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# On averaged angular time delay for two-body scattering

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*Abstract.* In this note, we introduce and discuss the concept of averaged angular time delay, in analogy with averaged total cross-sections.

## I. Introduction

Recently, there has been a renewed interest in the study of time delay in scattering theory. (See Ref. [1–6], and the references in [6] for earlier results). These papers discuss the notion of global time delay and its relation to the scattering matrix, using a time-dependent approach.

In this note, we rigorously study the angular time delay from a stationary point of view. Heuristic discussions in this spirit can be found in Ref. [7, 8] and the references cited therein.

In particular, we introduce and discuss averaged angular time delay in analogy with recent treatments of averaged total cross-sections [9–11]. We show that it is equal to the trace of the on-shell global time-delay operator. In establishing this relation, we first have to prove the differentiability of the scattering matrix such that we know that time delay exists as a bounded operator. Furthermore, we have to show that the time-delay operator is trace class. This is done in Theorem 3 for the class of potentials  $(1 + |\mathbf{x}|)V(\mathbf{x}) \in L^1 \cap R$ . This condition roughly implies that  $V(\mathbf{x}) = O(|\mathbf{x}|^{-4-\varepsilon})$ ,  $\varepsilon > 0$  as  $|\mathbf{x}| \rightarrow \infty$ . Note that the differentiability of the scattering matrix has also been proved in Ref. [1, 4] for a more general class of potentials, roughly allowing a  $|\mathbf{x}|^{-1-\varepsilon}$  behavior at infinity. The proof we present here is extremely simple and, in addition, yields the trace class property of the time-delay operator. Finally, in Theorem 4, we establish the continuity of the averaged angular time delay (and averaged total cross-section) with respect to interactions.

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## II. Some results on two-body scattering

Let  $V: \mathbb{R}^3 \rightarrow \mathbb{R}$  be a real measurable function and assume  $V \in R$ , where  $R$  denotes the Rollnik class (i.e.  $\|V\|_R^2 = \int d^3x d^3y |V(\mathbf{x})| |V(\mathbf{y})| |\mathbf{x} - \mathbf{y}|^{-2} < \infty$ ). If  $H_0 = -\Delta$  denotes the usual self-adjoint realization of the kinetic energy operator, we define the Hamiltonian  $H = H_0 + V$  through the method of forms [12]. Introducing

$$v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}, \quad u(\mathbf{x}) = v(\mathbf{x}) \operatorname{sign} V(\mathbf{x}), \quad (2.1)$$

the symmetrized resolvent operator  $uG_k v$  defined as the norm limit

$$uG_{\pm k} v = n - \lim_{\varepsilon \rightarrow 0^+} u(H_0 - k^2 \mp i\varepsilon)^{-1} v, \quad k \geq 0, \quad (2.2)$$

is Hilbert-Schmidt and satisfies

$$uG_{-k} v = (uG_k v)^*. \quad (2.3)$$

The scattering wave functions  $\Phi^\pm$  obey the following inhomogeneous Lippmann-Schwinger equations

$$\Phi^\pm(k\omega, \mathbf{x}) = \Phi_0^\pm(k\omega, \mathbf{x}) - (uG_{\mp k} v \Phi^\pm)(k\omega, \mathbf{x}), \quad k^2 \notin \mathcal{E}, \quad (2.4a)$$

which can also be written in the form

$$\Phi^\pm(k\omega, \mathbf{x}) = ((1 + uG_{\mp k} v)^{-1} \Phi_0^\pm)(k\omega, \mathbf{x}), \quad k^2 \notin \mathcal{E}, \quad (2.4b)$$

where the  $\Phi_0^\pm$  are defined by

$$\Phi_0^-(k\omega, \mathbf{x}) = u(\mathbf{x}) e^{ik\omega \cdot \mathbf{x}}, \quad \Phi_0^+(k\omega, \mathbf{x}) = v(\mathbf{x}) e^{ik\omega \cdot \mathbf{x}}, \quad (2.5)$$

and where  $\omega \in S^2$ , the unit sphere in  $\mathbb{R}^3$ . The set  $\mathcal{E}$  is defined as

$$\mathcal{E} = \{k^2 \geq 0 \mid uG_k v \psi = -\psi \text{ for some } \psi \in L^2(\mathbb{R}^3), k \geq 0\}. \quad (2.6)$$

$\mathcal{E}$  is a closed subset of  $[0, \infty)$  with Lebesgue measure zero containing the singular continuous spectrum and positive eigenvalues of  $H$  [12]. Then we have (for a proof see [12, 13]).

**Theorem 1.** *Let  $V \in L^1(\mathbb{R}^3) \cap R$ , then the scattering operator  $S$ , associated with the pair  $(H, H_0)$  is unitary, commutes with  $H_0$  and in the spectral representation of  $H_0$ , the corresponding on-shell operator  $S(k)$  reads*

$$(S(k)\phi)(\omega) = \phi(\omega) - \frac{k}{2\pi i} \int_{S^2} d\omega' f(k, \omega, \omega') \phi(\omega'), \quad k^2 \notin \mathcal{E}, \quad \phi \in L^2(S^2), \quad (2.7)$$

where  $f(k, \omega, \omega')$  represents the on-shell scattering amplitude. For  $k^2 \notin \mathcal{E}$ ,  $S(k) - 1$  is a trace class-operator in  $L^2(S^2)$ , i.e.  $[S(k) - 1] \in \mathcal{B}_1(L^2(S^2))$ ,  $k^2 \notin \mathcal{E}$ , and continuous in trace norm with respect to  $k$ . Furthermore, an explicit characterization of  $f(k, \omega, \omega')$  is obtained from

$$\begin{aligned} f(k, \omega, \omega') &= -(4\pi)^{-1} (\Phi_0^+(k\omega), \Phi^-(k\omega')) \\ &= -(4\pi)^{-1} (\Phi^+(k\omega), \Phi_0^-(k\omega')), \quad k^2 \notin \mathcal{E}. \end{aligned} \quad (2.8)$$

Finally,  $f(k, \omega, \omega')$  is uniformly continuous in all variables if  $k^2$  varies in compact intervals not intersecting  $\mathcal{E}$ .

Following [9–11], the averaged total cross-section  $\bar{\sigma}(k)$  is defined as

$$\begin{aligned}\bar{\sigma}(k) &= (4\pi)^{-1} \int_{S^2} d\omega \int_{S^2} d\omega' |f(k, \omega, \omega')|^2 \\ &= \frac{\pi}{k^2} \|S(k) - 1\|_2^2 = -\frac{2\pi}{k} \operatorname{Re} (\operatorname{Tr} (S(k) - 1)).\end{aligned}\quad (2.9)$$

Theorem 1 then immediately leads to

**Theorem 2.** Assume  $V \in L^1(\mathbb{R}^3) \cap R$ . Then for  $k^2 \notin \mathcal{E}$ ,  $\bar{\sigma}(k)$  is finite and continuous in  $k$ . In addition, the optical theorem

$$\sigma(k, \omega) = \int_{S^2} d\omega' |f(k, \omega, \omega')|^2 = \frac{4\pi}{k} \operatorname{Im} f(k, \omega, \omega) \quad (2.10)$$

is valid.

### 3. Averaged angular time delay

In the following we discuss the necessary modifications of the above approach in order to study the concept of averaged angular time delay in a rigorous way. We start with

**Lemma 1.** Let  $(1 + |\mathbf{x}|)V \in L^1(\mathbb{R}^3) \cap R$  and  $k^2 \notin \mathcal{E}$ . Then  $(1 + |\mathbf{x}|)^{1/2} \Phi^\pm(k\omega, \mathbf{x})$  is strongly continuous in  $L^2(\mathbb{R}^3)$  with respect to  $k$  for all  $\omega \in S^2$  and  $f(k, \omega, \omega')$  is continuously differentiable with respect to  $k$  for all  $\omega, \omega' \in S^2 \times S^2$ . In particular,  $S(k)$  is continuously differentiable in trace norm, i.e.  $S'(k) \in \mathcal{B}_1(L^2(S^2))$  and  $(-k(2\pi i)^{-1} f(k, \omega, \omega'))'$  is the kernel of  $S'(k)$ . (The ' denotes the derivative with respect to  $k$ ).

*Proof.* The strong continuity of  $(1 + |\mathbf{x}|)^{1/2} \Phi^\pm(k\omega, \mathbf{x})$  is clear from (2.4) and the condition on  $V$ . (cf. [12], Section IV.5). To prove the rest of the lemma, we first note that, using (2.4) and (2.8)

$$\begin{aligned}f'(k, \omega, \omega') &= i(4\pi)^{-1} (|\omega \cdot \mathbf{x}|^{1/2} \Phi_0^+(k\omega), \operatorname{sign}(\omega \cdot \mathbf{x}) |\omega \cdot \mathbf{x}|^{1/2} \Phi^-(k\omega')) \\ &\quad + (4\pi)^{-1} (\Phi^+(k\omega), (uG'_k v \Phi^-)(k\omega')) \\ &\quad - i(4\pi)^{-1} (|\omega' \cdot \mathbf{x}|^{1/2} \Phi^+(k\omega), \operatorname{sign}(\omega' \cdot \mathbf{x}) |\omega' \cdot \mathbf{x}|^{1/2} \Phi_0^-(k\omega')), \\ &\quad k^2 \notin \mathcal{E},\end{aligned}\quad (3.1)$$

where  $uG'_k v$  denotes the Hilbert–Schmidt operator represented by the kernel

$$(uG'_k v)(\mathbf{x}, \mathbf{y}) = i(4\pi)^{-1} u(\mathbf{x}) e^{ik|\mathbf{x}-\mathbf{y}|} v(\mathbf{y}). \quad (3.2)$$

Equation (3.1) proves the assertions for the on-shell scattering amplitude  $f(k, \omega, \omega')$ . Next, following Ref. [13], we introduce the maps

$$A_V(k): \begin{cases} L^2(\mathbb{R}^3) \rightarrow L^2(S^2) \\ g(\mathbf{x}) \rightarrow (A_V(k)g)(\omega) = \int_{\mathbb{R}^3} d^3x u(\mathbf{x}) e^{-ik\omega \cdot \mathbf{x}} g(\mathbf{x}) \end{cases} \quad (3.3)$$

and

$$A_V(k)^*: \begin{cases} L^2(S^2) \rightarrow L^2(\mathbb{R}^3) \\ \phi(\omega) \rightarrow (A_V(k)^* \phi)(\mathbf{x}) = \int_{S^2} d\omega u(\mathbf{x}) e^{ik\omega \cdot \mathbf{x}} \phi(\omega). \end{cases} \quad (3.4)$$

Since  $A_{|V|}(k) \in \mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$  we have that

$$S(k) - 1 = \frac{-k}{2\pi i} A_{|V|}(k) (1 + uG_k v)^{-1} A_V(k)^* \in \mathcal{B}_1(L^2(S^2)), \quad k^2 \notin \mathcal{E}. \quad (3.5)$$

For  $k_0^2 \notin \mathcal{E}$  and  $|k - k_0|$  small enough such that also  $k^2 \notin \mathcal{E}$ , we consider, using equations (2.4), (2.7) and (2.8),

$$\begin{aligned} [S(k) - S(k_0)]/(k - k_0) = & -\frac{1}{2\pi i} \left\{ A_{|V|}(k) (1 + uG_k v)^{-1} A_V(k)^* \right. \\ & + \left[ \frac{k_0}{k - k_0} (A_{|V|}(k) - A_{|V|}(k_0)) (1 + |\mathbf{x}|)^{-1/2} \right] \\ & \cdot [(1 + |\mathbf{x}|)^{1/2} (1 + uG_k v)^{-1} A_V(k)^*] \\ & + k_0 A_{|V|}(k_0) \frac{1}{k - k_0} [(1 + uG_k v)^{-1} - (1 + uG_{k_0} v)^{-1}] A_V(k)^* \\ & + [k_0 A_{|V|}(k_0) (1 + uG_{k_0} v)^{-1} (1 + |\mathbf{x}|)^{1/2}] \\ & \cdot \left. \left[ \frac{1}{k - k_0} (1 + |\mathbf{x}|)^{-1/2} (A_V(k)^* - A_V(k_0)^*) \right] \right\} \end{aligned} \quad (3.6)$$

Looking e.g. at the kernel of the operator  $(1 + |\mathbf{x}|)^{1/2} (1 + uG_k v)^{-1} A_V(k)^*$ , we infer from Lemma 1 that this operator is in  $\mathcal{B}_2(L^2(S^2), L^2(\mathbb{R}^3))$  and continuous with respect to  $k$  in Hilbert-Schmidt norm. Similarly  $[k_0/(k - k_0)] [A_{|V|}(k) - A_{|V|}(k_0)] (1 + |\mathbf{x}|)^{-1/2}$  converges in Hilbert-Schmidt norm to a  $\mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$ -valued operator with kernel  $-ik_0(\omega \cdot \mathbf{x})(1 + |\mathbf{x}|)^{-1/2} u(\mathbf{x}) \exp(-ik_0 \omega \cdot \mathbf{x})$ . Thus recalling also (3.5) we have that  $S'(k_0) \in \mathcal{B}_1(L^2(S^2))$  with kernel  $(-k_0(2\pi i)^{-1} f(k_0, \omega, \omega'))'$ . ■

In analogy with eq. (2.9) we now introduce the averaged angular time delay  $\bar{\tau}(k)$  by the following definition

$$\bar{\tau}(k) = \int_{S^2} d\omega \tau_a(k, \omega), \quad k > 0, \quad k^2 \notin \mathcal{E}, \quad (3.7)$$

where the angular time delay  $\tau_a$  is given by [7, 8]

$$\begin{aligned} \tau_a(k, \omega) = & \frac{1}{4\pi k} \frac{\partial}{\partial k} (\operatorname{Re} k f(k, \omega, \omega)) \\ & + \frac{k}{8\pi^2} \int_{S^2} d\omega' |f(k, \omega, \omega')|^2 \frac{\partial}{\partial k} (\arg f(k, \omega, \omega')), \quad k > 0, \quad k^2 \notin \mathcal{E}. \end{aligned} \quad (3.8)$$

We then have

**Theorem 3.** *Let  $(1 + |\mathbf{x}|)V \in L^1(\mathbb{R}^3) \cap R$ . Then, for  $k > 0$  and  $k^2 \notin \mathcal{E}$ , the averaged angular time delay  $\bar{\tau}(k)$  is finite and continuous in  $k$ . Furthermore*

$$\bar{\tau}(k) = \operatorname{Tr}(\tau(k)), \quad k > 0, \quad k^2 \notin \mathcal{E}, \quad (3.9)$$

where  $\tau(k)$  is the on-shell time delay operator defined by

$$\tau(k) = -\frac{i}{2k} S^*(k)S'(k), \quad k > 0, \quad k^2 \notin \mathcal{E}. \quad (3.10)$$

*Proof.* the continuity of  $\bar{\tau}(k)$  and the fact that  $\tau(k) \in \mathcal{B}_1(L^2(S^2))$  follow from Lemma 1. Equation (3.9) is directly obtained from equations (2.7), (3.7)–(3.8) and unitarity of  $S$  which allows (3.10) to be written in the form

$$\tau(k) = \frac{1}{2} \left[ -\frac{i}{2k} S^*(k)S'(k) + \frac{i}{2k} S^{*'}(k)S(k) \right], \quad k > 0, \quad k^2 \notin \mathcal{E}. \quad \blacksquare \quad (3.11)$$

*Remark 1.* a) If  $V$  is spherically symmetric, then

$$\bar{\tau}(k) = \sum_l (2l+1) \frac{1}{k} \frac{\partial}{\partial k} \delta_l(k),$$

where  $\delta_l(k)$  is the partial wave phase shift.

b) For a connection between  $\bar{\tau}(k)$  and two-body Levinson's theorem in a generalised form we refer to [14–16] and [6].

c) Definition (3.7) and Theorem 3 easily extend to relative time delay (as discussed e.g. in [3]).

*Remark 2.* Under appropriate additional assumptions on  $V$ , the singular continuous spectrum of  $H$  is empty (cf. [13] and references therein). In that case  $\mathcal{E}$  consists of the positive point spectrum of  $H$ . It has been shown in [15] that under suitable conditions on  $V$ , the positive eigenvalues of  $H$  decouple from the scattering phenomena in the sense that the scattering amplitude remains continuous at these points. However, the threshold point  $k = 0$  needs a separate discussion. If  $0 \in \mathcal{E}$ , one has to distinguish whether it is a bound state of  $H$ , or a resonance, or both. (cf. e.g. Ref. [17]). Only if  $0 \notin \mathcal{E}$  or if  $0$  is a bound state of  $H$ , the averaged total cross-section  $\bar{\sigma}(0_+)$  remains finite. On the contrary, the limit of the averaged time delay,  $\lim_{k \rightarrow 0_+} \bar{\tau}(k)$ , never exists, irrespective whether  $0 \in \mathcal{E}$  or not.

Finally, we state a continuity result for the averaged angular time delay and the averaged total cross-section with respect to the interactions. Assume  $V, V_n \in L^1(\mathbb{R}^3) \cap \mathcal{R}$  to be real valued,  $n = 1, 2, \dots$  and denote by  $\mathcal{E}_n$  the exceptional sets corresponding to  $H_n = H_0 + V_n$  (cf. equation (2.6)). Furthermore, let  $\bar{\sigma}_n(k)$  and  $\bar{\tau}_n(k)$  be the averaged total cross-section respectively the averaged angular time delay for the interaction  $V_n$ . Then we have

**Theorem 4.** a) Suppose  $V, V_n \in L^1(\mathbb{R}^3) \cap \mathcal{R}$ ,  $n = 1, 2, \dots$  and

$$\lim_{n \rightarrow \infty} \|V_n - V\|_1 = 0, \quad \lim_{n \rightarrow \infty} \|V_n - V\|_{\mathcal{R}} = 0. \quad \text{Then} \\ \lim_{n \rightarrow \infty} \bar{\sigma}_n(k) = \bar{\sigma}(k), \quad k^2 \notin \mathcal{E}. \quad (3.12)$$

b) Assume  $(1 + |\mathbf{x}|)V, (1 + |\mathbf{x}|)V_n \in L^1(\mathbb{R}^3) \cap \mathcal{R}$ ,  $n = 1, 2, \dots$  and

$$\lim_{n \rightarrow \infty} \|(1 + |\mathbf{x}|)(V_n - V)\|_1 = 0, \quad \lim_{n \rightarrow \infty} \|(1 + |\mathbf{x}|)(V_n - V)\|_{\mathcal{R}} = 0. \quad \text{Then} \\ \lim_{n \rightarrow \infty} \bar{\tau}_n(k) = \bar{\tau}(k), \quad k^2 \notin \mathcal{E}. \quad (3.13)$$

*Proof.* Continuity of the scattering amplitude  $f_n(k, \omega, \omega')$  and the scattering matrix  $S_n(k)$  associated with the pair of Hamiltonians  $(H_n, H_0)$  under the hypothesis a) has been proved in Ref. [18]. By exactly the same methods, assumption b) implies continuity of  $f'_n(k, \omega, \omega')$  and  $S'_n(k)$ , completing the proof. (We remark that by the joint continuity of  $uG_k v$  in  $k$  and  $V$ ,  $k^2 \in \mathcal{E}$  implies  $k^2 \notin \mathcal{E}_n$  for  $n$  large enough). ■

Of course, the concept of time delay is not restricted to the simple case of two-body potential scattering. In the literature, one finds discussion of time delay for multiparticle systems ([7], [19] and [6]). Also nonlocal [20], dissipative [6, 21] and Coulomb interactions [3] were investigated. Finally, we also mention a recent treatment of the Lax–Philips scattering theory [5]. It is clear that our approach presented above could be extended to all these situations.

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