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On averaged angular time delay for two-body scattering

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Abstract. In this note, we introduce and discuss the concept of averaged angular time delay, in analogy with averaged total cross-sections.

I. Introduction

Recently, there has been a renewed interest in the study of time delay in scattering theory. (See Ref. [1–6], and the references in [6] for earlier results). These papers discuss the notion of global time delay and its relation to the scattering matrix, using a time-dependent approach.

In this note, we rigorously study the angular time delay from a stationary point of view. Heuristic discussions in this spirit can be found in Ref. [7, 8] and the references cited therein.

In particular, we introduce and discuss averaged angular time delay in analogy with recent treatments of averaged total cross-sections [9–11]. We show that it is equal to the trace of the on-shell global time-delay operator. In establishing this relation, we first have to prove the differentiability of the scattering matrix such that we know that time delay exists as a bounded operator. Furthermore, we have to show that the time-delay operator is trace class. This is done in Theorem 3 for the class of potentials $(1+|\mathbf{x}|)V(\mathbf{x}) \in L^1 \cap R$. This condition roughly implies that $V(\mathbf{x}) = 0(|\mathbf{x}|^{-4-\varepsilon})$, $\varepsilon > 0$ as $|\mathbf{x}| \to \infty$. Note that the differentiability of the scattering matrix has also been proved in Ref. [1, 4] for a more general class of potentials, roughly allowing a $|\mathbf{x}|^{-1-\varepsilon}$ behavior at infinity. The proof we present here is extremely simple and, in addition, yields the trace class property of the time-delay operator. Finally, in Theorem 4, we establish the continuity of the averaged angular time delay (and averaged total cross-section) with respect to interactions.

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II. Some results on two-body scattering

Let $V:\mathbb{R}^3 \to \mathbb{R}$ be a real measurable function and assume $V \in R$, where R denotes the Rollnik class (i.e. $\|V\|_R^2 = \int d^3x\, d^3y\, |V(\mathbf{x})|\, |V(\mathbf{y})|\, |\mathbf{x}-\mathbf{y}|^{-2} < \infty$). If $H_0 = -\Delta$ denotes the usual self-adjoint realization of the kinetic energy operator, we define the Hamiltonian $H = H_0 \dot{+} V$ through the method of forms [12]. Introducing

$$v(\mathbf{x}) = |V(\mathbf{x})|^{1/2}, \qquad u(\mathbf{x}) = v(\mathbf{x}) \operatorname{sign} V(\mathbf{x}), \tag{2.1}$$

the symmetrized resolvent operator $uG_k v$ defined as the norm limit

$$uG_{\pm k} v = n - \lim_{\varepsilon \to 0^+} u(H_0 - k^2 \mp i\varepsilon)^{-1} v, \qquad k \geq 0, \tag{2.2}$$

is Hilbert–Schmidt and satisfies

$$uG_{-k} v = (uG_k v)^*. \tag{2.3}$$

The scattering wave functions $\Phi^\pm$ obey the following inhomogeneous Lippmann–Schwinger equations

$$\Phi^\pm(k\boldsymbol{\omega}, \mathbf{x}) = \Phi_0^\pm(k\boldsymbol{\omega}, \mathbf{x}) - (uG_{\mp k} v \Phi^\pm)(k\boldsymbol{\omega}, \mathbf{x}), \qquad k^2 \notin \mathscr{E}, \tag{2.4a}$$

which can also be written in the form

$$\Phi^\pm(k\boldsymbol{\omega}, \mathbf{x}) = ((1 + uG_{\mp k} v)^{-1} \Phi_0^\pm)(k\boldsymbol{\omega}, \mathbf{x}), \qquad k^2 \notin \mathscr{E}, \tag{2.4b}$$

where the $\Phi_0^\pm$ are defined by

$$\Phi_0^-(k\boldsymbol{\omega}, \mathbf{x}) = u(\mathbf{x}) e^{ik\boldsymbol{\omega}\cdot\mathbf{x}}, \qquad \Phi_0^+(k\boldsymbol{\omega}, \mathbf{x}) = v(\mathbf{x}) e^{ik\boldsymbol{\omega}\cdot\mathbf{x}}, \tag{2.5}$$

and where $\boldsymbol{\omega} \in S^2$, the unit sphere in $\mathbb{R}^3$. The set $\mathscr{E}$ is defined as

$$\mathscr{E} = \{k^2 \geq 0 \mid uG_k v\psi = -\psi \text{ for some } \psi \in L^2(\mathbb{R}^3),\ k \geq 0\}. \tag{2.6}$$

$\mathscr{E}$ is a closed subset of $[0, \infty)$ with Lebesgue measure zero containing the singular continuous spectrum and positive eigenvalues of H [12]. Then we have (for a proof see [12, 13]).

Theorem 1. *Let $V \in L^1(\mathbb{R}^3) \cap R$, then the scattering operator S, associated with the pair (H, H_0) is unitary, commutes with H_0 and in the spectral representation of H_0, the corresponding on-shell operator $S(k)$ reads*

$$(S(k)\phi)(\boldsymbol{\omega}) = \phi(\boldsymbol{\omega}) - \frac{k}{2\pi i} \int_{S^2} d\omega' f(k, \boldsymbol{\omega}, \boldsymbol{\omega}') \phi(\boldsymbol{\omega}'), \qquad k^2 \notin \mathscr{E}, \qquad \phi \in L^2(S^2), \tag{2.7}$$

where $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ represents the on-shell scattering amplitude. For $k^2 \notin \mathscr{E}$, $S(k) - 1$ is a trace class-operator in $L^2(S^2)$, i.e. $[S(k) - 1] \in \mathscr{B}_1(L^2(S^2))$, $k^2 \notin \mathscr{E}$, and continuous in trace norm with respect to k. Furthermore, an explicit characterization of $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ is obtained from

$$\begin{aligned} f(k, \boldsymbol{\omega}, \boldsymbol{\omega}') &= -(4\pi)^{-1} (\Phi_0^+(k\boldsymbol{\omega}), \Phi^-(k\boldsymbol{\omega}')) \\ &= -(4\pi)^{-1} (\Phi^+(k\boldsymbol{\omega}), \Phi_0^-(k\boldsymbol{\omega}')), \qquad k^2 \notin \mathscr{E}. \end{aligned} \tag{2.8}$$

Finally, $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ is uniformly continuous in all variables if k^2 varies in compact intervals not intersecting $\mathscr{E}$.

Following [9–11], the averaged total cross-section $\bar{\sigma}(k)$ is defined as

$$\bar{\sigma}(k) = (4\pi)^{-1}\int_{S^2} d\omega \int_{S^2} d\omega' \, |f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')|^2$$
$$= \frac{\pi}{k^2}\|S(k)-1\|_2^2 = -\frac{2\pi}{k}\operatorname{Re}(\operatorname{Tr}(S(k)-1)). \tag{2.9}$$

Theorem 1 then immediately leads to

Theorem 2. *Assume $V \in L^1(\mathbb{R}^3) \cap R$. Then for $k^2 \notin \mathscr{E}$, $\bar{\sigma}(k)$ is finite and continuous in k. In addition, the optical theorem*

$$\sigma(k, \boldsymbol{\omega}) = \int_{S^2} d\omega' \, |f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')|^2 = \frac{4\pi}{k}\operatorname{Im} f(k, \boldsymbol{\omega}, \boldsymbol{\omega}) \tag{2.10}$$

is valid.

3. Averaged angular time delay

In the following we discuss the necessary modifications of the above approach in order to study the concept of averaged angular time delay in a rigorous way. We start with

Lemma 1. *Let $(1+|\mathbf{x}|)V \in L^1(\mathbb{R}^3) \cap R$ and $k^2 \notin \mathscr{E}$. Then $(1+|\mathbf{x}|)^{1/2}\,\Phi^{\pm}(k\boldsymbol{\omega}, \mathbf{x})$ is strongly continuous in $L^2(\mathbb{R}^3)$ with respect to k for all $\boldsymbol{\omega} \in S^2$ and $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ is continuously differentiable with respect to k for all $\boldsymbol{\omega}, \boldsymbol{\omega}' \in S^2 \times S^2$. In particular, $S(k)$ is continuously differentiable in trace norm, i.e. $S'(k) \in \mathscr{B}_1(L^2(S^2))$ and $(-k(2\pi i)^{-1}f(k, \boldsymbol{\omega}, \boldsymbol{\omega}'))'$ is the kernel of $S'(k)$. (The ′ denotes the derivative with respect to k).*

Proof. The strong continuity of $(1+|\mathbf{x}|)^{1/2}\Phi^{\pm}(k\boldsymbol{\omega}, \mathbf{x})$ is clear from (2.4) and the condition on V. (cf. [12], Section IV.5). To prove the rest of the lemma, we first note that, using (2.4) and (2.8)

$$\begin{aligned} f'(k, \boldsymbol{\omega}, \boldsymbol{\omega}') = {} & i(4\pi)^{-1}(|\boldsymbol{\omega}\cdot\mathbf{x}|^{1/2}\Phi_0^{+}(k\boldsymbol{\omega}), \operatorname{sign}(\boldsymbol{\omega}\cdot\mathbf{x})\,|\boldsymbol{\omega}\cdot\mathbf{x}|^{1/2}\Phi^{-}(k\boldsymbol{\omega}')) \\ & + (4\pi)^{-1}(\Phi^{+}(k\boldsymbol{\omega}), (uG_k'v\Phi^{-})(k\boldsymbol{\omega}')) \\ & - i(4\pi)^{-1}(|\boldsymbol{\omega}'\cdot\mathbf{x}|^{1/2}\Phi^{+}(k\boldsymbol{\omega}), \operatorname{sign}(\boldsymbol{\omega}'\cdot\mathbf{x})\,|\boldsymbol{\omega}'\cdot\mathbf{x})|^{1/2}\Phi_0^{-}(k\boldsymbol{\omega}')), \\ & \qquad k^2 \notin \mathscr{E}, \end{aligned} \tag{3.1}$$

where $uG_k'v$ denotes the Hilbert–Schmidt operator represented by the kernel

$$(uG_k'v)(\mathbf{x}, \mathbf{y}) = i(4\pi)^{-1}u(\mathbf{x})e^{ik|\mathbf{x}-\mathbf{y}|}v(\mathbf{y}). \tag{3.2}$$

Equation (3.1) proves the assertions for the on-shell scattering amplitude $f(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$. Next, following Ref. [13], we introduce the maps

$$A_V(k): \begin{cases} L^2(\mathbb{R}^3) \to L^2(S^2) \\ g(\mathbf{x}) \to (A_V(k)g)(\boldsymbol{\omega}) = \int_{\mathbb{R}^3} d^3x\,u(\mathbf{x})e^{-ik\boldsymbol{\omega}\cdot\mathbf{x}}g(\mathbf{x}) \end{cases} \tag{3.3}$$

and

$$A_V(k)^*:\begin{cases} L^2(S^2)\to L^2(\mathbb{R}^3) \\ \phi(\boldsymbol{\omega})\to (A_V(k)^*\phi)(\mathbf{x})=\int_{S^2} d\omega u(\mathbf{x})e^{ik\boldsymbol{\omega}\cdot\mathbf{x}}\phi(\boldsymbol{\omega}). \end{cases} \tag{3.4}$$

Since $A_{|V|}(k)\in\mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$ we have that

$$S(k)-1=\frac{-k}{2\pi i}A_{|V|}(k)(1+uG_kv)^{-1}A_V(k)^*\in\mathcal{B}_1(L^2(S^2)),\qquad k^2\notin\mathscr{E}. \tag{3.5}$$

For $k_0^2\notin\mathscr{E}$ and $|k-k_0|$ small enough such that also $k^2\notin\mathscr{E}$, we consider, using equations (2.4), (2.7) and (2.8),

$$\begin{aligned}[S(k)-S(k_0)]/(k-k_0) = -\frac{1}{2\pi i}\Bigg\{ & A_{|V|}(k)(1+uG_kv)^{-1}A_V(k)^* \\ & +\left[\frac{k_0}{k-k_0}(A_{|V|}(k)-A_{|V|}(k_0))(1+|\mathbf{x}|)^{-1/2}\right] \\ & \cdot[(1+|\mathbf{x}|)^{1/2}(1+uG_kv)^{-1}A_V(k)^*] \\ & +k_0A_{|V|}(k_0)\frac{1}{k-k_0}[(1+uG_kv)^{-1}-(1+uG_{k_0}v)^{-1}]A_V(k)^* \\ & +[k_0A_{|V|}(k_0)(1+uG_{k_0}v)^{-1}(1+|\mathbf{x}|)^{1/2}] \\ & \cdot\left[\frac{1}{k-k_0}(1+|\mathbf{x}|)^{-1/2}(A_V(k)^*-A_V(k_0)^*)\right]\Bigg\} \end{aligned} \tag{3.6}$$

Looking e.g. at the kernel of the operator $(1+|\mathbf{x}|)^{1/2}(1+uG_kv)^{-1}A_V(k)^*$, we infer from Lemma 1 that this operator is in $\mathcal{B}_2(L^2(S^2), L^2(\mathbb{R}^3))$ and continuous with respect to k in Hilbert–Schmidt norm. Similarly $[k_0/(k-k_0)][A_{|V|}(k)-A_{|V|}(k_0)]$ $(1+|\mathbf{x}|)^{-1/2}$ converges in Hilbert–Schmidt norm to a $\mathcal{B}_2(L^2(\mathbb{R}^3), L^2(S^2))$-valued operator with kernel $-ik_0(\boldsymbol{\omega}\cdot\mathbf{x})(1+|\mathbf{x}|)^{-1/2}u(\mathbf{x})\exp(-ik_0\boldsymbol{\omega}\cdot\mathbf{x})$. Thus recalling also (3.5) we have that $S'(k_0)\in\mathcal{B}_1(L^2(S^2))$ with kernel $(-k_0(2\pi i)^{-1}f(k_0,\boldsymbol{\omega},\boldsymbol{\omega}'))'$. ■

In analogy with eq. (2.9) we now introduce the averaged angular time delay $\bar{\tau}(k)$ by the following definition

$$\bar{\tau}(k)=\int_{S^2} d\omega\tau_a(k,\boldsymbol{\omega}),\qquad k>0,\qquad k^2\notin\mathscr{E}, \tag{3.7}$$

where the angular time delay τ_a is given by [7, 8]

$$\begin{aligned}\tau_a(k,\boldsymbol{\omega}) &= \frac{1}{4\pi k}\frac{\partial}{\partial k}(\operatorname{Re} kf(k,\boldsymbol{\omega},\boldsymbol{\omega})) \\ &\quad +\frac{k}{8\pi^2}\int_{S^2} d\omega'\,|f(k,\boldsymbol{\omega},\boldsymbol{\omega}')|^2\frac{\partial}{\partial k}(\arg f(k,\boldsymbol{\omega},\boldsymbol{\omega}')),\qquad k>0,\qquad k^2\notin\mathscr{E}. \end{aligned} \tag{3.8}$$

We then have

Theorem 3. *Let $(1+|\mathbf{x}|)V\in L^1(\mathbb{R}^3)\cap R$. Then, for $k>0$ and $k^2\notin\mathscr{E}$, the averaged angular time delay $\bar{\tau}(k)$ is finite and continuous in k. Furthermore*

$$\bar{\tau}(k)=\operatorname{Tr}(\tau(k)),\qquad k>0,\qquad k^2\notin\mathscr{E}, \tag{3.9}$$

where $\tau(k)$ *is the on-shell time delay operator defined by*

$$\tau(k) = -\frac{i}{2k} S^*(k) S'(k), \qquad k > 0, \qquad k^2 \notin \mathscr{E}. \tag{3.10}$$

Proof. the continuity of $\bar{\tau}(k)$ and the fact that $\tau(k) \in \mathscr{B}_1(L^2(S^2))$ follow from Lemma 1. Equation (3.9) is directly obtained from equations (2.7), (3.7)–(3.8) and unitarity of S which allows (3.10) to be written in the form

$$\tau(k) = \frac{1}{2}\left[-\frac{i}{2k} S^*(k) S'(k) + \frac{i}{2k} S^{*\prime}(k) S(k)\right], \qquad k > 0, \qquad k^2 \notin \mathscr{E}. \quad \blacksquare \tag{3.11}$$

Remark 1. a) If V is spherically symmetric, then
$\bar{\tau}(k) = \sum_l (2l+1) \frac{1}{k} \frac{\partial}{\partial k} \delta_l(k),$

where $\delta_l(k)$ is the partial wave phase shift.

b) For a connection between $\bar{\tau}(k)$ and two-body Levinson's theorem in a generalised form we refer to [14–16] and [6].

c) Definition (3.7) and Theorem 3 easily extend to relative time delay (as discussed e.g. in [3]).

Remark 2. Under appropriate additional assumptions on V, the singular continuous spectrum of H is empty (cf. [13] and references therein). In that case $\mathscr{E}$ consists of the positive point spectrum of H. It has been shown in [15] that under suitable conditions on V, the positive eigenvalues of H decouple from the scattering phenomena in the sense that the scattering amplitude remains continuous at these points. However, the threshold point $k = 0$ needs a separate discussion. If $0 \in \mathscr{E}$, one has to distinguish whether it is a bound state of H, or a resonance, or both. (cf. e.g. Ref. [17]). Only if $0 \notin \mathscr{E}$ or if 0 is a bound state of H, the averaged total cross-section $\bar{\sigma}(0_+)$ remains finite. On the contrary, the limit of the averaged time delay, $\lim_{k \to 0_+} \bar{\tau}(k)$, never exists, irrespective whether $0 \in \mathscr{E}$ or not.

Finally, we state a continuity result for the averaged angular time delay and the averaged total cross-section with respect to the interactions. Assume $V, V_n \in L^1(\mathbb{R}^3) \cap R$ to be real valued, $n = 1, 2, \ldots$ and denote by $\mathscr{E}_n$ the exceptional sets corresponding to $H_n = H_0 + V_n$ (cf. equation (2.6)). Furthermore, let $\bar{\sigma}_n(k)$ and $\bar{\tau}_n(k)$ be the averaged total cross-section respectively the averaged angular time delay for the interaction V_n. Then we have

Theorem 4. a) *Suppose* $V, V_n \in L^1(\mathbb{R}^3) \cap R$, $n = 1, 2, \ldots$ *and*
$\lim_{n \to \infty} \|V_n - V\|_1 = 0, \qquad \lim_{n \to \infty} |V_n - V\|_R = 0.$ *Then*

$$\lim_{n \to \infty} \bar{\sigma}_n(k) = \bar{\sigma}(k), \qquad k^2 \notin \mathscr{E}. \tag{3.12}$$

b) *Assume* $(1 + |\mathbf{x}|) V$, $(1 + |\mathbf{x}|) V_n \in L^1(\mathbb{R}^3) \cap R$, $n = 1, 2, \ldots$ *and*
$\lim_{n \to \infty} \|(1 + |\mathbf{x}|)(V_n - V)\|_1 = 0, \qquad \lim_{n \to \infty} \|(1 + |\mathbf{x}|)(V_n - V)\|_R = 0.$ *Then*

$$\lim_{n \to \infty} \bar{\tau}_n(k) = \bar{\tau}(k), \qquad k^2 \notin \mathscr{E}. \tag{3.13}$$

Proof. Continuity of the scattering amplitude $f_n(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ and the scattering matrix $S_n(k)$ associated with the pair of Hamiltonians (H_n, H_0) under the hypothesis a) has been proved in Ref. [18]. By exactly the same methods, assumption b) implies continuity of $f_n'(k, \boldsymbol{\omega}, \boldsymbol{\omega}')$ and $S_n'(k)$, completing the proof. (We remark that by the joint continuity of uG_kv in k and V, $k^2 \in \mathscr{E}$ implies $k^2 \notin \mathscr{E}_n$ for n large enough). ■

Of course, the concept of time delay is not restricted to the simple case of two-body potential scattering. In the literature, one finds discussion of time delay for multiparticle systems ([7], [19] and [6]). Also nonlocal [20], dissipative [6, 21] and Couloumb interactions [3] were investigated. Finally, we also mention a recent treatment of the Lax–Philips scattering theory [5]. It is clear that our approach presented above could be extended to all these situations.

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