Zeitschrift: Helvetica Physica Acta
Band: 55 (1982)
Heft: 1

Artikel: Some applications of the Birman-Schwinger principle
Autor: Klaus, Martin
DOI: https://doi.org/10.5169/seals-115278

Helvetica Physica Acta, Vol. **55** (1982) 49–68 0018-0238/82/010049-20$1.50 + 0.20/0

Some applications of the Birman–Schwinger principle

By Martin Klaus[1][2]) Institut für Theoretische Physik der Universität Zürich, Schönberggasse 9, CH-8001 Zürich

(12. XII. 1981; rev. 4. III. 1982)

Abstract. We discuss applications of the Birman–Schwinger principle to eigenvalue problems of operators of the form $A+\lambda B$ (A, B self-adjoint, $\lambda \in \mathbb{R}$). In particular, we prove some general theorems about existence and threshold behavior of eigenvalues and investigate some special cases in more detail: the threshold behavior of $H_0+V+\lambda W$, periodic Hamiltonians under perturbation and potentials like $\sin x/x$.

1. Introduction

In this paper we study spectral properties of operators which are of the form $A+\lambda B$, with A, B self-adjoint (B being A-compact, say) and $\lambda \in \mathbb{R}$ being a varying coupling constant. Our main tool will be the so-called Birman–Schwinger principle. We have selected a few sample problems which are intended to illustrate different aspects of the method and show what it is good for. We will be looking into problems which are not so widely known, referring the reader to the literature (e.g. [5] and references therein) for an account of other applications (like e.g. the estimates on the number of bound states).

In Section 2 we prove some general results on the behavior of the eigenvalues of $A+\lambda B$ which lie in a spectral gap of A. We prove existence of eigenvalues in the gap (Theorem (2.2)) and investigate the threshold behavior (Theorem (2.4)). These results will be of use in the subsequent sections.

In Section 3 we discuss the low energy limit of the resolvent of $-d^2/dx^2+V$ and use the results to study the threshold behavior of $-d^2/dx^2+V+\lambda W$. We see that one dimension is fairly complicated.

Section 4 deals with the Hamiltonian $-d^2/dx^2+\lambda V_p+\sigma W$ where V_p is periodic and W short range and negative, say. As $\lambda \downarrow 0$, the gaps disappear, but as we shall see, there is always at least one eigenvalue in any given gap, provided σ goes to 0 not slower than $c \cdot |\ln \lambda|^{-1}$ ($c>0$).

The last section is devoted to the Hamiltonian $-d^2/dx^2+\lambda \sin x/x$. We show how one can decide the finiteness or infinitude of the number of negative eigenvalues with the help of a Birman–Schwinger principle. We would like to

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[1]) *Work supported by the Swiss National Science Foundation*

[2]) *Address after October* 1, 1981: *Institut für Theoretische Physik, Lehrstuhl A, RWTH Aachen, Erweiterungsgebäude Seffent–Melaten,* 5100 *Aachen, Germany*

thank the Institute for Theoretical Physics at the University of Zürich for its kind hospitality during this work would like to thank Prof. G. Scharf for making our stay at Zürich possible. We are also indebted to Prof. W. Amrein for some useful remarks concerning this paper.

2. The Birman–Schwinger principle and some general results

2.1. *The Birman–Schwinger principle*

Let A and B be two self-adjoint operators and B be relatively A-compact (then $\sigma_{\text{ess}}(A+B)=\sigma_{\text{ess}}(A)$). Suppose that A has a gap in the spectrum, i.e. there exist numbers a and b such that $a<b$, $(a,b)\in\rho(A)$ and $a, b\in\sigma(A)$. Consider the eigenvalue problem

$$(A+B)\psi=E\psi, \qquad E\in(a,b) \tag{2.1}$$

Let $B=U\,|B|$ be the polar decomposition of B and let ($B^{1/2}=U\,|B|^{1/2}$)

$$K_E=|B|^{1/2}(A-E)^{-1}B^{1/2}, \tag{2.2}$$

be the so-called *Birman–Schwinger kernel*, for in applications K_E has an integral kernel. K_E is compact. the Birman–Schwinger principle says: $A+B$ has eigenvalue E with multiplicity m if and only if K_E has eigenvalue -1 with *geometric* multiplicity m.

To see this note that if ψ satisfies (2.1) then $f=|B|^{1/2}\psi$ obeys $K_Ef=-f$, and, conversely, if f satisfies the latter equation, then $\psi=(A-E)^{-1}B^{1/2}f$ obeys (2.1). Since the kernels of these two transformations on the eigenspaces spanned by the ψ's and f's, respectively, are trivial, the assertion about the multiplicities follows. If $A=-\Delta$ and $B=V$ (or, more generally, if A is semibounded from below, say) and $E<0$ ($E<\inf\sigma(A)$), one knows that the geometric and algebraic multiplicities of the nonzero eigenvalues of K_E are equal and that K_E is isospectral to $(A-E)^{1/2}B(A-E)^{-1/2}$, which is self-adjoint [7]. (If $\sigma(A)\setminus\{0\}=\sigma(B)\setminus\{0\}$ we say that A and B are isospectral). However, in a gap situation multiplicities are important. The following example shows what can happen. Let

$$A=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \qquad B=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

The two eigenvalues of $A+\lambda B$, $\lambda\in[0,\infty)$ are

$$E_\pm(\lambda)=\tfrac{1}{2}(1\pm(1-4\lambda+8\lambda^2)^{1/2})$$

and both have a "turning point" at $\lambda=\frac{1}{4}$, i.e. $E_\pm(\frac{1}{4})=\frac{1}{2}(1\pm\frac{1}{2}\sqrt{2})$. K_E takes the form

$$-\frac{1}{E}\begin{pmatrix}1 & 1\\ \dfrac{-E}{1-E} & \dfrac{E}{1-E}\end{pmatrix}$$

which has eigenvalues $-\frac{1}{2}E^{-1}(1-E)^{-1}(1\pm(8E^2-8E+1)^{1/2})$. Both eigenvalues are real if either $0<E<E_-$ or $E_+<E<1$. They coalesce at $E=E_\pm$ and are complex for $E\in(E_-,E_+)$. At $E=E_\pm$ the *algebraic multiplicity is* 2 *but the geometric* *multiplicity is* 1. Obviously, if B is either positive or negative, K_E is self-adjoint and, as we shall see shortly, there are no turning points. What still can happen is that an eigenvalue of $A+\lambda B$ converges to a limit in the gap as $\lambda\uparrow\infty$. For example, take A as above and

$$B=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

Then $B\geq 0$ and the eigenvalue that emanates from 0 is $\lambda+\frac{1}{2}-\frac{1}{2}(1+4\lambda^2)^{1/2}$, converging to $\frac{1}{2}$ as $\lambda\uparrow\infty$. The kernel K_E is equal to $(1-E)^{-1}(1-\frac{1}{2}E^{-1})B$ and has eigenvalue 0 for all $E\in(0,1)$, and another eigenvalue at $(2E-1)/E(1-E)$, which is negative (positive) for $E<\frac{1}{2}$ $(E>\frac{1}{2})$. At $E=E(\infty)=\frac{1}{2}$, $K_E=0$!

Moreover, we have the elementary

Lemma (2.1). *Suppose that $B\geq 0$ $(B\leq 0)$. Then the nonzero eigenvalues of K_E are strictly monotone increasing (decreasing).*

Proof. If $B\geq 0$, then $(d/dE)K_E=|B|^{1/2}(A-E)^{-2})\,|B|^{1/2}\geq 0$ $(\leq 0, \text{if } B\leq 0)$ which, along with analyticity in E (or by perturbation theory) implies the result. ■

The following result was found in response to a question of Mr. Hinz (Munich): Suppose $A=-d^2/dx^2+\text{periodic potential}$, $B=V\leq 0$, another smooth potential of compact support, say. Does for some $\lambda\in\mathbb{R}$, the operator $A+\lambda B$ have an eigenvalue in any preassigned gap? The answer is 'yes' and contained in

Theorem (2.2). *Suppose $B\geq 0$ or $B\leq 0$, $B\neq 0$. Let*

$$S=\{E\in(a,b)\mid E\notin\sigma(A+\lambda B)\ \textit{for any}\ \lambda\in\mathbb{R}\}$$

Then S is either empty or a single point.

Proof. $E\in S\Leftrightarrow K_E=0$. Suppose that E_1, E_2 $(E_1<E_2)$ are two points in S. If $B\geq 0$, then by Lemma (2.1), K_E has no positive eigenvalues for $a<E\leq E_2$, and no negative eigenvalues for $E_1\leq E<b$. Hence $K_E=0$ for $E\in(E_1,E_2)$. Therefore, on (E_1,E_2), $(d/dE)K_E=0$, implying $(A-E)^{-1}\,|B|^{1/2}=0$. But $\operatorname{Ker}(A-E)^{-1}=\{0\}$ and $B\neq 0$. Thus we arrived at a contradiction. A similar argument works if $B\leq 0$. ■

2.2. *Threshold behavior of operators with spectral gap*

We continue to study the operator $A+\lambda B$ under the general assumptions made at the beginning of the last section. As λ varies the number of eigenvalues in the spectral gap (a,b) of A will change, since eigenvalues may leave or enter the gap at its endpoints. If $\lambda=\lambda_0$ is such that we can find an eigenvalue $E(\lambda)$ of $A+\lambda B$ with the property that $E(\lambda)\uparrow b$ as $\lambda\downarrow\lambda_0$ (or $\lambda\uparrow\lambda_0$) we call λ_0 *coupling constant threshold* (c.c. threshold). Conversely, as $\lambda\uparrow\lambda_0+\varepsilon$ $(\lambda\downarrow\lambda_0-\varepsilon)$ a new eigenvalue (or possibly more than one) enters the gap at b. Of course, there is nothing special about the point b, we could have taken a as well. By 'threshold behavior' we mean the behavior of $E(\lambda)$ as a function of λ. We are going to prove some general facts about such threshold situations. Let $E_\Delta(A)$ denote the spectral projection of A associated with the interval Δ. Moreover, without loss we may assume $\lambda_0 = 0$. Then we have

Lemma (2.3). *Suppose* $B \leq 0$ $(B \neq 0)$. *Then*

(i) $\lambda = 0$ *is a c.c. threshold of* $A + \lambda B \Rightarrow \|K_\varepsilon\|$ *is unbounded as* $E \uparrow b$
(ii) *If* $\|K_\varepsilon\|$ *stays bounded as* $E \uparrow b$, *then* $S - \lim_{E \uparrow b} K_E \equiv K_b$ *exists.*
(iii) K_b *compact* $\Leftrightarrow K_E \to K_b$ *in norm.*
(iv) *If* $\dim E_{(a,b)}(A + \lambda B) = \infty$ *for some* $\lambda = \lambda_1 > 0$ *then* $\dim E_{(a,b)}(A + \lambda B) = \infty$ *for all* $\lambda > \lambda_1$.
(v) K_b *compact* $\Leftrightarrow \dim E_{(a,b)}(A + \lambda B) < \infty$ *for all* $\lambda > 0$.
(vi) *Suppose* $\lambda = 0$ *is not a c.c. threshold and* $\dim E_{(a,b)}(A + \lambda B) < \infty$ *for all* $\lambda > 0$. *Then* K_b *is compact.*
(vii) K_b *exists and is not compact* $\Leftrightarrow \lambda = 0$ *is not a c.c. threshold and there exists* $\lambda^* > 0$ *such that* $\dim E_{(a,b)}(A + \lambda B) = \infty$ *for* $\lambda > \lambda^*$ *and* $\dim E_{(a,b)}(A + \lambda B) < \infty$ *for* $\lambda < \lambda^*$.

Proof. (i)$\Rightarrow$As $E \uparrow b$ the positive part of the spectrum of K_E stays bounded (Lemma (2.1)). If $\lambda = 0$ is to be a c.c. threshold it is necessary that $\sigma_E = \inf \sigma(K_E) \to -\infty$ as $E \uparrow b$. But $\sigma_E = -\|K_E\|$ for E close to b proving the result.
$\Leftarrow$If $\|K_E\|$ blows up, $\sigma_E \to -\infty$, i.e. $\lambda\sigma_E = -1$ is an implicit equation for $E(\lambda)$.

(ii) This follows from the montonicity of K_E in conjunction with Theorem (3.3) in ([6], p. 454).
(iii) $\Rightarrow$By Theorem (3.5) ([6], p. 455).$\Leftarrow K_E$ is compact, so $K_E \to K_b$ in norm implies that K_b is compact.
(iv) From the Birman–Schwinger principle along with analyticity and monotonicity of K_E, it follows that $\dim E_{(a,b)}(A + \lambda_1 B) = \infty$ if and only if $\dim E_{(-\infty,-1)}(\lambda_1 K_E) \to \infty$ as $E \uparrow b$. This proves (iv) if we note that $\dim E_{(-\infty,-1)}(\lambda K_E) \geq \dim E_{(-\infty,-1)}(\lambda_1 K_E)$ for $\lambda \geq \lambda_1$.
(v) This follows from the inequality $\dim E_{(a,b)}(A + \lambda B) \leq \dim E_{(-\infty,-1)}(\lambda K_b)$ which is a consequence of the Birman–Schwinger principle and Lemma (2.1).
(vi) By (i) K_b exists. Suppose there exists $\alpha > 0$, $\alpha \in \sigma_{\text{ess}}(K_b)$. Then $\dim E_{(\alpha-\varepsilon,\alpha+\varepsilon)}(K_b) = \infty$ for any $\varepsilon > 0$. Hence $\dim E_{(\alpha-\varepsilon,\infty)}(K_E) = \infty$, for $K_E \geq K_b$. This contradicts the compactness of K_E. Now suppose that $\alpha < 0$, $\alpha \in \sigma_{\text{ess}}(K_b)$. Then $\dim E_{(\alpha-\varepsilon,\alpha+\varepsilon)}(K_E) \to \infty$ as $E \uparrow b$. Let $\lambda_1 = -(\varepsilon + \alpha)^{-1}$ (assuming $\varepsilon < -\alpha$). Then $\dim E_{(-\infty,-1)}(\lambda_1 K_E) \to \infty$ as $E \uparrow b$, or $\dim E_{(a,b)}(A + \lambda, B) = \infty$, contrary to our assumption. Hence $\sigma_{\text{ess}}(K_b) = \phi$, i.e. K_b is compact.
(vii) $\Rightarrow K_b$ can only exist if $\|K_E\|$ stays bounded as $E \uparrow b$. Hence $\lambda = 0$ is not a c.c. threshold.

By (iv) and since K_b is not compact we have that $\dim E_{(a,b)}(A + \lambda B) = \infty$ for some $\lambda > 0$. Define λ^* to be the infimum over all such λ. Then $\lambda^* > 0$, for $\lambda = 0$ is not a c.c. threshold. Appealing to (iv) completes the proof.
$\Leftarrow K_b$ exists and is not compact on account of (i) and (v). ■

The threshold behavior is described in

Theorem (2.4). *Suppose $B \leqslant 0$ $(B \neq 0)$. Then we have*

(i) *Suppose $\lambda = 0$ is a threshold. Then b is not eigenvalue of $A \Leftrightarrow$ all eigenvalues $E_i(\lambda)$ which are absorbed at $\lambda = 0$ obey $(E_i(\lambda) - b)/\lambda \to 0$.*

(ii) *b is eigenvalue of A with eigenprojection P, and $\dim E_{(-\infty,0)}(PBP) = m$ $(m \geqslant 1) \Leftrightarrow$ for small enough λ we can find exactly m eigenvalues $E_i(\lambda)$ $(i = 1 \ldots m)$ of $A + \lambda B$ obeying $E_i(\lambda) \in (b - c_1\lambda, b - c_2\lambda)$ for suitable positive constants c_1, c_2 $(c_2 < c_1)$. If $m = 1$, then $(E_1(\lambda) - b)/\lambda \to (f, Bf)$ where f $(\|f\| = 1)$ satisfies $Pf = f$.*

Remark. 1. This theorem is an extension of a theorem of B. Simon [8]. Our proof is based entirely on the Birman–Schwinger principle, a possibility that has already been conjectured in [8]. 2. In (ii), besides the m eigenvalues of $0(\lambda)$, there may be other eigenvalues of $o(\lambda)$ that also converge to b.

Proof. Pick $\delta > 0$ and write

$$K_E = -|B|^{1/2} P (A - E)^{-1} |B|^{1/2} - |B|^{1/2} E_{(b,b+\delta)}(A)(A - E)^{-1} |B|^{1/2} - |B|^{1/2} E_{[b+\delta,\infty)}(A)(A - E)^{-1} |B|^{1/2} - |B|^{1/2} E_{(-\infty,a]}(A)(A - E)^{-1} |B|^{1/2} \tag{2.3}$$

The first term on the r.h.s. of (2.3) equals

$$-\frac{1}{b - E} |B|^{1/2} P |B|^{1/2}$$

which is isospectral to

$$-\frac{1}{b - E} P |B| P = \frac{1}{b - E} PBP \tag{2.4}$$

The norm of the second term is bounded by

$$\frac{1}{b - E} \| |B|^{1/2} E_{(b,b+\delta)}(A) |B|^{1/2} \|$$

As $\delta \downarrow 0$, $E_{(b,b+\delta)}(A) \to 0$ strongly, and, since B is A-compact,

$$\| |B|^{1/2} E_{(b,b+\delta)}(A) |B|^{1/2} \| = \| E_{(b,b+\delta)}(A) B E_{(b,b+\delta)}(A) \| = \| E_{(b,b+\delta)}(A)(A + i)(A + i)^{-1} B E_{(b,b+\delta)}(A) \|$$

goes to zero also. The third and forth term remain bounded in norm as $E \uparrow b$. Thus the norm of the sum of the last three terms on the right side of (2.3) is, for any $\varepsilon > 0$, less than

$$\frac{\varepsilon}{b - E} + c_\varepsilon \tag{2.5}$$

where c_ε is a suitable constant which may blow up as $\varepsilon \downarrow 0$.

(i) $\Rightarrow$ by (2.5) and (2.3), since $P = 0$. Conversely, if not all eigenvalues are $0(\lambda)$, $P \neq 0$ and b must be eigenvalue of A.

(ii) In order to describe the spectrum of K_E we write $K_E = (b-E)^{-1}(-|B|^{1/2}P\,|B|^{1/2}+R_E)$ and infer from (2.5) that $\|R_E\|\to 0$ as $E\uparrow b$. Thus by standard eigenvalue perturbation theory [5], [6], the spectrum of K_E splits up into two parts as E approaches b. One part, consisting of exactly m eigenvalues, is contained in an interval of the form $(-c_1(b-E)^{-1}, -c_2(b-E)^{-1})$ where c_1, c_2 are suitable positive constants and $c_1 > c_2$. The other part of the spectrum is confined to a region $\{z \mid |z| < h(E)\}$ where $h(E) > 0$ and $(b-E)h(E)\to 0$ as $E\uparrow b$. If $(b-E)h(E) < c_2$, i.e. if E is sufficiently close to b, the two parts are separated.

Letting $E_2^*(\lambda) = b - c_2\lambda$ and looking at the spectrum of λK_E we observe that $\lambda K_{E_2^*(\lambda)}$ has exactly m eigenvalues in $(-c_1/c_2, -1)$ while the rest of the spectrum lies above $-\lambda h(E_2^*(\lambda))$ where $\lambda h(E_2^*(\lambda)) = \lambda(b-E_2^*(\lambda))^{-1}(b-E_2^*(\lambda))h(E_2^*(\lambda)) = c_2^{-1}(b-E_2^*(\lambda))h(E_2^*(\lambda))\to 0$ as $\lambda\to 0$. Similarly, putting $E_1^*(\lambda) = b - c_1\lambda$ we see that $\lambda K_{E_1^*(\lambda)}$ has exactly m eigenvalues in $(-1, -c^2/c_1)$. With the aid of Lemma (2.1) we now conclude that, if we follow the eigenvalues of λK_E as E increases from $E_1^*(\lambda)$ to $E_2^*(\lambda)$ (λ held fixed), exactly m eigenvalues must pass through the value -1. If this happens at points $E = E_i(\lambda)$ $(i = 1 \ldots m)$, then, by the Birman–Schwinger principle, $E_i(\lambda)$ is eigenvalue of $A+\lambda B$. Moreover, $E_i(\lambda)\in (E_1^*(\lambda), E_2^*(\lambda)) = (b-c_1\lambda, b-c_2\lambda)$. If $m=1$, then λK_E has a negative eigenvalue obeying $\lambda(b-E)^{-1}(f, Bf) + 0((b-E)^{-1})$ which translates into the result that $A+\lambda B$ has eigenvalue $E_1(\lambda)$ obeying $E_1(\lambda) = b + \lambda(f, (Bf) + 0(\lambda)$. Thus all conclusions of part (ii) of Theorem (2.4) are established. ∎

The reason for Remark 2 to Theorem (2.4) is the following: Any further eigenvalue, call it $E_{m+1}(\lambda)$, of $A+\lambda B$ must be such that $\lambda h(E_{m+1}(\lambda)) > 1$. The aforementioned properties of $h(E)$ immediately imply then that $b - E_{m+1}(\lambda) = o(\lambda)$.

Theorem (2.4) also holds for general B. In fact, we have

Theorem (2.5). *If in Theorem* (2.4) *the restriction* $B \leqslant 0$ *is dropped, then* (i) *and* (ii) *hold.*

Proof. We have to find a substitute for (2.2). This will be done by 'reducing the problem to one band', as it is familiar from solid state physics. Let $P_+ = E_{[b,\infty)}(A)(P_- = 1 - P_+ = E_{(-\infty,a]}(A))$ and set $A_\pm = P_\pm A P_\pm$. Multiply the equation $(A+\lambda B)\psi = E\psi$ on both sides by P_+ and P_-, to get

$$(A_+ + \lambda P_+BP_+ + \lambda P_+BP_-)\psi = EP_+\psi \tag{2.6}$$

$$(A_- + \lambda P_-BP_+ + \lambda P_-BP_-)\psi = EP_-\psi \tag{2.7}$$

Now solve (2.7) for $P_-\psi$, obtaining

$$P_-\psi = -(A_- + \lambda P_-BP_- - E)^{-1}\lambda P_-BP_+\psi \tag{2.8}$$

Plugging this into (2.6) and solving for $\phi \equiv P_+\psi$, yields

$$(A_+ + \lambda P_+BP_+ - \lambda^2 P_+BP_-(A_- + \lambda P_-BP_- - E)^{-1}P_-BP_+)\phi = E\phi \tag{2.9}$$

We can now reformulate (2.9) in terms of the Birman–Schwinger kernel

$$\tilde{K}_E = (A_+ - E)^{-1/2}(-\lambda P_+BP_+ + \lambda^2 P_+BP_- \times (A_- + \lambda P_-BP_- - E)^{-1}P_-BP_+)(A_+ - E)^{-1/2} \tag{2.10}$$

to the effect that (2.9) holds if and only if $\tilde{K}_E$ has eigenvalue 1 (and multiplicities are equal). Also, if for an eigenfunction we had $P_+\psi = 0$ then $(A_- + \lambda P_-B)\psi = E\psi$ with $P_-\psi = \psi$. But if λ is sufficiently small, and E near b, this is impossible (except if $P_-\psi = 0$, so $\psi \equiv 0$). This means that the familiar relation between the eigenvalue problem $(A + \lambda B)\psi = E\psi$ and the kernel $\tilde{K}_E$ is also valid. Now (2.10) can be treated in the same way as (2.3), by inserting $E_{(b,b+\delta)}$, etc. In addition, we use the fact that one term in (2.10) is $0(\lambda^2)$ while the dominant term is $0(\lambda)$. Also $(A_- + \lambda P_-BP_- - E)^{-1}$ remains bounded as $E \uparrow b$. Now, if (i) holds, the right hand side of (2.10) is bounded by $\varepsilon \cdot \lambda 0((b-E)^{-1})(c_1 + \lambda c_2)$, and if (ii) holds, there exist m eigenvalues of order $\lambda \cdot 0((b-E)^{-1})$. This proves the Theorem. ■

We end this section with a lemma which will be useful in Section 4.

Lemma (2.6). *Suppose that $B \leq 0$ and that $E \in (a, b)$ is such that $E \notin \sigma(A + \lambda B)$ for all $0 \leq \lambda \leq 1$. Then*

$$(A - E)^{-1} \leq (A + B - E)^{-1}$$

Proof. By assumption on E and the Birman–Schwinger principle, $|B|^{1/2}(A - E)^{-1}|B|^{1/2} < 1$. Upon writing

$$(A + B - E)^{-1} = (A - E)^{-1} + (A - E)^{-1}|B|^{1/2} \times (1 - |B|^{1/2}(A - E)^{-1}|B|^{1/2})^{-1}|B|^{1/2}(A - E)^{-1}$$

the result follows. ■

Remark. The condition that B be A-compact can be relaxed. See [18], [19].

3. $-d^2/dx^2 + V$ under perturbation

In this section we study the low energy behavior of the resolvent of $H = H_0 + V$, and say something about the threshold properties of $H_0 + V + \lambda W$. This work was motivated by a question of G. Scharf, who pointed out that the study of thresholds of $H_0 + V + \lambda W$ would be relevant to the stability probelm of solutions of the Korteweg–de Vries equation. Again, properties of some Birman–Schwinger kernel will play an important role. The type of kernels involved will belong to the class referred to in Lemma (2.3)(i). We begin with some notation. Let

$$K_\alpha = |V|^{1/2}(H_0 + \alpha^2)^{-1}V^{1/2} \tag{3.1}$$

($V^{1/2} = |V|^{1/2}\,\mathrm{sgn}\, V$, $H_0 = -d^2/dx^2$) which has the kernel

$$K_\alpha(x, y) = (2\alpha)^{-1}|V(x)|^{1/2}e^{-\alpha|x-y|}V(y)^{1/2} \tag{3.2}$$

Then $K_\alpha = L_\alpha + M_\alpha$, where

$$L_\alpha(x, y) = (2\alpha)^{-1} |V(x)|^{1/2} V(y)^{1/2} \tag{3.3}$$

$$\begin{aligned} M_\alpha(x, y) &= (2\alpha)^{-1} |V(x)|^{1/2}(e^{-\alpha|x-y|} - 1) V(y)^{1/2} \quad (3.4)\\ &= -\tfrac{1}{2} |V(x)|^{1/2} |x-y| \; V(y)^{1/2} \\ &\quad + \frac{\alpha}{4} |V(x)|^{1/2} (x-y)^2 V(y)^{1/2} + 0(\alpha^2) \\ &= M_0 + \alpha M_0^{(1)} + 0(\alpha^2) \quad (3.5) \end{aligned}$$

If $d = \int V(x)\, dx \neq 0$, we set

$$L_\alpha = \frac{d}{2\alpha} L \tag{3.6}$$

so that $L^2 = L$. Let $Q = 1 - L$.

If $d = 0$, the analysis becomes more complicated. We give a few details in an appendix. Throughout this section we assume $d \neq 0$ and $V \in C_0^\infty(\mathbb{R})$. Of course, local singularities could be handled easily. A fall-off condition like $\int |V(x)| (1 + |x|)\, dx < \infty$ would suffice to prove the main results, except where a series appears and one needs exponential fall-off to have all terms finite. See the remarks at the end of this section for a subtlety in case we merely require $\int |V(x)| (1 + |x|)\, dx < \infty$.

In this section, an operator $H = H_0 + V$ is said to be critical if $\lambda = 1$ is a c.c. threshold of $H_0 + \lambda V$. We emphasize that the threshold eigenvalue appears as $\lambda \uparrow 1 + \varepsilon$ and never as $\lambda \downarrow 1 - \varepsilon$ [5, p. 79].

As a first result we have

Theorem (3.1). *The following statements are equivalent:*

(i) *$H = H_0 + V$ is critical*
(ii) *There exists a unique $\psi \in L^\infty$ such that $H\psi = 0$.*
(iii) *There exists f so that $QM_0Qf = -f$ (and the nonzero eigenvalues of QM_0Q are simple, so f is unique).*
(iv) *The integral equation $\psi(x) = 1 + \int_x^\infty (x-t) V(t) \psi(t)\, dt$ has a L^∞-solution, or equivalently, has a solution obeying $(V, \psi) = 0$.*

Remarks. 1. If $d = 0$, a similar theorem holds, but Q changes (Appendix). 2. For related results see Deift–Trubowitz [14], Simon [15] and [16]. In particular we added (iv) in order to establish the equivalence with the results of Deift–Turbowitz. Note that (iii) is self-contained, whereas (iv) involves subsidiary conditions on V and ψ. 3. A Birman–Schwinger kernel of the type considered here also appears in Klaus–Simon [17] in the two dimensional case, but our analysis here is different.

Concerning the threshold behavior of $H_0 + V + \lambda W$ we have

Theorem (3.2). *Suppose that $H_0 + V$ is critical and that $\psi \in L^\infty$ solves $H\psi = 0$. Then, if*

$$\int_{-\infty}^{\infty} W\psi^2\, dx < 0 \tag{3.7}$$

the operator $H_0+V+\lambda W$, $\lambda \geq 0$, has c.c. threshold $\lambda=0$, and a unique threshold eigenvalue $E(\lambda)$, obeying

$$E(\lambda)=-\lambda^2\frac{(\int W\psi^2\,dx)^2}{(\psi(+\infty)^2+\psi(-\infty)^2)^2}+0(\lambda^3) \tag{3.8}$$

where $\psi(\pm\infty)=\lim_{x\to\pm\infty}\psi(x)$. Moreover, $E(\lambda)$ is analytic at $\lambda=0$. If $\int W\psi^2\,dx>0$, $\lambda=0$ is not a c.c. threshold. If $\int W\psi^2\,dx=0$ and supp W *lies between two consecutive zeros of ψ, then there exists a bound state near zero for all small enough λ (positive and negative).*

The following expansion of the resolvent of K_α is crucial

Lemma (3.3).

$$(K_\alpha-z)^{-1}=Q(QM_0Q-z)^{-1}Q+\sum_{n=1}^{\infty}\alpha^nB(n;z) \tag{3.9}$$

where

$$\begin{aligned}B(1;z)=\frac{2L}{d}-\frac{2}{d}(QM_0Q-z)^{-1}QM_0L+(QM_0Q-z)^{-1}(-QM_0^{(1)}Q\\+\frac{2}{d}QM_0LM_0Q)(QM_0Q-z)^{-1}-\frac{2}{d}LM_0Q(QM_0Q-z)^{-1}Q\end{aligned} \tag{3.10}$$

and the series converges in norm for α small, and, with respect to z, uniformly on compact subsets of $\rho(QM_0Q)$.

Proof. By applying the formula $(B+C)^{-1}=(1+B^{-1}C)^{-1}B^{-1}$ three times, first with $B=\sigma L-z$ (using $(\sigma L-z)^{-1}=(\sigma-z)^{-1}L-z^{-1}Q$), then with $B=1-z^{-1}QM_\alpha L$ (using $B^{-1}=2-B$) and finally with $B=1-z^{-1}QM_\alpha Q$, we get $(\sigma=d/2\alpha)$

$$\begin{aligned}(K_\alpha-z)^{-1}&=(\sigma L+M_\alpha-z)^{-1}\\&=(1+D_\alpha)^{-1}\left(1-\frac{QM_\alpha Q}{z}\right)^{-1}\left(1+\frac{QM_\alpha L}{z}\right)\left(\frac{L}{\sigma-z}-\frac{Q}{z}\right)\end{aligned} \tag{3.11}$$

where

$$D_\alpha=\left(1-\frac{QM_\alpha Q}{z}\right)^{-1}\left(\frac{LM_\alpha}{\sigma-z}+\frac{QM_\alpha LM_\alpha}{z(\sigma-z)}\right) \tag{3.12}$$

We observe that $\|D_\alpha\|=0(\sigma^{-1})=0(\alpha)$, so that (3.11) can be further expanded. It is obvious, that we get a norm convergent series in powers of α. That's how we got (3.9). In particular, as $\alpha\downarrow 0$, the r.h.s. of (3.11) tends to

$$Q(QM_0Q-z)^{-1}Q \tag{3.13}$$

in norm, uniformly for z in compact subsets of $\rho(QM_0Q)$. ■

Proof of Theorem (3.1). (i)$\Rightarrow$(ii) If H is critical, then $H_0+(1+\varepsilon)V$ has (at least) one negative bound state which is absorbed as $\varepsilon\downarrow 0$. Equivalently, $(1+\varepsilon)K_\alpha$ has eigenvalue -1 for some $\alpha=\alpha(\varepsilon)$, and $\alpha(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$. Thus $K_{\alpha(\varepsilon)}$ has eigenvalue $-(1+\varepsilon)^{-1}$ or, equivalently, $(K_{\alpha(\varepsilon)}-z)^{-1}$ has eigenvalue $(-(1+\varepsilon)^{-1}-z)^{-1}$ where z is any number with $\operatorname{Im} z \neq 0$. As $\varepsilon \downarrow 0$, this eigenvalue tends to $(-1-z)^{-1}$ which, by Lemma (3.3), must then be an eigenvalue of $Q(QM_0Q-z)^{-1}Q$. This is the same as saying that QM_0Q has eigenvalue -1. That -1 is a simple eigenvalue (and therefore exactly one bound state is absorbed as $\varepsilon \downarrow 0$) will be shown at the end of the entire proof. Furthermore, an explicit expression for the corresponding eigenfunction will be given when we prove (ii)$\Rightarrow$(iii). To establish (iii)$\Rightarrow$(i) we observe that QM_0Q having eigenvalue -1 implies that $K\alpha$ has eigenvalue $\mu(d)$, such that $(\mu(\alpha)-z)^{-1} \to (-1-z)^{-1}$ as $\alpha \downarrow 0$, i.e. $\mu(\alpha) \to -1$. In the usual way, the equation $\lambda\mu(\alpha)=-1$ determines $\alpha(\lambda)$ and thus the threshold eigenvalue $E(\lambda)=-\alpha^2(\lambda)$.

(iii)$\Rightarrow$(ii). Let f obey $QM_0Qf=-f$ and let $g=-\frac{1}{2}\int |x-y|\, V(y)^{1/2}f(y)\, dy$. Then $g''=-V^{1/2}f=V^{1/2}QM_0Qf=V^{1/2}M_0Qf-(V/d)(V^{1/2}, M_0Qf)=V^{1/2}M_0f-(V/d)(V^{1/2}, M_0f)=Vg-(V/d)(V^{1/2}, M_0f)$. Let $\psi=g-d^{-1}(V^{1/2}, M_0f)$. Then $\psi''=V\psi$ and $\psi \in L^\infty$, for $g \in L^\infty$ by inspection, $\lim_{x\to\pm\infty} g(x)=\pm\frac{1}{2}(y, V^{1/2}f)$. We used that $Qf=f$ implies $(V^{1/2}, f)=0$. Clearly, ψ is unique.

(ii)$\Rightarrow$(iii). Given ψ let $f=|V|^{1/2}\psi$. Then $\int_{-\infty}^{\infty} V\psi\, dx=\psi'|_{-\infty}^{\infty}=0$, and $(V^{1/2}, f)=0$, i.e. $Qf=f$. Define g as above. Then $g''=-V\psi=-\psi''$ from which we infer that $g=-\psi+c_1x+c_2$. Since $g \in L^\infty$ and $\psi \in L^\infty$, we get $c_1=0$. Moreover, $(V, g)=dc_2$. Thus $QM_0f=M_0f-d^{-1}|V|^{1/2}(V^{1/2}, M_0f)=|V|^{1/2}g-|V|^{1/2}d^{-1}(V, g)=|V|^{1/2}(-\psi)=-f$.

(iii)$\Rightarrow$(iv). It follows from the definition of g and $(V^{1/2}, f)=0$ that $g(x)=g(+\infty)+\int_x^\infty (x-y)V(y)^{1/2}f(y)\, dy$. But $g=-\psi+c_2$ and $V^{1/2}f=V\psi$, so that the assertion is proved (with $\psi(\infty)=1$).

(iv)$\Leftrightarrow$(ii) is well known.

It remains to show that QM_0Q has only simple nonzero eigenvalues. This, in turn, rules out the possibility of a simultaneous absorption of more than an eigenvalue at some c.c. threshold. If QM_0Q had eigenvalue -1 with algebraic multiplicity $m \geq 1$, then K_α would have m eigenvalues converging to -1 as $\varepsilon \downarrow 0$, or, equivalently, $H_0+(1+\varepsilon)V$ would have m eigenvalues converging to 0. Assume $\operatorname{supp} V \subseteq (a, b)$ and let $\psi_\varepsilon(x)(\varepsilon \geq 0)$ denote the unique solution of $(H_0+(1+\varepsilon)V)\psi_\varepsilon=0$ obeying $\psi_\varepsilon(x)=1$ for $x<a$. The number of zeros of $\psi_\varepsilon(x)$ equals the number of bound states of $H_0+(1+\varepsilon)V$ ([16], Lemma (2.3)). Moreover, ψ_0 is constant for $x>b$ ([16], Lemma (2.4)) and $\psi_\varepsilon(x)$ is linear there, and thus can vanish at most once in (b, ∞). As a function of ε, $\psi_\varepsilon(x)$ is uniformly continuous on compact x-intervals. This implies that for sufficiently small $\varepsilon>0$ $\psi_\varepsilon(x)$ can have at most one zero more than $\psi_\varepsilon(x)$. Hence $m=1$. ■

In the proof of Theorem (3.2) we will study the Birman–Schwinger kernel

$$|W|^{1/2}(H_0+V+\alpha^2)^{-1}W^{1/2} \tag{3.14}$$

which can be expanded as

$$|W|^{1/2}(H_0+\alpha^2)^{-1}W^{1/2}-|W|^{1/2}(H_0+\alpha^2)^{-1}V^{1/2}(1+K_\alpha)^{-1}|V|^{1/2}(H_0+\alpha^2)^{-1}W^{1/2} \tag{3.15}$$

If H_0+V is critical there exists an eigenvalue $\mu(\alpha)$ of K_α approaching -1 as $\alpha \downarrow 0$. Call its spectral projection $P(\alpha)$. Thus $(1+K_\alpha)^{-1}$ diverges, and we want to know how. To this end, we have to recall another important property of K_α, namely, that it has a 'unique large eigenvalue' denoted by $\mu_1(\alpha)$ such that $\mu_1(\alpha)=d/2\alpha+0(1)$ for α small (see e.g. [3, Lemma 4]). Let $P_1(\alpha)$ denote its spectral projection, and let $P_2(\alpha)=1-P_1(\alpha)-P(\alpha)$. Then we can write

$$(K_\alpha+1)^{-1}=\frac{P_1(\alpha)}{\mu_1(\alpha)+1}+\frac{P(\alpha)}{\mu(\alpha)+1}+(K_\alpha+1)^{-1}P_2(\alpha). \tag{3.16}$$

By (3.9) $(K_\alpha-z)^{-1}\to Q(QM_0Q-z)^{-1}Q$ in norm as $\alpha\downarrow 0$. Thus norm $\lim P(\alpha)=P(0)\equiv P$ $(\alpha\downarrow 0)$ exists. We can even find an explicit representation for P: $P\cdot=f((\operatorname{sgn} V)f,\cdot)((\operatorname{sgn} V)f,f)^{-1}$ where f obeys $QM_0Qf=-f$, $((\operatorname{sgn} V)f,f)=(V\psi,\psi)<0$ with ψ obeying $(H_0+V)\psi=0$. To see this note that P must be of the form $f(g,\cdot)$. Moreover, $[P,QM_0Q]=0$ leads us to $(QM_0Q)^*g=-g$. Since $(QM_0Q)^*=UQM_0QU$ where $U=\operatorname{sgn} V$ is unitary we deduce $g=(f,Uf)^{-1}\cdot Uf$. For α small we have the expansions

Lemma (3.4).

(a)

$$P_1(\alpha)=L+\frac{2\alpha}{d}(LM_0Q+QM_0L)+0(\alpha^2)$$

$$\mu_1(\alpha)=d/2\alpha+0(1) \tag{3.17}$$

(b)

$$\mathrm{P}(\alpha)=\mathrm{P}-\frac{2\alpha}{\mathrm{d}}(\mathrm{PM_0L}+\mathrm{LM_0P}+\text{terms of the form }\mathrm{P}\cdots\mathrm{P})$$

$$+\alpha^2\left(\frac{4}{d^2}LM_0PM_0L+\cdots\right)+0(\alpha^3) \tag{3.18}$$

$$\mu(\alpha)=-1-c\alpha+0(\alpha^2)$$

where

$$c=\tfrac{1}{2}\left((x,V^{1/2}f)^2+\frac{4}{d^2}(V^{1/2},M_0f)^2\right)\cdot(\operatorname{sgn} Vf,f)^{-1} \tag{3.19}$$

(c)

$$(1+K_\alpha)^{-1}P_2(\alpha)\to(1+Q_2M_0Q_2)^{-1}Q_2+0(\alpha) \tag{3.20}$$

and $Q_2^2=Q_2$, $Q_2L=LQ_2=0$, $Q_2P=PQ_2=0$.

Proof. (a) $(d/2\alpha)L+M_\alpha=(d/2\alpha)(L+(2\alpha/d)M_\alpha)$ and we may write

$$P_1(\alpha)=\frac{1}{2\pi i}\int(z-L-(2\alpha/d)M_\alpha)^{-1}\,dz,$$

integrating around a small circle about the point 1. The result now follows by expanding the resolvent. Note that we cannot simply integrate (3.11), for z was supposed to be fixed there.

(b) Here we may integrate (3.11) around a circle about the point -1. The coefficient c follows from first order perturbation theory applied to the eigenvalue $(\mu(\alpha)-z)^{-1}$ of $(K_\alpha-z)^{-1}$.

(c) By using a contour integral surrounding all other eigenvalues except μ_1 and μ. ■

Armed with these expansions we return to (3.15) and plug them in. Obviously, the L and M operators contained in $(H_0+\alpha^2)^{-1}$ will hit the various P, Q and L terms contained in $(K_\alpha+1)^{-1}$, and there will be lots of cancellations. For example, the singularity in $|W|^{1/2}(H_0+\alpha^2)^{-1}V^{1/2}$ is cancelled out by the L-term in $P_1(\alpha)=L+\ldots$. We spare the reader further details. One finds four rank one terms of $0(\alpha^{-1})$ (α^{-1} is in agreement with the fact that the threshold eigenvalue of $H_0+(1+\varepsilon)V$ goes like ε^2) which sum up to one such term, namely

$$|W|^{1/2}(H_0+V+\alpha^2)^{-1}W^{1/2} = \frac{1}{4\alpha\tilde{c}}|W|^{1/2}\left(h-\frac{(V,h)}{d}\right)\left(W^{1/2}\left(h-\frac{(V,h)}{d}\right),\cdot\right)+0(1) \tag{3.21}$$

where $\tilde{c}=c(\operatorname{sgn} Vf, f)$, and $h(x)=\int|x-y|\,V^{1/2}(y)g(y)\,dy$. It follows from the relations derived in the proof of Theorem (3.1) that $h-(V,h)/d=\psi$, and that $\tilde{c}=\frac{1}{4}(\psi(+\infty)^2+\psi(-\infty)^2)$. To see this, note that $h=-2g$, $\psi(\infty)+\psi(-\infty)=(4/d)(V^{1/2}, M_0 f)$, and $\psi(\infty)-\psi(-\infty)=2(x, V^{1/2}f)$. Of course ψ solves $(H_0+V)\psi=0$.

Proof of Theorem (3.2). (3.8) follows from (3.21), and analyticity can be proved as in the $V=0$ case, by writing down an implicit [17] equation for $E(\lambda)$. This proves the first two parts of the theorem. Now suppose that $\int W\psi^2\,dx=0$ and that supp W lies between two consecutive zeros x_1 and x_2 of $\psi\equiv\psi_0$. Let ψ_1 denote a second, linearly independent solution, and let $\psi_0\psi_1'-\psi_0'\psi_1=1$. The equation $(H_0+V+\lambda W)\chi_\lambda=0$ has a unique solution which agrees with ψ_0 for $x\to-\infty$. It can be obtained by iteration from

$$\chi_\lambda=\psi_0+\lambda\psi_1\int_{-\infty}^{x}\psi_0 W\chi_\lambda-\lambda\psi_0\int_{-\infty}^{x}\psi_1 W\chi_\lambda \tag{3.22}$$

Since the number of zeros of χ_λ equals the number of negative eigenvalues of $H_0+V+\lambda W$ [16], we may pick a point x_0 to the far right (outside of supp $V\cup$ supp W) and calculate $\chi_\lambda'(x_0)$. If this derivative is >0 (<0) and $\psi_0(x_0)<0$ ($\psi_0(x_0)>0$) we get absorption. Iterating (3.22) twice gives

$$\chi_\lambda'(x_0)=\lambda^2\psi_1'(x_0)\int_{-\infty}^{x_0}\psi_0 W\left\{\psi_1\int_{-\infty}^{x}\psi_0^2 W-\psi_0\int_{-\infty}^{x}\psi_1\psi_0 W\right\}+0(\lambda^3) \tag{3.23}$$

Since $\psi_1'(x_0)=1/\psi_0(x_0)$ (note that $\psi_0'(x_0)=0$) the existence of absorption is guaranteed if the integral in (3.23) is negative. Upon integrating by parts, this is equivalent to demanding that

$$I=\int_{-\infty}^{x_0}W\psi_0\psi_1\int_{-\infty}^{x}\psi_0^2 W<0 \tag{3.24}$$

ψ_1 has a unique zero τ between x_1 and x_2, and

$$\psi_1=\psi_0\int_{\tau}^{x}\psi_0^{-2}\,dx$$

Inserting this in (3.24) gives, after another integration by parts,

$$I=-\frac{1}{2}\int_{-\infty}^{x_0}\psi_0^{-2}\left(\int_{-\infty}^{x}\psi_0^2 W\right)^2<0, \tag{3.25}$$

proving Theorem (3.2). ■

Remarks. 1. If one only requires that $\int |V(x)|(1+|x|)\,dx<\infty$ (and $d\neq 0$) one can still prove that $(K_\alpha - z)^{-1}$ has a norm limit as $\alpha\downarrow 0$. One of the critical terms in the derivation of this is QM_0Q which is not well-defined anymore for $\alpha=0$. To get around this difficulty one can make a different decomposition of K_α following an idea of Simon [4], essentially replacing $V^{1/2}$ by $V^{1/2}e^{-\alpha|x|}$. Then QM_0Q has to be replaced by $Q\tilde{M}_0Q$ where $\tilde{M}_0=\frac{1}{2}|V|^{1/2}(|x|+|y|-|x-y|)V^{1/2}$. If one tries to convert $Q\tilde{M}_0Q$ to QM_0Q one meets terms of the form $|x|\,|V|^{1/2}(V^{1/2},\cdot)$, which cancel out but are not defined unless $|x|\,|V|^{1/2}\in L_2$.

2. The small α expansion of $(H_0+V+\alpha^2)^{-1}$ is relevant to other problems, like e.g. finding the large t behavior of $\exp(iHt)$, following Jensen–Kato [20].

3. If $V=0$ in the last part of Theorem (3.2), then $\psi_0=1$, $\psi_1=x$, and (3.25) is equivalent to the conditional positivity of the kernel $-\frac{1}{2}|x-y|$.

4. One could treat the case $\int W\psi^2\,dx=0$ of Theorem (3.2) in a different way, following [3] where the case $V=0$ has been considered. For this one would have to work out the 0(1) term in (3.21). We do not claim that our assumption on the location of supp W is necessary to have absorption at $\lambda=0$.

5. The bound state of $H_0+V+\lambda W$ to which Theorem (3.2) refers, may or may not be the ground state. More precisely, if $d<0$ then H_0+V necessarily has at least one negative bound state [3] and so does $H_0+V+\lambda W$ for λ small. In this case we are dealing with absorption of an excited state. However, if $d>0$, H_0+V may be critical without having a negative bound state (μ_1 is positive) so that Theorem (3.2) deals with the ground state of $H_0+V+\lambda W$. In this case 4 vanishes nowhere and the restriction concerning supp W in the last statement of Theorem (3.2) becomes redundant.

4. Periodic Hamiltonians under perturbation

Initially we had the intention to derive some results about the threshold behavior at $\lambda=0$ of $H_0+V_p+\lambda W$, where V_p is a periodic potential and W a suitable perturbation. But we were fortunate to find in due time two papers by V. A. Zheludev [1, 2] on this subject, and we find his work in many ways a valuable predecessor to the more recent literature on coupling constant thresholds. One of this results is ($H_0=-d^2/dx^2$).

Theorem (4.1). (V. A. Zheludev). *Suppose that* sgn $W=$ const. *and* $\int|W|(1+x^2)\,dx<\infty$ *(and V_p periodic and piecewise continuous), then in each gap sufficiently far out we can find an eigenvalue of H_0+V_p+W. If, in addition, W is bounded, then there exists exactly one eigenvalue of $H_0+V_p+\lambda W$ in any given gap, provided λ is sufficiently small.*

Let us remark that on the basis of the recent work on the non-periodic case [3, 4], we conjecture that one could relax the fall-off condition to $\int |W| (1+|x|)\, dx$ and drop the assumption that W be bounded in the second part. Maybe we come back to these questions elsewhere. The work of Zheludev caused us to think about a related question, which we would like to present in this section. Consider the operator

$$H_0 + \lambda V_p + \sigma W \tag{4.1}$$

and suppose that V_p is smooth and periodic, and $W \in C_0^\infty$, $W \leqslant 0$, and $\lambda, \sigma \geqslant 0$.

For λ fixed, Theorem (4.1) ensures the existence of an eigenvalue in any given gap if σ is sufficiently small, depending on λ. It says nothing about how large σ might be in order to be sure that at least one bound state is present. In particular, as $\lambda \downarrow 0$, one might like to know whether $\sigma = 0(\lambda)$ is permissible. If yes, we would know e.g. that $H_0 + \lambda(V_p + W)$ also has a bound state for λ small. At first blush one might think that $0(\lambda)$ is the optimum one can possibly hope for, because the width of a gap is also $0(\lambda)$ (or $o(\lambda)$ in exceptional cases). That the situation is actually much better is the content of our next theorem:

Theorem (4.2). *Suppose that $a(\lambda)$ and $b(\lambda)(a<b)$ are the edges of a gap of $H_0 + \lambda V_p$. Suppose that $b(\lambda) - a(\lambda) = 0(\lambda)$. Then we can find a point $\tilde{E}(\lambda) \in (a(\lambda), b(\lambda))$ such that $\lambda^{-1} \min\{\tilde{E}(\lambda) - a(\lambda), b(\lambda) - \tilde{E}(\lambda)\} \not\to 0$ as $\lambda \downarrow 0$, and a constant $c > 0$ (depending on W) such that*

$$(a(\lambda), \tilde{E}(\lambda)) \in \rho(H_0 + \lambda V_p + \sigma W)$$

but

$$(\tilde{E}(\lambda), b(\lambda)) \cap \sigma(H_0 + \lambda V_p + \sigma W) \neq \varnothing$$

for all $\sigma \leqslant c/|\ln \lambda|$ and all sufficiently small λ.

As $\lambda \downarrow 0$, the spectrum of $H_0 + \lambda V_p + \sigma W$ penetrates into the gap only very 'reluctantly', i.e. it doesn't reach down to the left gap edge even if σ is large compared to λ, i.e. if $\sigma/\lambda = 0(1/\lambda\, |\ln \lambda|)$.

The intuitive idea behind this result is that, as $\lambda \downarrow 0$, the behavior of the band energies $E_n(k)$ near the band edges, changes from parabolic to linear. In our first attempt of proving Theorem (4.2) we got entangled in some unpleasant estimates due to the two-fold degeneracy of the $E_n(k)$ at the band edge. In fact, we will need to know the location of these eigenvalues as a function of k near $\frac{1}{2}$ *and* of λ with error estimates that are uniform in these two variables. We figured that if the eigenvalue were simple this would be easier to get. So we invented a trick to remove the degeneracy. We shall use k-space notation and refer to [5] for basic results on periodic Hamiltonians. Suppose that V_p is 2π-periodic. $H_0 + \lambda V_p$ has a direct integral representation as

$$\int_{(-\frac{1}{2}, \frac{1}{2}]} H(k;\lambda)\, dk \tag{4.2}$$

where $H(k;\lambda)$ acts on l_2 according to

$$(H(k;\lambda)c)_j = (k+j)^2 c_j + \sum_{h=-\infty}^{\infty} \tilde{V}_n c_{j-n} \tag{4.3}$$

where $\{c_j) \in l_2$, and $\tilde{V}_n$ is given by

$$\tilde{V}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} V(x) e^{-inx}\, dx \tag{4.4}$$

Let $E_n(k;\lambda)$ denote the n-th eigenvalue of $H(k;\lambda)$.

In the sequel we will concentrate without loss of generality on the first energy gap. To this end, we suppose that $\tilde{V}_0 \neq 0$, $\tilde{V}_1 = \tilde{V}_{-1} \neq 0$. Then the doubly degenerate eigenvalue $E_1(\frac{1}{2}; 0) = E_2(\frac{1}{2}; 0)$ splits (under the perturbation λV_p) into two eigenvalues $E_\pm(\lambda)$ where $E_\pm(\lambda) = \frac{1}{4} + \lambda \tilde{V}_0 \pm \lambda\, |\tilde{V}_1| + 0(\lambda^2)$. This is an elementary perturbational calculation. Now define

$$\tilde{E}(\lambda) = E_-(\lambda) + (\lambda/2)\, |\tilde{V}_1|.$$

Then $\tilde{E}(\lambda)$ satisfies the condition required in Theorem (4.2). (More generally, $E_-(\lambda) + a\, |\tilde{V}_1|$ with any $a, 0 < a < 1$ could be chosen).

Proof of Theorem (4.2). By virtue of the Birman–Schwinger principle, if the positive part of the spectrum of

$$\sigma\, |W|^{1/2} (H_0 + \lambda V_p - \tilde{E})^{-1}\, |W|^{1/2} \tag{4.5}$$

is less than 1, then $\tilde{E}$ is not eigenvalue of $H_0 + \lambda V_p + \sigma' W$ for any $\sigma' \leq \sigma$. Suppose that we can show that this positive part is bounded by $c\, |\ln \lambda|$ $(c > 0)$. Then Theorem (4.2) follows immediately. So, all our work goes into estimating this positive spectral part.

Let P denote the projection onto the vector $\delta_{j,0} \in l_2$, and let $Q = 1 - P$. Let

$$H_\mu(k) = H(k;0) - \mu P \tag{4.6}$$

Then

$$\operatorname*{norm-resolvent-lim}_{\mu \uparrow K^2} H_\mu(k) = H(k;0) - P \equiv H_1(k) \tag{4.7}$$

and

$$\operatorname*{norm-resolvent-lim}_{\mu \uparrow K^2} H(k;\lambda) - \mu P = H_1(k) + \lambda V_p \tag{4.8}$$

$H_1(k)$ has eigenvalues 0 and $E_n(k;0)$ where $n \geq 2$, for the eigenvalue $E_1(k;0)$ has been removed (meaning, has been replaced by 0). The spectrum of $H_1(k) + \lambda V_p$ is located near that of $H_1(k)$, in particular, the eigenvalue near $E_1(k;0) = (k-1)^2$ is now approximately $(k-1)^2 + \lambda V_0 + 0(\lambda^2)$. Note that the eigenvalue $\frac{1}{4}$ of $H_1(\frac{1}{2})$ is now simple. From this we infer that as μ varies from 0 to k^2 no eigenvalue of $H_\mu(k) + \lambda V_p$ ever passes through the point $\tilde{E}$! Applying Lemma (2.6) with $A = H(k;\lambda)$ and $B = P$, and $E = \tilde{E}$, we obtain the estimate

$$(H_0 + \lambda V_p - \tilde{E})^{-1} \leq (H_1(k) + \lambda V_p - \tilde{E})^{-1} \tag{4.9}$$

The next step is to estimate the norm of $|W|^{1/2}$ (r.h.s. of (4.9)) $|W|^{1/2}$. Clearly, the point $k = \frac{1}{2}$ (and $k = -\frac{1}{2}$) is critical. By symmetry we need only consider $k = \frac{1}{2}$. Let $\sigma = \frac{1}{2} - k$. Then $H_1(k) + \lambda V_p$ has eigenvalue

$$\tfrac{1}{4} + \delta + \lambda \tilde{V}_0 + 0(\sqrt{\delta^2 + \lambda^2}) \tag{4.10}$$

with eigenvector

$$\delta_{j,-1}+0(\lambda)+0(\delta)+0(\sqrt{\delta^2+\lambda^2}) \tag{4.11}$$

Here we refer to Kato [6, Ch. II] for further details about the errors in (4.10) (4.11). Clearly, we could write down the $0(\lambda)$ term in (4.11) as $\sim\lambda S(k)V_p\delta_{j,-1}$ where $S(k)$ denotes the reduced resolvent of $H_1(k)$ with respect to the eigenvalue $(k-1)^2$. $S(k)$ can be further expanded into powers of δ, etc. Note that in (4.10) we have a $0(\delta)$-term, i.e the band energie of $H_1(k)$ is *linear* near $k=\frac{1}{2}$. Now pick $\delta_0>0$ such that for $\delta<\delta_0$ and λ sufficiently small

$$(4.10)-\tilde{E}(\dot{\lambda})=\left(\delta+\frac{\lambda}{2}|\tilde{V}_1|\right)\left(1+\frac{0(\sqrt{\delta^2+\lambda^2})}{\delta+\frac{\lambda}{2}|\tilde{V}_1|}\right)\geqslant\frac{1}{2}\left(\delta+\frac{\lambda}{2}|\tilde{V}_1|\right) \tag{4.12}$$

Then, if λ is sufficiently small, there exists a number $\gamma>0$ such that for $k\in[-\frac{1}{2}+\delta_0,\frac{1}{2}-\delta_0]$

$$\operatorname{dist}[\tilde{E},\sigma(H_1(k)+\lambda V_p)]\geqslant\gamma$$

Thus the leading contribution to the norm of the Birman–Schwinger kernel comes from the region $-\frac{1}{2}\leqslant k\leqslant-\frac{1}{2}+\delta_0$ and $\frac{1}{2}-\delta_0\leqslant k\leqslant\frac{1}{2}$. In x-space, $\delta_{j,-1}$ is $\sim\exp(ix(k-1))$ so that

$$\||W|^{1/2}(H_1(k)+\lambda V_p-\tilde{E})^{-1}|W|^{1/2}\|\leqslant c\,\|W\|_\infty\int_{1/2-\delta_0}^{1/2}\frac{dk}{\frac{1}{2}-k+\frac{\lambda}{2}|\tilde{V}_1|} \tag{4.13}$$

where we used (4.12) and note that other bands do not give a contribution to the leading order, which, as we now infer from (4.13), diverges logarithmically as $\lambda\downarrow 0$. So we are done. ■

Remark. Clearly, by the same method we can treat any other band, for we can always remove degenerate eigenvalues by subtracting a suitable projection.

5. On potentials like $\sin x/x$

It is a known fact that the operator $H_0+\lambda/1+x^2$ has exactly one negative eigenvalue if $-\frac{1}{4}\leqslant\lambda<0$, but an infinite number of eigenvalues if $\lambda<-\frac{1}{4}$. It seems to be less well known that e.g. the operator $H_0+\lambda\sin x/x$ shows a similar behavior. We shall prove this in Theorem (5.1). And to the best of our knowledge it is a novelty that proving this fact isn't that hard if one analyses a certain Birman–Schwinger kernel. Our interest in this problem was aroused by a remark of B. Simon. At the time when we had solved this problem in our way, we were unaware of the recent literature on this subject. However, we subsequently learned that D. Willet [9, 10] had solved this problem by proving some very general criteria for oscillation or nonoscillation of ordinary differential equations. But we were also surprised to see how big a machinery was needed to crack it (see also the article by J. S. Wong [11]). Although we can extend our results to a whole class of potentials, it seems unlikely that we could reach the same degree of generality as with the recent o.d.e. methods. Our method works well if we have sufficient control of the Fourier transform of the potential, in particular, in the case of $\sin x/x$, we shall benefit from the nice and unique fact that the Fourier transform of $\sin x/x$ is just a characteristic function. Since this potential will also be the subject of an appendix to [12] we merely give the main steps of the analysis and skip over some technical details.

We have

Theorem (5.1). *Let $H = H_0 + \lambda(\sin x/x)$ on $L_2(\mathbb{R})$. Then*
(i) *if $|\lambda| > 1/\sqrt{2}$, H has an infinite number of negative eigenvalues*
(ii) *if $|\lambda| < 1/\sqrt{2}$, H has at most a finite number of negative eigenvalues.*

Remark. $\lambda = 1/\sqrt{2}$ is included in (ii) by a result of Wong [11].

Sketch of proof. Instead of (2.2) we consider the momentum space version (which is isospectral to (2.2)) of

$$K_E = (H_0 - E)^{-1/2} V (H_0 - E)^{-1/2}, \tag{5.1}$$

which has the kernel

$$\hat{K}_E(p, p') = \frac{1}{(p^2 - E)^{1/2}} \frac{\hat{V}(p - p')}{\sqrt{2\pi}} \frac{1}{(p'^2 - E)^{1/2}} \tag{5.2}$$

where now

$$\hat{V}(p - p') = \sqrt{\frac{\pi}{2}}\, \chi_{[-1,1]}(p - p') \tag{5.3}$$

and χ_Δ denotes the characteristic function of the set Δ. $\hat{K}_E$ leaves the odd and even functions invariant, and by a simple argument [12] it suffices to consider only odd functions. For the kernel $\hat{K}_E^{\mathrm{odd}}$ we have

$$\hat{K}_E^{\mathrm{odd}}(p, p') = \hat{K}_E(p, p') + \hat{K}_E(p, -p') \tag{5.4}$$

and the underlying Hilbert space is now $L_2(0, \infty)$. Let

$$M_0(p, p') = \operatorname*{s\text{-lim}}_{E \uparrow 0} \hat{K}_E^{\mathrm{odd}}(p, p') = \frac{1}{2pp'} (\chi_{[-1,1]}(p - p') - \chi_{[-1,1]}(p + p')) \tag{5.5}$$

Then

$$M_0(p, p') = \begin{cases} 0 & \text{if } |p' - p| > 1 \\ 0 & \text{if } p + p' < 1 \\ \dfrac{1}{2pp'} & \text{otherwise} \end{cases}$$

We are interested in finding the spectrum of M_0, in particular the essential spectrum. To find the latter we work 'modulo compacts', i.e. we write $A \doteq B$ if $A = B + K$ with K compact. Then

$$M_0(p, p') \doteq \frac{1}{2p} D_1(p, p') + \frac{1}{2p'} D_2(p, p') \tag{5.6}$$

where D_1, resp. D_2, denotes the characteristic function of the set $\{p, p' \mid 0 \leq p \leq \frac{1}{2}, p' \in (1-p, 1+p)\}$, resp. of its reflection about the line $p=p'$. The compact part which has been thrown out in (5.6) is Hilbert–Schmidt. The operator on the right side of (5.6) is most conveniently viewed as a two-by-two matrix operator on $L_2(0, \frac{1}{2}) \oplus L_2(\frac{1}{2}, \frac{3}{2})$,

$$\begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix} \tag{5.7}$$

where $A=(1/2p)D_1(p, p')$ and $A^*=(1/2p')D_2(p, p')$ is its adjoint. Since the operator (5.7) annihilates vectors whose second component is odd with respect to the mid-point 1 of the interval $(\frac{1}{2}, \frac{3}{2})$, we can further restrict ourselves to even functions in this component. Thus we may as well view the operator as acting on $L_2(0, \frac{1}{2}) \oplus L_2(1, \frac{3}{2})$. If we now translate the variables of the second component by -1 we get an operator on $L_2(0, \frac{1}{2}) \oplus L_2(0, \frac{1}{2})$, which we denote by Q, and we have

$$Q = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix} \tag{5.8}$$

where $B(p, p') = 1/\sqrt{2}p$, $0<p'<p<\frac{1}{2}$. A little computation shows that

$$Q^2 = \begin{pmatrix} C & 0 \\ 0 & C-P \end{pmatrix} \tag{5.9}$$

where $C = BB^* = (B+B^*)/\sqrt{2}$ has kernel

$$C(p, p') = \begin{cases} \dfrac{1}{2p} & 0<p'<p<- \\ \dfrac{1}{2p'} & \frac{1}{2}>p'>p>0 \end{cases} \tag{5.10}$$

and $P = 1(1, \cdot)$ is rank one. We denote by $\tilde{C}$ the kernel (5.10) but without the restriction that $p, p' < \frac{1}{2}$. Then $\tilde{C}$ commutes with dilations, i.e. $(W_\delta f)(p) = \sqrt{\delta} f(\delta p)$, $\delta > 0$ obeys $[W_\delta, C] = 0$ for all $\delta > 0$, and moreover

$$W_\delta C W_\delta^{-1} \to \tilde{C} \tag{5.11}$$

strongly as $\delta \downarrow 0$.

One can calculate the spectrum of $\tilde{C}$ and one finds $\sigma(\tilde{C}) = [0, 2]$ [12, 13]. Since $\|C\| \leq \|\tilde{C}\|$, $\sigma(C) = \sigma(W_\delta C W_\delta^{-1})$, and we have (5.11), we conclude that $\sigma(C) = \sigma(\tilde{C}) = [0, 2]$. Hence $\sigma_{\text{ess}}(Q^2) = \sigma(Q^2) = [0, 2]$ where the last equality follows from the fact that adding $-P$ does not create any discrete eigenvalue, for $Q^2 \geq 0$. Since Q and $-Q$ are unitarily equivalent under the transformation $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we obtain finally

$$\sigma_{\text{ess}}(M_0) = \sigma_{\text{ess}}(Q) = [-\sqrt{2}, \sqrt{2}] \tag{5.12}$$

This, along with the Birman–Schwinger principle (see also the proof of Lemma (2.3) (iv)) implies (i) of Theorem (5.1). Part (ii) doesn't quite follow yet, for in (5.5) we have only strong convergence. But we have an additional argument [12] by which we can prove stability of the discrete eigenvalues of K_E as $E \uparrow 0$. This implies (ii). ■

Appendix to Section 3: The case $d=0$

It is a nuisance that this case requires a separate analysis. All we want to do here is to find the limiting operator of $(K_\alpha - z)^{-1}$ and the operator that corresponds to Q in the case $d \neq 0$. We start out from

$$K_\alpha = L_\alpha + M_\alpha \tag{A.1}$$

where now $L_\alpha^2 = 0$. Then

$$(L_\alpha - z)^{-1} = -\frac{L_\alpha}{z^2} - \frac{1}{z} \tag{A.2}$$

Let $L = 2\alpha L_\alpha$. Now we proceed as in Section 3 and convert $(K_\alpha - z)^{-1}$ to the form corresponding to the right side of (3.11). Obviously there will be terms of the form

$$LM_0 = |V|^{1/2}(V^{1/2}, M_0 \,\cdot) \tag{A.3}$$

which give rise to a projection

$$\tilde{L} = c^{-1} L M_0 \tag{A.4}$$

where $c = (V^{1/2}, M_0 |V|^{1/2}) > 0$. Let $\tilde{Q} = 1 - \tilde{L}$. We obtain ($\tilde{M}_\alpha = M_\alpha - M_0$)

$$\begin{aligned}(K_\alpha - z)^{-1} = &-\left(1 + \left(\tilde{Q} + \left(1 - \frac{c}{2\alpha z^2}\right)^{-1} \tilde{L}\right)\left(-\frac{L\tilde{M}_\alpha}{2\alpha z^2} - \frac{1}{z} M_\alpha\right)\right)^{-1} \\ &\times \left(\tilde{Q} + \left(1 - \frac{c}{2az^2}\right)^{-1} \tilde{L}\right)\left(\frac{L}{2\alpha z^2} + \frac{1}{z}\right)\end{aligned} \tag{A.5}$$

Now we use that $\tilde{L}L = L$, $L\tilde{L} = 0$, $\tilde{Q}L = 0$, and then let α go to zero. The result is

$$\operatorname*{norm-lim}_{\alpha \downarrow 0} (K_\alpha - z)^{-1} = \left(1 - \frac{\tilde{Q}M_0}{z}\right)^{-1}\left(-\frac{\tilde{Q}}{z} + \frac{L}{c}\right) \tag{A.6}$$

$$= \left(1 - \frac{\tilde{Q}M_0\tilde{Q}}{z}\right)^{-1}\left(-\frac{\tilde{Q}}{z} + \frac{L}{c} - \frac{\tilde{Q}M_0 L}{zc}\right) \tag{A.7}$$

where in the last step we used $(1 - \tilde{Q}M_0\tilde{L}/z) = (1 + \tilde{Q}M_0\tilde{L}/z)$. Now it is easy to verify that any eigenvalue of the operator (A.7) is of the form $(\sigma - z)^{-1}$ where σ is eigenvalue of $\tilde{Q}M_0\tilde{Q}$. The corresponding eigenfunction satisfies two conditions: $Lf = 0$, i.e. $(V^{1/2}, f) = 0$ and $\tilde{L}f = 0$, i.e. $(V^{1/2}, M_0 f) = 0$. Conversely, any nonzero eigenvalue of $\tilde{Q}M_0\tilde{Q}$ gives rise to a corresponding eigenvalue of (A.7). $\tilde{Q}$ replaces Q of Section 3. Note the bizarre fact that (A.7) is *not* of the form $R(RM_0R - z)^{-1}R$ for any projection R, as can be easily seen by looking at the limit $|z| \to \infty$.

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