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On some K-representations of the Poincaré and Einstein groups

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Abstract. Using some recently developed work on group representations on (topological direct) unions of Hilbert spaces (termed K-spaces), explicit examples of such representations are constructed for the Poincaré and the Einstein groups. These representations appear quite naturally in a formulation of relativistic mechanics on phase space, both in the classical and the quantal domains. Some physical consequences, relating to localization on phase space in particular, are discussed.

1. Introduction

The theory of representations of locally compact groups on K-spaces has been recently developed in a series of papers (cf. [1], [2] and references cited therein). The resulting construction uses a simple generalization of the standard Mackey theory of induced representations of locally compact groups on Hilbert spaces, to that of induced representations on spaces which are topological direct unions of such Hilbert spaces. Apart from a mathematical richness of the ensuing theory, its structure is broad enough to deal in a unified manner and in a unified language, both with the quantal and classical frameworks (when the unions are trivial, or over 1-dimensional Hilbert spaces, respectively) as well as with intermediate possibilities (allowing e.g. for, possibly continuous, superselection variables). This framework has been physically motivated by a fundamental discussion of the kinematics of elementary particles (classical or quantal), both in the relativistic and non-relativistic cases [3, 4], especially concerning the relations between states, observables and symmetry principles. For example, it has been shown [4] that the irreducible K-representations of the Newton and Einstein groups (then being the symmetry groups underlying the kinematics of elementary massive particles) lead exactly to two types of state spaces, as the unique solutions of an imprimitivity problem. These solutions correspond, in a very simple and direct way, to the state spaces of the classical and the quantal elementary particles (of arbitrary spin), respectively, both in the relativistic and in the non-relativistic contexts.

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In the present paper we apply this theory of group representations on K-spaces to a somewhat different way of looking at such problems, namely to that of a phase space formulation of classical and quantum mechanics, a subject which has been extensively developed in the last few years (cf. for example [5], [6] for a review). The point of view adopted in this approach is to start from a phase space Γ, each point $(\mathbf{q}, \mathbf{p})$ of which is equipped with probability densities $(\chi_{\mathbf{q}}, \hat{\chi}_{\mathbf{p}})$ – termed confidence functions – which correspond to the sharpness of localization of a particle at $(\mathbf{q}, \mathbf{p})$. On the basis of such a "stochastic" description of localization it has been possible to construct a consistent single particle relativistic quantum mechanics – among other things – with a proper probability interpretation [7] and conserved currents. The computation of the mass spectrum of baryons [15] as well as a description of a spin–1/2 particle in interaction with an external electromagnetic field by means of a covariant 2-component wave equation [8] have been some other successes of this approach. It is therefore of great interest to study representations of the space-time symmetry groups, – i.e., the Galilei or Poincaré groups – or of kinematical symmetry groups [3] – i.e., the Newton or Einstein groups – on Hilbert spaces built out of functions of the phase space variables. Some work in this direction has been reported in [9]–[12]. The present paper examines some representations of the Poincaré and the Einstein groups on K-spaces which are built up as direct unions of Hilbert spaces of phase space functions.

The rest of this paper is organized as follows: In Section 2 we briefly review some relevant definitions and results relating to projective group representations on K-spaces. In Section 3 we use the framework to explicitly construct an induced K-representation for the Poincaré group, and later extend it to the (larger) Einstein group, on the same K-space. Finally in Section 4, we briefly analyze some interesting physical properties of these representations, especially relating to a quantization procedure in this framework as well as to the localization operators for massive relativistic particles.

2. *K*-spaces and *K*-representations

We briefly recall in this section some mathematical ideas and results concerning K-spaces and K-representations, as developed in [1] and [2], in a form which is more suited to the purposes of the present paper. The notational convention will also be set up here. For further details on related notions (like irreducibility, equivalence, etc...) of K-representations, we refer to [1]–[4].

Let G be a separable locally compact group, H a closed subgroup of G, and S the quotient space

$$S = G/H \tag{2.1}$$

To each $s \in S$ let us associate a (separable, complex) Hilbert space $\mathscr{H}_s$ and assume, like in the irreducible case, that all these Hilbert spaces are isomorphic, so that there exists, for each $s \in S$, a linear isometry

$$i_s^{-1} : \mathscr{H}_s \to \mathscr{H} \tag{2.2}$$

from $\mathcal{H}_s$ onto some fixed Hilbert space $\mathcal{H}$. For vectors $\phi_s \in \mathcal{H}_s$, we shall make the convention of writing

$$\phi_s = i_s(\phi) \tag{2.3}$$

where $\phi \in \mathcal{H}$. We can now build up a K-space as the (topological) direct union

$$K = \bigcup_{s \in S} \mathcal{H}_s \tag{2.4}$$

an element $\Phi \in K$ being thus given by a pair consisting of an element $s_0 \in S$, together with a vector $\phi_{s_0} \in \mathcal{H}_{s_0}$.

Corresponding to H, let $g = k_g \cdot h_g$ be the (Mackey) left coset decomposition of an arbitrary $g \in G$ with $\{k_g\}$ a (fixed) set of coset representatives for G/H, and $h_g \in H$. We normalize this choice such that $k_\varepsilon = \varepsilon$, with ε the unit element of G. Let

$$\beta : S \to G \tag{2.5}$$

be the Borel section corresponding to this choice, i.e., $\forall s \in S$,

$$\beta(s) = k_{\beta(s)} \tag{2.6}$$

and denote by ν the associated (G-S-H)-cocycle [13]

$$\nu(g, s) = (\beta(g * s))^{-1} \cdot g \cdot \beta(s) \tag{2.7}$$

$\forall s \in S$, $\forall g \in G$, with $g * s$ the natural action of g on G/H.

Let $\mathcal{H}$ be the carrier space of a unitary projective representation $h \to L(h)$ of H, with arbitrary multiplier $\omega \in Z^2(H, U(1))$, i.e.,

$$L(h_1)L(h_2) = \omega(h_1, h_2)L(h_1 h_2) \tag{2.8}$$

where $\omega : H \times H \to U(1)$.

We then define a unitary projective K-representation of G on (2.4), $g \to U^K(g)$, induced from the representation L of H, by

$$U_s^K(g)\phi_s = (L(\nu(g, s))\phi)_{g*s} \tag{2.9}$$

where $U^K(g)$ is a family $\{U_s^K(g) \mid s \in S\}$ of isometries $U_s^K(g) : \mathcal{H}_s \to \mathcal{H}_{g*s}$.

It is important to note here that there is no phase factor in (2.9), unlike in the direct integral case, forcing the projectivity of U^K to be completely carried by H, as shown in [2]. This implies for example that the Planck's constant that turns out to label such an ω is necessarily zero in the classical case and non-zero in the quantal case [4].

The above construction can also be shown to be exhaustive [1] in the sense that every (irreducible) projective K-representation of G is equivalent to an induced representation of this type.

Let us finally briefly exhibit the observables in K. They are given by a slight generalization of the notion of systems of imprimitivity of Mackey: let T be any G-space and $B(T)$ the Borel sets in T. We consider the mappings

$$P : \mathcal{B}(T) \to \mathcal{P}(K) \tag{2.10}$$

with $\mathscr{P}(K)$ the projections in K, i.e., the families $\{P_s, s \in S\}$ of projectors in the corresponding Hilbert spaces $\mathscr{H}_s$. Assume further that the map (2.10) satisfy

$$\begin{aligned} &\text{(i)}\ P(\varnothing) = 0_K,\ P(T) = \mathbb{1}_K \\ &\text{(ii)}\ P(E)P(E') = P(E \cap E') \\ &\text{(iii)}\ P(UE_i) = \sum P(E_i) \text{ for } E_i \cap E_j = \varnothing \text{ if } i \neq j \text{ and } i, j \in I, \text{ a countable set} \end{aligned} \tag{2.11}$$

with sum and multiplications defined over S term by term.

We have called such maps supersystems of imprimitivity if they satisfy in addition the covariance conditions

$$(U^K(g)P(E)U^K(g)^{-1})_s = (P(\tau(g)E))_s \tag{2.12}$$

$\forall s \in S$ and with τ the G-action on T. In physical applications T is just the set of all possible values of a given observable [3].

A particular, very simple but important solution of (2.10), (2.11), (2.12) is given by $T = S$ and

$$(P(\Delta))_s = \chi_\Delta(s) \cdot \mathbb{1}_s \tag{2.13}$$

with χ_Δ the characteristic function of the Borel set $\Delta \subseteq S$ and $\mathbb{1}_s$ the identity operator on $\mathscr{H}_s$.

More generally, we have shown in [1] that all supersystems of imprimitivity can be canonically associated to the above induction procedure, and to the usual Mackey systems of imprimitivity based on $\mathscr{H}$, for the corresponding subgroups of H. The latter correspond of course to the quantal observables (as being related to self-adjoint operators and their spectral resolutions), whereas for example (2.13) gives an observable of the classical type (as it commutes with all other observables). We refer to [1] for more details on this construction and for the explicit form of the corresponding imprimitivity theorem.

3. A *K*-representation of the Poincaré group

Using the tools just presented, we now explicitly construct a K-representation for the Poincaré group $P_+^\uparrow$. The approach will first differ from the one in [3] [4] in that we shall start from a given set of physically motivated functions. The background Hilbert space will in fact consist of functions of the relativistic phase space variables q and p. On the other hand we shall show that this representation is related in a simple way to the ones studied in [7]–[11].

For any unit time-like 4-vector n ($n^2 = n_0^2 - \mathbf{n}^2 = 1$, $n_0 > 0$) and $\tau \in \mathbb{R}$, let

$$\sigma_{n,\tau} = \{q \in \mathscr{M} \mid n \cdot q = \tau\} \tag{3.1}$$

be the space-like surface, orthogonal to n (in the metric $g^{\mu\nu} = (1, -1, -1, -1)$) and having proper time τ ($\mathscr{M}$ is the Minkovski space). Physically $\sigma_{n,\tau}$ is the 3-dimensional *space* of an observer moving along n, at time τ. The set of all $\sigma_{n,\tau}$ then corresponds to the operational space of all possible space-time measurements if we a priori distinguish events measured with respect to different (inequivalent) sets of clocks, i.e., corresponding to experimentally distinguishable (moving with respect to each other) reference frames. We refer to [14] for more on this interpretation.

We further remark here that it follows from (3.1) that τ is a Lorentz invariant. Let then $V_m^\pm$ denote the forward ($p_0>0$) and backward ($p_0<0$) mass hyperboloids:

$$V_m^\pm = \{p \in \mathcal{M}^* \mid p^2 = p_0^2 - \mathbf{p}^2 = m^2c^2\} \tag{3.2}$$

and let

$$\textstyle\sum_{n,\tau}^{\pm,m} = \sigma_{n,\tau} \times V_m^\pm \tag{3.3}$$

We consider on $\sum_{n,\tau}^{\pm,m}$ the following natural measure

$$d\textstyle\sum = n \cdot p \, d\sigma \dfrac{d^3\mathbf{p}}{p_0} \tag{3.4}$$

where

$$d\sigma(\Delta_{n,\tau}) = d^3\mathbf{q}(L_n^{-1}\,\Delta_{n,\tau}) \tag{3.5}$$

for $\Delta_{n,\tau}$ any Borel set in $\sigma_{n,\tau}$ and $\{L_n\}$ a (fixed) set of Lorentz transformations satisfying, for each n,

$$L_n u_0 = n \tag{3.6}$$

with $u_0 = (1, \mathbf{0})$. In particular one has for (3.4), for $n = u_0$

$$d\textstyle\sum_{u_0,\tau}^{\pm,m} = d^3\mathbf{p}\, d^3\mathbf{q} \tag{3.7}$$

which is the usual phase space measure.

Let $(a, \Lambda) \in P_+^\uparrow$ be a Poincaré transformation, and (q, p) a point in some $\sum_{n,\tau}^{\pm,m}$. The action

$$(a, \Lambda)(q, p) = (\Lambda q + a, \Lambda p) \tag{3.8}$$

transforms $\sum_{n,\tau}^{\pm,m}$ into $\sum_{n',\tau}^{\pm,m}$ with

$$\begin{aligned} \tau' &= \tau + a \cdot \Lambda n \\ n' &= \Lambda n \end{aligned} \tag{3.9}$$

but it leaves the measure (3.4) invariant.

For $j = 0, \frac{1}{2}, 1, \ldots$, let $\mathcal{K}^j$ be the usual $(2j+1)$-dimensional spinor space, and set

$$\mathcal{H}_{n,\tau}^j = \mathcal{K}^j \otimes \mathcal{L}^2(\textstyle\sum_{n,\tau}^{\pm,m}, d\sum) \tag{3.10}$$

Our purpose is now to use these Hilbert spaces to construct a K-space. To do so however, we first have to show that it is possible to identify the set

$$S = \{(n, \tau) \mid \tau \in \mathbb{R}, n \in V_{1/c^2}^+\} \tag{3.11}$$

with a homogeneous space of $P_+^\uparrow$.

Consider therefore the Euclidean subgroup $\mathbb{E}^3$ of $P_+^\uparrow$, consisting of all space translations and rotations. We write $h \in \mathbb{E}^3$ as

$$h = ((0, \mathbf{b}), \alpha). \tag{3.12}$$

with $\alpha \in J$ the rotation group. Since any Lorentz transformation $\Lambda \in \mathcal{L}_+^\uparrow$ can be written as

$$\Lambda = L_n \cdot \alpha(\Lambda) \tag{3.13}$$

for some uniquely determined n (and corresponding L_n from (3.6)) and $\alpha(\Lambda) \in J$, we can decompose any $(a, \Lambda) \in P_+^\uparrow$ as

$$(a, \Lambda) = (\tau n, L_n)((0, \mathbf{b}), \alpha(\Lambda)) \tag{3.14}$$

with*)

$$\begin{aligned} \tau &= a \cdot n \\ \mathbf{b} &= \underline{L_n^{-1} a} \end{aligned} \tag{3.15}$$

This decomposition obviously correspond to the left coset decomposition of $P_+^\uparrow$ along $\mathbb{E}^3$, so that

$$S \cong P_+^\uparrow / \mathbb{E}^3 \tag{3.16}$$

S being given by (3.11), with coset representatives (2.5), given from (3.14) by

$$\beta(n, \tau) = (\tau n, L_n) \tag{3.17}$$

Furthermore the action of $P_+^\uparrow$ on S_+ is given from (3.9) by

$$(a, \Lambda) * (n, \tau) = (\Lambda n, \tau + a \cdot \Lambda n) \tag{3.18}$$

The ($P_+^\uparrow$-S-$\mathbb{E}^3$)-cocycle ν in (2.7) is now easily computed and gives

$$\nu((a, \Lambda), (n, \tau)) = ((0, \underline{L_{\Lambda n}^{-1} a}), L_{\Lambda n}^{-1} \cdot \Lambda \cdot L_n) \tag{3.19}$$

Consider now the Hilbert spaces in (3.10) and construct the direct union

$$K^j = \bigcup_{(n,\tau) \in S} \mathcal{H}_{n,\tau}^j \tag{3.20}$$

It follows from (3.18) that in (3.12) $\mathbb{E}^3$ is the stabilizer of $(n, \tau) = (u_0, 0)$, hence we can identify $\mathcal{H}$ with $\mathcal{H}_{u_0,0}^j$, where, by virtue of (3.10) and (3.7)

$$\mathcal{H}_{u_0,0}^j = \mathcal{K}^j \otimes \mathcal{L}^2(\mathbb{R}^3 \times V_m^\pm, d^3\mathbf{q}\, d^3\mathbf{p}) \tag{3.21}$$

On $\mathcal{H}_{u_0,0}^j$ we define the following representation of $\mathbb{E}^3$ (for h as in (3.12)):

$$(L(h)\phi)(\mathbf{q}, \mathbf{p}) = D^j(\alpha)\phi(\alpha^{-1}(\mathbf{q} - \mathbf{b}), \alpha^{-1}\mathbf{p}) \tag{3.22}$$

where D^j is the usual $(2j+1)$-dimensional projective unitary irreducible representation of J and where we have identified the pair $\{(0, \mathbf{q}), (p_0, \mathbf{p}) \in V_m^+ \text{ (or } V_m^-)\}$ with $(\mathbf{q}, \mathbf{p})$.

We can now define with (2.9) the following K-representation, on the space (3.20)

$$W_{n,\tau}^K(a, \Lambda)\phi_{n,\tau} = (L(\nu((a, \Lambda), (n, \tau)))\phi)_{\Lambda n, \tau + \alpha \cdot \Lambda n} \tag{3.23}$$

with $\nu((a, \Lambda), (n, \tau))$ as in (3.19) and L from (3.22). We can thus also write

$$W_{n,\tau}^K(a, \Lambda) = i_{\Lambda n, \tau + a \cdot \Lambda n} \cdot L(\nu((a, \Lambda), (n, \tau))) \cdot i_{n,\tau}^{-1} \tag{3.24}$$

Since the representation $h \to L(h)$ of $\mathbb{E}^3$ in (3.22) is clearly reducible, the representation W^K is also reducible [1]. We shall come back later to this point and indicate why it should be so, and how it can correspondingly be made irreducible.

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*) Boldface or underlined means: the 3 spatial components.

Let us for the moment prove a simple relationship between the representation (3.23) of $P_+^\uparrow$ with the one studied in [9]–[11] (cf. for example Eq. (2.86) in [11]). For simplicity we assume $j=0$ in the definition of the space $\mathcal{H}^j_{n,\tau}$.

Let us therefore introduce the embeddings i_s in (2.2) or (3.24) by

$$\mathcal{H}=\mathcal{H}_{u_0,0}=\mathcal{L}^2(\mathbb{R}^3\times V_m^\pm,\, d^3\mathbf{p}\, d^3\mathbf{q}) \tag{3.25}$$

and

$$i_{n,\tau}:\mathcal{H}\to\mathcal{H}_{n,\tau} \tag{3.26}$$

defined by

$$\phi_{n,\tau}(\gamma_{n,\tau})=\phi(\gamma) \tag{3.27}$$

where $\gamma=((0,\mathbf{q}),p_0,\mathbf{p}))$ and

$$\gamma_{n,\tau}=(\tau n, L_n)\cdot\gamma \tag{3.28}$$

with the action (3.8). Consider now the space $\Omega^\pm=\mathcal{M}\times V_m^\pm$ and let $\mathcal{H}(\Omega^\pm)$ be the space of all maps

$$\Phi:\Omega^\pm\to\mathbb{C} \tag{3.29}$$

defined as

$$\Phi(q,p)=\phi_{u_0,\tau=q^0}(\mathbf{q},\mathbf{p}) \tag{3.30}$$

$\forall(q,p)=((q_0,\mathbf{q}),(p_0,\mathbf{p}))\in\Omega^\pm$, $\phi\in\mathcal{H}$ (cf. (3.21)). Obviously $\mathcal{H}(\Omega^\pm)$ is not the same as in (3.20). However, we can consider, for any $\Phi\in\mathcal{H}(\Omega^\pm)$ the restriction of Φ to $\sum_{n,\tau}^{\pm,m}$ (denoted $\Phi_{n,\tau}$) and let then

$$[\mathcal{H}(\Omega^\pm)]_{n,\tau}=\{\Phi_{n,\tau}\mid\Phi\in\mathcal{H}(\Omega^\pm)\}\cap\mathcal{H}_{n,\tau} \tag{3.31}$$

then it is obvious that

$$\mathcal{H}_{n,\tau}=[\mathcal{H}(\Omega^\pm)]_{n,\tau} \tag{3.32}$$

We can rewrite the representation (3.23) on $\mathcal{H}(\Omega^\pm)$ as

$$\begin{aligned}(W^K(a,\Lambda)\Phi)_{n,\tau}(\gamma_{n,\tau})&=L(\nu((a,\Lambda),(n',\tau')))(i^{-1}_{n',\tau'}\Phi_{n',\tau'})(\gamma)\\&=(i^{-1}_{n',\tau'}\Phi_{n',\tau'})(\gamma')\end{aligned} \tag{3.33}$$

with

$$\begin{aligned}\gamma'&=\nu((a,\Lambda),(n',\tau'))^{-1}\cdot\gamma\\&=(L^{-1}_{\Lambda^{-1}n}\Lambda^{-1}L_n(\mathbf{q}-\underline{L_n^{-1}a}),\ \underline{L^{-1}_{\Lambda^{-1}n}\Lambda^{-1}L_n p})\end{aligned} \tag{3.34}$$

and $(n',\tau')=(\Lambda^{-1},\tau-an)$.

Since (by (3.14), (3.15) and (3.28)) one has

$$(a,\Lambda)^{-1}\gamma_{n,\tau}=(\tau'n',L_{n'})\gamma' \tag{3.35}$$

we obtain simply

$$(W^K(a,\Lambda)\Phi)_{n,\tau}(\gamma_{n,\tau})=\Phi_{(a,\Lambda)^{-1}*(n,\tau)}((a,\Lambda)^{-1}\gamma_{n,\tau}) \tag{3.36}$$

so that if we restrict (q,p) in $\Omega_\pm$ in one fixed surface $\sum_{n,0}^{\pm,m}$, then we may rewrite (3.36) as

$$(W^K(a,\Lambda)\Phi)(q,p)=\Phi(\Lambda^{-1}(q-a),\Lambda^{-1}p) \tag{3.37}$$

where $W(a,\Lambda)$ is now supposed to imply a unitary map between $\mathcal{H}_{n,0}$ and $\mathcal{H}_{\Lambda^{-1}n,-a\cdot n}$. This is exactly the form of equation (2.86) in [11].

Let us now come to the reducibility of the representation W^K in (3.23). From the form of the representation it is clear that one has to decompose the representation (3.22) of $\mathbb{E}^3$ into its irreducible components to get the irreducible subrepresentations of W^K. Moreover, it follows from (3.37) that a decomposition technique similar to the one worked out in [11] would apply here as well.

There is however a profound reason for W^k to be reducible. As can be seen at the observables (p, q), given in K by the characteristic function in p and q in each $\mathcal{H}^j_{n,\tau}$ (and which thus commute with each other), the system under consideration is a classical one (possibly with spin). In view of Section 2 (and [1–4]), only the Dirac functions in the dual $(\mathcal{H}^j_{n,\tau})'$ are classically pure states and one should strictly speaking consider in place of (3.10) the following K-space

$$K^i_{n,\tau} = \bigcup_{\Sigma_{n,\tau}} \mathcal{K}^j \tag{3.38}$$

The choice of (3.20) with (3.10) is however justified by practical reasons, as we shall see in particular in the next section.

Still, in order to get irreducibility on (3.38) of the corresponding representation, it is easy to see at the p-dependence that one necessarily has to extend first the classical representation to the (larger) Einstein group $E^\uparrow_+$ discussed in [3]–[4].

We therefore have to show that W^K can be extended to a representation of $E^\uparrow_+$.

The Einstein group $E^\uparrow_+$ consists of all space-time translations a, 4-momentum translations w and Lorentz transformations Λ. The product of two elements (a_i, w_i, Λ_i), $i = 1, 2$, is given by

$$(a_1, w_1, \Lambda_1)(a_2, w_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, w_1 + \Lambda_1 w_2, \Lambda_1\Lambda_2) \tag{3.39}$$

Obviously $P^\uparrow_+ \subset E^\uparrow_+$. Let $\mathbb{E}^7$ be the subgroup of $E^\uparrow_+$ which consists of elements $\tilde{h}$ of the form

$$\tilde{h} = ((0, \mathbf{b}), (w_0, \mathbf{w}), \alpha) \tag{3.40}$$

Similarly to (3.14)–(3.15) one can write the following coset decomposition of $E^\uparrow_+$

$$(a, w, \Lambda) = (\tau n, 0, L_n)((0, \underline{L_n^{-1} a}), ((L_n^{-1} w)^0, \underline{L_n^{-1} w}), \alpha(\Lambda)) \tag{3.41}$$

with n, τ and $\alpha(\Lambda)$ as in (3.14).

The coset space $E^\uparrow_+/\mathbb{E}^7$ is thus the same as S in (3.11) and can again be parametrized by the same (n, τ). The action of $E^\uparrow_+$ on S is given by

$$(a, w, \Lambda) * (n, \tau) = (\Lambda n, \tau + a \cdot \Lambda n) \tag{3.42}$$

corresponding to (3.9). The $(E^\uparrow_+, S, \mathbb{E}^7)$-cocycle ν is now given by

$$\nu((a, w, \Lambda), (n, \tau)) = ((0, \underline{L_{\Lambda n}^{-1} a}), L_{\Lambda n}^{-1} w, L_{\Lambda n}^{-1} \Lambda L_n) \tag{3.43}$$

Let us now extend the representation $h \to L(h)$ of (3.22) to $\tilde{h} \to \tilde{L}(\tilde{h})$ on the same space, by

$$(\tilde{L}(\tilde{h})\phi)(\mathbf{q}, \mathbf{p}) = D^j(\alpha)\phi(\alpha^{-1}(\mathbf{q} - \mathbf{b}), \alpha^{-1}(\mathbf{p} - \mathbf{w})) \tag{3.44}$$

with $\tilde{h}$ as in (3.40) and $w = (w_0, \mathbf{w})$. Using now the fact that (3.43) does not depend on τ, we can now induce, similarly to (3.23), to a representation $\tilde{W}^K$ on the same K as before, by this $\tilde{L}$

$$\tilde{W}^K_{n,\tau}(a, w, \Lambda)\phi_{n,\tau} = (\tilde{L}(\nu((a, w, \Lambda), (n, \tau)))\phi)_{\Lambda n, \tau + a \cdot \Lambda n} \tag{3.45}$$

which thus extends (3.23) to $E^{\uparrow}_{+}$.

4. Some physical implications

In this section we shall exhibit how the previous theory can be quantized using the phase-space interpretation, and we shall discuss the corresponding (localization) observables.

Let us therefore go back to the representation $h \to L(h)$ of $\mathbb{E}^3$ in (3.22) and let us restrict ourselves for simplicity to $j = 0$. We define next a projection operator $\mathbb{P}_e$ on $\mathscr{H}$, by means of a reproducing kernel $e(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}')$

$$(\mathbb{P}_e\phi)(\mathbf{q}, \mathbf{p}) = \int e(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}')\phi(\mathbf{q}', \mathbf{p}')\, d^3\mathbf{q}' \cdot d^3\mathbf{p}' \tag{4.1}$$

where $e(q, p; q', p')$ is supposed to satisfy the following properties [12]

(a) $e : (\mathbb{R}^3 \times V^{\pm}_m) \times (\mathbb{R}^3 \times V^{\pm}_m) \to \mathbb{C}$ (4.2)

(b) $e(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') = \overline{e(\mathbf{q}', \mathbf{p}'; \mathbf{q}, \mathbf{p})}$ (4.3)

(c) $e((h(\mathbf{q}, \mathbf{p}); h(\mathbf{q}', \mathbf{p}')) = e(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}'), \quad \forall h \in \mathbb{E}^3$ (4.4)

(d) $e(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') = \int e(\mathbf{q}, \mathbf{q}; \mathbf{q}'', \mathbf{p}'')e(\mathbf{q}'', \mathbf{p}''; \mathbf{q}', \mathbf{p}')\, d^3\mathbf{q}''\, d^3\mathbf{p}''$ (4.5)

A more detailed discussion of such kernels has been given in [9], [11] and [12]. We just exhibit here an explicit construction procedure for such a kernel. Let therefore $\tilde{e} \in \mathscr{L}^2(\mathbb{R}^3, d^3\mathbf{k})$ be such that

(a) $\|\tilde{e}\|^2 = \int |\tilde{e}(\mathbf{k})|^2\, d^3\mathbf{k} = 1$ (4.6)

(b) $\tilde{e}(\alpha\mathbf{k}) = \tilde{e}(\mathbf{k}), \qquad \forall \alpha \in J$ (4.7)

For each $(\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times V^{\pm}_m$, we define a vector $\tilde{e}_{\mathbf{q},\mathbf{p}} \in \mathscr{L}^2(\mathbb{R}^3, d^3k)$ by

$$\tilde{e}_{\mathbf{q},\mathbf{p}}(\mathbf{k}) = \tilde{e}(\mathbf{k} - \mathbf{p}) \exp\left(-\frac{i}{\hbar}\mathbf{k} \cdot \mathbf{q}\right) \tag{4.8}$$

and let then

$$e(\mathbf{q}, \mathbf{p}; \mathbf{q}', \mathbf{p}') = \frac{1}{h^3}\langle \tilde{e}_{\mathbf{q},\mathbf{p}} \mid \tilde{e}_{\mathbf{q}',\mathbf{p}'}\rangle \tag{4.9}$$

It is easily verified that this function satisfies all required conditions (4.2)–(4.5).

Consider in addition on this space $\mathscr{L}^2(\mathbb{R}^3, d^3k)$ the irreducible unitary representation $\hat{L}$ of $\mathbb{E}^3$, viz.,

$$(\hat{L}(h)\tilde{\varphi})(\mathbf{k}) = \exp\left(-\frac{i}{\hbar}\mathbf{k} \cdot \mathbf{b}\right)\tilde{\varphi}(\alpha^{-1}\mathbf{k}) \tag{4.10}$$

with h as in (3.12). Then the mapping $\tilde{\varphi} \to \phi_e \in \mathcal{H}$, with

$$\phi_e(\mathbf{q}, \mathbf{p}) = \frac{1}{h^{3/2}} \langle \tilde{e}_{\mathbf{q},\mathbf{p}} \mid \tilde{\varphi} \rangle \tag{4.11}$$

embeds $\mathcal{L}^2(\mathbb{R}^2, d^3\mathbf{k})$ isometrically onto $\mathcal{H}_e = \mathbb{P}_e\mathcal{H}$, and the representation $\hat{L}$ onto a (projected) subrepresentation L_e of L as defined by (3.22).

Consider now the reduced K-space

$$K_e = \bigcup_{(n,\tau)\in S} \mathcal{H}_{e,n,\tau} \tag{4.12}$$

with

$$\mathcal{H}_{e,n,\tau} = i_{n,\tau}\mathcal{H}_e \tag{4.13}$$

Obviously K_e carries an irreducible unitary K-representation W_e^K of $P_+^\uparrow$, with the same procedure as before.

It is now possible to give the so-called stochastic phase space interpretation [5] [6] to the wave functions ϕ_e, i.e., $|\phi_e(\mathbf{q}, \mathbf{p})|^2$ is the probability density for finding the particle localized at the stochastic phase space point $((\mathbf{q}, \chi_\mathbf{q}), (\mathbf{p}, \tilde{\chi}_\mathbf{p}))$ with [6]

$$\chi_\mathbf{q}(\mathbf{q}') = |e(\mathbf{q}' - \mathbf{q})|^2 \tag{4.14}$$

and

$$\tilde{\chi}_\mathbf{p}(\mathbf{p}') = |\tilde{e}(\mathbf{p}' - \mathbf{p})|^2 \tag{4.15}$$

$\tilde{e}$ being the Fourier transform of e.

The classical projection operators $P(\Delta)$ defining the phase space localization properties in the space $\mathcal{H}$ of the last section, corresponding to the set $\Delta \subset \mathbb{R}^3 \times V_m^\pm$ are obviously given by

$$(P(\Delta)\phi)(\gamma) = \chi_\Delta(\gamma)\phi(\gamma) \tag{4.16}$$

and on $\mathcal{H}_e$, they become

$$P_e(\Delta) = \mathbb{P}_e P(\Delta)\mathbb{P}_e \tag{4.17}$$

On K we can define the following system of projection valued measures giving rise to a supersystem of imprimitivity. Let

$$(P_{n,\tau}(\Delta)\phi_{n,\tau})(\gamma_{n,\tau}) = \chi_\Delta((\tau n, L_n)^{-1}\gamma_{n,\tau})\phi_{n,\tau}(\gamma_{n,\tau}) \tag{4.18}$$

$\forall \phi_{n,\tau} \in \mathcal{H}_{n,\tau}$ and (almost) all $\gamma_{n,\tau} \in \sum_{n,\tau}^{\pm,m}$. Let then

$$P^K(\Delta) = \{P_{n,\tau}(\Delta) \mid (n, \tau) \in S\} \tag{4.19}$$

the corresponding projection in K (cf. (2.10)); then it is easily computed that

$$(W^K(a, \Lambda)P^K(\Delta)(W^K(a, \Lambda))^{-1})_{n,\tau} = P_{n,\tau}(((0, \underline{L_n^{-1}a}), L_n^{-1}\Lambda L_{\Lambda^{-1}n})\Delta) \tag{4.20}$$

In particular for $n = u_0$ this yields

$$(W^K(a, \Lambda)P^K(\Delta)(W^K(a, \Lambda))^{-1})_{u_0,\tau} = P_{u_0,\tau}(\alpha(\Lambda)\Delta + \mathbf{a}) \tag{4.21}$$

The effect of (a, Λ) on the localization operator $P^K(\Delta)$ corresponding to an observer on the surface $\sum_{n,\tau}^{\pm,m}$, is thus simply to transform the set $(\tau n, L_n)\Delta \subseteq \sum_{n,\tau}^{\pm,m}$ by the Euclidean subgroup part of (a, Λ), as seen by this observer. This important relationship clearly remains the same for the projected operators $P_e(\Delta)$ in (4.17).

Finally we note that if $\mathbf{Q}$ and $\mathbf{P}$ are the corresponding observables, i.e., the operators of component wise multiplication by $\mathbf{q}$ and $\mathbf{p}$ on $\mathscr{H}$ then the operators

$$\mathbf{Q}_e = \mathbb{P}_e \mathbf{Q} \mathbb{P}_e \tag{4.22}$$

and

$$\mathbf{P}_e = \mathbb{P}_e \mathbf{P} \mathbb{P}_e \tag{4.23}$$

satisfy the canonical commutation relations [12]

$$[\mathbf{Q}_e, \mathbf{P}_e]_{n,\tau} = i\hbar \mathbb{1}_{n,\tau} \tag{4.24}$$

exhibiting thus again the fact that $\mathbb{P}_e$ quantizes the theory.

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