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On Callan's proof of the BPHZ theorem

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Abstract. We give an elementary proof of the BPHZ theorem in the case of the Euclidean $\lambda\varphi^4$ theory. The method of proof is based on an idea of Callan [3]. It relies on a detailed analysis of the skeleton structure of graphs and estimates based on the Callan–Symanzik equations.

I. Introduction

The main result of renormalization theory, the BPHZ theorem [1], [5], [9] states that the renormalized Green's functions are free of ultraviolet divergences. There exist quite a lot of more or less sophisticated proofs of this theorem. We want to add to this collection another one, which seems to be particularly pedagogical. It applies to any *massive renormalizable* theory, although for the sake of simplicity we restrict ourselves to the $\lambda\varphi^4$ theory with non-zero mass μ^2. Our guide is a beautiful idea of Callan [3]. He suggested that the existence of renormalized Green's functions should follow from the fact that they satisfy the Callan–Symanzik equations.

There are some advantages of the present approach. It contains no combinatorial part, and the problem of overlapping divergences is circumvented altogether. Moreover, we do not merely prove the existence of Green's functions, but simultaneously we give some bounds on the behavior of the Euclidean p-space Green's functions. Finally, we show that the Green's functions satisfy the Callan–Symanzik equations.

Our approach is purely Euclidean. To come back to the Minkowski space one may use the results of [4].

The paper is organized as follows. In Section II we summarize in considerable detail some standard definitions and fix the notations. In Section III we introduce the regularized Callan–Symanzik equations. Section IV contains the formulation of our main result. The proofs are carried out in Sections V and VI. The appendix contains a sketch of the proof of the Callan–Symanzik equations.

II. BPHZ renormalization [7], [9]

We assume that the reader has some familiarity with Zimmermann's formulation of renormalization theory. The aim of this section is to fix the notation, and not to provide an introduction to the subject.

(a) *Green's functions.* We are concerned with the following three Green's functions: $\Gamma^{(n)}(\mathbf{p}, \mu, \lambda)$, $\Delta_0\Gamma^{(n)}(\mathbf{p}, \mu, \lambda)$ and $\Delta_0^2\Gamma^{(n)}(\mathbf{p}, \mu, \lambda)$, where $\mathbf{p}=(p_1, \ldots, p_n)$, $\sum_{i=1}^n p_i = 0$ are external momenta, μ is the mass, and λ is the coupling constant. $\Gamma^{(n)}$ is the (Fourier transform of the) n-point vertex function, i.e. the proper amputated part of the full Green's function. $\Delta_0\Gamma^{(n)}$ is the n-point vertex function with an $\frac{1}{2}\varphi^2$ insertion at zero external momentum, and $\Delta_0^2\Gamma^{(n)}$ is the n-point vertex function with two $\frac{1}{2}\varphi^2$ insertions at zero external momenta. All these functions are defined perturbatively via the (formal power series) Gell-Mann-Low expansion. For example

$$\Gamma^{(n)}(\mathbf{p}, \mu, \lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \Gamma_k^{(n)}(\mathbf{p}, \mu),$$

where $\Gamma_k^{(n)}(\mathbf{p}, \mu)$, the kth order vertex function is a sum of the corresponding renormalized Feynman amplitudes $\Gamma_k^{(n)}(\mathbf{p}, \mu) = \sum_G \Gamma_k^{(n)}(\mathbf{p}, \mu \mid G)$. In a similar fashion one defines $\Delta_0\Gamma_k^{(n)}(\mathbf{p}, \mu)$ and $\Delta_0^2\Gamma_k^{(n)}(\mathbf{p}, \mu)$. Let us note that

$$\Gamma^{(2)}(p, -p, \mu, \lambda) = 0(\lambda^0), \tag{II.1}$$

$$\Gamma^{(4)}(p_1, \ldots, p_4, \mu, \lambda) = 0(\lambda^1), \tag{II.2}$$

$$\Delta_0\Gamma^{(2)}(p, -p, \mu, \lambda) = 0(\lambda^0), \tag{II.3}$$

$$\Delta_0\Gamma^{(4)}(p_1, \ldots, p_4, \mu, \lambda) = 0(\lambda^2), \tag{II.4}$$

$$\Delta_0^2\Gamma^{(2)}(p, -p, \mu, \lambda) = 0(\lambda^1). \tag{II.5}$$

(b) *Graphs.* Let G be a (proper) graph corresponding to one of the above listed vertex functions. By $\mathscr{L}(G)$ we denote the set of its (internal) lines, by $\mathscr{V}(G)$ the set of its vertices. The numbers of elements of these sets are denoted by L and V respectively. To each $\sigma \in \mathscr{L}(G)$ we assign a momentum l_σ (internal momentum). To each $v_i \in \mathscr{V}(G)$ we assign a momentum q_i (external momentum), which is the sum of those of basic external momenta $p_1, \ldots, p_n$, which enter v_i. l's and q's obey the momentum conservation law at each vertex: $\sum^{(i)} l_\sigma = q_i$, where $\sum^{(i)}$ means summation over all σ's entering v_i. l_σ has the structure $l_\sigma = k_\sigma + q_\sigma$, where the k_σ's are (non-uniquely) determined by the equations $\sum^{(i)} k_\sigma = 0$, and the q_σ's are (non-uniquely) determined by $\sum^{(i)} q_\sigma = q_i$. k_σ is a linear combination of linearly independent vectors $k_1, \ldots, k_m$ (loop momenta), the integration variables: $k_\sigma = k_\sigma(\mathbf{k})$. q_σ is a linear combination of $q_1, \ldots, q_V: q_\sigma = q_\sigma(\mathbf{q})$ (and hence $q_\sigma = q_\sigma(\mathbf{p})$).

(c) *Unrenormalized Feynman amplitudes.* To a graph G we assign its unrenormalized regularized Feynman amplitude, defined as the integral:

$$\begin{aligned} J(\mathbf{p}; \kappa \mid G) &= (2\pi)^{-4m} \int d\mathbf{k} \prod_{\sigma \in \mathscr{L}(G)} \frac{\theta(l_\sigma^2 - \kappa^2)}{l_\sigma^2 + \mu^2} \\ &\equiv \int d\mathbf{k} \prod_{\sigma \in \mathscr{L}(G)} \theta(l_\sigma^2 - \kappa^2) I(\mathbf{K}(\mathbf{k}), \mathbf{q}(\mathbf{p}) \mid G) \\ &\equiv \int_\kappa d\mathbf{k} I(\mathbf{K}(\mathbf{k}), \mathbf{q}(\mathbf{p}) \mid G), \end{aligned} \tag{II.6}$$

where $d\mathbf{k} = d^4k_1 \cdots d^4k_m$, $\mathbf{K} = \{k_\sigma\}_{\sigma \in \mathscr{L}(G)}$, and $\kappa^2 < \infty$. We assign to G its dimension

$d(G)$ defined as $d(G)=4m-2L$. It is easy to see that

$$d(G)=4-n-2t, \tag{II.7}$$

where t is the number of the $\frac{1}{2}\varphi^2$ insertions.

(d) *Subgraphs.* A subgraph γ of a graph G is determined by the set of its vertices $\mathcal{V}(\gamma)\subset\mathcal{V}(G)$, and the set $\mathcal{L}(\gamma)\subset\mathcal{L}(G)$ of its lines containing all lines joining elements of $\mathcal{V}(\gamma)$. To lines and vertices of γ we assign momenta l_σ^γ, q_i^γ, k_σ^γ, and q_σ^γ in a similar manner as in case of G. We require $l_\sigma^\gamma=l_\sigma$. Then, clearly $q_i^\gamma=q_i^\gamma(\mathbf{q},\mathbf{K})$, $k_\sigma^\gamma=k_\sigma^\gamma(\mathbf{q},\mathbf{K})$, where the dependence is linear. q_i^γ is a sum of external momenta p_{ij}^γ entering v_i (p_{ij}^γ is either one of the p's or, if the line $\sigma\in\mathcal{L}(G)$ is external to γ, $p_{ij}^\gamma=l_\sigma$). It turns out [7], [9], that there exist such assignments of momenta that $k_\sigma^\gamma=k_\sigma^\gamma(\mathbf{K})$. Generally, we say that an assignment of momenta is admissible, if $\tau\subset\gamma$ implies $k_\sigma^\tau=k_\sigma^\tau(\mathbf{K}^\gamma)$, for $\sigma\in\mathcal{L}(\tau)$. We assume henceforth that our assignment of momenta is admissible. The meaning of the dimension $d(\gamma)$ of γ is clear. We say that G is a skeleton graph, if it contains no proper subgraph γ with $d(\gamma)\geqq 0$. A graph G has a skeleton expansion, if it contains pairwise disjoint proper subgraph $\gamma_1,\ldots,\gamma_s$ with $d(\gamma_i)\geqq 0$, $i=1,\ldots,s$, such that the reduced graph $\bar{G}=G/\{\gamma_1,\ldots,\gamma_s\}$ (i.e. the graph obtained from G by shrinking $\gamma_1,\ldots,\gamma_s$ to points) is a skeleton graph.

(e) *Renormalized Feynman amplitudes.* We define a renormalized Feynman amplitude corresponding to (II.6) by $RJ(\mathbf{p};\kappa\mid G)=\int_\kappa d\mathbf{k}R(\mathbf{K}(\mathbf{k}),\mathbf{q}(\mathbf{p})\mid G)$, where $R(\mathbf{K}(\mathbf{k}),\mathbf{q}(\mathbf{p})\mid G)$ is given by Zimmermann's forest formula [7]:

$$R(\mathbf{K}(\mathbf{k}),\mathbf{q}(\mathbf{p})\mid G)=S_G\sum_{u\in\mathcal{U}}\prod_{\gamma\in u}(-t_\gamma^{d(\gamma)}S_\gamma)I(\mathbf{K}(\mathbf{k}),\mathbf{q}(\mathbf{p})\mid u). \tag{II.8}$$

Here $\mathcal{U}$ is the set of all G-forests (a G-forest = a collection of non-overlapping proper subgraphs of G with non-negative dimensions). S_γ is a substitution operator: $S_\gamma:\mathbf{K}^\tau\to\mathbf{K}^\tau(\mathbf{K}^\gamma)$, $\mathbf{q}^\tau\to\mathbf{q}^\tau(\mathbf{q}^\gamma,\mathbf{K}^\gamma)$, if $\tau\subset\gamma$. $I(\mathbf{K},\mathbf{q}\mid u)$ is equal to $I(\mathbf{k},\mathbf{q}\mid G)$, but the momenta are labelled in a special way. Namely, if $\gamma\in u$ is the smallest element of u containing $\sigma\in\mathcal{L}(G)$, then $l_\sigma=l_\sigma^\gamma=k_\sigma^\gamma+q_\sigma^\gamma(\mathbf{q}^\gamma)$. If for some $\sigma\in\mathcal{L}(G)$ there is no $\gamma\in u$ containing it, then we put $l_\sigma=k_\sigma+q_\sigma(\mathbf{q})$. $t_\gamma^{d(\gamma)}$ is the Taylor operator, i.e. the operator which expands a function $F(\mathbf{q}^\gamma)$ (say) around zero up to order $d(\gamma)$ with respect to those of $\mathbf{q}^\gamma$ which are independent (if $N(\gamma)$ is the number of q^γ's, then $N(\gamma)-1$ of them are independent). The ordering of factors in (II.8) is such that if $\tau\subset\gamma$, then $-t_\gamma^{d(\gamma)}S_\gamma$ stands left to $-t_\tau^{d(\tau)}S_\tau$; if $\tau\cap\gamma=\emptyset$, then the order is arbitrary.

Let us conclude this subsection with some simple properties of renormalized Feynman amplitudes.

(i) $RJ(\mathbf{p};\kappa\mid G)$ is independent of the choice of admissible assignment of momenta.

(ii) If $d(G)\geqq 0$, then $R(\mathbf{K}(\mathbf{k}),\mathbf{q}(\mathbf{p})\mid G)$ has at $\mathbf{p}=0$ a zero of order $d(G)+1$.

(iii) If G has a skeleton expansion, then

$$R(\mathbf{K}(\mathbf{k}),\mathbf{q}(\mathbf{p})\mid G)=S_G I(\mathbf{K}^{\bar{G}}(\mathbf{k}),\mathbf{q}^{\bar{G}}(\mathbf{p})\mid\bar{G})\prod_{i=1}^{s}R(\mathbf{K}^{\gamma_i}(\mathbf{k}),\mathbf{q}^{\gamma_i}(\mathbf{p}^{\gamma_i})\mid\gamma_i) \tag{II.9}$$

with natural definitions of $\mathbf{K}^{\bar{G}}$ and $\mathbf{q}^{\bar{G}}$. We choose the integration variables

$(k_1, \ldots, k_m) = (k'_1, \ldots, k'_a, k_{11}, \ldots, k_{1a_1}, \ldots, k_{s1}, \ldots, k_{sa_s})$ in such a way that $k'_1, \ldots, k'_a$ are integration variables for $I(\mathbf{K}^{\bar{G}}(\mathbf{k}), \mathbf{q}^{\bar{G}}(\mathbf{p}) \mid \bar{G})$, and $k_{i1}, \ldots, k_{ia_i}$ are integration variables for $R(\mathbf{K}^{\gamma_i}(\mathbf{k}), \mathbf{q}^{\gamma_i}(\mathbf{p}^{\gamma_i}) \mid \gamma_i)$, $i = 1, \ldots, s$.

All this is well known and follows directly from (II.8).

III. Callan–Symanzik equations

Denote by $\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$, $\Delta_0\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$, and $\Delta_0^2\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$ the sums of the corresponding renormalized regularized Feynman amplitudes. Define

$$r(\mu, \lambda; \kappa) = -\left[\frac{d}{dp^2}\Delta_0\Gamma^{(2)}(p, -p, \mu, \lambda; \kappa)\right]_{p=0}, \tag{III.1}$$

$$s(\mu, \lambda; \kappa) = \Delta_0\Gamma^{(4)}(\mathbf{0}, \mu, \lambda; \kappa), \tag{III.2}$$

$$t(\mu, \lambda; \kappa) = \Delta_0^2\Gamma^{(2)}(0, 0, \mu, \lambda; \kappa). \tag{III.3}$$

Observe that $r(\lambda) = 0(\lambda^2)$, $s(\lambda) = 0(\lambda^2)$, $t(\lambda) = 0(\lambda^1)$. Introduce also the following functions

$$\alpha(\mu, \lambda; \kappa) = \{1 - \mu^2 r(\mu, \lambda; \kappa)\}^{-1}, \tag{III.4}$$

$$\beta(\mu, \lambda; \kappa) = \mu^2(2\lambda r(\mu, \lambda; \kappa) - s(\mu, \lambda; \kappa))\{1 - \mu^2 r(\mu, \lambda; \kappa)\}^{-1}, \tag{III.5}$$

$$\gamma(\mu, \lambda; \kappa) = \tfrac{1}{2}\mu^2 r(\mu, \lambda; \kappa)\{1 - \mu^2 r(\mu, \lambda; \kappa)\}^{-1}. \tag{III.6}$$

A scaling argument shows that α, β, γ, and $\mu^2 t$ are independent of μ. Moreover, we have (cf. (III.1)–(III.3) and (II.3)–(II.5))

$$\alpha(\lambda) = 0(\lambda^0), \tag{III.7}$$

$$\beta(\lambda) = 0(\lambda^2), \tag{III.8}$$

$$\gamma(\lambda) = 0(\lambda^2). \tag{III.9}$$

Proposition. *The regularized Green's functions obey the following (Callan–Symanzik [2], [8]) equations:*

$$\left\{\mu^2\frac{\partial}{\partial\mu^2} + \beta(\lambda; \kappa)\frac{\partial}{\partial\lambda} - n\gamma(\lambda; \kappa)\right\}\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -\mu^2\alpha(\lambda; \kappa)\,\Delta_0\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa), \tag{III.10}$$

$$\left\{\mu^2\frac{\partial}{\partial\mu^2} + \beta(\lambda; \kappa)\frac{\partial}{\partial\lambda} - (n-2)\gamma(\lambda; \kappa) - \mu^2\alpha(\lambda; \kappa)t(\mu, \lambda; \kappa)\right\} \times \Delta_0\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -\mu^2\alpha(\lambda; \kappa)\,\Delta_0^2\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa). \tag{III.11}$$

We sketch the proof of (III.10) and (III.11) in the appendix.

Remark. The choice of the regularization is crucial for proving the regularized Callan–Symanzik equations. In particular, the Green's functions with the Pauli–Villars regularization used originally by Callan [3] *do not* satisfy the regularized Callan–Symanzik equations.

IV. Main result

Theorem. *Let $\mu > 0$. The Green's functions $\Gamma^{(n)}(\mathbf{p}, \mu, \lambda)$, $\Delta_0\Gamma^{(n)}(\mathbf{p}, \mu, \lambda)$, and $\Delta_0^2\Gamma^{(n)}(\mathbf{p}, \mu, \lambda)$ (defined as the $\kappa \to \infty$ limits of the corresponding regularized objects) exist and satisfy the Callan–Symanzik equations (equations (III.10), (III.11) with $\kappa \to \infty$). Moreover*

(a) *For any $k \geqq 2$ and arbitrary $0 < \varepsilon < 1$ there is a constant $C_1 = C_1(\varepsilon, k) > 0$, such that for all* $\mathbf{p}$

$$|\Gamma_k^{(4)}(\mathbf{p}, \mu)| \leqq C_1 \, |\mathbf{p}/\mu|^\varepsilon,$$

where $|\mathbf{p}|^2 = p_1^2 + \cdots + p_4^2$.

(b) *For any $k \geqq 1$ and arbitrary $0 < \varepsilon < 1$ there is a constant $C_2 = C_2(\varepsilon, k) > 0$, such that for all p*

$$|\Delta_0\Gamma_k^{(2)}(p, -p, \mu)| \leqq C_2 \, |p/\mu|^\varepsilon.$$

(c) *For any $k \geqq 2$ and arbitrary $0 < \varepsilon < 1$ there is a constant $C_3 = C_3(\varepsilon, k) > 0$, such that for all p*

$$|\Gamma_k^{(2)}(p, -p, \mu)| \leqq C_3 p^2 \, |p/\mu|^\varepsilon.$$

(d) *Let $F(\mathbf{p}, \mu)$ denotes either $\Gamma_k^{(n)}(\mathbf{p}, \mu)$ with $n \geqq 6$, or $\Delta_0\Gamma_k^{(n)}(\mathbf{p}, \mu)$ with $n \geqq 4$, or $\Delta_0^2\Gamma_k^{(n)}(\mathbf{p}, \mu)$ with $n \geqq 2$. For arbitrary $0 < \varepsilon < 1$ there is a constant $C_4 = C_4(\varepsilon, k) > 0$, such that for all* $\mathbf{p}$

$$|F(\mathbf{p}, \mu)| \leqq C_4 \mu^{4-n-2t}(1 + |\mathbf{p}/\mu|^\varepsilon).$$

The constants C_i, $i = 1, \ldots, 4$ are independent of μ.

Remark. The $F(\mathbf{p}, \mu)$ of part (d) are just the Green's functions with negative dimensions.

The proof of the theorem is inductive. Let us make the following

Inductive assumption. Let $r \geqq 1$.

(a) For any $2 \leqq k \leqq r+1$ the limits $\Gamma_k^{(4)}(\mathbf{p}, \mu) = \lim_{\kappa \to \infty} \Gamma_k^{(4)}(\mathbf{p}, \mu; \kappa)$ exist. For arbitrary $0 < \varepsilon < 1$ there is a constant $C_1 = C_1(\varepsilon, k) > 0$, such that for all $\mathbf{p}$

$$|\Gamma_k^{(4)}(\mathbf{p}, \mu; \kappa)| \leqq C_1 \, |\mathbf{p}/\mu|^\varepsilon, \tag{IV.1}$$

uniformly in μ and κ.

(b) For any $1 \leqq k \leqq r$ the limits $\Delta_0\Gamma_k^{(2)}(p, -p, \mu) = \lim_{\kappa \to \infty} \Delta_0\Gamma_k^{(2)}(p, -p, \mu; \kappa)$ exist. For arbitrary $0 < \varepsilon < 1$ there is a constant $C_2 = C_2(\varepsilon, k) > 0$, such that for all p

$$|\Delta_0\Gamma_k^{(2)}(p, -p, \mu; \kappa)| \leqq C_2 \, |p/\mu|^\varepsilon, \tag{IV.2}$$

uniformly in μ and κ.

(c) For any $2 \leqq k \leqq r$ the limits $\Gamma_k^{(2)}(p, -p, \mu) = \lim_{\kappa \to \infty} \Gamma_k^{(2)}(p, -p, \mu; \kappa)$ exist. For arbitrary $0 < \varepsilon < 1$ there is a constant $C_3 = C_3(\varepsilon, k) > 0$, such that for all p

$$|\Gamma_k^{(2)}(p, -p, \mu; \kappa)| \leqq C_3 p^2 \, |p/\mu|^\varepsilon, \tag{IV.3}$$

uniformly in μ and κ.

Remark. It follows from the second part of the induction hypothesis that $r_k(\mu)$ exists for all $k \leqq r$.

V. Proof of the theorem

Lemma 1. *Let G be a skeleton graph. Define*

$$J_\eta(\mathbf{p} \mid G) = (2\pi)^{-4m} \int d\mathbf{k} \prod_{\sigma \in \mathscr{L}(G)} \frac{1}{(l_\sigma^2 + \mu^2)^\eta}.$$

If $1 - 1/2L < \eta \leqq 1$, then J_η exists and

$$J_\eta(\mathbf{p} \mid G) \leqq C\mu^{d(G)+2L(1-\eta)}, \tag{V.1}$$

for all $\mathbf{p}$ and $\mu > 0$, with $C > 0$ independent of $\mathbf{p}$ and μ.

Proof. The lemma follows directly from an elementary application of the method of Hepp sectors [5]. Let us describe this standard argument. By scaling ($l_\sigma \to \mu l_\sigma$) we extract the factor $\mu^{d(G)+2L(1-\eta)}$. Using the α-representation

$$(l^2+1)^{-\eta} = \frac{1}{\Gamma(\eta)} \int_0^\infty d\alpha \alpha^{\eta-1} e^{-\alpha(l^2+1)},$$

changing the order of $\mathbf{k}$ and $\boldsymbol{\alpha}$ integrations (this is legitimate, since the integrand is positive) and performing the $\mathbf{k}$ integration explicitly, we find [1]:

$$J_\eta(\mathbf{p} \mid G) = (4\pi)^{-2m} \Gamma(\eta)^{-L} \mu^{d(G)+2L(1-\eta)} K_\eta(\mathbf{p}/\mu \mid G),$$

where

$$K_\eta(\mathbf{p} \mid G) = \int_0^\infty \cdots \int_0^\infty d\boldsymbol{\alpha} \prod_{\sigma \in \mathscr{L}(G)} \alpha_\sigma^{\eta-1} D(\boldsymbol{\alpha})^{-2} \exp\left\{ -\frac{A(\boldsymbol{\alpha}, \mathbf{p})}{D(\boldsymbol{\alpha})} - \sum_{\sigma \in \mathscr{L}(G)} \alpha_\sigma \right\}.$$

Here $A(\boldsymbol{\alpha}, \mathbf{p})$ and $D(\alpha)$ are the Symanzik polynomials [1]. Recall, that $A(\boldsymbol{\alpha}, \mathbf{p})$, $D(\boldsymbol{\alpha}) \geqq 0$ in the domain $\alpha_\sigma \geqq 0$ ($\sigma \in \mathscr{L}(G)$), and $A(\boldsymbol{\alpha}, \mathbf{0}) = 0$. Hence $K_\eta(\mathbf{p}/\mu \mid G)$ may be estimated by $K_\eta(\mathbf{0} \mid G)$. To prove the lemma it is thus sufficient to show the existence of $K_\eta(\mathbf{0} \mid G)$. To this end, represent $K_\eta(\mathbf{0} \mid G)$ as a sum of integrals over the sectors $\{\boldsymbol{\alpha} \mid \alpha_{\pi(1)} \leqq \cdots \leqq \alpha_{\pi(L)}\}$, where π runs through the permutations of $\{1, \ldots, L\}$ (we choose some ordering of the elements of $\mathscr{L}(G)$). Consider the sector $\{\boldsymbol{\alpha} \mid \alpha_1 \leqq \cdots \leqq \alpha_L\}$ (other sectors are dealt with in a similar manner). Substituting $\alpha_l = t_l t_{l+1} \cdots t_L$, $l = 1, \ldots, L$, we find that the integral over our sector becomes

$$\int_0^1 \cdots \int_0^1 \int_0^\infty dt_1 \cdots dt_{L-1}\, dt_L \prod_{l=1}^{L} t_l^{l(\eta-1)-\frac{1}{2}d(G_l)-1} [1+P(\mathbf{t})]^{-2} e^{-t_L[1+Q(\mathbf{t})]},$$

where $P(\mathbf{t})$, $Q(\mathbf{t}) \geqq 0$ (in the domain of integration) are some polynomials. G_l is the subgraph of G formed by the lines $\sigma_1, \ldots, \sigma_l$. By our assumptions $l(\eta-1) - \frac{1}{2} d(G_l) > -\frac{1}{2}(1 + d(G_l)) \geqq 0$, so the integral is convergent. Q.E.D.

Lemma 2. *If a graph G corresponds to $\Gamma^{(n)}$ with $n \geqq 6$, or $\Delta_0\Gamma^{(n)}$ with $n \geqq 4$, or to $\Delta_0^2\Gamma^{(n)}$, with $n \geqq 2$, then G is either a skeleton graph, or has a skeleton expansion (in terms of $\Gamma^{(4)}$, $\Gamma^{(2)}$ and $\Delta_0\Gamma^{(2)}$).*

Proof. Otherwise there are two overlapping maximal proper subgraphs γ_1, γ_2 with $d(\gamma_1)$, $d(\gamma_2) \geqq 0$. Since γ_1 and γ_2 are proper, $\gamma_1 \cap \gamma_2$ has at least four external legs. Hence, by (II.7) $d(\gamma_1 \cap \gamma_2) \leqq 0$. This, however, implies that $d(\gamma_1 \cup \gamma_2) = d(\gamma_1) + d(\gamma_2) - d(\gamma_1 \cap \gamma_2) \geqq 0$ in contradiction with maximality of the γ's. Q.E.D.

Assume now that the statements of the inductive hypothesis hold true for all $r \geqq 1$.

Proof of the theorem. The theorem is obviously true for $\Gamma^{(4)}$, $\Delta_0\Gamma^{(2)}$ and $\Gamma^{(2)}$. By Lemma 2 all the other Green's functions have skeleton expansions. Let $F(\mathbf{p}, \mu)$ be such a function (it corresponds to the l'th, say, order of the perturbation expansion). Using (iii) of Section II (e), and factorizing the integrals (which is legitimate for $\kappa^2 < \infty$), we can write

$$F(\mathbf{p}, \mu) = \sum_{\bar{G}} \int_\kappa d\mathbf{k}' I(\mathbf{K}^{\bar{G}}(\mathbf{k}'), \mathbf{q}^{\bar{G}}(\mathbf{p}) \mid \bar{G}) \times \prod_{i=1}^{s} F_i^{\bar{G}}(\mathbf{p}_i(\mathbf{k}', \mathbf{p}), \mu; \kappa).$$

Here $\sum_{\bar{G}}$ denotes the summation over all skeleton graph corresponding to $F(\mathbf{p}, \mu; \kappa)$. $F_i^{\bar{G}}$ denotes one of the functions $\Gamma_k^{(4)}$, $\Gamma_k^{(2)}$ or $\Delta_0\Gamma_k^{(2)}$ with $k \leqq l$. $\mathbf{k}'$ is the set of the loop momenta of $\bar{G}$. Let $F^{\bar{G}}(\mathbf{p}, \mu; \kappa)$ be any summand in the above sum. Choose $0 < \varepsilon < 1/L$ (L is now the number of lines of a graph corresponding to $F(\mathbf{p}, \mu; \kappa)$). Using (IV.1)–(IV.3) we bound the integrand of $F^{\bar{G}}(\mathbf{p}, \mu; \kappa)$ by

$$C\mu^{-L_1\varepsilon} \left\{ \prod_{\sigma \in \mathscr{L}(G_1)} \frac{1}{(l_\sigma^2 + \mu^2)^{1-\varepsilon/2}} \right\} \prod_i{}' (p_i^2 + \mu^2)^{\varepsilon/2} \mu^{-\varepsilon},$$

where we have used the fact that $p^2 |p/\mu|^\varepsilon \leqq (p^2 + \mu^2)^{1+\varepsilon/2}$ and $(p_1^2 + \cdots + p_4^2)^{\varepsilon/2} \mu^{-\varepsilon} \leqq \prod_{i=1}^4 (p_i^2 + \mu^2)^{\varepsilon/2} \mu^{-\varepsilon}$. G_1 is the graph obtained from $\bar{G}$ by cancelling vertices which come from the $\Gamma^{(2)}$ insertions, L_1 is the number of its lines. Clearly G_1 is a skeleton graph. $\prod_i'$ is the product over those external momenta of $F(\mathbf{p}, \mu; \kappa)$ which are also external momenta of some of $F_i^{\bar{G}}(\mathbf{p}_i, \mu; \kappa)$, $i = 1, \ldots, s$. The last expression is integrable by Lemma 1. Hence $F^{\bar{G}}(\mathbf{p}, \mu) = \lim_{\kappa \to \infty} F^{\bar{G}}(\mathbf{p}, \mu; \kappa)$ exists. This implies the existence of $F(\mathbf{p}, \mu)$. Using Lemma 1 ($\eta = 1 - \varepsilon/2$) we obtain

$$\begin{aligned} |F^{\bar{G}}(\mathbf{p}, \mu; \kappa)| &\leqq C_4 \mu^{4-n-2t} \prod_i{}' (p_i^2 + \mu^2)^{\varepsilon/2} \mu^{-\varepsilon} \\ &\leqq C_4 \mu^{4-n-2t} (|\mathbf{p}|^2 + \mu^2)^{n'\varepsilon/2} \mu^{-n'\varepsilon} \end{aligned} \tag{V.2}$$

where n' is the number of factors in $\prod_i'$, and C_4 does not depend on κ. This implies the bound on $F(\mathbf{p}, \mu)$. Q.E.D.

VI. From r to r+1

We come now to the proof of the inductive step.

1. Let us start with $\Gamma^{(4)}$. Using (II.2), (II.4), and (III.7)–(III.9) we may write (III.10) in the $(r+2)$th order as

$$\begin{aligned} \mu^2 \frac{\partial}{\partial \mu^2} \Gamma_{r+2}^{(4)}(\mathbf{p}, \mu; \kappa) = &-\mu^2 \sum_{i=0}^{r} \alpha_{r-i}(\kappa)\, \Delta_0 \Gamma_{i+2}^{(4)}(\mathbf{p}, \mu; \kappa) \\ &- \sum_{i=0}^{r} [(i+1)\beta_{r+2-i}(\kappa) - 4\gamma_{r+1-i}(\kappa)] \Gamma_{i+1}^{(4)}(\mathbf{p}, \mu; \kappa). \end{aligned} \tag{VI.1}$$

We claim that all the quantities on the RHS of this equation have limits as $\kappa \to \infty$. Indeed, by the induction hypothesis $\Gamma^{(4)}_{i+1}(\mathbf{p}, \mu)$, $i = 0, \ldots, r$ exist. By Lemma 1 graphs corresponding to $\Delta_0\Gamma^{(4)}_{i+2}$, $i = 0, \ldots, r$ have skeleton expansions. Hence, using the argument of Section V one proves that $\Delta_0\Gamma^{(4)}_{i+2}(\mathbf{p}, \mu)$, $i \leqq r$ exist and $|\Delta_0\Gamma^{(4)}_{i+2}(\mathbf{p}, \mu)| \leqq C\mu^{-2} |\mathbf{p}/\mu|^{\varepsilon}$ for $|\mathbf{p}/\mu| \geqq 1$. By the definitions and the inductive hypothesis α_k, β_k, and γ_k, $k = 1, \ldots, r$ exist. The only potentially dangerous terms are β_{r+1}, γ_{r+1}, and β_{r+2}. However, looking at the definitions we see that β_{r+1} involves only s_k $(k \leqq r+1)$, which exist (because they have skeleton expansions), and r_k $(k \leqq r)$, which exist by the induction hypothesis. γ_{r+1} and β_{r+2} involve also s_{r+2} (which has a skeleton expansion), and γ_{r+1}, whose existence we shall state below. From the induction hypothesis and what we have said above it follows that

$$\left|\mu \frac{\partial}{\partial \mu} \Gamma^{(4)}_{r+2}(\mathbf{p}, \mu; \kappa)\right| \leqq C_5 |\mathbf{p}/\mu|^{\varepsilon}, \tag{VI.2}$$

for $|\mathbf{p}/\mu| \geqq 1$. However, it follows from (ii) Section II (e) that $\Gamma^{(4)}_{r+2}(\mathbf{p}, \mu; \kappa)$ has a simple zero at $\mathbf{p} = \mathbf{0}$. Hence, (VI.2) holds for all $\mathbf{p}$. Denoting the RHS of (VI.1) by $\mathcal{F}_{r+2}(\mathbf{p}, \mu; \kappa)$, using again (ii) Section II (e) and a scaling argument, we find

$$\begin{aligned}
\Gamma^{(4)}_{r+2}(\mathbf{p}, \mu; \kappa) &= \Gamma^{(4)}_{r+2}(\mathbf{p}/\mu, 1; \kappa/\mu) \\
&= \int_{\mu}^{\infty} d\xi \frac{\partial}{\partial \xi} \Gamma^{(4)}_{r+2}(\mathbf{p}/\xi, 1; \kappa/\mu) \\
&= 2\int_{\mu}^{\infty} \frac{d\xi}{\xi} \mathcal{F}_{r+2}(\mathbf{p}, \xi; \kappa).
\end{aligned}$$

It follows from (VI.2) that

$$|\Gamma^{(4)}_{r+2}(\mathbf{p}, \mu; \kappa)| \leqq C_5 |\mathbf{p}|^{\varepsilon} \int_{\mu}^{\infty} \frac{d\xi}{\xi^{1+\varepsilon}} \leqq \frac{1}{\varepsilon} C_5 |\mathbf{p}/\mu|^{\varepsilon},$$

for all κ. Hence, $\Gamma^{(4)}_{r+2}(\mathbf{p}, \mu)$ exists and (IV.1) holds.

2. Let us consider $\Delta_0\Gamma^{(2)}_{r+1}$. Using (II.3), (II.5), and (III.3) we find that (III.11) in the $(r+1)$-th order takes the form:

$$\begin{aligned}
&\mu^2 \frac{\partial}{\partial \mu^2} \Delta_0\Gamma^{(2)}_{r+1}(p, -p, \mu; \kappa) \\
&\quad = -\mu^2 \sum_{i=0}^{r} \alpha_{r-i}(\kappa)[\Delta_0^2\Gamma^{(2)}_{i+1}(p, -p, \mu; \kappa) - \Delta_0^2\Gamma^{(2)}_{i+1}(0, 0, \mu; \kappa)] \\
&\qquad - \sum_{i=0}^{r} [i\beta_{r+1-i}(\kappa) - \Lambda_{r-i}(\kappa)] \Delta_0\Gamma^{(2)}_{i+1}(p, -p, \mu; \kappa),
\end{aligned}$$

where $\Lambda(\lambda) = \mu^2\alpha(\lambda)t(\lambda)$. The only term on the RHS of the above equation, which is not obviously convergent is β_{r+1}. However, in the first part of the proof we have shown that its existence follows from the induction hypothesis. As in the case of $\Gamma^{(4)}_{r+2}$ we find that

$$\left|\mu \frac{\partial}{\partial \mu} \Delta_0\Gamma^{(2)}_{r+1}(p, -p, \mu; \kappa)\right| \leqq C |p/\mu|^{\varepsilon},$$

and hence, $\Delta_0\Gamma^{(2)}_{r+1}(p, -p, \mu)$ exists and (IV.2) holds.

3. Finally, let us discuss $\Gamma_{r+1}^{(2)}$. In this case (III.10) takes the form

$$\mu^2 \frac{\partial}{\partial \mu^2} \Gamma_{r+1}^{(2)}(p, -p, \mu; \kappa) = -\mu^2 \sum_{i=0}^{r+1} \alpha_{r+1-i}(\kappa)\, \Delta_0 \Gamma_i^{(2)}(p, -p, \mu; \kappa)$$

$$- \sum_{i=0}^{r} (i\beta_{r+2-i}(\kappa) - 2\gamma_{r+1-i}(\kappa)) \Gamma_i^{(2)}(p, -p, \mu; \kappa).$$

Using the second part of the proof (to conclude the existence of $\Delta_0 \Gamma_i^{(2)}(p, -p, \mu)$, $i \leqq r+1$), the induction hypothesis, and the definitions of α, β, and γ we find that everything converges. Moreover,

$$\left| \mu \frac{\partial}{\partial \mu} \Gamma_{r+1}^{(2)}(p, -p, \mu; \kappa) \right| \leqq Cp^2 \, |p/\mu|^{\varepsilon} \tag{VI.3}$$

for $|p/\mu| \geqq 1$. But $\Gamma_{r+1}^{(2)}(p, -p, \mu; \kappa)$ has a zero of the third order at $p = 0$. Hence, (VI.3) holds for all p's and we may repeat the argument of the first part of the proof. This completes our proof.

Appendix

Sketch of the proof of the Callan–Symanzik equations. We need three other vertex functions (see [6]): $\Delta_2 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$, $\Delta_3 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$, and $\Delta_1 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$. $\Delta_2 \Gamma^{(n)}$ and $\Delta_3 \Gamma^{(n)}$ are the n-point vertex functions with the $-\frac{1}{2}(\partial\varphi)^2$ and φ^4 insertions respectively. $\Delta_1 \Gamma^{(n)}$ is essentially the same thing as $\Delta_0 \Gamma^{(n)}$ but the renormalization prescription is different. Graphs corresponding to it are oversubtracted, i.e. to each (sub)graph γ containing the vertex $\frac{1}{2}\varphi^2$ we assign dimension $\delta(\gamma) = 4 - n(\gamma)$ (instead of $d(\gamma) = 2 - n(\gamma)$). Similarly we define $\Delta_2\, \Delta_0 \Gamma^{(n)}$, $\Delta_3\, \Delta_0 \Gamma^{(n)}$, and $\Delta_1\, \Delta_0 \Gamma^{(n)}$. The following relations hold:

(i) $$\frac{\partial}{\partial \mu^2} \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -\Delta_1 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

$$\frac{\partial}{\partial \mu^2} \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -\Delta_1\, \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

(ii) $$\frac{\partial}{\partial \lambda} \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -\Delta_3 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

$$\frac{\partial}{\partial \lambda} \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -\Delta_3\, \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

(iii) $$n\Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -[2\mu^2\, \Delta_1 + 2\Delta_2 + 4\lambda\, \Delta_3] \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

$$(n-2)\, \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = -[2\mu^2\, \Delta_1 + 2\Delta_2 + 4\lambda\, \Delta_3]\, \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

(iv) $$\Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = [\Delta_1 + r(\mu, \lambda; \kappa)\, \Delta_2 + s(\mu, \lambda; \kappa)\, \Delta_3] \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa),$$

$$\Delta_0^2 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa) = [\Delta_1 + r(\mu, \lambda; \kappa)\, \Delta_2 + s(\mu, \lambda; \kappa)\, \Delta_3]\, \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa)$$
$$+ t(\mu, \lambda; \kappa)\, \Delta_0 \Gamma^{(n)}(\mathbf{p}, \mu, \lambda; \kappa).$$

Now, (III.10) and (III.11) follow from (i)–(iv) by algebraic operations on formal power series.

Properties (i)–(iii) follow by inspection of graphs. The proof of (iv) is just the standard proof of Zimmermann's identities, and can be found in many places, see e.g. [3], [7], [10].

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