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On averaged angular time delay for two-body scattering

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Abstract. In this note, we introduce and discuss the concept of averaged angular time delay, in analogy with averaged total cross-sections.

I. Introduction

Recently, there has been a renewed interest in the study of time delay in scattering theory. (See Ref. [1–6], and the references in [6] for earlier results). These papers discuss the notion of global time delay and its relation to the scattering matrix, using a time-dependent approach.

In this note, we rigorously study the angular time delay from a stationary point of view. Heuristic discussions in this spirit can be found in Ref. [7, 8] and the references cited therein.

In particular, we introduce and discuss averaged angular time delay in analogy with recent treatments of averaged total cross-sections [9–11]. We show that it is equal to the trace of the on-shell global time-delay operator. In establishing this relation, we first have to prove the differentiability of the scattering matrix such that we know that time delay exists as a bounded operator. Furthermore, we have to show that the time-delay operator is trace class. This is done in Theorem 3 for the class of potentials $(1+|\mathbf{x}|)V(\mathbf{x}) \in L^1 \cap R$. This condition roughly implies that $V(\mathbf{x}) = 0(|\mathbf{x}|^{-4-\varepsilon})$, $\varepsilon > 0$ as $|\mathbf{x}| \to \infty$. Note that the differentiability of the scattering matrix has also been proved in Ref. [1, 4] for a more general class of potentials, roughly allowing a $|\mathbf{x}|^{-1-\varepsilon}$ behavior at infinity. The proof we present here is extremely simple and, in addition, yields the trace class property of the time-delay operator. Finally, in Theorem 4, we establish the continuity of the averaged angular time delay (and averaged total cross-section) with respect to interactions.

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II. Some results on two-body scattering

Let $V:\mathbb{R}^3 \to \mathbb{R}$ be a real measurable function and assume $V \in \mathbb{R}$, where \mathbb{R} denotes the Rollnik class (i.e. $\|V\|_{\mathbb{R}}^2 = \int d^3x \, d^3y \, |V(\mathbf{x})| \, |V(\mathbf{y})| \, |\mathbf{x}-\mathbf{y}|^{-2} < \infty$). If $H_0 = -\Delta$ denotes the usual self-adjoint realization of the kinetic energy operator, we define the Hamiltonian $H = H_0 + V$ through the method of forms [12]. Introducing

$$v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}, \qquad u(\mathbf{x}) = v(\mathbf{x}) \operatorname{sign} V(\mathbf{x}), \tag{2.1}$$

the symmetrized resolvent operator $uG_k v$ defined as the norm limit

$$uG_{\pm k}v = n - \lim_{\varepsilon \to 0^+} u(H_0 - k^2 \mp i\varepsilon)^{-1}v, \qquad k \ge 0,$$
(2.2)

is Hilbert-Schmidt and satisfies

$$uG_{-k}v = (uG_{k}v)^{*}.$$
 (2.3)

The scattering wave functions Φ^{\pm} obey the following inhomogeneous Lippmann–Schwinger equations

$$\Phi^{\pm}(k\boldsymbol{\omega}, \mathbf{x}) = \Phi_0^{\pm}(k\boldsymbol{\omega}, \mathbf{x}) - (uG_{\mp k}v\Phi^{\pm})(k\boldsymbol{\omega}, \mathbf{x}), \qquad k^2 \notin \mathscr{E},$$
(2.4a)

which can also be written in the form

$$\Phi^{\pm}(k\boldsymbol{\omega}, \mathbf{x}) = ((1 + uG_{\mp k}v)^{-1}\Phi_0^{\pm})(k\boldsymbol{\omega}, \mathbf{x}), \qquad k^2 \notin \mathscr{E},$$
(2.4b)

where the Φ_0^{\pm} are defined by

$$\Phi_0^-(k\boldsymbol{\omega}, \mathbf{x}) = u(\mathbf{x})e^{ik\boldsymbol{\omega}\cdot\mathbf{x}}, \qquad \Phi_0^+(k\boldsymbol{\omega}, \mathbf{x}) = v(\mathbf{x})e^{ik\boldsymbol{\omega}\cdot\mathbf{x}}, \qquad (2.5)$$

and where $\omega \in S^2$, the unit sphere in \mathbb{R}^3 . The set \mathscr{E} is defined as

$$\mathscr{E} = \{k^2 \ge 0 \mid uG_k v\psi = -\psi \text{ for some } \psi \in L^2(\mathbb{R}^3), k \ge 0\}.$$
(2.6)

 \mathscr{E} is a closed subset of $[0, \infty)$ with Lebesgue measure zero containing the singular continuous spectrum and positive eigenvalues of H [12]. Then we have (for a proof see [12, 13]).

Theorem 1. Let $V \in L^1(\mathbb{R}^3) \cap \mathbb{R}$, then the scattering operator S, associated with the pair (H, H_0) is unitary, commutes with H_0 and in the spectral representation of H_0 , the corresponding on-shell operator S(k) reads

$$(S(k)\phi)(\boldsymbol{\omega}) = \phi(\boldsymbol{\omega}) - \frac{k}{2\pi i} \int_{S^2} d\omega' f(k, \boldsymbol{\omega}, \boldsymbol{\omega}') \phi(\boldsymbol{\omega}'), \qquad k^2 \notin \mathcal{E}, \qquad \phi \in L^2(S^2),$$
(2.7)

where $f(k, \omega, \omega')$ represents the on-shell scattering amplitude. For $k^2 \notin \mathcal{E}$, S(k)-1 is a trace class-operator in $L^2(S^2)$, i.e. $[S(k)-1] \in \mathcal{B}_1(L^2(S^2))$, $k^2 \notin \mathcal{E}$, and continuous in trace norm with respect to k. Furthermore, an explicit characterization of $f(k, \omega, \omega')$ is obtained from

$$f(k, \boldsymbol{\omega}, \boldsymbol{\omega}') = -(4\pi)^{-1}(\Phi_0^+(k\boldsymbol{\omega}), \Phi^-(k\boldsymbol{\omega}'))$$

= $-(4\pi)^{-1}(\Phi^+(k\boldsymbol{\omega}), \Phi_0^-(k\boldsymbol{\omega}')), \quad k^2 \notin \mathscr{E}.$ (2.8)

Finally, $f(k, \omega, \omega')$ is uniformly continuous in all variables if k^2 varies in compact intervals not intersecting \mathcal{E} .

Following [9–11], the averaged total cross-section $\bar{\sigma}(k)$ is defined as

$$\bar{\sigma}(k) = (4\pi)^{-1} \int_{S^2} d\omega \int_{S^2} d\omega' |f(k, \omega, \omega')|^2$$
$$= \frac{\pi}{k^2} ||S(k) - 1||_2^2 = -\frac{2\pi}{k} \operatorname{Re} \left(\operatorname{Tr} \left(S(k) - 1\right)\right).$$
(2.9)

Theorem 1 then immediately leads to

Theorem 2. Assume $V \in L^1(\mathbb{R}^3) \cap \mathbb{R}$. Then for $k^2 \notin \mathcal{E}$, $\bar{\sigma}(k)$ is finite and continuous in k. In addition, the optical theorem

$$\sigma(k,\boldsymbol{\omega}) = \int_{S^2} d\boldsymbol{\omega}' |f(k,\boldsymbol{\omega},\boldsymbol{\omega}')|^2 = \frac{4\pi}{k} \operatorname{Im} f(k,\boldsymbol{\omega},\boldsymbol{\omega})$$
(2.10)

is valid.

3. Averaged angular time delay

In the following we discuss the necessary modifications of the above approach in order to study the concept of averaged angular time delay in a rigorous way. We start with

Lemma 1. Let $(1+|\mathbf{x}|) V \in L^1(\mathbb{R}^3) \cap \mathbb{R}$ and $k^2 \notin \mathscr{E}$. Then $(1+|\mathbf{x}|)^{1/2} \Phi^{\pm}(k\omega, \mathbf{x})$ is strongly continuous in $L^2(\mathbb{R}^3)$ with respect to k for all $\omega \in S^2$ and $f(k, \omega, \omega')$ is continuously differentiable with respect to k for all $\omega, \omega' \in S^2 \times S^2$. In particular, S(k) is continuously differentiable in trace norm, i.e. $S'(k) \in \mathfrak{B}_1(L^2(S^2))$ and $(-k(2\pi i)^{-1}f(k, \omega, \omega'))'$ is the kernel of S'(k). (The ' denotes the derivative with respect to k).

Proof. The strong continuity of $(1+|\mathbf{x}|)^{1/2}\Phi^{\pm}(k\omega, \mathbf{x})$ is clear from (2.4) and the condition on V. (cf. [12], Section IV.5). To prove the rest of the lemma, we first note that, using (2.4) and (2.8)

$$f'(k, \boldsymbol{\omega}, \boldsymbol{\omega}') = i(4\pi)^{-1}(|\boldsymbol{\omega} \cdot \mathbf{x}|^{1/2}\Phi_0^+(k\boldsymbol{\omega}), \operatorname{sign}(\boldsymbol{\omega} \cdot \mathbf{x})|\boldsymbol{\omega} \cdot \mathbf{x}|^{1/2}\Phi^-(k\boldsymbol{\omega}')) + (4\pi)^{-1}(\Phi^+(k\boldsymbol{\omega}), (uG'_k v \Phi^-)(k\boldsymbol{\omega}')) - i(4\pi)^{-1}(|\boldsymbol{\omega}' \cdot \mathbf{x}|^{1/2}\Phi^+(k\boldsymbol{\omega}), \operatorname{sign}(\boldsymbol{\omega}' \cdot \mathbf{x})|\boldsymbol{\omega}' \cdot \mathbf{x})|^{1/2}\Phi_0^-(k\boldsymbol{\omega}')), k^2 \notin \mathscr{E},$$
(3.1)

where $uG'_k v$ denotes the Hilbert-Schmidt operator represented by the kernel

$$(\mathbf{u}G_{k}'\mathbf{v})(\mathbf{x},\mathbf{y}) = i(4\pi)^{-1}\mathbf{u}(\mathbf{x})e^{ik|\mathbf{x}-\mathbf{y}|}\mathbf{v}(\mathbf{y}).$$
(3.2)

Equation (3.1) proves the assertions for the on-shell scattering amplitude $f(k, \omega, \omega')$. Next, following Ref. [13], we introduce the maps

$$A_{\mathbf{V}}(k): \begin{cases} L^{2}(\mathbb{R}^{3}) \to L^{2}(S^{2}) \\ g(\mathbf{x}) \to (A_{\mathbf{V}}(k)g)(\boldsymbol{\omega}) = \int_{\mathbb{R}^{3}} d^{3}x u(\mathbf{x}) e^{-ik\boldsymbol{\omega}\cdot\mathbf{x}} g(\mathbf{x}) \end{cases}$$
(3.3)

and

$$A_{V}(k)^{*}:\begin{cases} L^{2}(S^{2}) \to L^{2}(\mathbb{R}^{3}) \\ \phi(\boldsymbol{\omega}) \to (A_{V}(k)^{*}\phi)(\mathbf{x}) = \int_{S^{2}} d\omega u(\mathbf{x}) e^{ik\boldsymbol{\omega}\cdot\mathbf{x}}\phi(\boldsymbol{\omega}). \end{cases}$$
(3.4)

Since $A_{|V|}(k) \in \mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$ we have that

$$S(k) - 1 = \frac{-k}{2\pi i} A_{|V|}(k) (1 + uG_k v)^{-1} A_V(k)^* \in \mathcal{B}_1(L^2(S^2)), \qquad k^2 \notin \mathscr{E}.$$
(3.5)

For $k_0^2 \notin \mathcal{E}$ and $|k - k_0|$ small enough such that also $k^2 \notin \mathcal{E}$, we consider, using equations (2.4), (2.7) and (2.8),

$$\begin{split} \left[S(k) - S(k_0) \right] / (k - k_0) &= -\frac{1}{2\pi i} \left\{ A_{|V|}(k) (1 + uG_k v)^{-1} A_V(k)^* \\ &+ \left[\frac{k_0}{k - k_0} \left(A_{|V|}(k) - A_{|V|}(k_0) \right) (1 + |\mathbf{x}|)^{-1/2} \right] \\ &\cdot \left[(1 + |\mathbf{x}|)^{1/2} (1 + uG_k v)^{-1} A_V(k)^* \right] \\ &+ \left[k_0 A_{|V|}(k_0) \frac{1}{k - k_0} \left[(1 + uG_k v)^{-1} - (1 + uG_{k_0} v)^{-1} \right] A_V(k)^* \\ &+ \left[k_0 A_{|V|}(k_0) (1 + uG_{k_0} v)^{-1} (1 + |\mathbf{x}|)^{1/2} \right] \\ &\cdot \left[\frac{1}{k - k_0} \left(1 + |\mathbf{x}| \right)^{-1/2} \left(A_V(k)^* - A_V(k_0)^* \right) \right] \right\} \end{split}$$

Looking e.g. at the kernel of the operator $(1+|\mathbf{x}|)^{1/2}(1+uG_kv)^{-1}A_V(k)^*$, we infer from Lemma 1 that this operator is in $\mathcal{B}_2(L^2(S^2), L^2(\mathbb{R}^3))$ and continuous with respect to k in Hilbert-Schmidt norm. Similarly $[k_0/(k-k_0)][A_{|V|}(k)-A_{|V|}(k_0)]$ $(1+|\mathbf{x}|)^{-1/2}$ converges in Hilbert-Schmidt norm to a $\mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$ -valued operator with kernel $-ik_0(\boldsymbol{\omega} \cdot \mathbf{x})(1+|\mathbf{x}|)^{-1/2}u(\mathbf{x}) \exp(-ik_0\boldsymbol{\omega} \cdot \mathbf{x})$. Thus recalling also (3.5) we have that $S'(k_0) \in \mathcal{B}_1(L^2(S^2))$ with kernel $(-k_0(2\pi i)^{-1}f(k_0, \boldsymbol{\omega}, \boldsymbol{\omega}'))'$.

In analogy with eq. (2.9) we now introduce the averaged angular time delay $\overline{\tau}(k)$ by the following definition

$$\bar{\tau}(k) = \int_{\mathbf{S}^2} d\omega \tau_a(k, \boldsymbol{\omega}), \qquad k > 0, \qquad k^2 \notin \mathcal{E},$$
(3.7)

where the angular time delay τ_a is given by [7, 8]

$$\tau_{a}(k,\boldsymbol{\omega}) = \frac{1}{4\pi k} \frac{\partial}{\partial k} (\operatorname{Re} kf(k,\boldsymbol{\omega},\boldsymbol{\omega})) + \frac{k}{8\pi^{2}} \int_{S^{2}} d\omega' |f(k,\boldsymbol{\omega},\boldsymbol{\omega}')|^{2} \frac{\partial}{\partial k} (\arg f(k,\boldsymbol{\omega},\boldsymbol{\omega}')), \quad k > 0, \quad k^{2} \notin \mathscr{E}.$$
(3.8)

We then have

Theorem 3. Let $(1+|\mathbf{x}|) V \in L^1(\mathbb{R}^3) \cap \mathbb{R}$. Then, for k > 0 and $k^2 \notin \mathcal{E}$, the averaged angular time delay $\overline{\tau}(k)$ is finite and continuous in k. Furthermore

$$\bar{\tau}(k) = \operatorname{Tr}(\tau(k)), \quad k > 0, \quad k^2 \notin \mathscr{E},$$
(3.9)

where $\tau(k)$ is the on-shell time delay operator defined by

$$\tau(k) = -\frac{\iota}{2k} S^*(k) S'(k), \qquad k > 0, \qquad k^2 \notin \mathscr{E}.$$
(3.10)

Proof. the continuity of $\bar{\tau}(k)$ and the fact that $\tau(k) \in \mathcal{B}_1(L^2(S^2))$ follow from Lemma 1. Equation (3.9) is directly obtained from equations (2.7), (3.7)–(3.8) and unitarity of S which allows (3.10) to be written in the form

$$\tau(k) = \frac{1}{2} \left[-\frac{i}{2k} S^*(k) S'(k) + \frac{i}{2k} S^{*'}(k) S(k) \right], \qquad k \ge 0, \qquad k^2 \notin \mathscr{E}. \quad \blacksquare \quad (3.11)$$

Remark 1. a) If V is spherically symmetric, then

$$\bar{\tau}(k) = \sum_{l} (2l+1) \frac{1}{k} \frac{\partial}{\partial k} \delta_{l}(k),$$

where $\delta_l(k)$ is the partial wave phase shift.

b) For a connection between $\overline{\tau}(k)$ and two-body Levinson's theorem in a generalised form we refer to [14–16] and [6].

c) Definition (3.7) and Theorem 3 easily extend to relative time delay (as discussed e.g. in [3]).

Remark 2. Under appropriate additional assumptions on V, the singular continuous spectrum of H is empty (cf. [13] and references therein). In that case \mathscr{E} consists of the positive point spectrum of H. It has been shown in [15] that under suitable conditions on V, the positive eigenvalues of H decouple from the scattering phenomena in the sense that the scattering amplitude remains continuous at these points. However, the threshold point k = 0 needs a separate discussion. If $0 \in \mathscr{E}$, one has to distinguish whether it is a bound state of H, or a resonance, or both. (cf. e.g. Ref. [17]). Only if $0 \notin \mathscr{E}$ or if 0 is a bound state of H, the averaged total cross-section $\overline{\sigma}(0_+)$ remains finite. On the contrary, the limit of the averaged time delay, $\lim_{k\to 0_+} \overline{\tau}(k)$, never exists, irrespective whether $0 \in \mathscr{E}$ or not.

Finally, we state a continuity result for the averaged angular time delay and the averaged total cross-section with respect to the interactions. Assume $V, V_n \in L^1(\mathbb{R}^3) \cap R$ to be real valued, n = 1, 2, ... and denote by \mathcal{E}_n the exceptional sets corresponding to $H_n = H_0 + V_n$ (cf. equation (2.6)). Furthermore, let $\bar{\sigma}_n(k)$ and $\bar{\tau}_n(k)$ be the averaged total cross-section respectively the averaged angular time delay for the interaction V_n . Then we have

Theorem 4. a) Suppose
$$V, V_n \in L^1(\mathbb{R}^3) \cap R, n = 1, 2, ...$$
 and

$$\lim_{n \to \infty} ||V_n - V||_1 = 0, \qquad \lim_{n \to \infty} ||V_n - V||_R = 0. \qquad \text{Then}$$

$$\lim_{n \to \infty} \bar{\sigma}_n(k) = \bar{\sigma}(k), \qquad k^2 \notin \mathscr{E}. \qquad (3.12)$$
b) Assume $(1 + |\mathbf{x}|)V, \ (1 + |\mathbf{x}|)V_n \in L^1(\mathbb{R}^3) \cap R, \ n = 1, 2, ... \text{ and}$

$$\lim_{n \to \infty} ||(1 + |\mathbf{x}|)(V_n - V)||_1 = 0, \qquad \lim_{n \to \infty} ||(1 + |\mathbf{x}|)(V_n - V)||_R = 0. \qquad \text{Then}$$

$$\lim_{n \to \infty} \bar{\tau}_n(k) = \bar{\tau}(k), \qquad k^2 \notin \mathscr{E}. \qquad (3.13)$$

Proof. Continuity of the scattering amplitude $f_n(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ and the scattering matrix $S_n(k)$ associated with the pair of Hamiltonians (H_n, H_0) under the hypothesis a) has been proved in Ref. [18]. By exactly the same methods, assumption b) implies continuity of $f'_n(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ and $S'_n(k)$, completing the proof. (We remark that by the joint continuity of $uG_k v$ in k and V, $k^2 \in \mathscr{E}$ implies $k^2 \notin \mathscr{E}_n$ for n large enough).

Of course, the concept of time delay is not restricted to the simple case of two-body potential scattering. In the literature, one finds discussion of time delay for multiparticle systems ([7], [19] and [6]). Also nonlocal [20], dissipative [6, 21] and Couloumb interactions [3] were investigated. Finally, we also mention a recent treatment of the Lax-Philips scattering theory [5]. It is clear that our approach presented above could be extended to all these situations.

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