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On Bose condensation

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Abstract. For infinite Bose systems in equilibrium we derive a generalization of the Bogoliubov condensate equation and we prove rigorously that Bose–Einstein condensation occurs if and only if there is spontaneous breaking of the gauge symmetry.

I. Introduction

In this paper we study Bose–Einstein condensation for systems with two-body interactions. We do not prove the existence of condensation but we assume to have an infinite system equilibrium state with possible occurrence of condensation. The type of condensation we consider is into the $k=0$ mode only. We do not consider other types of condensation (see e.g. [1]) although they are not excluded.

In the first part we derive an equation representing the extremality of the free energy density as a function of the condensation parameters. We call this equation the condensate equation, since it is a generalization of the condensate equation introduced by Bogoliubov [2]. It is an important equation in the sense that if only zero condensate densities satisfy it then condensation is excluded while on the other hand non-zero solutions show that condensation is present in the equilibrium state being considered.

In the last part of the paper we give a rigorous and complete proof of the fact that there is condensation if and only if the gauge symmetry is broken. One implication being trivial, the other, namely that condensation yields necessarily the breaking of the gauge symmetry, is less clear, although it is generally believed to hold [3].

We consider the usual framework of Bose systems on $\mathbf{R}^\nu (\nu \geqslant 1)$. The algebra of observables is $\mathscr{A} = \bigcup_\Lambda \mathscr{A}_\Lambda$, where Λ stands for any open, connected, bounded region of $\mathbf{R}^\nu$, and where $\mathscr{A}_\Lambda$ is the algebra $\mathscr{B}(\mathscr{F}(L^2(\Lambda)))$ of bounded linear operators on the Fock space $\mathscr{F}(L^2(\Lambda))$ of symmetric functions with support in Λ. On each local Fock space we take the Hamiltonian H_Λ which on the n-particle subspace is given by

$$H_\Lambda^n = T_\Lambda^n + V_\Lambda^n \tag{1}$$

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where

$$T_\Lambda^n = -\tfrac{1}{2}\sum_{i=1}^{n} \Delta_i \quad (\Delta = \text{Laplacian with Dirichlet boundary conditions})$$

and V_Λ^n is the multiplication operator by $\sum_{1\leq i\leq j\leq n} v(|x_i - x_j|)$. The potential v is supposed to be absolutely integrable and superstable [4]. The latter condition ensures the existence of local Gibbs states.

We want to study properties of the equilibrium states corresponding to the Bose system described by the Hamiltonian (1). Different definitions of an equilibrium state in the thermodynamic limit are available. In what follows the most convenient way is to define the state through the correlation inequalities [5]. In particular a state ω of $\mathscr{A}$ is an equilibrium state at inverse temperature $\beta = 1$ and chemical potential $\mu \in \mathbf{R}$ if it satisfies:

$$\lim_{\Lambda} \omega(X^*[H_\Lambda - \mu N_\Lambda, X]) \geq \omega(X^*X) \ln \frac{\omega(X^*X)}{\omega(XX^*)} \tag{2}$$

where X is any observable of $\mathscr{A}$ such that $X \in \mathscr{D}([H_\Lambda - \mu N_\Lambda, .])$ for Λ large enough; N_Λ is the number operator for the volume Λ and lim Λ tending to infinity is always understood in the sense of an increasing sequence of cubes Λ with volume $|\Lambda|$.

For any state ω satisfying (2) we assume a number of conditions:

(a) The state is space translation invariant

(b) The state is characterized by a family of reduced density matrices, i.e. there exist twice continuously differentiable complex functions $\rho_{n,m}(x; y)$ on $\mathbf{R}^{\nu(n+m)}$ for all $n, m \in \mathbf{N}$, such that for $f_i, g_j \in L^2(\mathbf{R}^\nu)$:

$$\omega(a^*(f_1) \cdots a^*(f_n)a(g_m) \cdots a(g_1))$$
$$= \int dx_1 \cdots dx_n\, dy_m \cdots dy_1 f_1(x_1) \cdots f_n(x_n)\bar{g}_m(y_m) \cdots \bar{g}_1(y_1)$$
$$\rho_{n,m}(x_1, \ldots, x_n; y_m, \ldots, y_1)$$

where $a^{(*)}$ are the Fock creation and annihilation operators. Furthermore, we assume the following bounds:

$$|\rho_{n,m}(x; y)| \leq AB^{n+m} n!\, m!$$

with A and B positive constants.

This condition implies that the state is locally normal and extends to polynomials in the local fields. In the following we use the same notation for the Fock fields as well as for their representatives under the state.

(c) From condition (b) it follows that for all polynomials P, Q and R in the fields

$$\lim_{\Lambda} \omega(P[H_\Lambda - \mu N_\Lambda, Q]R)$$

exists e.g.

$$\lim_{\Lambda} \omega(a^*(f_1)\cdots a^*(f_n)[H_\Lambda-\mu N_\Lambda,\ a(f)]a(g_m)\cdots a(g_1))$$

$$=\int dy\, dx_1\cdots dx_n\, dy_m\cdots dy_1 f_1(x_1)\cdots f_n(x_n)f(y)\bar{g}_m(y_m)\cdots \bar{g}_1(y_1)$$

$$\times[(\tfrac{1}{2}\Delta_y+\mu)\rho_{n,m+1}(x_1,\ldots,x_n;\, y, y_m,\ldots,y_1)$$

$$-\int dz\, v(y-z)\rho_{n+1,m+2}(x_1,\ldots,x_n, z;\, z, y, y_m,\ldots,y_1)]$$

We assume that these matrix elements define operators $\delta(Q)$ on a common core $\mathscr{D}$ containing the polynomials in the fields. We assume that the map $Q\to\delta(Q)$ satisfies

$$\delta(PQ)=\delta(P)Q+P\delta(Q)$$

$$(\delta P)^*=-\delta(P^*)$$

(d) Let

$$\alpha_\Lambda=\frac{a(\chi_\Lambda)}{|\Lambda|}\quad\text{and}\quad \alpha_\Lambda^*=\frac{a^*(\chi_\Lambda)}{|\Lambda|}$$

where χ_Λ is the characteristic function of the volume Λ, then we suppose that in the limit $\Lambda\to\infty$ these operators converge strongly on $\mathscr{D}$ to the operators α and α^*, affiliated to the centre of the representation. Moreover we assume that ordered monomials in α_Λ and α_Λ^* also converge strongly to the corresponding monomials in α and α^*. Next we impose conditions related to the derivation δ. If P is any ordered monomial we assume that $s\text{-lim}\,\delta(P(\alpha_\Lambda, \alpha_\Lambda^*))$ exists and defines the extension of δ to the polynomials in α, α^* and the fields. Finally it is assumed that

$$s\text{-}\lim_{\Lambda}\delta(P(\alpha_\Lambda, \alpha_\Lambda^*))P'(\alpha_\Lambda, \alpha_\Lambda^*))=\delta(P(\alpha, \alpha^*))P'(\alpha, \alpha^*).$$

The conditions (d) are a characterization of the condensation level.

II. The condensate equation

We start with the proof of the main result of this section, namely the condensate equation, which we derive for the equilibrium states of the infinite system under fairly general conditions. The equation expresses the extremality of the free energy as a function of the condensate parameters. In this sense it is a generalization of the one derived by Ginibre [2].

Theorem II.1. *If ω is an equilibrium state, satisfying* (2) *and the conditions* (a)–(d), *then for each polynomial P in the fields and in the operators α and α^*, and for each polynomial Q in the α, α^* one has*

$$\omega(P\,\delta(Q))=0 \tag{3}$$

Proof. Take $\lambda \in \mathbf{C}$ and $X = \lambda P^* + Q$ in the inequality (2). Using

$$a \ln \frac{a}{b} \geqslant a - b \quad \text{for} \quad a, b \in \mathbf{R}^+$$

as Q is affiliated to the center one gets:

$$|\lambda|^2 \, \omega(P\,\delta(P^*) + \bar{\lambda}\omega(P\,\delta(Q)) + \lambda\omega(Q^*\,\delta(P^*)) + \omega(Q^*\,\delta(Q))$$
$$\geqslant |\lambda|^2 \, \omega(PP^* - P^*P)$$

If we show that $\omega(Q^*\,\delta(Q)) = 0$ the theorem follows from the observation that only the left hand side of this inequality contains a linear term in λ. Finally we prove that the constant terms vanishes. Take $\lambda = 0$, then $\omega(Q^*\,\delta(Q)) \geqslant 0$. After substitution of Q by Q^* one finds also that $\omega(Q\,\delta(Q^*)) \geqslant 0$. Furthermore, using the time invariance of the state ω, which is an immediate consequence of (2):

$$o = \omega(\delta(Q^*Q)) = \omega(\delta(Q^*)Q + \omega(Q^*\,\delta(Q)) = \omega(Q\,\delta(Q^*)) + \omega(Q^*\,\delta(Q)).$$

From the positivity of both terms one gets $\omega(Q^*\,\delta(Q)) = 0$. ■

As a special case of (3) taking $Q = \alpha$ and $P = \alpha^*$ one gets

$$\omega(\alpha^*\,\delta(\alpha)) = 0 \tag{4}$$

We rewrite this equation in terms of the reduced density matrices $\rho_{n,m}$ defined in condition (b)

$$\omega(\alpha^*\,\delta(\alpha)) = \mathcal{M}_x\mathcal{M}_y\left\{\mu\rho_{1,1}(x;y) - \int dz\, v(y-z)\rho_{2,2}(x,z;z,y)\right\} = 0 \tag{5}$$

where the space means $\mathcal{M}$ are well defined due to condition (d).

Remark that equation (4) is obtained from the inequality (2). The latter one is an upper bound [6] for the change in free energy under semigroups of completely positive maps. In the case of equation (4) the semigroup is given by

$$\gamma_\lambda = \exp \lambda\Gamma$$
$$\Gamma = \lim_{\Lambda} \int_{\Lambda} dx \frac{1}{|\Lambda|^2} \{[a(\chi_{\Lambda+x}), .\,]a^*(\chi_{\Lambda+x}) + a(\chi_{\Lambda+x})[.\,, a^*(\chi_{\Lambda+x})]$$

Therefore equation (4) expresses the differentiability of the free energy with respect to the parameter λ, with derivative zero.

In condition (a) we supposed already that the state ω is space translation invariant. Now we suppose further that the state ω is space clustering and hence extremal invariant. In this case the operator α is a multiple of the identity, in particular

$$\alpha = \omega(\alpha)\mathbf{1}$$

Let τ be the transformation defined by

$$\tau(W(h)) = W(h) \exp\{-2i \operatorname{Re}(\omega(\alpha^*)\hat{h}(0))\}$$

where $\hat{h}$ is the Fourier transform of a function h, element of $L^2(\Lambda)$ for some $\Lambda \subset \mathbf{R}^\nu$ and where

$$W(h) = \exp i[a(h) + a^*(h)]$$

is a Weyl operator.

Denote by $\rho^\alpha_{n,m}$ the reduced density matrices defining the state $\omega \circ \tau$. The transformation removes the condensate part of the fields in the state $\omega \circ \tau$, e.g. $\tilde{a}(f) = a(f) - \alpha\hat{f}(0)$ represents the annihilation operator with wave function f of the excitation from the condensate.

Using the clustering property of ω:

$$\mathcal{M}_x\mathcal{M}_y\rho_{1,1}(x; y) = |\rho_{1,0}(0)|^2 = |\alpha|^2$$

$$\mathcal{M}_x\rho_{2,2}(x, z; z, y) = \rho_{1,0}(0)\rho_{1,2}(z; z, y)$$

and after expressing the $\rho_{n,m}$ in terms of the $\rho^\alpha_{n,m}$ one gets from equation (5):

$$\mu\,|\alpha|^2 - \bar{\alpha}\int dy\,\rho^\alpha_{1,2}(0; 0, y)v(y) - \bar{\alpha}^2\int dy\,\rho^\alpha_{0,2}(0, y)v(y)$$

$$-|\alpha|^2\left[\left[\int dy\,\rho^\alpha_{1,1}(0; y)v(y) + \hat{v}(0)\rho^\alpha_{1,1}(0; 0)\right] - |\alpha|^4\,\hat{v}(0) = 0 \qquad (6)$$

This equation (6) coincides with the condensate equation of [2] after the thermodynamic limit is taken. We emphasize that the condensate equation in [2] is obtained by expressing the extremality of the pressure in the Bogoliubov approximation for the finite system, while (6) is obtained immediately for the infinite system without the use of any approximation.

The next objective is to prove the exactness of the Bogoliubov approximation in the thermodynamic limit as far as the state is concerned. This would complement the result of Ginibre [2] where it is proved that this approximation is exact as far as the pressure is concerned. For a given Bose model (i.e. for given potential v) one should solve the condensate equation and obtain the spectrum of the operator α and hence of $n_0 = \alpha^*\alpha$ the condensate density. What one should show is the following. Let $\alpha = \int \lambda\, dE(\lambda)$ be the spectral resolution of α, and let λ be an element out of the spectrum of α then one can prove from (2) that the measure $d\omega(E(\lambda)X)$ is absolutely continuous with respect to $d\omega(E(\lambda))$ so that we can define

$$\omega_\lambda(X) = \frac{d\omega(E(\lambda)X)}{d\omega(E(\lambda))} \quad \text{a.e. in } \lambda.$$

We should prove that the map $X \to \omega_\lambda(X)$ is a state satisfying the equilibrium condition (2). This would provide us a central decomposition of the state with respect to α. Unfortunately $\omega_\lambda(X)$ may not be uniquely defined for λ in a set S with measure $\omega(E(S)) = 0$ where S may depend on the observable X, and so we are unable to prove the above property in full, but we have the following partial result.

Theorem II.2. *Let Q be any polynomial in the operators α and α^*, define the state*

$$\omega_Q(Y) = \frac{\omega(Q^*QY)}{\omega(Q^*Q)}, \qquad Y \in \mathcal{A}$$

where ω satisfies (2), *then also ω_Q is an equilibrium state in the sense of inequality* (2).

Proof. Take $X = QY$ in the inequality (2) then one gets:

$$\omega(Q^* Y^* \delta(QY) \geq \omega(Q^*QY^*Y) \ln \frac{\omega(Q^*QY^*Y)}{\omega(Q^*QYY^*)}$$

From Theorem II.1

$$\omega(Q^* Y^* \delta(QY)) = \omega(Q^*QY^* \delta(Y))$$

and hence the result. ■

If the operator α is bounded, for any spectral projection $E(\Delta)$, Δ a measurable set in the spectrum of α, there is a sequence of polynomials Q_n such that $\lim_n Q_n = E(\Delta)$ and then it follows from the theorem that the state $\omega_\Delta = \omega(E(\Delta)\,.)/\omega(E(\Delta))$ is an equilibrium state in the sense of (2).

If ω is gauge invariant we shall see in the next section that we can realize the decomposition of the state with respect to the argument of α. If n_0 has a discrete spectrum this yields a full decomposition of the kind discussed before Theorem II.2.

III. Condensation and gauge symmetry breaking

Now we treat rigorously the connection between condensation and breaking of the gauge symmetry. It is well known that breaking of gauge symmetry implies condensation: if $\omega(\alpha) \neq 0$ then from Schwartz inequality it follows $\omega(n_0) \neq 0$ and so $n_0 \neq 0$, i.e. occurence of Bose–Einstein condensation.

The converse statement namely that $n_0 \neq 0$ implies $\omega(\alpha) \neq 0$, i.e. spontaneous breaking of gauge symmetry, is much less clear, although this property is usually taken for granted in the physics literature. Roepstorff [3] has provided some nonrigorous arguments for the inequality $\omega(n_0) \leq |\omega(\alpha)|^2$.

First we prove two lemmas about states on the CCR-algebra $\mathcal{A}(\mathcal{H})$, where $\mathcal{H}$ is any Hilbert space. The first one is for regular states ω i.e. the map $\lambda \in \mathbf{R} \to \omega(W(\lambda h))$ is continuous for all $h \in \mathcal{H}$; the second one is for analytic states i.e. the map $\lambda \in \mathbf{C} \to \omega(W(\lambda h))$ is analytic in an open strip around the real axis.

Lemma III.1. *Let ω be a regular state on the CCR-algebra $\mathcal{A}(\mathcal{H})$ and $(\pi_\omega, \Omega_\omega, \mathcal{H}_\omega)$ the corresponding GNS-representation. If U is any unitary operator in the center of the von Neumann algebra $\pi_\omega(\mathcal{A})''$, for any $h \in \mathcal{H}$ define the self-adjoint operator $\tilde{\phi}(h)$ on $\mathcal{H}_\omega$ by*

$$\tilde{\phi}(h) = (U^* a_\omega(h) + U a^*_\omega(h))^*$$

where $a_\omega(h)$, $a^*_\omega(h)$ are the annihilation and creation operators for the state ω, then:

(i) *the map $\tilde{W}: \mathcal{H} \to \mathcal{B}(\mathcal{H}_\omega)$*

$$\tilde{W}(h) = \exp i\tilde{\phi}(h), \qquad h \in \mathcal{H}$$

is a representation of the canonical commutation relations on $\mathcal{H}$

(ii) *define the functional* ω_θ, $\theta \in [0, 2\pi)$, *on* $\mathscr{A}(\mathscr{H})$:

$$\omega_\theta(W(h)) = (\Omega_\omega, \tilde{W}(e^{i\theta}h)\Omega_\omega), \qquad h \in \mathscr{H}$$

then ω_θ *is a state.*

(iii) *if* $\{\tau_\phi \mid \phi \in [0, 2\pi)\}$ *is the one-parameter group of gauge transformations i.e.* $\tau_\phi\, W(h) = W(e^{i\phi}h)$ *then*

$$\omega_\theta \tau_\phi = \omega_{\theta+\phi}, \qquad \theta, \phi \in [0, 2\pi)$$

Proof. As the state ω is regular there exist field operators $\phi(h)$, $\phi(ih)$, $h \in \mathscr{H}$ and creation and annihilation operators $a_\omega(h)$, $a_\omega^*(h)$ with domain $\mathscr{D} = \mathscr{D}(\phi(h)) \cap \mathscr{D}(\phi(ih))$, such that $a_\omega(h)^* = a_\omega^*(h)$, $a_\omega^*(h)^* = a_\omega(h)$ (see e.g; [7] Lemma 5.2.12). As U is a unitary element of the center, the operator $U^*a(h) + Ua^*(h)$ is essentially self-adjoint on $\mathscr{D}$.

Therefore $\tilde{\phi}(h)$ is self-adjoint and clearly the field $\tilde{\phi}$ satisfies the same commutation relations as the field ϕ. Hence $\tilde{W}$ is a representation of the canonical commutation relations on the Hilbert space $\mathscr{H}_\omega$, proving (i). The properties (ii) and (iii) are then immediate. ■

Lemma III.2. *Let* ω *be an analytic state on the CCR-algebra* $\mathscr{A}(\mathscr{H})$. *If* U *is any unitary operator in the center of the algebra* $\pi_\omega(\mathscr{A})''$, *let* ω_θ $(\theta \in [0, 2\pi))$ *be as in Lemma* III.1, *then*

(i) *if there exists an alement* $h \in \mathscr{H}$ *such that* $\omega(U^*a(h)) \neq 0$ *then for all* θ, $\phi \in [0, 2\pi)$ *with* $\theta \neq \phi$ *we have* $\omega_\theta \neq \omega_\phi$.

(ii) *if* ω *is gauge invariant, then*

$$\omega = \frac{1}{2\pi}\int_0^{2\pi} d\theta\omega_\theta$$

Proof. From the definition the states ω_θ $(\theta \in [0, 2\pi))$ are analytic if ω is analytic. Therefore it is sufficient to prove the statement of the lemma on monomials in the field operators [7; p. 39]. To prove (i) one remarks that

$$\omega_\theta(a(h)) = e^{-i\theta}\omega(U^*a(h)), \qquad h \in \mathscr{H}, \qquad \theta \in [0, 2\pi)$$

Therefore

$$(\omega_\theta - \omega_\phi)(a(h)) = (e^{-i\theta} - e^{-i\phi})\omega(U^*a(h)) \neq 0$$

and the result follows.

Furthermore, for all $n, m \in \mathbf{N}$ and $f \in \mathscr{H}$:

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\omega_\theta(a_\omega^*(f)^n a_\omega(f)^m)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{i\theta(n-m)}\omega(U^{n-m}a_\omega^*(f)^n a_\omega(f)^m)$$

$$= \omega(a_\omega^*(f)^n a_\omega(f)^m)$$

proving (ii). ■

Theorem III.3. *Suppose that* ω *is a gauge invariant equilibrium state in the sense of the inequality* (2) *satisfying the conditions* (a)–(d). *If Bose–Einstein* *condensation occurs (i.e. $n_0 \neq 0$) then there exists states ω_θ ($\theta \in [0, 2\pi)$) not gauge invariant satisfying:*

(i) *for all $\theta, \phi \in [0, 2\pi)$ such that $\theta \neq \phi$ then $\omega_\theta \neq \omega_\phi$*

(ii) *the state ω has the decomposition*

$$\omega = \frac{1}{2\pi} \int_0^{2\pi} d\theta \omega_\theta$$

(iii) *for each $\theta \in [0, 2\pi)$, the state ω_θ is an equilibrium state.*

(iv) *for each polynomial Q in the operators α and α^* and for each $\theta \in [0, 2\pi)$:*

$$\omega_\theta(Q(\alpha, \alpha^*)(X)) = \omega_\theta(Q(e^{i\theta} n_0^{1/2}, e^{-i\theta} n_0^{1/2})X); \qquad X \in \mathscr{A}$$

Proof. Consider the polar decomposition $\alpha = U n_0^{1/2}$ of the normal operator α (see condition (d)); U is the unitary extension in the center of $\pi_\omega(\mathscr{A})''$ of the partial isometry defined by the polar decomposition [8, p. 935]. As $n_0 \neq 0$ there exists some $f \in \bigcup_\Lambda L^2(\Lambda)$ such that

$$\omega(U^* a(f)) \neq 0$$

Indeed suppose that it vanishes for all functions, then, since $\alpha_\Lambda \Omega_\omega$ tends to $\alpha \Omega_\omega$,

$$\omega(n_0^{1/2}) = \omega(U^* \alpha) = \lim_\Lambda \omega(U^* \alpha_\Lambda) = 0$$

which due to the separating character of the state ω contradicts $n_0 \neq 0$. Hence the existence of the states ω_θ ($\theta \in [0, 2\pi)$) with the properties (i) and (ii) follow from the Lemmas III.1 and 2.

To prove (iii) and (iv) we introduce the *-isomorphic map J of the Weyl algebra into $\mathscr{B}(\mathscr{H}_\omega)$ defined by:

$$J(W(h)) = \tilde{W}(h)$$

where $\tilde{W}$ is defined as in Lemma III.1. Then the states ω_θ ($\theta \in [0, 2\pi)$) are given by

$$\omega_\theta = \omega \circ J \circ \tau_\theta$$

As ω is analytic, also ω_θ is analytic and the latter extends to the algebra generated by $\mathscr{A}$ and the polynomial algebra $\mathscr{P}(\mathscr{A})$ in the creation and annihilation operators. The map J extends in a similar way. In particular

$$Ja(f)^n = U^{*n} a(f), \qquad f \in \bigcup_\Lambda L^2(\Lambda), \qquad n \in \mathbf{N}$$

Therefore from the condition (d) by continuity one has

$$J\alpha^n = U^{*n} \alpha^n$$

Let $\mathscr{P}(\mathscr{A})_g$ be the gauge invariant polynomial subalgebra, then the restriction of J to $\mathscr{P}(\mathscr{A})_g$ is the identity and therefore J extends trivially to the linear operators affiliated to the von Neumann algebra $\mathscr{P}(\mathscr{A})_g''$. First we prove (iii).

For all $X \in \mathcal{A} \cap \mathcal{D}([H_\Lambda - \mu N_\Lambda, .])$, Λ large enough, using the gauge invariance of the Hamiltonian and the inequality (2) for the state ω:

$$\lim_\Lambda \omega_\theta(X^*[H_\Lambda - \mu N_\Lambda, X])$$

$$= \lim_\Lambda \omega(J\tau_\theta(X)[H_\Lambda - \mu N_\Lambda, J\tau_\theta(X)])$$

$$\geqslant \omega(J\tau_\theta(X^*)J\tau_\theta(X)) \ln \frac{\omega(J\tau_\theta(X^*)J\tau_\theta(X))}{\omega(J\tau_\theta(X)J\tau_\theta(X^*))}$$

$$= \omega_\theta(X^*X) \ln \frac{\omega_\theta(X^*X)}{\omega_\theta(XX^*)}$$

This proves that ω_θ satisfies the inequality (2) for equilibrium states. Finally, as $J\alpha = J\bar{\alpha}^* = n_0^{1/2}$:

$$\omega_\theta(Q(\alpha, \alpha^*)X)$$

$$= \omega(Q(e^{i\theta}J\alpha, e^{-i\theta}J\alpha^*)J\tau_\theta(X))$$

$$= \omega(Q(e^{i\theta}n_0^{1/2}, e^{-i\theta}n_0^{1/2})J\tau_\theta(X))$$

$$= \omega_\theta(Q(e^{i\theta}n_0^{1/2}, e^{-i\theta}n_0^{1/2})X) \quad \blacksquare$$

States obtained by taking the thermodynamic limit of Gibbs states for Bose systems with local Hamiltonians H_Λ (1) are always gauge invariant and will always satisfy the equilibrium inequality (2). The main new result proved in the theorem is that those limit Gibbs states, for which there is condensation in the ground state, can be decomposed with respect to the gauge group into distinct equilibrium states. This is what is meant by spontaneous breaking of the gauge symmetry. The technical conditions (a)–(d) are reasonably believed to be satisfied for realistic Bose systems. They can be checked for the exactly solvable models.

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