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On a non-unitary evolution of quantum systems

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Abstract. A notion of deterministic evolution of quantum systems is discussed within the axiomatic framework of the propositional system formalism. It is shown that in general, the deterministic evolution of a quantum system does not need to preserve the orthogonality of states. An example of non-linear Schrödinger-like equation governing such an evolution is indicated.

1. Introduction

Following Piron [1] we shall describe a quantum system by means of the collection of its properties (propositions), which are identified with equivalence classes of questions (elementary observables, yes-no experiments). It was shown in [1] that the structure of this set is that of a complete, orthocomplemented, weakly modular, atomic lattice satisfying the covering law (see Appendix for the definition). Such a lattice is called a propositional system (PROP). A pure quantum system (with no superselection rules) is described by an irreducible PROP (see Appendix), which was proved to be isomorphic to the lattice of closed subspaces of a Hilbert space over a certain field K, [1]. A case $K=\mathbb{C}$ gives the realization of standard PROP, namely the lattice $\mathscr{L}(\mathscr{H})$ of all closed subspaces of a complex Hilbert space $\mathscr{H}$.

A proposition $a \in \mathscr{L}$ is said to be true for the actual physical system, when one may affirm that in the event of an experiment the result will be yes, for certain (and hence for any) question $\alpha \in a$. Such proposition corresponds to an actual property of a physical system ("element of reality"), [1].

A state of a quantum system is defined in [1] as the set of all propositions which are actually true for the system. Equivalently the state of a quantum system may be represented by the greatest lower bound of this set, which turned to be an atom of PROP (see Appendix), [1].

Let an atom p represent the state of the quantum system. If for a given proposition $a \in \mathscr{L}$, $p < a$ then "a is true" in this state. If we know only that a is compatible with p (see Appendix), then either "a is true" or "a' is true". If a is not compatible with p, then a corresponds to no actual property or "element of reality" of the system in the state p, [1].

Evolution of the system is an alteration of state in time. This means that the set of actually true propositions i.e. "elements of reality" changes: certain properties become actual, and others disappear in potentiality, [2]. In the following we shall consider the deterministic evolution of a quantum system given by a semi-group of mappings $\{T_t\}_{t\geq 0}$ of the set of states (atoms of the PROP $\mathscr{L}$) $\mathscr{A}$, i.e.

$$\begin{aligned} &T_t : \mathscr{A} \to \mathscr{A}, \quad t \geq 0, \\ &T_0 = I \\ &T_t T_{t'} = T_{t+t'}, \quad t, t' \geq 0, \end{aligned} \tag{1}$$

where I denotes the identity mapping. The interpretation is that $T_t p$ is a state of the system at time $t_0 + t$, when the state of the system at time t_0 was p. If for each $t \geq 0$, T_t^{-1} exists, then by putting $T_{-t} := T_t^{-1}$ one gets a one-parameter group of mappings $\{T_t\}_{t\in R}$, of $\mathscr{A}$ onto itself.

However not every one-parameter semigroup of mappings of the set of states of a quantum system describes a physically meaningful dynamics. The aim of the present note is to propose a certain axiom for mappings T_t in order to obtain the very general class of physically meaningfull semi-groups of mappings, which describe the deterministic evolution of the quantum system. It will be shown that a class of groups of mappings obtained in such a way, obeys not only the unitary evolution, but also a certain type of evolution which does not preserve the orthogonality of states.

2. The state space of a quantum system

Let $\mathscr{L}$ be a PROP and $\mathscr{A}$ a set of states (atoms of $\mathscr{L}$). For any set $M \subset \mathscr{A}$ define $M^\perp = \{p \in \mathscr{A} : p \perp q \text{ for each } q \in M\}$. In the collection of all subsets of $\mathscr{A}$ we may define the mapping

$${}^{\perp\perp} : M \mapsto M^{\perp\perp}, \quad M \subset \mathscr{A}.$$

Using the definition one immediately verifies that for any $M \subset \mathscr{A}$, $N \subset \mathscr{A}$,

$$\begin{aligned} &M \subset N \Rightarrow M^{\perp\perp} \subset N^{\perp\perp} \\ &M \subset M^{\perp\perp} \\ &(M^{\perp\perp})^{\perp\perp} = M^{\perp\perp}, \end{aligned}$$

i.e. ${}^{\perp\perp}$ is a closure operation. Since the collection of all subsets of given set closed with respect to a certain closure operation forms a complete lattice with respect to the set-theoretical inclusion [3], the set

$$\mathscr{L}(\mathscr{A}) := \{M \subset \mathscr{A} : M = M^{\perp\perp}\}$$

is a complete lattice. For any family $\{M_i\}_{i\in I}$, $M_i \subset \mathscr{A}$

$$\bigwedge_i M_i = \bigcap_i M_i, \qquad \bigvee_i M_i = \Big(\bigcup_i M_i\Big)^{\perp\perp}.$$

Obviously $\mathscr{A} \in \mathscr{L}(\mathscr{A})$, $\phi \in \mathscr{L}(\mathscr{A})$. Since singleton sets are closed the lattice $\mathscr{L}(\mathscr{A})$ is atomic. It is also orthocomplemented, since as one easily verifies ${}^\perp : \mathscr{L}(\mathscr{A}) \to \mathscr{L}(\mathscr{A})$ is an orthocomplementation on $\mathscr{L}(\mathscr{A})$.

Let us define for any $a \in \mathscr{L}$

$$\alpha(a) := \{p \in \mathscr{A} : p < a\} \subset \mathscr{A}.$$

Clearly $a = \bigvee_{\alpha(a)} p$, since $\mathscr{L}$ is atomistic and $\alpha(a') = \alpha(a)^{\perp}$.

Lemma 1. *For any* $M \subset \mathscr{A}$, $\alpha(\bigvee_M p) = M^{\perp\perp}$.

Proof. We have: $q \in \alpha[(\bigvee_M r)']$ iff $q < (\bigvee_M r)' = \bigwedge_M r' < r'$ for any $r \in M$, i.e. $\alpha(\bigvee_M r)' = M^{\perp}$. Consequently

$$\alpha\left(\bigvee_M r\right) = \alpha\left\{\left[\left(\bigvee_M r\right)'\right]'\right\} = \alpha\left[\left(\bigvee_M r\right)'\right]^{\perp} = M^{\perp\perp}. \qquad \square$$

Proposition 1. *The mapping* $\mathscr{L} \ni a \to \alpha(a)$ *defines an orthoisomorphism of the* PROP $\mathscr{L}$ *onto* $\mathscr{L}(\mathscr{A})$.

Proof. Since $\alpha(a) = \alpha(a'') = \alpha(a)^{\perp\perp}$, $\alpha(a) \in \mathscr{L}(\mathscr{A})$ and α is well defined. α preserves the ordering: $a < b$ iff $\alpha(a) \subset \alpha(b)$. Consequently it preserves the glb and lub. Since α preserves the orthocomplementation it is an orthoinjection. By the Lemma 1 for any $M \in \mathscr{L}(\mathscr{A})$, $\alpha(\bigvee_M p) = M$ what shows that α is onto and consequently it is an orthoisomorphism. $\square$

Let us remark that Lemma 1 means that for any $M \subset \mathscr{A}$

$$q < \bigvee_M p \quad \text{iff} \quad q \in M^{\perp\perp}. \tag{2}$$

The inequality $q < \bigvee_M p$ reads: a state q is a superposition of states from the set M, [4], [5]. In other words a state $q \in \mathscr{A}$ is said to be a superposition of states from the set $M \subset \mathscr{A}$, whenever for a proposition $a \in \mathscr{L}$, "a is true" in each state $p \in M$ implies that "a is true" in the state q. Thus elements of $\mathscr{L}(\mathscr{A})$ are exactly subsets of the set of states closed with respect to the superposition of states. Taking the closure M of any set $M \subset \mathscr{A}$ means in fact that one adds to M all the states which are superpositions of states from M. The conclusion of the Proposition 1 was called in [5] the superposition principle. It means that given a set of states M there exists a proposition which is true if and only if the system is in a state which is a superposition of states from the set M.

3. The deterministic evolution of a quantum system

Let the quantum system undergo evolution given by a semi-group (1) of mappings of the set of states. As a state of the system alters, the set of actual properties changes. Since the evolution we consider is deterministic, there should exist certain relations between properties which are actual in different moments of time ("elements of reality"). In other words we assume, that no property may become actual in a stochastic way, i.e. for any property $b \in \mathscr{L}$, which is actual in a given moment of time, say t, there is in the set of properties which were actual in any time $t_1 < t$, a property $a \in \mathscr{L}$ which conditions b in the following sense:

Definition. Let $t_1, t_2, t_1 < t_2$, be two fixed moments of time. We say that property a conditions property b iff whenever one affirms a to be actual at t_1, then one may also affirm that b will be actual at t_2 and conversely, whenever one affirms that b is actual at t_2, then one may affirm that a was actual at t_1.

Therefore the deterministic nature of the evolution is guaranteed by the following axiom.

Axiom (D). For any property $b \in \mathscr{L}$ which is affirmed to be actual at a certain moment t, at any moment $t_1 < t$ there is a property $a \in \mathscr{L}$ which conditions b.

If the evolution of the system under consideration is described by a one-parameter semi-group $\{T_t\}_{t \geq 0}$, then in order to be consistent with the above axiom, mappings T_t must satisfy certain additional requirement. Let $p \in \mathscr{A}$ be the initial state at the moment t_0. If one affirms certain $b \in \mathscr{L}$ to be actual at the moment $t_0 + t$, then it is necessary that $T_t p < b$ and according to the Axiom (D) there should exist $a \in \mathscr{L}$ which conditions b. This means that $p < a$ if and only if $T_t p < b$, what is equivalent to

$$\alpha(a) = T_t^{-1} \alpha(b) \tag{3}$$

We formalize it as follows.

Definition. A semi-group (group) of mappings $\{T_t\}_{t \geq 0}$ ($\{T_t\}_{t \in R}$) of the set of states $\mathscr{A}$ into itself is called a dynamical semi-group (dynamical group) iff the Axiom (D) is satisfied i.e. iff for any t and for any $b \in \mathscr{L}$ there exists $a \in \mathscr{L}$ such that (3) is satisfied.

Proposition 2. *A semi-group of mappings of $\mathscr{A}$ into itself $\{T_t\}_{t \geq 0}$ is a dynamical semi-group if and only if*

$$(T_t^{-1} M)^{\perp\perp} = T_t^{-1} M \tag{4}$$

for any $t \geq 0$ and for any $M \subset \mathscr{A}$, such that $M^{\perp\perp} = M$.

Proof. It is an immediate consequence of the Proposition 1. □

If for any T_t, $t \geq 0$, T_t^{-1} exists then the semi-group $\{T_t\}_{t \geq 0}$ may be extended to the one-parameter group. In this case we can prove the following proposition.

Proposition 3. *A one parameter group $\{T_t\}_{t \in R}$ of mappings of $\mathscr{A}$ onto itself is a dynamical group if and only if for any $t \in R$, $M \subset \mathscr{A}$*

$$T_t^{-1}(M^{\perp\perp}) = (T_t^{-1} M)^{\perp\perp} \tag{5}$$

or equivalently

$$T_t(M^{\perp\perp}) = (T_t M)^{\perp\perp}. \tag{5'}$$

Proof. If $\{T_t\}_{t \in R}$ is a dynamical group then for any $t \in R$, $M \in \mathscr{L}(\mathscr{A})$ $(T_t^{-1} M)^{\perp\perp} = T_t^{-1} M$ by the Proposition 2, and this is equivalent to $(T_t M)^{\perp\perp} = T_t M$ since $T_t^{-1} = T_{-t}$. For any $M \subset \mathscr{A}$, $M^{\perp\perp} \in \mathscr{L}(\mathscr{A})$, hence $(T_t^{-1} M^{\perp\perp})^{\perp\perp} = T_t^{-1}(M^{\perp\perp})$.

But $M \subset M^{\perp\perp}$ implies that $T_t^{-1}M \subset T_t^{-1}(M^{\perp\perp})$ and consequently $(T_t^{-1}M)^{\perp\perp} \subset (T_t^{-1}(M^{\perp\perp}))^{\perp\perp} = T_t^{-1}(M^{\perp\perp})$. Therefore $(T_t^{-1}M)^{\perp\perp} \subset T_t^{-1}(M^{\perp\perp})$. Similarly $T_t^{-1}M \subset (T_t^{-1}M)^{\perp\perp}$ implies $M \subset T_t(T_t^{-1}M)^{\perp\perp}$ and consequently $M^{\perp\perp} \subset (T_t(T_t^{-1}M)^{\perp\perp})^{\perp\perp} = T_t(T_t^{-1}M)^{\perp\perp}$. Hence $T_t^{-1}(M^{\perp\perp}) \subset (T_t^{-1}M)^{\perp\perp}$ what shows that $T_t^{-1}(M^{\perp\perp}) = (T_t^{-1}M)^{\perp\perp}$. Let us assume now that (5) holds. Take any $b \in \mathscr{L}$; then $T_t^{-1}\alpha(b) = T_t^{-1}\alpha(b)^{\perp\perp} = (T_t^{-1}\alpha(b))^{\perp\perp}$. Thus $T_t^{-1}\alpha(b) \in \mathscr{L}(\mathscr{A})$ and according to the Proposition 1 there is a unique $a \in \mathscr{L}$ such that $\alpha(a) = T_t^{-1}\alpha(b)$ what shows that (3) is satisfied and $\{T_t\}_{t \in R}$ is a dynamical group. Finally the equivalence of (5) and (5′) follows easily from the fact that T_t is bijective on $\mathscr{A}$. □

The above considerations were based on the assumption that the deterministic evolution establishes certain relation between actual properties ("elements of reality") in different moments of time. We shall make it more precise.

Proposition 4. *Let $\{T_t\}_{t \geqslant 0}$ be a dynamical semi-group. Then the mappings $\tilde{T}_t$, $t \geqslant 0$, defined by $\alpha(\tilde{T}_t b) = T_t^{-1}\alpha(b)$, i.e. $\tilde{T}_t b = \bigvee_{T_t^{-1}\alpha(b)} p$, forms the one parameter semi-group of mappings of $\mathscr{L}$ into itself with the following properties:*

(i) $\tilde{T}_t\left(\bigwedge_i b_i\right) = \bigwedge_i \tilde{T}_t b_i$

(ii) $\tilde{T}_t\left(\bigvee_i b_i\right) < \bigvee_i \tilde{T}_t b_i,$

for any $t \geqslant 0$, and for any family $\{b_i\}_{i \in I} \subset \mathscr{L}$.

Proof. Owing to the Proposition 1 these mappings are well defined. Obviously $T_0 = I$(identity), and the semi-group property follows immediately from the definition. We have:

$$\alpha\left(\tilde{T}_t\left(\bigwedge_i b_i\right)\right) = T_t^{-1}\alpha\left(\bigwedge_i b_i\right) = \bigcap_i T_t^{-1}\alpha(b_i) = \alpha\left(\bigwedge_i \tilde{T}_t b_i\right)$$

and

$$\alpha\left(\tilde{T}_t\left(\bigvee_i b_i\right)\right) = T_t^{-1}\alpha\left(\bigvee_i b_i\right) = T_t^{-1}\left(\bigcup_i \alpha(b_i)\right)^{\perp\perp} \subset \left(T_t^{-1}\bigcup_i \alpha(b_i)\right)^{\perp\perp}$$
$$= \alpha\left(\bigvee_i \tilde{T}_t b_i\right).$$ □

The mapping $\tilde{T}_t$ from the above proposition associates with any property actual at a certain moment $t \geqslant 0$ the property which conditions it in $t = 0$.

Proposition 4. *Let $\{T_t\}_{t \in R}$ be a dynamical group. Then there is a unique one-parameter group of lattice automorphisms (i.e. bijective mappings of $\mathscr{L}$ onto itself which preserve the glb and lub) $\{\tilde{T}_t\}_{t \in R}$ such that $\tilde{T}_{t|\mathscr{A}} = T_t$ for any $t \in R$. Conversely, if $\{\tilde{T}_t\}_{t \in R}$ is a one parameter group of lattice automorphisms then the restriction of mappings T_t to the set of atoms, i.e. $T_t := \tilde{T}_{t|\mathscr{A}}$ forms a dynamical group.*

Proof. Let $\{T_t\}_{t \in R}$ be a dynamical group. For any $t \in R$ define $\tilde{T}_t : \mathscr{L} \to \mathscr{L}$ by $\alpha(\tilde{T}_t a) = T_t\alpha(a)$. Owing to the Proposition 3 mappings $\tilde{T}_t$ are well defined and we have explicitly $\tilde{T}_t a = \bigvee_{T_t\alpha(a)} p$. Since each T_t is bijective on $\mathscr{A}$, each $\tilde{T}_t$ is bijective on $\mathscr{L}$. Obviously $\{\tilde{T}_t\}_{t\in R}$ is a one parameter group. For $a, b \in \mathscr{L}$, $a < b$ iff $T_t\alpha(a) \subset T_t\alpha(b)$ iff $\tilde{T}_t a < \tilde{T}_t b$ and consequently $\tilde{T}_t$ preserves lub and glb. Thus $\tilde{T}_t$ is a lattice automorphism. From the very definition it follows that for any $p \in \mathscr{A}$, $\tilde{T}_t p = T_t p$. Now let $\{\tilde{T}_t\}_{t\in R}$ be a one-parameter group of lattice automorphisms of $\mathscr{L}$. Consider $T_t := \tilde{T}_{t|\mathscr{A}}$. Each T_t is bijective on $\mathscr{A}$ and $\{T_t\}_{t\in R}$ is a one parameter group. For any $t \in R$, $b \in \mathscr{L}$, $T_t^{-1}\alpha(b) = \alpha(\tilde{T}_{-t} b)$, since $T_t p < b$ iff $p < \tilde{T}_t^{-1} b = \tilde{T}_{-t} b$. This shows that for any $b \in \mathscr{L}$, $t \in R$ there is $\tilde{T}_{-t} b$ which determines it. □

It follows that for a given dynamical group $\{T_t\}_{t\in R}$ the mappings $\tilde{T}_t$, $t \in R$ establish a one-one correspondence between actual properties ("elements of reality") in the moment t_0 and $t_0 + t$. Conversely if once such correspondence is established then there is a unique dynamical group which generates it.

Corollary. *A one parameter group $\{T_t\}_{t\in R}$ of mappings of the set of states $\mathscr{A}$ into itself is a dynamical group if and only if there is a group of lattice automorphisms $\{\tilde{T}_t\}_{t\in R}$ of $\mathscr{L}$, such that $T_t = \tilde{T}_{t|\mathscr{A}}$ for any $t \in R$.*

From Proposition 3 and remarks in the end of Section 2 it follows that a dynamical group is a group of mappings which preserve the superposition of states in the following sense. A state $p \in \mathscr{A}$ is a superposition of states from the set $M \subset \mathscr{A}$ if and only if $T_t p$ is a superposition of states from the image of M under T_t, i.e. $p < \bigvee_M r$ iff $T_t p < \bigvee_{T_t M} q$.

A particular case of lattice automorphisms are symmetries, i.e. lattice automorphisms of $\mathscr{L}$ which preserve orthocomplementation. Due to the theorem of Piron ([1] Th. 2.46) every symmetry of $\mathscr{L}$ restricted to $\mathscr{A}$ is a bijective mapping of $\mathscr{A}$ onto itself (this mapping by definition preserves orthogonality of atoms) and conversely every bijective mapping of $\mathscr{A}$ onto itself which preserve orthogonality of atoms may be uniquely extended to a symmetry of $\mathscr{L}$. Therefore an evolution described by a group of symmetries of PROP is in fact related to a particular dynamical group which preserve the orthogonality of states. It is worth to remark that in this case the stronger axiom than (D) is satisfied, namely

Axiom (D′). For any property $b \in \mathscr{L}$ which is affirmed to be actual at a certain moment t, for any $t_1 \leq t$ there is a property $a \in \mathscr{L}$ such that a conditions b and a' conditions b'.

4. Examples

We are going to show how the above described dynamical groups may arise in the usual Hilbert space formulation of quantum theory. Actually the PROP under consideration is $\mathscr{L}(\mathscr{H})$, i.e. the lattice of closed subspaces of a complex Hilbert space $\mathscr{H}$. If $\{h_i\}_{i\in I}$ is any family of closed subspaces of $\mathscr{H}$, i.e. $h_i \in \mathscr{L}(\mathscr{H})$, then lattice operations are defined in the following way:

$$\bigvee_i h_i = \overline{\left(\bigcup_i h_i\right)}$$

$$\bigwedge_i h_i = \bigcap_i h_i.$$

States i.e. atoms of $\mathscr{L}(\mathscr{H})$ are now rays i.e. one dimensional subspaces of $\mathscr{H}$, which form the so called projective Hilbert space $\mathbb{H}$,. Let $p, q \in \mathbb{H}$. If $(\cdot,\cdot)$ denotes the scalar product in $\mathscr{H}$, then $p \perp q$ iff $(x, y) = 0$ for any vectors $x \in p$ and $y \in q$.

The reversible evolution of an isolated system is described by a Schrödinger equation ($\hbar = 1$):

$$\dot{x} = -iHx, \tag{6}$$

where H is a self-adjoint hamiltonian of the system. It is well known that the operator $-iH$ is a generator of a one parameter group $\{U_t\}_{t\in R}$ of unitary operators in $\mathscr{H}$, such that $x(t) := U_t x$, $x \in \mathscr{H}$ is a solution of the equation (6) with the initial condition $x(0) = x$. Let $p = [x]$ be a ray spanned by a vector $x \in \mathscr{H}$. For any $t \in R$ define $T_t p := U_t[x] = [U_t x]$. Owing to the linearity of U_t such defined T_t forms a one parameter group of mappings of $\mathbb{H}$ onto itself. Moreover since U_t preserve the scalar product, T_t preserve the orthogonality of states. Therefore $\{T_t\}_{t\in R}$ is a dynamical group (since (5) is trivially satisfied) and according to the theorem of Piron mentioned in the end of the preceding section, may be uniquely extended to a group of symmetries of $\mathscr{L}(\mathscr{H})$.

The unitary evolution of a quantum system described briefly above is usually considered in quantum theory. However there are physically interesting situations when the orthogonality of states is not preserved during the evolution. This is the case when the evolution is described by an equation with a "perturbed" generator:

$$\dot{x} = -iHx - kBx, \quad k \in R, \tag{7}$$

where $k \in R$ and B is a bounded self-adjoint operator on $\mathscr{H}$. The operator $-iH - kB$ is a generator of a one parameter group of bounded operators on $\mathscr{H}$, $\{V_t\}_{t\in R}$, such that $x(t) := V_t(x)$, $x \in \mathscr{H}$ is a solution of (7) with the initial condition $x(0) = x$, [6]. It follows that each V_t is bijective and maps a linear subspace of $\mathscr{H}$ onto a linear subspace. Moreover since for any $t \in R$ V_t is continuous, it maps a closed subspace onto a closed subspace. Therefore for any closed subspace $h \in \mathscr{L}(\mathscr{H})$ we may define $\tilde{T}_t h = V_t h = \{V_t x : x \in h\}$. It is obvious that $\{\tilde{T}_t\}_{t\in R}$ is a one parameter group of mappings of $\mathscr{L}(\mathscr{H})$ onto itself. These mappings preserve lattice operations in $\mathscr{L}(\mathscr{H})$:

$$\tilde{T}_t\Big(\bigwedge_i h_i\Big) = V_t\Big(\bigcap_i h_i\Big) = \bigcap_i V_t h_i = \bigwedge_i \tilde{T}_t h_i$$

$$\tilde{T}_t\Big(\bigvee_i h_i\Big) = V_t\Big(\overline{\bigcup_i h_i}\Big) = \overline{V_t \bigcup_i h_i} = \overline{\bigcup_i V_t h_i} = \bigvee_i \tilde{T}_t h_i.$$

We have used the fact that for any subset g of $\mathscr{H}$, $V_t \bar{g} = \overline{V_t(g)}$, since V_t and V_{-t} are both continuous. Thus $\{\tilde{T}_t\}_{t\in R}$ is a one parameter group of lattice automorphisms of $\mathscr{L}(\mathscr{H})$ and according to the Corollary of the preceding section $\{\tilde{T}_{t|\mathbb{H}}\}_{t\in R}$ is a dynamical group. In fact for any ray $p = [x]$

$$T_t p := \tilde{T}_t p = V_t[x] = [V_t x]. \tag{8}$$

Equation (7) has an unpleasant feature that it does not preserve the norm of a vector, which is the usual demand for the evolution equation in quantum theory. However as direct computation shows, if $x(t) = V_t x$ is a solution of the equation

(7) then the normalized solution $x(t)=\frac{V_t x}{\|V_t x\|}$ satisfies the equation:

$$\dot{x}=-iHx+k(\langle B\rangle_x-B)x, \tag{9}$$

where $\langle B\rangle_x=(x, Bx)/(x, x)$. Conversely, if $x(t)$ is a solution of the equation (9) with the initial condition $x(0)=x$ (provided it exists and is unique), then from the uniqueness it follows that $x(t)=V_t x/\|V_t x^{\cdot}\|$, where V_t is a group generated by the operator $-iH-kB$.

We see that from the physical point of view both of the equations (7) and (9) describes the same deterministic evolution, since they generate the same dynamical group (8).

The equation (9) with $B=H$ was proposed in [7]. It was shown there that the evolution governed by this equation is dissipative, since – provided the initial state is not an eigenstate of the hamiltonian – the expectation value of the hamiltonian (energy) decreases during the evolution. Another interesting property is that if H has a purely point spectrum, then each eigenstate is stationary, but only the ground state is stable, [7]. Gisin's equation has been successfully applied to the damped harmonic oscillator, spin 1/2 system [7] and to the description of the quantum measurement [8]. Recently it was shown that the equation (9) may be derived using the general formalism of master equations [9].

5. Conclusion

It was demonstrated that defining in a quite general way in terms of "elements of reality" of a quantum system the notion of deterministic evolution, one obtains really a generalization of a unitary evolution (evolution described by symmetries). This evolution has the property that the orthogonality of states is not preserved, although the superposition of states is. It was indicated also, that this generalization is "non-empty" in a sense that there is a certain type of physically meaningfull non-linear Schrödinger-like equations which govern such an evolution. From the fundamental point of view it is interesting that this evolution is dissipative, although not stochastic, [7], [9].

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Appendix

A set $\mathscr{L}$ is called a *propositional system* (PROP) iff:

1. $\mathscr{L}$ is a complete lattice, i.e. a partially ordered set $(\mathscr{L}, <)$ in which for every family of elements $\{a_i\}$ there exists a greatest lower bound (glb) $\bigwedge_i a_i$ and a least upper bound (lub) $\bigvee_i a_i$.

2. $\mathscr{L}$ is orthocomplemented, i.e. there is a mapping: $\mathscr{L} \to \mathscr{L}$ such that for $a, b \in \mathscr{L}$
   (i) $a'' = a$
   (ii) $b < a \Rightarrow a' < b'$
   (iii) $a \vee a' = \mathbb{1}$, $a \wedge a' = \mathbb{0}$, where $\mathbb{1}$ and $\mathbb{0}$ are maximal and minimal elements in $\mathscr{L}$ respectively.

3. $\mathscr{L}$ is weakly modular, i.e. $a < b$ implies that the sublattice generated by $\{a, a', b, b'\}$ is distributive.

4. $\mathscr{L}$ is atomic, what means that for any $a \in \mathscr{L}$ there exists an atom $q \in \mathscr{L}$ such that $q < a$. An atom of $\mathscr{L}$ is an element $q \in \mathscr{L}$ such that $b \in \mathscr{L}$ and $\mathbb{0} < b < q$ implies $b = \mathbb{0}$ or $b = q$.

5. If $q \in \mathscr{L}$ is an atom and for $a, b \in \mathscr{L}$, $a \wedge q = \mathbb{0}$ and $a < b < a \vee q$, then $b = q$ or $b = a \vee q$ (covering law).

Two propositions $a, b \in \mathscr{L}$ are said to be *compatible* iff the sublattice generated by $\{a, a', b, b'\}$ is distributive.

Two propositions are said to be *orthogonal*, $a \perp b$, iff $a < b'$.

A propositional system $\mathscr{L}$ is said to be *irreducible* iff the set of elements which are compatible with all the elements of $\mathscr{L}$ contains only $\mathbb{0}$ and $\mathbb{1}$.

The full formulation and explanation of the above definitions may be found in [1].