neq \bar{X}_1 \subset X_1 \\ \emptyset \neq \bar{X}_2 \subset X_2}} \rho^T(\bar{X}_1 \bar{X}_2^{\lambda})\rho^T(X_1/\bar{X}_1, (X_2/\bar{X}_2)^{\lambda})$$

we have:

$$\lim_{\lambda \to \infty} \lambda^{\gamma}[\rho(X_1 X_2^{\lambda}) - \rho(X_1)\rho(X_2^{\lambda})] = -\beta\, d(\hat{u}) \Big[\sum_{\emptyset \neq \bar{X}_1 \subset X_1} H^T(\bar{X}_1)\rho(X_1/\bar{X}_1) \Big] \Big[\sum_{\emptyset \neq \bar{X}_2 \subset X_2} \tilde{H}^T(\bar{X}_2)\tilde{\rho}(X_2/\bar{X}_2) \Big]$$

But

$$\begin{aligned}
&\sum_{\emptyset \neq \bar{X}_1 \subset X_1} H^T(\bar{X}_1)\rho(X_1/\bar{X}_1) \\
&= \sum_{\emptyset \neq \bar{X}_1 \subset X_1} \Big[|\bar{X}_1|\, \rho^T(\bar{X}_1)\rho(X_1/\bar{X}_1) + \int dy \rho^T(y\bar{X}_1)\rho(X_1/\bar{X}_1) \Big] \\
&= \sum_{x \in X_1} \sum_{X \subset X_1/x} \rho^T(x\tilde{X})\rho(X_1/\tilde{X}x) + \int dy \sum_{\emptyset \neq \bar{X}_1 \subset X_1} \rho^T(y\bar{X}_1)\rho(X_1/\bar{X}_1) \\
&= \sum_{x \in X_1} [\rho(x_1) - \rho(x)\rho(X_1/x) + \rho(x)\rho(X_1/x)] + \int dy(\rho(yX_1) - \rho(y)\rho(X_1)) \\
&= |X_1|\, \rho(X_1) + \int_{\mathscr{D}} dy[\rho(yX_1) - \rho(y)\rho(X_1)] = H(X_1)
\end{aligned}$$

and hence the result of the corollary.

Appendix B

Proof of Lemma 3

With $\mathscr{B}^{\lambda} = \{y \mid |y| \leq \lambda/8\}$ we decompose the domain of integration $\mathscr{D}^n$ of the $Y^{(n)}$ into $(n+2)$ disjoint domains:

$$\mathscr{D}^n = \mathscr{D}_{\emptyset}^{(\lambda)} \bigcup_{i=1}^{n} \mathscr{D}_i^{(\lambda)} \bigcup \mathscr{D}_{n+1}^{(\lambda)}$$

where

$$\mathscr{D}_{\emptyset}^{(\lambda)} = \Big\{ Y^{(n)} \subset \mathscr{D}^{(n)}; |y_j| > \frac{\lambda}{8} \quad \text{for all} \quad j = 1 \cdots n \Big\}$$

$$\mathscr{D}_i^{(\lambda)} = \Big\{ Y^{(n)} \subset \mathscr{D}^{(n)}; |y_i| \leq \frac{\lambda}{8}, |y_j| > \frac{\lambda}{8} \quad \text{for all} \quad j \neq i \Big\}$$

We then have:

$$\int_{\mathscr{D}^n} dY^{(n)} = \int_{\mathscr{D}_{\emptyset}^{(\lambda)}} dY^{(n)} + \sum_{i=1}^{n} \int_{\mathscr{D}_i^{(\lambda)}} dY^{(n)} + \int_{\mathscr{D}_{n+1}^{(\lambda)}} dY^{(n)}$$

In the following, we take

$$\lambda > \lambda_0 = 32 \cdot \sup_{z \in xX_1X_2} |z|$$

1) Let us first consider the contribution due to $\mathscr{D}_{\Phi}^{(\lambda)}$. Since $|y_j| > \lambda/8$ for all $j = 1, \ldots, n$, we have $d(X_1; X_2^\lambda Y) \geq \lambda/8 - \lambda/32 > \lambda/11$;

$$\lambda^\gamma \int_{\mathscr{D}_{\Phi}^{(\lambda)}} dY^{(n)} K(x + \lambda\hat{u}; Y)[\rho(X_1, X_2^\lambda Y) - \rho(X_1)\rho(X_2^\lambda Y)]$$

$$= \int_{\mathbb{R}^\nu} dY^{(n)} K(x; Y)\chi_{\mathscr{D}-\lambda\hat{u}}(Y) \cdot \prod_{j=1}^{n} \theta\left[|y_j + \lambda\hat{u}| - \frac{\lambda}{8}\right].$$

$$\cdot \lambda^\gamma \cdot [\rho(X_1, X_2^\lambda, Y^\lambda) - \rho(X_1)\rho(X_2^\lambda Y^\lambda)]$$

with

$$\theta[x] = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0 \end{cases}$$

Using the condition 3 of Proposition 2 and the fact that $d(X_1; X_2^\lambda Y^\lambda) > \lambda/11$, the integrand is thus uniformly bounded by

$$(11)^\gamma \xi^{|X_1| + |X_2| + n} |K(x; Y)|$$

which is integrable, and it converges point wise to

$$-\beta g_{\hat{u}}(X_1; X_2 Y) \cdot K(x; Y) \cdot \chi_{\tilde{\mathscr{D}}}(Y)$$

Therefore by dominated convergence this contribution yields in the limit $\lambda \to \infty$:

$$-\beta \int_{\tilde{\mathscr{D}}} dY^{(n)} K(x; Y) g_{\hat{u}}(X_1; X_2 Y)$$

Furthermore for all $\lambda > \lambda_0$ the integral is bounded by

$$(11)^\gamma \xi^{|X_1| + |X_2| + n} b^n$$

with b defined by (2b)

2) Let us consider then the contribution due to $\mathscr{D}_1^{(\lambda)}$, i.e. $|y_1| \leq \lambda/8$; by symmetry we shall obtain the same result for all $\mathscr{D}_i^{(\lambda)}$. In this case we have to distinguish the two situations, y_1 close to X_1 and y_1 far from X_1; therefore we shall decompose the integral over y_1 into two parts $|y_1| \leq \lambda/16$ and $|y_1| > \lambda/16$. Writing $\bar{Y} = Y/y_1$ i.e. $Y = y_1\bar{Y}$, we have:

$$\lambda^\gamma \int_{\mathscr{D}_1^{(\lambda)}} dY^{(n)} K(x + \lambda\hat{u}; Y)[\rho(X_1, X_2^\lambda, Y) - \rho(X_1)\rho(X_2^\lambda Y)]$$

$$= \int_{|y_1| \leq \lambda/16} dy_1 K(x + \lambda\hat{u}; y_1) \int_{\mathbb{R}^\nu} d\bar{Y} K(x; \bar{Y}) \chi_{\mathscr{D}}(y_1) \chi_{\mathscr{D}-\lambda\hat{u}}(\bar{Y}) \prod_{j=1}^{n-1} \theta\left[|\bar{y}_j + \lambda\hat{u}| = \frac{\lambda}{8}\right]$$

$$\times \{\lambda^\gamma[\rho(X_1 y_1 X_2^\lambda \bar{Y}^\lambda) - \rho(X_1 y_1)\rho(X_2^\lambda \bar{Y}^\lambda)] \tag{I}$$

$$+ \lambda^\gamma[\rho(X_1 y_1) - \rho(X_1)\rho(y_1)]\rho(X_2^\lambda \bar{Y}^\lambda) \tag{II}$$

$$+ \lambda^\gamma[\rho(y_1)\rho(X_2^\lambda \bar{Y}^\lambda) - \rho(y_1 X_2^\lambda \bar{Y}^\lambda)]\rho(X_1)\} \tag{III}$$

$$+ \int_{\lambda/16 \leq |y_1 + \lambda\hat{u}| \leq \lambda/8} dy_1 K(x; y_1) \int_{\mathbb{R}^\nu} d\bar{Y} K(x; \bar{Y}) \chi_{\mathscr{D}-\lambda\hat{u}}(y_1\bar{Y}) \prod_{j=1}^{n-1} \theta\left[|\bar{y}_j + \lambda\hat{u}| - \frac{\lambda}{8}\right]$$

$$\times \lambda^\gamma[\rho(X_1 X_2^\lambda Y^\lambda) \times \rho(X_1)\rho(X_2^\lambda Y^\lambda)] \tag{IV}$$

Since for $|y_1| \leq \lambda/16$, $d(X_1 y_1; X_2^\lambda \bar{Y}^\lambda) \geq \lambda/16$, and $|x + \lambda\hat{u} - y_1| \geq \lambda/4 \geq |y_1|$, Equation (2a) yields

$$K(x+\lambda\hat{u}; y_1) \leq \frac{C}{|y_1|^\gamma + 1}$$

Therefore conditions 2 and 3 of Propositon 2 imply that the integrand (I) + (III) is uniformly bounded by the integrable function

$$2.16^\gamma \xi^{|X_1|+|X_2|+n} \, |K(x; \bar{Y})| \frac{C}{|y_1|^\gamma + 1}$$

and converges pointwise to zero because of $K(x+\lambda\hat{u}; y_1)$. The contribution (I) + (III) will thus give zero in the limit $\lambda \to \infty$, and this contribution is bounded by

$$2\,.\,16^\gamma \xi^{|X_1|+|X_2|+n} b^{n-1} C_2$$

$$C_2 = C \int dy \frac{1}{|y|^\gamma + 1}$$

On the other hand, since $|x + \lambda\hat{u} - y_1| \geq \lambda/4$ implies $\lambda^\gamma \, |K(x+\lambda\hat{u}; y_1) \leq M$, the integrand (II) is uniformly bounded by

$$M\,|K(x; \bar{Y})| \, \xi^{|X_2|+n-1} \, |\rho(X_1 y_1) - \rho(X_1)\rho(y_1)|$$

which is integrable, and converges pointwise to

$$-\beta\, d(\hat{u}) K(x; \bar{Y}) \chi_{\mathscr{D}}(y_1) \chi_{\bar{\mathscr{D}}}(\bar{Y}) \tilde{\rho}(X_2 \bar{Y})[\rho(X_1 y_1) - \rho(X_1)\rho(y_1)]$$

Therefore (II) will give in the limit $\lambda \to \infty$

$$-\beta\, d(\hat{u}) \int_{\mathscr{D}} dy_1 [\rho(X_1 y_1) - \rho(X_1)\rho(y_1)] \int_{\bar{\mathscr{D}}} d\bar{Y} K(x; \bar{Y}) \tilde{\rho}(X_2 \bar{Y})$$

Furthermore the integral of (II) is uniformly bounded by:

$$\beta\, |d(\hat{u})| \, b^{n-1} \xi^{|X_1|+|X_2|+n} C_3 \, |X_1|$$

where $\int dy \, |\rho(yX_1) - \rho(y)\rho(X_1)| \leq \xi^{|X_1|+1} \, |X_1| \, C_3$ This last inequality follows from the following lemma:

Lemma 7. *If the correlation functions satisfy the condition*

$$|\rho(X, Y) - \rho(X)\rho(Y)| \leq \frac{\xi^{|X|+|Y|}}{d(X; Y)^\alpha + 1} \qquad \alpha > \nu$$

then

$$\int_{\mathscr{D}} dy \, |\rho(X, Y+y) - \rho(X)\rho(Y+y)| \leq \xi^{|X|+|Y|} \, |X| \cdot |Y| \cdot C_3$$

Proof. Let us decompose $\mathscr{D}$ as union of domains $\mathscr{D}_{ij}$

$$\mathscr{D} = \bigcup_{i=1}^{|X|} \bigcup_{j=1}^{|Y|} \mathscr{D}_{ij}$$

$$\mathscr{D}_{ij} = \{y \mid d(X; y+Y) = |x_i - y - y_j|\}$$

$$\int_{\mathscr{D}} dy\, |\rho(X, Y+y) - \rho(X)\rho(Y+y)|$$

$$\leqslant \sum_{i=1}^{|X|} \sum_{j=1}^{|Y|} \int_{\mathscr{D}_{ij}} dy \frac{\xi^{|X|+|Y|}}{|x_i - y_j - y|^\alpha + 1} \leqslant |X| \cdot |Y|\, \xi^{|X|+|Y|} \int_{\mathbb{R}^\nu} \frac{dy}{|y|^\alpha + 1}$$

$$\leqslant |X|\,|Y|\, \xi^{|X|+|Y|} C_3$$

Finally, for the contribution (IV) $d(X_1; X_2^\lambda Y^\lambda) \geqslant \lambda/16 - \lambda/32 = \lambda/32$ and $|x - y_1| \geqslant \lambda/2$; using the same argument as above, this contribution will be zero in the limit $\lambda \to \infty$ and the integral is bounded by

$$32^\gamma \xi^{|X_1|+|X_2|+n} b^n$$

In conclusion the contribution due to $\mathscr{D}_\varnothing^{(\lambda)} \cup_{i=1}^n \mathscr{D}_i^{(\lambda)}$ gives the desired result in the limit $\lambda \to \infty$ and for all $\lambda > \lambda_0$ this contribution is bounded by

$$Anb^n \xi^{|X_1|+|X_2|+n} |X_1|$$

3) It remains to show that the contribution $\mathscr{D}_{n+1}^{(\lambda)}$ is zero in the limit $\lambda \to \infty$ and satisfies a bound of the type (36).

Let us write $\mathscr{D}_{n+1}^{(\lambda)}$ as union of non-disjoint sets in the following manner:

$$\mathscr{D}_{n+1}^{(\lambda)} = \bigcup_{\substack{I \subset \{1,\ldots,n\} \\ |I| \geqslant 2}} \mathscr{D}_I^{(\lambda)}$$

with

$$\mathscr{D}_I^{(\lambda)} = \bigcup_{k=0}^{|I|-2} \mathscr{D}_{I,k}^{(\lambda)}$$

$$\mathscr{D}_{I,k}^{(\lambda)} = \{Y^{(n)} \subset \mathscr{D}^n;\ Y_I \subset \mathscr{B}_k^\lambda,\ Y_J \cap \mathscr{B}_{k+1}^\lambda = \phi\}$$

$$Y_I = \{y_i;\ i \in I\} \qquad Y_J = Y^{(n)}/Y_I$$

$$\mathscr{B}_k^\lambda = \left\{y;\ |y| \leqslant \frac{\lambda}{8}\left(1 + \frac{k}{n}\right)\right\}$$

i.e. $\mathscr{D}_I^{(\lambda)}$ is a set of configuration, with Y_I inside the ball of radius $\lambda/8(1 + |I|/n)$ separated from Y_J by a distance $\lambda/8n$.

$$\lambda^\gamma \int_{\mathscr{D}_{I,k}^{(\lambda)}} dYK(x + \lambda\hat{u}; Y)[\rho(X_1 X_2^\lambda Y) - \rho(X_1)\rho(X_2^\lambda Y)]$$

$$= \lambda^\gamma \int_{|y_i| \leqslant \lambda/8(1+k/n)} dY_I K(x + \lambda\hat{u}; Y_I) \int dY_J K(x; Y_J)$$

$$\cdot \prod_{j \in J} \theta\left[|y_j + \lambda\hat{u}| - \frac{\lambda}{8}\left(1 + \frac{k+1}{n}\right)\right]$$

$$\times \{[\rho(X_1 Y_I X_2^\lambda Y_J^\lambda) - \rho(X_1 Y_I)\rho(X_2^\lambda Y_J^\lambda)] + \qquad \text{(I)}$$

$$+ [\rho(X_1 Y_I) - \rho(X_1)\rho(Y_I)]\rho(X_2^\lambda Y_J^\lambda)] + \qquad \text{(II)}$$

$$+ [\rho(Y_I)\rho(X_2^\lambda Y_I^\lambda) - \rho(Y_I X_2^\lambda X_J^\lambda)]\rho(X_1)\} \qquad \text{(III)}$$

Since

$$d(X_1 Y_I; X_2^\lambda Y_J^\lambda) \geq \frac{\lambda}{8n}$$

and since $|x+\lambda\hat{u}-y_i| \geq |y_i|$ implies

$$|K(x+\lambda\hat{u}; y_i)| \leq \frac{C}{|y_i|^\gamma + 1}$$

it follows that the integrand (I)+(III) is uniformly bounded by the integrable function,

$$2 \,.\, 8^\gamma n^\gamma \xi^{|X_1|+|X_2|+n} |K(x; Y_J)| \prod_{i \in I} \frac{C}{|y_i|^\gamma + 1}$$

and it converges pointwise to zero because of $K(x+\lambda\hat{u}; Y_I)$.

Therefore (I)+(III) gives zero in the limit $\lambda \to \infty$ and for all $\lambda > \lambda_0$ this contribution is bounded by

$$2 \cdot 8^\gamma \cdot n^\gamma \xi^{|X_1|+|X_2|+n} b^{n-|I|} C_2^{|I|}$$

To discuss the contribution (II) we choose one variable y in Y_I and introduce $|I|-1$ new integration variables $\eta = (\eta_1, \ldots, \eta_{|I|-1})$ with $Y_I/y = \eta + y$. The contribution (II) is majorized by

$$\lambda^\gamma \int_{|y| \leq \lambda/8(1+k/n)} dy\, |K(x+\lambda\hat{u}; y)| \int_{|y+\eta_i| \leq \lambda/8(1+k/n)} d\eta_1 \cdots d\eta_{|I|-1} \, |K(x+\lambda\hat{u}; \eta+y)| \int_{\mathbb{R}^\nu} dY_J\, |K(x; Y_J)|$$

$$|\rho(X_1, y, (\eta+y)) - \rho(X_1)\rho(y, (\eta+y))| \, \rho(X_2^\lambda Y_J^\lambda). \qquad \text{(II}')$$

Since

$$|\eta_i| \leq \frac{\lambda}{2} \quad \text{and} \quad |x+\lambda\hat{u}-y| \geq \frac{3}{4}\lambda - |x|$$

we have

$$|x+\lambda\hat{u}-y-\eta_i| \geq \frac{3}{4}\lambda - |x| - \frac{\lambda}{2} \geq \frac{1}{4}|\eta_i|$$

which implies that

$$|K(x+\lambda\hat{u}; y+\eta_i)| \leq \frac{C}{(\frac{1}{4}|\eta_i|)^\gamma + 1}$$

Therefore the integrand (II′) is bounded by

$$C_3 \prod_{i=1}^{|I|-1} \left(\frac{C}{\frac{1}{4}|\eta_i|^\gamma + 1} \right) K(x; Y_J) \, |\rho(X_1, y, \eta+y) - \rho(X_1)\rho(y, \eta+y)|$$

which is jointly integrable in all variables η, Y_J and y since one has the bound

(Lemma 7)

$$\int dy\, |\rho(X_1, y, \eta + y) - \rho(X_1)\rho(y, \eta + y)| \leq C_3 \xi^{|X_1|+|I|}(|X_1| + |I|)$$

uniform with respect to η.

Moreover, this integrand converges pointwise to zero because of the factor $K(x + \lambda\hat{u}; y + \eta_i)$ (recall that $|I| - 1 \geq 1$), and thus this contribution vanishes by dominated convergence.

Furthermore, for all $\lambda > \lambda_0$ to this contribution is bounded by

$$C_4 \xi^{|X_1|+|X_2|+n} C_0^n(|X_1| + n)$$

We have thus shown that $\mathscr{D}_{n+1}^{(\lambda)}$ does not give any contribution is the limit $\lambda \to \infty$ and this contribution is bounded for all $\lambda > \lambda_0$ by

$$n^{\gamma} \xi^{|X_1|+|X_2|+n} C_5^n |X_1|.$$

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