Zeitschrift: Helvetica Physica Acta
Band: 55 (1982)
Heft: 1

Artikel: On the S-operator for the external field problem of QED
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DOI: https://doi.org/10.5169/seals-115276

Helvetica Physica Acta, Vol. **55** (1982) 1–28 0018–0238/82/010001–28$1.50 + 0.20/0

On the S-operator for the external field problem of QED

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(22. XII. 1981)

Abstract. The scattering operator associated with the quantized electron-positron field interacting with a time-dependent external electromagnetic field is investigated. It is shown that it depends analytically on the strength of the external field up to arbitrarily high values. As a consequence, there exists no threshold for the occurrence of spontaneous pair production.

1. Introduction

Looking at the Hamiltonian $H = H_0 + \varkappa V$, where H_0 is the free Dirac Hamiltonian and V an electrostatic potential associated with a distribution of positive charges, one observes the following pecularities: when $\varkappa$ increases, bound states appear at the upper continuous part of the spectrum ($E = m$), move across the energy gap and dive into the lower continuum at $E = -m$. This is usually illustrated by a gedanken experiment, considering an atomic nucleus of charge Z, assuming that Z increases infinitely slowly [1]. When the $1s$ level reaches the lower continuum, its binding energy reaches the value $2m$ and the creation of an electron–positron pair becomes energetically favorable. The value Z at which this happens is called the critical charge Z_c. When $Z > Z_c$, the creation of an electron–positron pair even reduces the total energy of the system. This phenomenon is interpreted as 'decay of the neutral vacuum' to a 'charged vacuum' by 'spontaneous pair production' and is expected to arise in strong fields, since the calculated value of Z_c for a 'realistic' potential is equal to about 173. (Notice that the potential of a point charge is not suitable because the Hamiltonian ceases to be essentially selfadjoint at $Z = 137$.)

Today the only practical means of creating strong electromagnetic fields are experiments involving heavy-ion scattering. Provided that two colliding nuclei approach each other closely enough so that their joint electric potential exceeds the critical strength, the occurring spontaneous pair production could be detected. However the electromagnetic field created during the collision is time-dependent, so pair production induced by the changing field is also expected to occur. A criterion is needed which makes it possible to distinguish between 'induced' and 'spontaneous' pair creation.

Most of the results that can be found in the literature concerning QED in strong fields (for a review, see [2, 3]) are based on approximate calculations, so the notion of spontaneous pair creation remains ambiguous.

If radiative corrections are neglected, which is an adequate first approximation, the problem can be treated rigororously in the framework of the external field problem of QED, i.e. the theory of the quantized electron–positron field interacting with an external classical electromagnetic field [4–8]. In the case of certain regular static electromagnetic potentials, a vacuum can be defined that becomes charged discontinuously as the potential increases. This has been discussed by Klaus and Scharf in [9, 10] (see also [11, 12]). A basic requirement of these investigations is the existence of a dressing transformation on Fock space which converts the 'bare' vacuum into a 'dressed' vacuum. However this construction refers to a static external field and cannot be used to describe the situation present in heavy-ion collisions. Furthermore it is known that in static fields no pair creation occurs.

Since pair creation is a scattering phenomenon it is most natural to discuss the problem in the framework of scattering theory. At the same time it yields an unambiguous particle interpretation of the asymptotically free incoming and outgoing states [13]. One observes that there is a resemblance between the Fock space S-operator with time-dependent external field and the dressing transformation in the static case (for a comparison, see [14]). At certain values of the field strength, the S-operator has singularities which are analogous to the discontinuities of the dressing transformation corresponding to the occurrence of the charged vacuum. For this reason one might suspect that these singularities are related to the occurrence of spontaneous pair creation.

It is the aim of our investigations to answer the following questions: Is it possible to distinguish between

(i) undercritical and overcritical external fields
(ii) induced and spontaneous pair creation
on the basis of the Fock space S-operator?

We will show that the singularities mentioned above are spurious. Moreover the S-operator depends analytically on the strength of the external field up to arbitrarily high values. Therefore, contrary to the presumptions, the answer to questions (i) and (ii) is negative.

2. Some results of classical Dirac theory

In the following we fix some basic notions of classical, i.e. nonquantized Dirac theory, and collect several results which we will use in Section 3 and most of which are well known [4, 5, 15]. The observations stated in Theorems 2 and 4 seem to be new and lead to a clarification of the situation.

The underlying Hilbert space $\mathscr{H} = (L^2(\mathbb{R}^3))^4$ is the space of Dirac four-spinors. We will only need its structure as an abstract Hilbert space later on. The set of bounded operators on $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$.

Because we frequently are dealing with holomorphic operator-valued functions $\mathbb{C} \to \mathscr{B}(\mathscr{H})$ we recall the convenient criterion [16]: Let $T(\varkappa) \in \mathscr{B}(\mathscr{H})$ be defined on a domain G of the complex plane. $T(\varkappa)$ is holomorphic in G if and only if each $\varkappa \in G$ has a neighbourhood in which $\|T(\varkappa)\|$ is bounded and $\langle f \mid T(\varkappa)g\rangle$ is holomorphic for any f, g in a fundamental subset of $\mathscr{H}$ (i.e. a subset which has a dense linear span in $\mathscr{H}$). We note that $T(\varkappa)^{-1}$ (if it exists) and $T(\bar{\varkappa})^*$

are holomorphic functions of $\varkappa$ if $T(\varkappa)$ is holomorphic, whereas $T(\varkappa)^*$ is not holomorphic (notice that the inner product $\langle\cdot|\cdot\rangle$ is conjugate linear in the first argument).

The Dirac equation with time-dependent external field reads

$$H(t)f(t)=i\frac{d}{dt}f(t), \qquad f\in\mathcal{H} \tag{2.1}$$

where

$$\begin{aligned} H(t) &= H_0+\varkappa V(t)\\ H_0 &= -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+m\beta \\ V(t) &\equiv V(\mathbf{x},t)=eA_0(\mathbf{x},t)-e\boldsymbol{\alpha}\cdot\mathbf{A}(\mathbf{x},t)\end{aligned} \tag{2.2}$$

(2.1) is solved by the unitary propagator $U(t, s)$

$$f(t)=U(t,s)f(s) \tag{2.3}$$

The spectrum of H_0 is $\sigma(H_0)=(-\infty,-m]\cup[m,\infty)$ and we introduce the spectral resolution $H_0=\int\lambda\, dE(\lambda)$.

The spectral projections corresponding to the positive and negative part of the spectrum are given by

$$\begin{aligned} P_+ &= \int_m^\infty dE(\lambda)=\frac{1}{2}+\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi(H_0-i\xi)^{-1}\\ P_- &= \int_{-\infty}^{-m} dE(\lambda)=\frac{1}{2}-\frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi(H_0-i\xi)^{-1}\end{aligned} \tag{2.4}$$

They satisfy

$$P_++P_-=1, \qquad P_+P_-=0 \tag{2.5}$$

and give rise to the direct sum decomposition

$$\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-, \qquad \mathcal{H}_\pm=P_\pm\mathcal{H} \tag{2.6}$$

of the Hilbert space in 'electron' and 'positron' subspace.

(2.1) can be converted into the interaction picture integral equation for the propagator $U(t, s)$

$$\tilde{U}(t,s)=1-i\varkappa\int_s^t dt_1\,\tilde{V}(t_1)\tilde{U}(t_1,s) \tag{2.7}$$

where

$$\begin{aligned}\tilde{U}(t,s) &= e^{itH_0}U(t,s)e^{-isH_0}\\ \tilde{V}(t) &= e^{itH_0}V(t)e^{-itH_0}\end{aligned} \tag{2.8}$$

If $V(t)$ is assumed to be a bounded (and of course selfadjoint) operator for each t, and strongly continuous in t, one obtains the norm converging Dyson expansion of $\tilde{U}(t, s)$ by iterating (2.7)

$$\tilde{U}(t,s)=\sum_{n=0}^{\infty}(-i\varkappa)^n\int_s^t dt_1\int_s^{t_1}dt_2\cdots\int_s^{t_{n-1}}dt_n\,\tilde{V}(t_1)\cdots\tilde{V}(t_n) \tag{2.9}$$

If furthermore $\int_{-\infty}^{\infty} dt\, \|V(t)\| < \infty$, the scattering operator exists by the strong limit

$$S = \lim_{\substack{t\to\infty \\ s\to-\infty}} \tilde{U}(t, s) \tag{2.10}$$

and according to (2.9) we have

$$\begin{aligned} S(\varkappa) &= \sum_{n=0}^{\infty} (-i\varkappa)^n S_n \\ S_n &= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \tilde{V}(t_1) \cdots \tilde{V}(t_n) \end{aligned} \tag{2.11}$$

We take (2.11) as the definition of the classical S-operator. Since the series has an infinite radius of convergence, S is an entire function of $\varkappa$. Moreover S is unitary for real $\varkappa$, thus

$$S(\varkappa)^* S(\varkappa) = S(\varkappa) S(\varkappa)^* = 1, \qquad \varkappa \in \mathbb{R} \tag{2.12}$$

This relation can be extended to the whole complex plane by analytic continuation since $S(\varkappa)$ and $S(\bar{\varkappa})^*$ are entire. One has

$$S(\bar{\varkappa})^* S(\varkappa) = S(\varkappa) S(\bar{\varkappa})^* = 1, \qquad \varkappa \in \mathbb{C} \tag{2.13}$$

For simplicity we will omit the argument $\varkappa$ resp. $\bar{\varkappa}$ whenever this does not give rise to confusion. It has to be chosen in such a way that the operators become holomorphic functions of $\varkappa$.

Using the projections $P_\pm$ (2.4) we define

$$\begin{aligned} &S_{++} = P_+ S P_+, \ S_{--} = P_- S P_- \\ &S_{+-} = P_+ S P_-, \ S_{-+} = P_- S P_+ \end{aligned} \tag{2.14}$$

and for the adjoints we adopt the convention

$$\begin{aligned} &S^*_{+-} = (S_{+-})^* = P_- S^* P_+ \\ &S^*_{-+} = (S_{-+})^* = P_+ S^* P_- \end{aligned} \tag{2.15}$$

Taking into account (2.5) one derives from (2.13)

$$\begin{aligned} &S^*_{++} S_{++} + S^*_{-+} S_{-+} = P_+ \\ &S_{++} S^*_{++} + S_{+-} S^*_{+-} = P_+ \\ &S^*_{--} S_{--} + S^*_{+-} S_{+-} = P_- \\ &S_{--} S^*_{--} + S_{-+} S^*_{-+} = P_- \end{aligned} \tag{2.16}$$

$$\begin{aligned} &S^*_{++} S_{+-} + S^*_{-+} S_{--} = 0 \\ &S_{++} S^*_{-+} + S_{+-} S^*_{--} = 0 \\ &S^*_{--} S_{-+} + S^*_{+-} S_{++} = 0 \\ &S_{--} S^*_{+-} + S_{-+} S^*_{++} = 0 \end{aligned} \tag{2.17}$$

The existence of the implemented S-operator on Fock space depends essentially on the fact that S_{+-} and S_{-+} are Hilbert–Schmidt operators (cf. Section 3, Theorem 5). According to Theorem 7 in the appendix, potentials $V(t)$ which give rise to this property exist under quite general conditions. Therefore it is justified to assume

$$S_{+-}(\varkappa), S_{-+}(\varkappa) \in HS, \qquad \forall \varkappa \in \mathbb{C} \tag{2.18}$$

Hence the operators

$$\begin{aligned} &S_{+-}(\bar{\varkappa})^* S_{+-}(\varkappa), \qquad S_{+-}(\varkappa) S_{+-}(\bar{\varkappa})^* \\ &S_{-+}(\bar{\varkappa})^* S_{-+}(\varkappa), \qquad S_{-+}(\varkappa) S_{-+}(\bar{\varkappa})^* \end{aligned} \tag{2.19}$$

are trace class for all complex $\varkappa$.

We define the operators[1])

$$F_+ = S_{++} + P_-, \qquad F_- = S_{--} + P_+ \tag{2.20}$$

They satisfy the following relations which can be verified using (2.16) and (2.17)

$$\begin{aligned} F_+^* F_+ &= 1 - S_{-+}^* S_{-+} \\ F_+ F_+^* &= 1 - S_{+-} S_{+-}^* \\ F_-^* F_- &= 1 - S_{+-}^* S_{+-} \\ F_- F_-^* &= 1 - S_{-+} S_{-+}^* \end{aligned} \tag{2.21}$$

$$\begin{aligned} F_+ S_{-+}^* + S_{+-} F_-^* &= 0 \\ F_- S_{+-}^* + S_{-+} F_+^* &= 0 \\ F_+^* S_{+-} + S_{-+}^* F_- &= 0 \\ F_-^* S_{-+} + S_{+-}^* F_+ &= 0 \end{aligned} \tag{2.22}$$

$$\begin{aligned} F_+ S_{-+}^* S_{-+} &= S_{+-} S_{+-}^* F_+ \\ F_- S_{+-}^* S_{+-} &= S_{-+} S_{-+}^* F_- \\ F_+^* S_{+-} S_{+-}^* &= S_{-+}^* S_{-+} F_+^* \\ F_-^* S_{-+} S_{-+}^* &= S_{+-}^* S_{+-} F_-^* \end{aligned} \tag{2.23}$$

In the following the kernel of an operator T is denoted by $N(T)$, its eigenspace corresponding to the eigenvalue 1 by $E_1(T)$ and its range by $R(T)$.

$$\begin{aligned} N(T) &= N(T^*T) = \{f \in \mathcal{H} \mid Tf = 0\} \\ E_i(T) &= N(T-1) = \{f \in \mathcal{H} \mid Tf = f\} \end{aligned} \tag{2.24}$$

$\mathcal{H}$ decomposes according to

$$\mathcal{H} = N(T) \oplus \overline{R(T^*)} = N(T^*) \oplus \overline{R(T)} \tag{2.25}$$

where $\overline{R(T)}$ means the closure of $R(T)$.

We briefly summarize some facts about the theory of Fredholm operators which can be found in [17, 18, 19]: An operator $T \in \mathcal{B}(\mathcal{H})$ is called Fredholm if and only if $\dim N(T)$ and $\dim N(T^*)$ are finite. The set of Fredholm operators is

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[1]) In literature usually the restrictions of S_{++} resp. S_{--} defined on $\mathcal{H}_+$ resp. $\mathcal{H}_-$ are used, but it is more convenient to consider the operators (2.20) because in this way one has not to worry about domains.

denoted by $\mathscr{F}(\mathscr{H})$. The integer ind $T = \dim N(T) - \dim N(T^*)$ is called the index of T. It follows from the definition that

$$\text{ind } T^* = -\text{ind } T. \tag{2.26}$$

Since $\dim N(T^*) < \infty$, the range of a Fredholm operator is closed [17].

Let M be a topological space. One says two elements $x, y \in M$ can be joined by a path in M if and only if there exists a continuous mapping (a path) $\pi : [a, b] \to M$ of some closed interval in the real line into M, such that $\pi(a) = x$ and $\pi(b) = y$. An equivalence relation $\sim$ can be defined on M, writing $x \sim y$ if and only if x and y can be jointed by a path in M. The equivalence classes are called the path components of M.

Now consider $\mathscr{F}(\mathscr{H})$ as a topological space endowed with the operator norm topology. One has

Theorem 1 [17, 18, 19]. *The index is constant on each path component of $\mathscr{F}(\mathscr{H})$. If two Fredholm operators have the same index, then they are in the same path component of $\mathscr{F}(\mathscr{H})$.*

Corollary. *A Fredholm operator T has index zero if and only if $T \sim 1$.*

We are now ready to prove

Theorem 2. *$F_+(\varkappa)$ and $F_-(\varkappa)$ (2.20) are Fredholm operators of index zero for every complex $\varkappa$.*

Proof. From (2.21) we conclude

$$\begin{aligned} &N(F_+) \subset E_1(S^*_{-+}S_{-+}) \\ &N(F^*_+) \subset E_1(S_{+-}S^*_{+-}) \\ &N(F_-) \subset E_1(S^*_{+-}S_{+-}) \\ &N(F^*_-) \subset E_1(S_{-+}S^*_{-+}) \end{aligned} \tag{2.27}$$

Each kernel on the left-hand side is finite dimensional because it is contained in the eigenspace corresponding to a nonzero eigenvalue of a compact operator. Hence $F_+, F_- \in \mathscr{F}(\mathscr{H})$. Let $\pi : [0, 1] \to \mathbb{C}$ be a path in the complex plane such that $\pi(0) = \varkappa$ and $\pi(1) = 0$. Then $F_+(\pi(\cdot))$ is a path joining $F_+(\varkappa)$ and 1 in $\mathscr{F}(\mathscr{H})$. $t \to F_+(\pi(t))$ is continuous since F_+ is holomorphic in $\varkappa$.

$$\begin{aligned} &F_+(\pi(t)) \in \mathscr{F}(\mathscr{H}), \qquad \forall t \in [0, 1]. \\ &F_+(\pi(0)) = F_+(\varkappa) \\ &F_+(\pi(1)) = F_+(0) = 1 \quad \text{since} \quad S(0) = 1. \end{aligned}$$

Thus $F_+(\varkappa)$ has index zero for every $\varkappa \in \mathbb{C}$ by the corollary of Theorem 1.

The same argument holds for $F_-(\varkappa)$. □

The kernels are explicitly given by

$$\begin{aligned} &N(F_+(\varkappa)) = \mathscr{H}_+ \cap S(\bar{\varkappa})^*\mathscr{H}_- \\ &N(F_+(\bar{\varkappa})^*) = \mathscr{H}_+ \cap S(\varkappa)\mathscr{H}_- \\ &N(F_-(\varkappa)) = \mathscr{H}_- \cap S(\bar{\varkappa})^*\mathscr{H}_+ \\ &N(F_-(\bar{\varkappa})^*) = \mathscr{H}_- \cap S(\varkappa)\mathscr{H}_+ \end{aligned} \tag{2.28}$$

Notice that the subspaces $S\mathcal{H}_\pm$, $S^*\mathcal{H}_\pm$ are closed due to the invertibility of S resp. S^* guaranteed by (2.13).

We give the proof for $N(F_+)$: $\mathcal{H}_+\cap S^*\mathcal{H}_-\subset N(F_+)$ is verified directly: let $f\in\mathcal{H}_+\cap S^*\mathcal{H}_-$, hence $f\in\mathcal{H}_+, f\in S^*\mathcal{H}_-$ and by (2.13) $Sf\in\mathcal{H}_-$. Thus $F_+f=S_{++}f+P_-f=P_+Sf=0$. Now suppose $F_+f=0, f\neq 0$. From $S_{++}f+P_-f=S_{++}f+f-P_+f=0$ it follows that $f=P_+f-S_{++}f$, hence $f\in\mathcal{H}_+$. Thus we have $F_+f=P_+Sf=0$ and therefore $Sf\in\mathcal{H}_-, f\in S^*\mathcal{H}_-$. We conclude $N(F_+)\subset\mathcal{H}_+\cap S^*\mathcal{H}_-$. □

Inspecting the expressions (2.28) we observe that

$$\begin{aligned} S(\varkappa)N(F_+(\varkappa)) &= N(F_-(\bar\varkappa)^*)\\ S(\varkappa)N(F_-(\varkappa)) &= N(F_+(\bar\varkappa)^*)\end{aligned} \tag{2.29}$$

Because S is nonsingular it follows that

$$\begin{aligned} \dim N(F_+(\varkappa)) &= \dim N(F_-(\bar\varkappa)^*)\\ \dim N(F_-(\varkappa)) &= \dim N(F_+(\bar\varkappa)^*)\end{aligned} \tag{2.30}$$

Furthermore by Theorem 2 we have

$$\begin{aligned} \dim N(F_+(\varkappa)) &= \dim N(F_+(\varkappa)^*)\\ \dim N(F_-(\varkappa)) &= \dim N(F_-(\varkappa)^*)\end{aligned} \tag{2.31}$$

We conclude from (2.30) and (2.31)

$$\begin{aligned} \dim N(F_+(\varkappa)) &= \dim N(F_+(\varkappa)^*) = \dim N(F_-(\bar\varkappa)) = \dim N(F_-(\bar\varkappa)^*)\\ \dim N(F_-(\varkappa)) &= \dim N(F_-(\varkappa)^*) = \dim N(F_+(\bar\varkappa)) = \dim N(F_+(\bar\varkappa)^*)\\ &\qquad\qquad \forall \varkappa\in\mathbb{C}\end{aligned} \tag{2.32}$$

For certain values of $\varkappa$ the upper four resp. the lower four kernels (2.32) appear simultaneously with the same finite dimension. The sets of these exceptional points are defined by

$$\begin{aligned} \Sigma &= \{\varkappa\in\mathbb{C} \mid N(F_+(\varkappa))\neq 0\}\\ \bar\Sigma &= \{\varkappa\in\mathbb{C} \mid N(F_-(\varkappa))\neq 0\}\end{aligned} \tag{2.33}$$

(2.32) shows that $\bar\Sigma$ is the complex conjugate of Σ.

Definition. The points $\kappa\in\Sigma\cup\bar\Sigma$ are called singular, all other points of $\mathbb{C}$ are called regular.

Theorem 3. *The set Σ (and hence $\bar\Sigma$) is discrete in $\mathbb{C}$ (i.e. there is only a finite number of singular points in each compact subset of $\mathbb{C}$) and lies outside the disk*

$$\left\{z\in\mathbb{C}\;\middle|\;|z|<\frac{\ln 2}{a}\right\},\quad \textit{where}\quad a=\int_{-\infty}^{\infty} dt\,\|V(t)\|.$$

Proof. It follows from (2.27) that $S_{-+}(\bar\varkappa)^*S_{-+}(\varkappa)$ has an eigenvalue 1 if $\varkappa\in\Sigma$, but by the analytic Fredholm theorem [20] (see also [16] Chapter VII) this is the case only on a discrete set.

By estimating

$$\begin{aligned}\|S_{-+}\| &= \left\|\sum_{n=1}^{\infty}(-i\varkappa)^n P_- S_n P_+\right\|\\ &\leqq \sum_{n=1}^{\infty}|\varkappa|^n\,\|S_n\|\\ &\leqq \sum_{n=1}^{\infty}|\varkappa|^n\frac{a^n}{n!}=e^{a|\varkappa|}-1\end{aligned}$$

we see, that if $e^{a|\varkappa|}-1<1$, i.e. $|\varkappa|<(\ln 2)/a$ holds, $S^*_{-+}S_{-+}$ cannot have an eigenvalue 1. □

In the rest of this section we restrict our considerations to real values of $\varkappa$. If $\varkappa$ is real, the operators (2.19) are selfadjoint and the spectrum of each is a subset of the interval [0, 1]. In this case it follows from (2.21) and (2.28) that

$$\begin{aligned}N(F_+)&=N(F_+^*F_+)=E_1(S^*_{-+}S_{-+})=\mathscr{H}_+\cap S^*\mathscr{H}_-\\ N(F_+^*)&=N(F_+F_+^*)=E_1(S_{+-}S^*_{+-})=\mathscr{H}_+\cap S\mathscr{H}_-\\ N(F_-)&=N(F_-^*F_-)=E_1(S^*_{+-}S_{+-})=\mathscr{H}_-\cap S^*\mathscr{H}_+\\ N(F_-^*)&=N(F_-F_-^*)=E_1(S_{-+}S^*_{-+})=\mathscr{H}_-\cap S\mathscr{H}_+\end{aligned}\tag{2.34}$$

Theorem 4. *If $\varkappa$ is real, the operators $S^*_{+-}S_{+-}$, $S_{+-}S^*_{+-}$, $S^*_{-+}S_{-+}$, $S_{-+}S^*_{-+}$ have the same spectrum including multiplicity.*

Proof.

$$\begin{aligned}\sigma(S^*_{+-}S_{+-})&=\sigma(S_{+-}S^*_{+-})\\ \sigma(S^*_{-+}S_{-+})&=\sigma(S_{-+}S^*_{-+})\end{aligned}\tag{2.35}$$

holds in any case [16, 20]. If $\varkappa$ is regular, $F_+(\varkappa)$ and $F_-(\varkappa)$ are nonsingular and (2.23) can be written

$$\begin{aligned}S_{+-}S^*_{+-}&=F_+S^*_{-+}S_{-+}F_+^{-1}\\ S_{-+}S^*_{-+}&=F_-S^*_{+-}S_{+-}F_-^{-1}\end{aligned}\tag{2.36}$$

Thus we have

$$\begin{aligned}\sigma(S_{+-}S^*_{+-})&=\sigma(S^*_{-+}S_{-+})\\ \sigma(S_{-+}S^*_{-+})&=\sigma(S^*_{+-}S_{+-})\end{aligned}\tag{2.37}$$

due to the similarity of the operators. If $\varkappa$ is a singular point, the four operators (2.19) have an eigenvalue 1 of the same multiplicity, as a consequence of (2.32) and (2.34). The similarities (2.36) still hold with the restricted operators

$$\begin{aligned}\dot F_+&=F_+\restriction N(F_+)^\perp\\ \dot F_-&=F_-\restriction N(F_-)^\perp\end{aligned}\tag{2.38}$$

for which the inverses exist

$$\begin{aligned}\dot F_+^{-1}&:N(F_+^*)^\perp\to N(F_+)^\perp\\ \dot F_-^{-1}&:N(F_-^*)^\perp\to N(F_-)^\perp\end{aligned}\tag{2.39}$$

(cf. (2.25)). Hence

$$S_{+-}S_{+-}^{*} = \dot{F}_{+}S_{-+}^{*}S_{-+}\dot{F}_{+}^{-1} \quad \text{on} \quad N(F_{+}^{*})^{\perp} = E_1(S_{+-}S_{+-}^{*})^{\perp}$$
$$S_{-+}S_{-+}^{*} = \dot{F}_{-}S_{+-}^{*}S_{+-}\dot{F}_{-}^{-1} \quad \text{on} \quad N(F_{-}^{*})^{\perp} = E_1(S_{-+}S_{-+}^{*})^{\perp}$$

so that (2.37) is true also in the singular case. Combining (2.35) with (2.37) proves the theorem. □

We remark that the symmetry expressed in Theorem 4 is independent of the existence of an isomorphism between $\mathscr{H}_{+}$ and $\mathscr{H}_{-}$ (e.g. charge conjugation).

Introducing explicitly the eigenvalues and eigenvectors, we write, in consideration of Theorem 4

$$\begin{aligned}
&S_{-+}^{*}S_{-+}\phi_n^{+} = \lambda_n\phi_n^{+}\\
&S_{+-}^{*}S_{+-}\phi_n^{-} = \lambda_n\phi_n^{-}\\
&S_{+-}S_{+-}^{*}\psi_n^{+} = \lambda_n\psi_n^{+}\\
&S_{-+}S_{-+}^{*}\psi_n^{-} = \lambda_n\psi_n^{-}\\
&0 \leqq \lambda_n \leqq 1
\end{aligned} \tag{2.40}$$

where the eigenvalues are counted with their multiplicites.

Each eigenvalue $\lambda_n(\varkappa)$ is a holomorphic function on the real axis as long as it does not reach the value zero [16]. The eigenvectors (including those corresponding to the eigenvalue zero) constitute complete orthonormal systems in $\mathscr{H}_{+}$ resp. $\mathscr{H}_{-}$:

$$\begin{aligned}
&\phi_n^{+}, \psi_n^{+} \in \mathscr{H}_{+}, \qquad \phi_n^{-}, \psi_n^{-} \in \mathscr{H}_{-}\\
&\langle \phi_m^{+} \mid \phi_n^{+}\rangle = \langle \psi_m^{+} \mid \psi_n^{+}\rangle = \langle \phi_m^{-} \mid \phi_n^{-}\rangle = \langle \psi_m^{-} \mid \psi_n^{-}\rangle = \delta_{mn}
\end{aligned} \tag{2.41}$$

We call the systems

$$\{\phi_n^{+}\}, \{\phi_n^{-}\}, \{\psi_n^{+}\}, \{\psi_n^{-}\}$$

canonical bases.

Notice that the canonical bases are not uniquely determined when there are degenerate eigenvalues. Especially the eigenvalue zero has infinite multiplicity. But it is possible to choose the vectors as holomorphic functions of $\varkappa$ whenever the corresponding eigenvalues are holomorphic. Using the canonical bases we obtain the canonical expansions of the compact operators S_{+-} and S_{-+} [16, 20]:

$$\begin{aligned}
&S_{+-} = \sum_n \mu_n \langle \phi_n^{-} \mid \cdot \rangle \psi_n^{+}\\
&S_{-+} = \sum_n \mu_n \langle \phi_n^{+} \mid \cdot \rangle \psi_n^{-}\\
&S_{+-}^{*} = \sum_n \mu_n \langle \psi_n^{+} \mid \cdot \rangle \phi_n^{-}\\
&S_{-+}^{*} = \sum_n \mu_n \langle \psi_n^{-} \mid \cdot \rangle \phi_n^{+}\\
&\mu_n = +\sqrt{\lambda_n}
\end{aligned} \tag{2.42}$$

from which it follows that

$$\begin{aligned} S_{+-}\phi_n^- &= \mu_n\psi_n^+ \\ S_{-+}\phi_n^+ &= \mu_n\psi_n^- \\ S_{+-}^*\psi_n^+ &= \mu_n\phi_n^- \\ S_{-+}^*\psi_n^- &= \mu_n\phi_n^+ \end{aligned} \tag{2.43}$$

From (2.40) one derives

$$\begin{aligned} F_+^*F_+\phi_n^+ &= S_{++}^*S_{++}\phi_n^+ = (1-\lambda_n)\phi_n^+ \\ F_-^*F_-\phi_n^- &= S_{--}^*S_{--}\phi_n^- = (1-\lambda_n)\phi_n^- \\ F_+F_+^*\psi_n^+ &= S_{++}S_{++}^*\psi_n^+ = (1-\lambda_n)\psi_n^+ \\ F_-F_-^*\psi_n^- &= S_{--}S_{--}^*\psi_n^- = (1-\lambda_n)\psi_n^- \end{aligned} \tag{2.44}$$

The remaining properties we are interested in are more subtle.

Let us assume for a moment that $\varkappa$ is regular, i.e. $\lambda_n < 1, \forall n$. Multiplication of the first of the relations (2.44) by F_+ yields

$$F_+F_+^*F_+\phi_n^+ = (1-\lambda_n)F_+\phi_n^+$$

and comparison with

$$F_+F_+^*\psi_n^+ = (1-\lambda_n)\psi_n^+$$

shows that we can write

$$F_+\phi_n^+ = c_n\psi_n^+$$

by choosing the vectors ψ_n^+ properly (remember that the eigenvalues can be degenerate). Normalization leads to

$$|c_n|^2 = \|F_+\phi_n^+\|^2 = \langle\phi_n^+ \mid F_+^*F_+\phi_n^+\rangle = 1-\lambda_n$$

We fix the constant c_n by

$$c_n = e^{i\theta_n}\sqrt{1-\lambda_n}$$

hence

$$F_+\phi_n^+ = e^{i\theta_n}\sqrt{1-\lambda_n}\psi_n^+ \tag{i}$$

If we multiply (i) by F_+^* we obtain

$$F_+^*F_+\phi_n^+ = e^{i\theta_n}\sqrt{1-\lambda_n}F_+^*\psi_n^+ = (1-\lambda_n)\phi_n^+$$

thus

$$F_+^*\psi_n^+ = e^{-i\theta_n}\sqrt{1-\lambda_n}\phi_n^+ \tag{ii}$$

Multiplying this by S_{-+} yields

$$S_{-+}F_+^*\psi_n^+ = e^{-i\theta_n}\sqrt{1-\lambda_n}\mu_n\psi_n^-$$

but by (2.22) we have

$$S_{-+}F_+^*\psi_n^+ = -F_-S_{+-}^*\psi_n^+ = -\mu_nF_-\phi_n^-$$

hence

$$F_-\phi_n^- = -e^{-i\theta_n}\sqrt{1-\lambda_n}\psi_n^- \tag{iii}$$

Multiplication by F_-^* leads finally to

$$F_-^*\psi_n^- = -e^{i\theta_n}\sqrt{1-\lambda_n}\phi_n^- \tag{iv}$$

The phases are arbitrary and we put $\theta_n = 0$.

Now suppose that $\varkappa$ approaches a singular point $\varkappa_0$ and that for an eigenvalue λ_n we have $\lambda_n(\varkappa_0) = 1$. Thus the function $\sqrt{1-\lambda_n(\varkappa)}$ vanishes at that point and the question arises whether there is a unique analytic continuation if $\varkappa$ goes through $\varkappa_0$. To see what happens, we recall that $\lambda_n(\varkappa)$ is bounded by 1 hence its power series around $\varkappa_0$ is of the form

$$\lambda_n(\varkappa) = 1 - a_0(\varkappa-\varkappa_0)^{2m} + a_1(\varkappa-\varkappa_0)^{2m+1} + \cdots, \qquad a_0 > 0 \tag{2.45}$$

Therefore the function

$$\nu_n(\varkappa) = \sqrt{1-\lambda_n(\varkappa)} = (\varkappa-\varkappa_0)^m\sqrt{a_0 - a_1(\varkappa-\varkappa_0) - \cdots} \tag{2.46}$$

has no branch point at $\varkappa_0$ and has a unique analytic continuation. Notice however that its sign changes when $\varkappa$ goes through $\varkappa_0$ if m is odd.

We adopt the convention $\nu_n(0) = +1, \forall n$.

With the functions ν_n defined in this way we write the relations (i)–(iv)

$$\begin{aligned} F_+\phi_n^+ &= S_{++}\phi_n^+ = \nu_n\psi_n^+ \\ F_-\phi_n^- &= S_{--}\phi_n^- = -\nu_n\psi_n^- \\ F_+^*\psi_n^+ &= S_{++}^*\psi_n^+ = \nu_n\phi_n^+ \\ F_-^*\psi_n^- &= S_{--}^*\psi_n^- = -\nu_n\phi_n^- \end{aligned} \tag{2.47}$$

Each function ν_n is holomorphic where λ_n is holomorphic, i.e. ν_n may have no further analytic continuation if λ_n becomes zero at some $\varkappa$.

3. The Fock space S-operator

In this section we study the S-operator in the framework of quantized Dirac theory with external field, especially its dependence on the coupling parameter $\varkappa$.

The existence of the scattering operator on Fock space is guaranteed for every real $\varkappa$ by (2.18) and Theorem 5 below, and its properties can be derived from the results of the 'c-number' theory discussed in Section 2.

Explicit expressions have been constructed by several authors with different methods [9, 11, 21–24]. One observes that an exceptional form is obtained at the singular points and it is not clear whether this is related to discontinuities in the dependence on $\varkappa$.

Ruijsenaars [25] proved that the S-operator is analytic for small $\varkappa$ and has a possibly two-valued analytic continuation, the singular points being branch points.

We will improve this result by showing that there exists a unique analytic continuation along the real axis (Theorem 6). The exceptional form of the S-operator at each real singular point $\varkappa_0$ is the limit of the expression at regular $\varkappa$ as $\varkappa$ approaches $\varkappa_0$.

We first introduce the quantized free Dirac field. It is constructed by choosing the Hilbert space $\mathcal{H}$ introduced in Section 2 as test function space and representing the *-algebra generated by the field operators $\Psi(f)$, $f \in \mathcal{H}$, satisfying canonical anticommutation relations, on a second Hilbert space $\mathcal{H}_F$ (the physical state space):

$$\begin{aligned} &\Psi : \mathcal{H} \to \mathcal{B}(\mathcal{H}_F) \\ &\qquad f \mapsto \Psi(f) \quad \text{conjugate linear} \end{aligned} \tag{3.1}$$

$$\begin{aligned} \{\Psi(f), \Psi(g)^*\} &= \langle f \mid g \rangle \\ \{\Psi(f), \Psi(g)\} &= 0 \\ \{\Psi(f)^*, \Psi(g)^*\} &= 0 \end{aligned} \tag{3.2}$$

The annihilation operators $b(f), d(f)$ for electrons respectively positrons are defined by

$$\begin{aligned} b(f) &= \Psi(P_+ f) = b(P_+ f) \\ d(f) &= \Psi(P_- f)^* = d(P_- f) \\ \Psi(f) &= b(f) + d(f)^* \end{aligned} \tag{3.3}$$

and satisfy the canonical anticommutation relations

$$\begin{aligned} \{b(f), b(g)^*\} &= \langle f \mid P_+ g \rangle \\ \{d(f)^*, d(g)\} &= \langle f \mid P_- g \rangle \end{aligned} \tag{3.4}$$

$b(f)$ is conjugate linear and $d(f)$ linear in f. It is assumed that a unique vacuum exists, i.e. a vector $\Omega \in \mathcal{H}_F$ for which

$$b(f)\Omega = 0, \qquad d(f)\Omega = 0, \qquad \forall f \in \mathcal{H}, \qquad \|\Omega\| = 1 \tag{3.5}$$

holds.

Let $\{f_m\}, \{g_n\}$ be complete orthonormal systems in $\mathcal{H}_+$ resp. $\mathcal{H}_-$. Then a fundamental subset $D \subset \mathcal{H}_F$ is given by the orthonormal vectors

$$\begin{aligned} &b(f_{m_1})^* b(f_{m_2})^* \cdots b(f_{m_M})^* d(g_{n_1})^* \cdots d(g_{n_N})^* \Omega \\ &m_1 < m_2 < \cdots < m_M \\ &n_1 < n_2 < \cdots < n_N \\ &M, N = 0, 1, 2, \ldots \end{aligned} \tag{3.6}$$

The number operator N and charge operator Q (both unbounded)

$$N = \sum_n (b(f_n)^* b(f_n) + d(g_n)^* d(g_n)) \tag{3.7}$$

$$Q = \sum_n (b(f_n)^* b(f_n) - d(g_n)^* d(g_n)) \tag{3.8}$$

are well-defined on D, and are independent of the orthonormal bases chosen. The time evolution of the free field is given by

$$\Psi_t(f) = \Psi(e^{itH_0} f) = \mathbb{V}_0(t)^* \Psi(f) \mathbb{V}_0(t) \tag{3.9}$$

where H_0 is the free Dirac Hamiltonian on $\mathcal{H}$ and $\mathbb{V}_0(t)$ the implemented unitary group on $\mathcal{H}_F$ which now has a positive generator.

A representation of the Fermi field algebra as described here can always be constructed in the well known way, if $\mathcal{H}_F$ is chosen to be the antisymmetrized Fock space over $\mathcal{H}$.

Within the framework of scattering theory, two free asymptotic fields $\Psi_{\rm in}(f)$ and $\Psi_{\rm out}(f)$ are related by the formula

$$\Psi_{\rm out}(f)=\Psi_{\rm in}(S^*f) \tag{3.10}$$

S being the classical S-operator (2.11). We identify $\Psi_{\rm in}(f)$ with the field $\Psi(f)$ (3.1) on $\mathcal{H}_F$ and write to have a simpler notation

$$\Psi'(f)\equiv\Psi_{\rm out}(f)=\Psi_{\rm in}(S^*f)\equiv\Psi(S^*f) \tag{3.11}$$

From (3.2), (3.11) and the unitarity of S it follows that the out-field again satisfies the canonical anticommutation relation

$$\{\Psi'(f),\Psi'(g)^*\}=\langle f\mid g\rangle \tag{3.12}$$

Considering the decomposition into electron- and positron-annihilation operators

$$\begin{aligned} b'(f)&=\Psi'(P_+f)=b'(P_+f)\\ d'(f)&=\Psi'(P_-f)^*=d'(P_-f)\\ \Psi'(f)&=b'(f)+d'(f)^* \end{aligned} \tag{3.13}$$

we have

$$\begin{aligned} \{b'(f),b'(g)^*\}&=\langle f\mid P_+g\rangle\\ \{d'(f)^*,d'(g)\}&=\langle f\mid P_-g\rangle \end{aligned} \tag{3.14}$$

Writing finally (3.11) in terms of creation and annihilation operators, we obtain a Bogoliubov transformation on $\mathcal{H}_F$.

$$\begin{aligned} b'(f)&=b(S^*_{++}f)+d(S^*_{+-}f)^*\\ d'(f)^*&=b(S^*_{-+}f)+d(S^*_{--}f)^* \end{aligned} \tag{3.15}$$

The question about the implementability of (3.15) is answered by a well known theorem proved by Shale and Stinespring [26]. We state it here for the sake of completeness.

Theorem 5. *The Bogoliubov transformation* (3.15) *is implementable by a unitary operator* $\mathbb{S}$ *on* $\mathcal{H}_F$ *if and only if* S_{+-} *and* S_{-+} *are Hilbert–Schmidt.*

The existence of the operator $\mathbb{S}$ is equivalent to the existence of a vector $\Omega'\in\mathcal{H}_F$ which satisfies

$$b'(f)\Omega'=0,\qquad d'(f)\Omega'=0,\qquad \forall f\in\mathcal{H} \tag{3.16}$$

As a result we have

$$\begin{aligned} b'(f)&=\mathbb{S}^*b(f)\mathbb{S}\\ d'(f)&=\mathbb{S}^*d(f)\mathbb{S} \end{aligned} \tag{3.17}$$

$$\Omega'=\mathbb{S}^*\Omega \tag{3.18}$$

$\mathbb{S}$ is uniquely determined up to a phase factor due to the irreducibility of the representation of the field algebra. It is interpreted as the S-operator on the physical state space $\mathscr{H}_F$ transforming incoming into outgoing many-particle states at time zero. Ω (3.5) represents the incoming vacuum.

At this stage it is instructive to check the charge conservation. As a representative example we first calculate the charge expectation value of the state $\mathbb{S}\Omega$:

$$\begin{aligned}\langle \mathbb{S}\Omega \mid Q\mathbb{S}\Omega\rangle &= \langle \Omega \mid \mathbb{S}^* Q\mathbb{S}\Omega\rangle \\ &= \left\langle \Omega \mid \mathbb{S}^* \sum_n (b(f_n)^* b(f_n) - d(g_n)^* d(g_n))\mathbb{S}\Omega\right\rangle \\ &= \sum_n \langle \Omega \mid b'(f_n)^* b'(f_n)\Omega\rangle - \sum_n \langle \Omega \mid d'(g_n)^* d'(g_n)\Omega\rangle \\ &= \sum_n \langle \Omega \mid d(S_{+-}^* f_n) d(S_{+-}^* f_n)^*\Omega\rangle - \sum_n \langle \Omega \mid b(S_{-+}^* g_n) b(S_{-+}^* g_n)^*\Omega\rangle \\ &= \sum_n \langle S_{+-}^* f_n \mid S_{+-}^* f_n\rangle - \sum_n \langle S_{-+}^* g_n \mid S_{-+}^* g_n\rangle \\ &= \operatorname{tr} S_{+-} S_{+-}^* - \operatorname{tr} S_{-+} S_{-+}^*\end{aligned}$$

According to Theorem 4 the traces are equal

$$\operatorname{tr} S_{+-} S_{+-}^* = \operatorname{tr} S_{-+} S_{-+}^* = \sum_n \lambda_n \tag{3.19}$$

thus we have

$$\langle \mathbb{S}\Omega \mid Q\mathbb{S}\Omega\rangle = 0 \tag{3.20}$$

in agreement with the fact that particles and antiparticles are created in pairs. To verify the charge conservation in the general case we write the Bogoliubov transformation (3.15) in terms of the canonical bases (2.41). Putting $f = \psi_n^+$ resp. $f = \psi_n^-$ in (3.15) and using (2.43) and (2.47) we get

$$\begin{aligned} b'(\psi_n^+) &= \nu_n b(\phi_n^+) + \mu_n d(\phi_n^-)^* \\ d'(\psi_n^-)^* &= \mu_n b(\phi_n^+) - \nu_n d(\phi_n^-)^* \end{aligned} \tag{3.21}$$

With Q expanded as follows

$$Q = \sum_n (b(\psi_n^+)^* b(\psi_n^+) - d(\psi_n^-)^* d(\psi_n^-))$$

we calculate

$$\begin{aligned}\mathbb{S}^* Q\mathbb{S} &= \sum_n (b'(\psi_n^+)^* b'(\psi_n^+) - d'(\psi_n^-)^* d'(\psi_n^-)) \\ &= \sum_n (\nu_n b(\phi_n^+)^* + \mu_n d(\phi_n^-))(\nu_n b(\phi_n^+) + \mu_n d(\phi_n^-)^*) \\ &\quad - \sum_n (\mu_n b(\phi_n^+) - \nu_n d(\phi_n^-)^*)(\mu_n b(\phi_n^+)^* - \nu_n d(\phi_n^-)) \\ &= \sum_n ((1-\lambda_n) b(\phi_n^+)^* b(\phi_n^+) + \lambda_n d(\phi_n^-) d(\phi_n^-)^* \\ &\quad - \lambda_n b(\phi_n^+) b(\phi_n^+)^* - (1-\lambda_n) d(\phi_n^-)^* d(\phi_n^-)) \\ &= \sum_n (b(\phi_n^+)^* b(\phi_n^+) - d(\phi_n^-)^* d(\phi_n^-)) \\ &= Q \end{aligned} \tag{3.22}$$

Therefore the charge conservation is established.

We now collect some formulas by which it will be quite easy to handle the expressions we are dealing with in the following.

Let T be a bounded operator on $\mathscr{H}$ and $\{f_m\}, \{f'_m\}$ resp. $\{g_n\}, \{g'_n\}$ complete orthonormal systems in $\mathscr{H}_+$ resp. $\mathscr{H}_-$. On $\mathscr{H}_F$ we define formally the operators

$$\begin{aligned} Tb^*d^* &= \sum_{m,n} \langle f_m \mid Tg_n \rangle b(f_m)^* d(g_n)^* \\ Tdb &= \sum_{m,n} \langle g_m \mid Tf_n \rangle d(g_m) b(f_n) \\ Tb^*b &= \sum_{m,n} \langle f_m \mid Tf'_n \rangle b(f_m)^* b(f'_n) \\ Tdd^* &= \sum_{m,n} \langle g_m \mid Tg'_n \rangle d(g_m) d(g'_n)^* \end{aligned} \tag{3.23}$$

They are independent of the orthonormal systems chosen whenever they are well-defined. This is immediately seen by carrying out orthogonal transformations on the bases.

In the same sense we consider the exponentials

$$e^{Tb^*d^*}, \quad e^{Tdb}, \quad :e^{Tb^*b}:, \quad :e^{Tdd^*}: \tag{3.24}$$

where the dots mean normal ordering of the operators b, b^*, d, d^*. The adjoints are given by

$$\begin{aligned} (e^{Tb^*d^*})^* &= e^{T^*db} \\ (e^{Tdb})^* &= e^{T^*b^*d^*} \\ (:e^{Tb^*b}:)^* &= :e^{T^*b^*b}: \\ (:e^{Tdd^*}:)^* &= :e^{T^*dd^*}: \end{aligned} \tag{3.25}$$

It is straightforward to verify the following intertwining relations by expanding the exponentials and using (3.4).

$$\begin{aligned} &:e^{(T-1)b^*b}: b(f)^* = b(TP_+f)^*: e^{(T-1)b^*b}: \\ &:e^{(T-1)b^*b}: b(T^*P_+f) = b(f) :e^{(T-1)b^*b}: \\ &:e^{(1-T)dd^*}: d(f)^* = d(T^*P_-f)^* :e^{(1-T)dd^*}: \\ &:e^{(1-T)dd^*}: d(TP_-f) = d(f) :e^{(1-T)dd^*}: \end{aligned} \tag{3.26}$$

$$\begin{aligned} e^{Tdb}b(f)^* &= (b(f)^* + d(TP_+f))e^{Tdb} \\ e^{Tdb}d(f)^* &= (d(f)^* - b(T^*P_-f))e^{Tdb} \\ e^{Tb^*d^*}b(f) &= (b(f) - d(T^*P_+f)^*)e^{Tb^*d^*} \\ e^{Tb^*d^*}d(f) &= (d(f) + b(TP_-f)^*)e^{Tb^*d^*} \end{aligned} \tag{3.27}$$

If T has a bounded inverse and leaves the subspaces $\mathscr{H}_+$ and $\mathscr{H}_-$ invariant, one has in addition to (3.26)

$$\begin{aligned} &e^{(T^{-1}-1)b^*b}: b(Tf)^* = b(f)^* :e^{(T^{-1}-1)b^*b}: \\ &:e^{(T^{-1}-1)b^*b}: b(f) = b(T^*f) :e^{(T^{-1}-1)b^*b}: \\ &:e^{(1-T^{-1})dd^*}: d(T^*f)^* = d(f)^* :e^{(1-T^{-1})dd^*}: \\ &:e^{(1-T^{-1})dd^*}: d(f) = d(Tf) :e^{(1-T^{-1})dd^*}: \end{aligned} \tag{3.28}$$

Before we discuss the operator $\mathbb{S}$, it is worthwhile to consider the vector Ω' since it has a simple expression characterized by (3.16) and shows all essential features of the full implemented operator.

We show that Ω' for regular $\varkappa$ is given by

$$\Omega' = ce^{-Ab^*d^*}\Omega \tag{3.29}$$

$$A = F_+^{-1}S_{+-} = -S_{-+}^*F_-^{*-1} \tag{3.30}$$

The two expressions (3.30) for A are equal by (2.22). Because the inverse F_+^{-1}, F_-^{*-1} are bounded in the regular case, A is Hilbert–Schmidt. Its canonical expansion follows from the canonical expansion of S_{+-} (2.42) and (2.47)

$$\begin{aligned} A = F_+^{-1}S_{+-} &= \sum_n \mu_n \langle \phi_n^- \mid \cdot \rangle F_+^{-1}\psi_n^+ \\ &= \sum_n \alpha_n \langle \phi_n^- \mid \cdot \rangle \phi_n^+, \qquad \alpha_n = \frac{\mu_n}{\nu_n} \end{aligned} \tag{3.31}$$

If the operator Ab^*d^* is expanded in terms of the canonical bases, it diagonalizes according to

$$\begin{aligned} Ab^*d^* &= \sum_{m,n} \langle \phi_m^+ \mid A\phi_n^- \rangle b(\phi_m^+)^* d(\phi_n^-)^* \\ &= \sum_n \alpha_n b(\phi_n^+)^* d(\phi_n^-)^* \end{aligned} \tag{3.32}$$

Thus (3.29) can be written

$$\begin{aligned} \Omega' &= ce^{-\sum_n \alpha_n b(\phi_n{}^+)^* d(\phi_n{}^-)^*}\Omega \\ &= c\prod_n e^{-\alpha_n b(\phi_n{}^+)^* d(\phi_n{}^-)^*}\Omega \\ &= c\prod_n (1-\alpha_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \end{aligned} \tag{3.33}$$

In the last step it is used that $b(f)^*b(f)^* = 0$, $d(f)^*d(f)^* = 0$, $\forall f \in \mathcal{H}$, thus the expansion of the exponentials consists of two terms only. Using this result, the norm of Ω' is calculated

$$\begin{aligned} \|\Omega'\|^2 &= |c|^2 \left\langle \Omega \,\middle|\, \prod_n (1-\alpha_n d(\phi_n^-) b(\phi_n^+))(1-\alpha_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \right\rangle \\ &= |c|^2 \left\langle \Omega \,\middle|\, \prod_n (1+\alpha_n^2 d(\phi_n^-) b(\phi_n^+) b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \right\rangle \\ &= |c|^2 \prod_n (1+\alpha_n^2) \end{aligned} \tag{3.34}$$

The infinite product has a finite value $\prod_n (1+\alpha_n^2) < \infty$ since $\sum_n \alpha_n^2 < \infty$ holds true, the α_n^2 being the eigenvalues of the trace class operator A^*A. Consequently the vector Ω' (3.29) is well-defined. We normalize it choosing

$$c = \left(\prod_n (1+\alpha_n^2)\right)^{-1/2} \equiv \Delta^{1/2} \tag{3.35}$$

$$\Delta = \left(\prod_n (1+\alpha_n^2)\right)^{-1} = \prod_n (1-\lambda_n) \tag{3.36}$$

$\Delta = \Delta(\varkappa)$ defined by (3.36) is equal to the determinant

$$\Delta = \det(1 - S_{+-}^* S_{+-}) \tag{3.37}$$

which is a holomorphic function of $\varkappa$ in the whole complex plane, since $S_{+-}^* S_{+-}$ is holomorphic and trace class there [27]. It vanishes if and only if $S_{+-}^* S_{+-}$ has an eigenvalue 1. This is the case exactly at the singular points. Note that one has

$$\begin{aligned}\det(1 - S_{+-}^* S_{+-}) &= \det(1 - S_{+-} S_{+-}^*) = \det(1 - S_{-+}^* S_{-+}) \\ &= \det(1 - S_{-+} S_{-+}^*) \qquad \forall \varkappa \in \mathbb{C}\end{aligned} \tag{3.38}$$

by Theorem 4 and the unique analytic continuation property.

We stress that the determinant Δ as a whole behaves much better than the individual eigenvalues λ_n, which are holomorphic in restricted regions only, even on the real axis. This has to be kept in mind when the expression (3.36) is used.

The square root of Δ is given by

$$\Delta^{1/2} = \left(\prod_n (1 - \lambda_n)\right)^{1/2} = \prod_n \nu_n \tag{3.39}$$

where the phasefactor is fixed in such a way that $\Delta^{1/2} = 1$, if $\varkappa = 0$. Suppose that at a real singular point $\varkappa_0$ the functions $\nu_1, \ldots, \nu_{n_0}$ vanish. Then we have in a neighbourhood of $\varkappa_0$

$$\Delta^{1/2} = \prod_{n=1}^{n_0} \nu_n \left(\prod_{n=n_0+1}^{\infty} (1 - \lambda_n)\right)^{1/2} \tag{3.40}$$

Both factors are holomorphic according to the discussion at the end of Section 2. Since this holds at any real singular point $\varkappa_0$, $\Delta^{1/2}$ is a holomorphic function on the real axis.

To verify that Ω' satisfies (3.16) we take the expression (3.33) and the Bogoliubov transformation in the form (3.21).

$$\begin{aligned} b'(\psi_n^+)\Omega' &= c(\nu_n b(\phi_n^+) + \mu_n d(\phi_n^-)^*) \cdot \prod_m (1 - \alpha_m b(\phi_m^+)^* d(\phi_m^-)^*)\Omega \\ &= c \prod_{m \neq n} (1 - \alpha_m b(\phi_m^+)^* d(\phi_m^-)^*) \\ &\quad \cdot (\nu_n b(\phi_n^+) + \mu_n d(\phi_n^-)^*)(1 - \alpha_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \\ &= c \prod_{m \neq n} (1 - \alpha_m b(\phi_m^+)^* d(\phi_m^-)^*) \cdot (\mu_n d(\phi_n^-)^* - \mu_n b(\phi_n^+) b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \\ &= 0 \end{aligned}$$

$$\begin{aligned} d'(\psi_n^-)\Omega' &= c(\mu_n b(\phi_n^+)^* - \nu_n d(\phi_n^-)) \cdot \prod_m (1 - \alpha_m b(\phi_m^+)^* d(\phi_m^-)^*)\Omega \\ &= c \prod_{m \neq n} (1 - \alpha_m b(\phi_m^+)^* d(\phi_m^-)^*) \cdot (\mu_n b(\phi_n^+)^* - \nu_n d(\phi_n^-))(1 - \alpha_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \\ &= c \prod_{m \neq n} (1 - \alpha_m b(\phi_m^+)^* d(\phi_m^-)^*) \cdot (\mu_n b(\phi_n^+)^* + \mu_n d(\phi_n^-) b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \\ &= 0 \end{aligned}$$

Ω' has the required properties and is given by (3.29) or equivalently by (3.33), both forms being well-defined for regular $\varkappa$. If $\varkappa$ approaches a singular point, the normalization constant c vanishes and the operator A diverges and therefore the exponential expression (3.29) ceases to be meaningful. The product form (3.33) however can be written

$$\begin{aligned}\Omega' &= \prod_n \nu_n \prod_n (1-\alpha_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \\ &= \prod_n (\nu_n - \mu_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega\end{aligned} \tag{3.41}$$

and is defined for all $\varkappa \in \mathbb{R}$. At the singular points one has $\nu_n = 0$, $\mu_n = 1$ for some n.

The analyticity properties of Ω' are established as follows.

Let Φ be a vector of the fundamental subset D (3.6) and Ab^*d^* expanded in terms of a fixed (i.e. $\varkappa$-independent) basis.

Then

$$\begin{aligned}\varphi(\varkappa) &\equiv \langle \Phi \mid \Omega' \rangle \\ &= \Delta^{1/2} \langle \Phi \mid e^{-Ab^*d^*}\Omega \rangle \\ &= \Delta^{1/2} \langle e^{-A^*db}\Phi \mid \Omega \rangle\end{aligned} \tag{3.42}$$

consists of a finite sum of terms, each being a holomorphic function since $\Delta(\varkappa)^{1/2}$ and $A(\varkappa)$ are holomorphic. Hence $\varphi(\varkappa)$ is holomorphic and so is $\Omega'(\varkappa)$ by the criterion stated at the beginning of Section 2, which is also valid for vector-valued functions.

Now consider a neighbourhood G of a singular point $\varkappa_0 \in \mathbb{R}$. By the previous argument the function $\varphi(\varkappa)$ is holomorphic in G except at the point $\varkappa_0$ where it is not defined. Furthermore $\varphi(\varkappa)$ is bounded in $G - \{\varkappa_0\}$ since $\|\Omega'\| = 1$. Thus by a theorem of function theory [28] the limit $\lim_{\varkappa \to \varkappa_0} \varphi(\varkappa)$ exists in a unique way, and defining $\varphi(\varkappa_0) = \lim_{\varkappa \to \varkappa_0} \varphi(\varkappa)$, $\varphi(\varkappa)$ becomes a holomorphic function in the whole of G.

This argument holds for every singular point. We conclude that $\Omega'(\varkappa)$ is holomorphic on the whole real axis.

The limit $\varkappa \to \varkappa_0$ is trivial to evaluate starting from (3.41). Writing the regular factor in exponential form we obtain a more familiar expression. Suppose

$$\begin{aligned}\lambda_n(\varkappa_0) &= 1; \qquad n = 1, 2, \ldots, n_0 \\ \lambda_n(\varkappa_0) &< 1; \qquad n > n_0\end{aligned} \tag{3.43}$$

then we have

$$\begin{aligned}\Omega'(\varkappa_0) &= (-)^{n_0} \prod_{n=1}^{n_0} b(\phi_n^+)^* d(\phi_n^-)^* \cdot \prod_{n=n_0+1}^{\infty} (\nu_n - \mu_n b(\phi_n^+)^* d(\phi_n^-)^*)\Omega \\ &= (-)^{n_0} c' \prod_{n=1}^{n_0} b(\phi_n^+)^* d(\phi_n^-)^* \cdot \exp\left(-\sum_{n=n_0+1}^{\infty} \alpha_n b(\phi_n^+)^* d(\phi_n^-)^*\right)\Omega \\ &= (-)^{n_0} c' \prod_{n=1}^{n_0} b(\phi_n^+)^* d(\phi_n^-)^* e^{-A'b^*d^*}\Omega\end{aligned} \tag{3.44}$$

$$c' = \left(\prod_{n=n_0+1}^{\infty} (1-\lambda_n)\right)^{1/2} \tag{3.45}$$

$$A' = A(1-P) \tag{3.46}$$

where P is the projection on the subspace spanned by the vectors $\phi_1^-, \ldots, \phi_{n_0}^-$.

Expressions similar to (3.44) are found in literature. With the method described here, the exceptional creation operators $b(\phi_n^+)^*, d(\phi_n^-)^*, n = 1, 2, \ldots, n_0$ automatically occur in pairs, in agreement with charge conservation.

We now consider the implemented S-operator. Since its explicit expression is known, as mentioned at the beginning of this section, we don't give a deduction but merely a verification of the result in our notation.

For regular $\varkappa$ we write $\mathbb{S}^*$ as follows

$$\begin{aligned}\mathbb{S}^* &= ce^{A_4^*b^*d^*} :e^{(A_2^*-1)b^*b}: :e^{(1-A_3^*)dd^*}: e^{A_1^*db}\\ &= c :e^{A_4^*b^*d^*+(A_2^*-1)b^*b+(1-A_3^*)dd^*+A_1^*db}:\end{aligned} \tag{3.47}$$

Note that written between normal ordering signs, the operators b, b^*, d, d^* anticommute thus the quadratic expressions commute, a fact we will use frequently in the following. By (3.25) the adjoint is given by

$$\begin{aligned}\mathbb{S} &= ce^{A_1b^*d^*} :e^{(1-A_3)dd^*}: :e^{(A_2-1)b^*b}: e^{A_4db}\\ &= ce^{A_1b^*d^*} :e^{(A_2-1)b^*b}: :e^{(1-A_3)dd^*}: e^{A_4db}\\ &= c :e^{A_1b^*d^*+(A_2-1)b^*b+(1-A_3)dd^*+A_4db}:\end{aligned} \tag{3.48}$$

The operators introduced are

$$\begin{aligned}A_1 &= S_{+-}F_-^{-1} = -F_+^{*-1}S_{-+}^*\\ A_2 &= F_+^{*-1}\\ A_3 &= F_-^{-1}\\ A_4 &= F_-^{-1}S_{-+} = -S_{+-}^*F_+^{*-1} = -A^*\\ c &= \Delta^{1/2}\end{aligned} \tag{3.49}$$

Using the intertwining relations (3.26)–(3.28) it is not hard to show that $\mathbb{S}^*$ given by (3.47) and $\mathbb{S}$ given by (3.48) satisfy

$$\begin{aligned}\mathbb{S}^*b(f) &= b'(f)\mathbb{S}^*\\ \mathbb{S}^*d(f) &= d'(f)\mathbb{S}^*\end{aligned} \tag{3.50}$$

$$\begin{aligned}b(f)\mathbb{S} &= \mathbb{S}b'(f)\\ d(f)\mathbb{S} &= \mathbb{S}d'(f)\end{aligned} \tag{3.51}$$

We will do this for the first of the equations (3.50) only. The proof of the others is quite similar and may be omitted.

$b(f)$ commutes with $e^{A_1^*db}$ and $:e^{(1-A_3^*)dd^*}:$ thus we have

$$\mathbb{S}^*b(f) = ce^{A_4^*b^*d^*} :e^{(A_2^*-1)b^*b}: b(f): e^{(1+A_3)dd^*}: e^{A_1db}$$

By the second relation (3.28)

$$\begin{aligned}:e^{(A_2^*-1)b^*b}: b(f) &= :e^{(F_+^{-1}-1)b^*b}: b(f)\\ &= b(F_+^*f) :e^{(F_+^{-1}-1)b^*b}:\end{aligned}$$

which leads to

$$\mathbb{S}^*b(f) = ce^{A_4^*b^*d^*}b(F_+^*f) :e^{(A_2^*-1)b^*b}: : e^{(1-A_3^*)dd^*}: e^{A_1^*db}$$

Now using the third of the equations (3.27) and $F_+^*P_+ = P_+F_+^*$

$$\begin{aligned} e^{A_4^* b^* d^*} b(F_+^* f) &= e^{-F_+^{-1} S_{+-} b^* d^*} b(F_+^* f) \\ &= (b(F_+^* f) + d(S_{+-}^* F_+^{*-1} P_+ F_+^* f)^*) e^{-F_+^{-1} S_{+-} b^* d^*} \\ &= (b(S_{++}^* f) + d(S_{+-}^* f)^*) e^{A_4^* b^* d^*} \\ &= b'(f) e^{A_4^* b^* d^*} \end{aligned}$$

Hence $\mathbb{S}^* b(f) = b'(f)\mathbb{S}^*$ is verified. Since by (3.50)

$$\begin{aligned} &\|\mathbb{S}^* b(f_1)^* \cdots b(f_m)^* d(g_1)^* \cdots d(g_n)^* \Omega\| \\ &\quad = \|b'(f_1)^* \cdots b'(f_m)^* d'(g_1)^* \cdots d'(g_n)^* \Omega'\| = 1 \end{aligned} \tag{3.52}$$

holds for any vector of D, $\mathbb{S}^*$ is well-defined on D and extends to an isometry on $\mathcal{H}_F$. Moreover from (3.50) and (3.51) it follows that we have

$$\left.\begin{aligned} \mathbb{S}\mathbb{S}^* b(f) &= b(f)\mathbb{S}\mathbb{S}^* \\ \mathbb{S}\mathbb{S}^* d(f) &= d(f)\mathbb{S}\mathbb{S}^* \\ \mathbb{S}^*\mathbb{S} b'(f) &= b'(f)\mathbb{S}^*\mathbb{S} \\ \mathbb{S}^*\mathbb{S} d'(f) &= d'(f)\mathbb{S}^*\mathbb{S} \end{aligned}\right. \tag{3.53}$$

for every $f \in \mathcal{H}$. Because the representation of the field algebra is irreducible and generated by $\Psi(f)$ as well as by $\Psi'(f)$, we conclude from (3.52), (3.53) and Schur's lemma

$$\mathbb{S}^*\mathbb{S} = \mathbb{S}\mathbb{S}^* = 1 \tag{3.54}$$

Therefore the operators (3.47) and (3.48) have the required properties and implement the Bogoliubov transformation (3.15) in the sense of Theorem 5.

To discuss the analyticity properties of $\mathbb{S}$ we consider a matrix element

$$\xi(\varkappa) \equiv \langle \Phi \mid \mathbb{S}\Phi' \rangle; \qquad \Phi, \Phi' \in D \tag{3.55}$$

where $\mathbb{S}$ is taken in the form (3.48), the operators in the exponent expanded in a basis independent of $\varkappa$.

As in the case of Ω', (3.55) consists of a finite sum of terms and is a holomorphic function at the regular points since all operators contained in (3.48) are holomorphic, the constant c included. Let G be a neighbourhood of a singular point $\varkappa_0$. $\xi(\varkappa)$ is holomorphic in $G - \{\varkappa_0\}$ and bounded since $\mathbb{S}$ is unitary. Hence $\xi(\varkappa_0)$ is defined by $\lim_{\varkappa \to \varkappa_0} \xi(\varkappa)$ and the function becomes holomorphic in the whole of G. We conclude that the matrix element (3.55) is a holomorphic function on the real axis for every $\Phi, \Phi' \in D$, and state our main result in

Theorem 6. *The implemented S-operator $\mathbb{S}(\varkappa)$ is holomorphic on the whole real axis.* □

It remains to evaluate the expression for $\mathbb{S}$ at the singular points. For this purpose we expand the operators (3.49) in terms of the canonical bases and get

$$
\begin{aligned}
A_1 b^* d^* &= -\sum_n \alpha_n b(\psi_n^+)^* d(\psi_n^-)^* \\
A_2 b^* b &= \sum_n \frac{1}{\nu_n} b(\psi_n^+)^* b(\phi_n^+) \\
A_3 dd^* &= -\sum_n \frac{1}{\nu_n} d(\phi_n^-) d(\psi_n^-)^* \\
A_4 db &= -\sum_n \alpha_n d(\phi_n^-) b(\phi_n^+)
\end{aligned}
\tag{3.56}
$$

We consider these operators in the neighbourhood of a singular point $\varkappa_0$ at which (3.43) holds and separate the diverging terms. Introducing the projections P_1, P_2 on the subspaces spanned by the vectors $\psi_1^-, \ldots, \psi_{n_0}^-$ resp. $\phi_1^+, \ldots, \phi_{n_0}^+$ we have

$$
\begin{aligned}
A_1 b^* d^* &= -\sum_{n=1}^{n_0} \alpha_n b(\psi_n^+)^* d(\psi_n^-)^* + A_1' b^* d^* \\
A_2 b^* b &= \sum_{n=1}^{n_0} \frac{1}{\nu_n} b(\psi_n^+)^* b(\phi_n^+) + A_2' b^* b \\
A_3 dd^* &= -\sum_{n=1}^{n_0} \frac{1}{\nu_n} d(\phi_n^-) d(\psi_n^-)^* + A_3' dd^* \\
A_4 db &= -\sum_{n=1}^{n_0} \alpha_n d(\phi_n^-) b(\phi_n^+) + A_4' db
\end{aligned}
\tag{3.57}
$$

$$
\begin{aligned}
A_1' &= A_1(1-P_1) \\
A_2' &= A_2(1-P_2) \\
A_3' &= A_3(1-P_1) \\
A_4' &= A_4(1-P_2)
\end{aligned}
\tag{3.58}
$$

Inserting this into (3.48), the exponential containing the diverging terms is expanded as follows

$$
\begin{aligned}
&:\exp\Bigg(-\sum_{n=1}^{n_0} \alpha_n b(\psi_n^+)^* d(\psi_n^-)^* + \sum_{n=1}^{n_0} \frac{1}{\nu_n} b(\psi_n^+)^* b(\phi_n^+) \\
&\quad + \sum_{n=1}^{n_0} \frac{1}{\nu_n} d(\phi_n^-) d(\psi_n^-)^* - \sum_{n=1}^{n_0} \alpha_n d(\phi_n^-) b(\phi_n^+)\Bigg): \\
&= :\prod_{n=1}^{n_0} \exp\Bigg(-\alpha_n b(\psi_n^+)^* d(\psi_n^-)^* + \frac{1}{\nu_n} b(\psi_n^+)^* b(\phi_n^+) \\
&\quad + \frac{1}{\nu_n} d(\phi_n^-) d(\psi_n^-)^* - \alpha_n d(\phi_n^-) b(\phi_n^+)\Bigg): \\
&= :\prod_{n=1}^{n_0} \Bigg(1 - \alpha_n b(\psi_n^+)^* d(\psi_n^-)^* + \frac{1}{\nu_n} b(\psi_n^+)^* b(\phi_n^+) \\
&\quad + \frac{1}{\nu_n} d(\phi_n^-) d(\psi_n^-)^* - \alpha_n d(\phi_n^-) b(\phi_n^+) - b(\psi_n^+)^* d(\psi_n^-)^* d(\phi_n^-) b(\phi_n^+)\Bigg):
\end{aligned}
$$

Therefore, using (3.40) and (3.45), $\mathbb{S}$ can be written

$$\mathbb{S} = c' : \prod_{n=1}^{n_0} (\nu_n - \mu_n b(\psi_n^+)^* d(\psi_n^-)^*$$
$$+ b(\psi_n^+)^* b(\phi_n^+) + d(\phi_n^-) d(\psi_n^-)^* - \mu_n d(\phi_n^-) b(\phi_n^+)$$
$$- \nu_n b(\psi_n^+)^* d(\psi_n^-)^* d(\phi_n^-) b(\phi_n^+))$$
$$\cdot e^{A_1' b^* d^* + (A_2' - 1) b^* b + (1 - A_3') d d^* + A_4' d b} : \tag{3.59}$$

The limit $\varkappa \to \varkappa_0$ is evaluated putting $\nu_n = 0$, $\mu_n = 1$, $n = 1, 2, \ldots, n_0$

$$\mathbb{S} = c' : \mathbb{S}_0 e^{A_1' b^* d^* + (A_2' - 1) b^* b + (1 - A_3') d d^* + A_4' d b} :$$
$$\mathbb{S}_0 = : \prod_{n=1}^{n_0} (b(\psi_n^+)^* - d(\phi_n^-))(b(\phi_n^+) - d(\psi_n^-)^*): \tag{3.60}$$

This is the exceptional form mentioned at the beginning of this section. It implements the Bogoliubov transformation (3.15) in the singular case.

If all modes with $\lambda_n \neq 0$ are separated, a form similar to (3.59) is obtained which is defined for all real $\varkappa$.

Let

$$M = \{n \mid \lambda_n \neq 0\}$$
$$M_0 = \{n \mid \lambda_n = 0\} \tag{3.61}$$

Using (3.56) and proceeding as before, we find

$$\mathbb{S} = : \prod_{n \in M} (\nu_n - \mu_n b(\psi_n^+)^* d(\psi_n^-)^* + b(\psi_n^+)^* b(\phi_n^+)$$
$$+ d(\phi_n^-) d(\psi_n^-)^* - \mu_n d(\phi_n^-) b(\phi_n^+) - \nu_n b(\psi_n^+)^* d(\psi_n^-)^* d(\phi_n^-) b(\phi_n^+))$$
$$\cdot \exp\Bigg(\sum_{n \in M_0} b(\psi_n^+)^* b(\phi_n^+) + \sum_{n \in M_0} d(\phi_n^-) d(\psi_n^-)^*$$
$$- \sum_{\forall n} b(\phi_n^+)^* b(\phi_n^+) + \sum_{\forall n} d(\psi_n^-) d(\psi_n^-)^* \Bigg): \tag{3.62}$$

To conclude this section we point out that in spite of the fact that $\mathbb{S}(\varkappa)$ is holomorphic on the real axis, its power-series expansion around the origin has a finite radius of convergence when the function $\Delta(\varkappa)^{1/2}$ has complex branch points, which cannot be excluded.

Appendix

In this appendix we prove the implementability condition (2.18). For potentials $V(\mathbf{x}, t)$ the matrix elements of which belong to $\mathscr{S}(\mathbb{R}^4)$, a proof was given by Ruijsenaars [29]. Palmer [30] considered a larger class of external fields, but his proof is quite long and complicated. We will follow essentially the proof of a similar theorem concerning the unitary propagator, given by Fierz and Scharf [13].

Before stating the theorem we introduce Fourier transforms.

$$\hat{f}(\mathbf{k}) = (2\pi)^{-3/2} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x})$$

$$f \in \mathcal{H} = (L^2(\mathbb{R}^3))^4$$

$$\widehat{H_0 f}(\mathbf{k}) = (\boldsymbol{\alpha}\cdot\mathbf{k} + m\beta)\hat{f}(\mathbf{k}) \equiv H_0(\mathbf{k})\hat{f}(\mathbf{k}) \tag{A.1}$$

$$\widehat{P_\pm f}(\mathbf{k}) = \frac{1}{2} \pm \frac{\boldsymbol{\alpha}\cdot\mathbf{k} + m\beta}{2E(\mathbf{k})} \hat{f}(\mathbf{k}) \equiv P_\pm(\mathbf{k})\hat{f}(\mathbf{k}) \tag{A.2}$$

$$E(\mathbf{k}) = +\sqrt{\mathbf{k}^2 + m^2} \tag{A.3}$$

The free Dirac Hamiltonian H_0 and the projections $P_\pm$ satisfy

$$H_0(\mathbf{k}) = E(\mathbf{k})(P_+(\mathbf{k}) - P_-(\mathbf{k})) \tag{A.4}$$

$$H_0(\mathbf{k})P_\pm(\mathbf{k}) = P_\pm(\mathbf{k})H_0(\mathbf{k}) = \pm E(\mathbf{k})P_\pm(\mathbf{k}) \tag{A.5}$$

The external field $V(\mathbf{x}, t)$ is a real valued function. We write

$$(V(t)f)(\mathbf{x}) = V(\mathbf{x}, t)f(\mathbf{x})$$

$$\begin{aligned} \widehat{V(t)f}(\mathbf{k}) &= (2\pi)^{-3/2} \int d^3k' \hat{V}(\mathbf{k}-\mathbf{k}', t)\hat{f}(\mathbf{k}') \\ &= (2\pi)^{-3/2}(\hat{V}(t) * \hat{f})(\mathbf{k}) \\ \hat{V}(\mathbf{k}, t) &= (2\pi)^{-3/2} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x}, t) \end{aligned} \tag{A.6}$$

The norm of the 4×4-matrix $\hat{V}(\mathbf{k}, t)$ for each $\mathbf{k}$ and t is denoted by $|\hat{V}(\mathbf{k}, t)|$. If

$$\|\hat{V}(t)\|_p \equiv \left(\int d^3k\, |\hat{V}(\mathbf{k}, t)|^p\right)^{1/p} < \infty \tag{A.7}$$

we write

$$\hat{V}(\cdot, t) \in (L^p(\mathbb{R}^3))^{16}$$

Theorem 7. *Suppose that the external field $V(\mathbf{x}, t)$ satisfies the following conditions.*

(1) *$V(t)$ is strongly continuous on $\mathcal{H}$ and two times piecewise strongly differentiable such that $(d^2/dt^2)V(t)$ is piecewise strongly continuous.*

(2) $\hat{V}^{(q)}(\cdot, t) \in (L^2(\mathbb{R}^3))^{16} \cap (L^1(\mathbb{R}^3))^{16}$, $\forall t$; $q = 0, 1, 2$

where

$$\hat{V}^{(q)}(\mathbf{k}, t) = \frac{\partial^q}{\partial t^q} \hat{V}(\mathbf{k}, t)$$

(3) $\|\hat{V}^{(q)}(t)\|_p \leqq F(t)$; $\quad p = 1, 2; \quad q = 0, 1, 2$

with

$$b \equiv \int_{-\infty}^{\infty} dt F(t) < \infty$$

Then $S_{+-}(\varkappa)$ and $S_{-+}(\varkappa)$ are Hilbert–Schmidt for all $\varkappa \in \mathbb{C}$.

Proof. We deal first with the case where the derivatives of $V(t)$ have no discontinuities. According to (2.11) we have

$$\begin{aligned} S_{+-}(\varkappa) &= \sum_{n=1}^{\infty} (-i\varkappa)^n T_n \\ T_n &= \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n P_+ \tilde{V}(t_1) \cdots \tilde{V}(t_n) P_- \end{aligned} \tag{A.8}$$

The Fourier transform is written in the following form

$$\begin{aligned} \widehat{T_n f}(\mathbf{k}) &= \int d^3k' T_n(\mathbf{k}, \mathbf{k}') \hat{f}(\mathbf{k}') \\ T_n(\mathbf{k}, \mathbf{k}') &= \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n e^{it_1 E(\mathbf{k})} e^{it_n E(\mathbf{k}')} I_n(\mathbf{k}, \mathbf{k}'; t_1, \ldots, t_n) \\ I_n(\mathbf{k}, \mathbf{k}'; t_1, \ldots, t_n) &= (2\pi)^{-3n/2} \int d^3k_1 \cdots d^3k_{n-1} P_+(\mathbf{k}) \\ &\quad \cdot \hat{V}(\mathbf{k}-\mathbf{k}_1, t_1) e^{-i(t_1-t_2)H_0(\mathbf{k}_1)} \hat{V}(\mathbf{k}_1-\mathbf{k}_2, t_2) e^{-i(t_2-t_3)H_0(\mathbf{k}_2)} \cdots \\ &\quad \cdot e^{-i(t_{n-1}-t_n)H_0(\mathbf{k}_{n-1})} \hat{V}(\mathbf{k}_{n-1}-\mathbf{k}', t_n) P_-(\mathbf{k}') \end{aligned} \tag{A.9}$$

Introducing new time variables

$$\begin{aligned} s_1 &= \frac{1}{n}(t_1 + t_2 + \cdots + t_n) \\ s_j &= t_j - t_{j-1}; \qquad j = 2, 3, \ldots, n \end{aligned} \tag{A.10}$$

with

$$\det \frac{\partial s_j}{\partial t_l} = 1$$

and the inverse transformation

$$\begin{aligned} t_1 &= s_1 - \frac{1}{n} \sum_{j=2}^{n} (n-j+1) s_j \\ t_n &= s_1 + \frac{1}{n} \sum_{j=2}^{n} (j-1) s_j \\ t_l &= s_1 + \frac{1}{n} \sum_{j=2}^{l} (j-1) s_j - \frac{1}{n} \sum_{j=l+1}^{n} (n-j+1) s_j, \qquad 2 \leqq l \leqq n-1 \end{aligned} \tag{A.11}$$

we obtain

$$T_n(\mathbf{k},\mathbf{k}')=\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{0} ds_2 \cdots \int_{-\infty}^{0} ds_n e^{is_1(E(\mathbf{k})+E(\mathbf{k}'))}$$
$$\cdot e^{-i/n\sum_{j=2}^{n} s_j((n-j+1)E(\mathbf{k})-(j-1)E(\mathbf{k}'))} I_n(\mathbf{k},\mathbf{k}'; s_1,\ldots,s_n) \tag{A.12}$$

Two successive integrations by parts with respect to s_1 lead to (the boundary terms vanish)

$$T_n(\mathbf{k},\mathbf{k}')=-\int_{-\infty}^{\infty} ds_1 \int_{-\infty}^{0} ds_2 \cdots \int_{-\infty}^{0} ds_n \frac{e^{is_1(E(\mathbf{k})+E(\mathbf{k}'))}}{(E(\mathbf{k})+E(\mathbf{k}'))^2}$$
$$\cdot e^{-i/n\sum_{j=2}^{n} s_j((n-j+1)E(\mathbf{k})-(j-1)E(\mathbf{k}'))} \frac{\partial^2}{\partial s_1^2} I_n(\mathbf{k},\mathbf{k}'; s_1,\ldots,s_n) \tag{A.13}$$

Differentiation of I_n with respect to s_1 acts on the potentials only. Since $\partial t_j/\partial s_1 = 1$, $j=1,2,\ldots,n$, (A.13) is equal to

$$T_n(\mathbf{k},\mathbf{k}')=-\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n \frac{e^{it_1E(\mathbf{k})}e^{it_nE(\mathbf{k}')}}{(E(\mathbf{k})+E(\mathbf{k}'))^2} I_n^{(2)}(\mathbf{k},\mathbf{k}'; t_1,\ldots,t_n)$$
$$I_n^{(2)}(\mathbf{k},\mathbf{k}'; t_1,\ldots,t_n)=(2\pi)^{-3n/2}\sum_{q_1\ldots q_n}\int d^3k_1\cdots d^3k_{n-1}P_+(\mathbf{k})$$
$$\cdot \hat{V}^{(q_1)}(\mathbf{k}-\mathbf{k}_1,t_1)e^{-i(t_1-t_2)H_0(\mathbf{k}_1)}\hat{V}^{(q_2)}(\mathbf{k}_1-\mathbf{k}_2,t_2)$$
$$\cdot e^{-i(t_2-t_3)H_0(\mathbf{k}_2)}\cdots e^{-i(t_{n-1}-t_n)H_0(\mathbf{k}_{n-1})}\hat{V}^{(q_n)}(\mathbf{k}_{n-1}-\mathbf{k}',t_n)P_-(\mathbf{k}') \tag{A.14}$$

where

$$q_j=0,1,2; \qquad \sum_{j=1}^{n} q_j = 2$$

Thus the sum in (A.14) contains n^2 terms.
We estimate the kernel

$$\left|\frac{e^{it_1E(\mathbf{k})}e^{it_nE(\mathbf{k}')}}{(E(\mathbf{k})+E(\mathbf{k}'))^2} I_n^{(2)}(\mathbf{k},\mathbf{k}'; t_1,\ldots,t_n)\right|$$
$$\leq \frac{1}{E(\mathbf{k})^2}|I_n^{(2)}(\mathbf{k},\mathbf{k}'; t_1,\ldots,t_n)|$$
$$\leq (2\pi)^{-3n/2}\frac{1}{E(\mathbf{k})^2}\sum_{q_1\cdots q_n}\int d^3k_1\cdots d^3k_{n-1}|\hat{V}^{(q_1)}(\mathbf{k}-\mathbf{k}_1,t_1)|$$
$$\cdot|\hat{V}^{(q_2)}(\mathbf{k}_1-\mathbf{k}_2,t_2)|\cdots|\hat{V}^{(q_n)}(\mathbf{k}_{n-1}-\mathbf{k}',t_n)|$$
$$=(2\pi)^{-3n/2}\frac{1}{E(\mathbf{k})^2}\sum_{q_1\cdots q_n}\int d^3p_1\cdots d^3p_{n-1}|\hat{V}^{(q_1)}(\mathbf{k}-\mathbf{k}'-\mathbf{p}_1,t_1)|$$
$$\cdot|\hat{V}^{(q_2)}(\mathbf{p}_1-\mathbf{p}_2,t_2)|\cdots|\hat{V}^{(q_n)}(\mathbf{p}_{n-1},t_n)|$$
$$=(2\pi)^{-3n/2}\frac{1}{E(\mathbf{k})^2}\sum_{q_1\cdots q_n}(|\hat{V}^{(q_1)}(t_1)|*|\hat{V}^{(q_2)}(t_2)|*\cdots*|\hat{V}^{(q_n)}(t_n)|)(\mathbf{k}-\mathbf{k}') \tag{A.15}$$

where we introduced the variables

$$\mathbf{p}_m = \mathbf{k}_m - \mathbf{k}'; \qquad m = 1, 2, \ldots, n-1$$

With this result we have

$$\begin{aligned}
&\|P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-\|_{\mathrm{HS}} \\
&\leqq (2\pi)^{-3n/2}\|E(\cdot)^{-2}\|_2 \sum_{q_1\cdots q_n} \||\hat{V}^{(q_1)}(t_1)| * \cdots * |\hat{V}^{(q_n)}(t_n)|\|_2 \\
&\leqq (2\pi)^{-3n/2}\|E(\cdot)^{-2}\|_2 \sum_{q_1\cdots q_n} \|\hat{V}^{(q_1)}(t_1)\|_1 \|\hat{V}^{(q_2)}(t_2)\|_1 \cdots \|\hat{V}^{(q_{n-1})}(t_{n-1})\|_1 \|\hat{V}^{(q_n)}(t_n)\|_2 \\
&\leqq (2\pi)^{-3n/2}\|E(\cdot)^{-2}\|_2 n^2 F(t_1)F(t_2)\cdots F(t_n)
\end{aligned} \tag{A.16}$$

where the convolutions are estimated using Young's inequality [31].

Let $\{\varphi_m\}$ be a complete orthonormal system in $\mathscr{H}$. From (A.8) it follows that

$$\begin{aligned}
\|T_n\|_{\mathrm{HS}}^2 &= \sum_m \|T_n\varphi_m\|^2 \\
&= \sum_m \left\| \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-\varphi_m \right\|^2 \\
&\leqq \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n \int_{-\infty}^{\infty} dt_1' \cdots \int_{-\infty}^{t_{n-1}'} dt_n' \\
&\cdot \sum_m \|P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-\varphi_m\| \cdot \|P_+\tilde{V}(t_1')\cdots\tilde{V}(t_n')P_-\varphi_m\|
\end{aligned} \tag{A.17}$$

and applying Schwarz's inequality to the sum

$$\begin{aligned}
\|T_n\|_{\mathrm{HS}}^2 &\leqq \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n \int_{-\infty}^{\infty} dt_1' \cdots \int_{-\infty}^{t_{n-1}'} dt_n' \\
&\quad \cdot \left(\sum_m \|P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-\varphi_m\|^2\right)^{1/2} \left(\sum_m \|P_+\tilde{V}(t_1')\cdots\tilde{V}(t_n')P_-\varphi_m\|^2\right)^{1/2} \\
&= \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n \int_{-\infty}^{\infty} dt_1' \cdots \int_{-\infty}^{t_{n-1}'} dt_n' \\
&\quad \cdot \|P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-\|_{\mathrm{HS}} \|P_+\tilde{V}(t_1')\cdots\tilde{V}(t_n')P_-\|_{\mathrm{HS}} \\
&= \left(\int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n \|P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-\|_{\mathrm{HS}}\right)^2
\end{aligned} \tag{A.18}$$

we conclude using (A.16)

$$\begin{aligned}
\|T_n\|_{\mathrm{HS}} &\leqq (2\pi)^{-3n/2}\|E(\cdot)^{-2}\|_2 n^2 \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{t_{n-1}} dt_n F(t_1)\cdots F(t_n) \\
&\leqq (2\pi)^{-3n/2}\|E(\cdot)^{-2}\|_2 n^2 \frac{b^n}{n!}
\end{aligned} \tag{A.19}$$

Calculating

$$\begin{aligned}
\|E(\cdot)^{-2}\|_2 &= \left(\int d^3k \frac{1}{(\mathbf{k}^2+m^2)^2}\right)^{1/2} \\
&= \left(4\pi \int_0^{\infty} dk k^2 \frac{1}{(k^2+m^2)^2}\right)^{1/2} = \frac{\pi}{\sqrt{m}}
\end{aligned} \tag{A.20}$$

we have finally

$$\|T_n\|_{HS} \leqq (2\pi)^{-3n/2} \frac{\pi}{\sqrt{m}} n^2 \frac{b^n}{n!} \tag{A.21}$$

Now suppose that $(d/dt)V(t)$ is discontinuous at N points. Then the integration by parts has to be carried out separately in each interval of continuity, leading to $2N$ boundary terms of the form

$$\int_{-\infty}^{0} ds_2 \cdots \int_{-\infty}^{0} ds_n \frac{e^{is_1(E(\mathbf{k})+E(\mathbf{k}'))}}{(E(\mathbf{k})+E(\mathbf{k}'))^2} \cdot e^{-i/n\sum_{j=2}^{n} s_j((n-j+1)E(\mathbf{k})-(j-1)E(\mathbf{k}'))} \frac{\partial}{\partial s_1} I_n(\mathbf{k}, \mathbf{k}'; s_1, \ldots, s_n) \tag{A.22}$$

in addition to (A.13).

By the same arguments we used to conclude (A.21), the HS-norm of the kernel (A.22) is estimated to be less than or equal to

$$(2\pi)^{-3n/2} \frac{\pi}{\sqrt{m}} n \frac{b^{n-1}}{(n-1)!} \tag{A.23}$$

$[(\partial/\partial s_1)I_n$ consists of n terms]. Consequently

$$\|T_n\|_{HS} \leqq (2\pi)^{-3n/2} \frac{\pi}{\sqrt{m}} \left(n^2 \frac{b^n}{n!} + 2Nn \frac{b^{n-1}}{(n-1)!}\right) \tag{A.24}$$

From this it follows that the series (A.8) converges in the HS-norm for every $\varkappa$, hence $S_{+-}(\varkappa)$ is Hilbert–Schmidt. The same holds for $S_{-+}(\varkappa)$ as can be seen by interchanging P_+ and P_-. $\square$

Looking back at the proof, we notice that $S_{+-}(\varkappa)$ and $S_{-+}(\varkappa)$ are even HS-entire functions (complex differentiable with respect to the HS-norm in the whole complex plane) if the external field satisfies the conditions of Theorem 7. However, to prove Theorem 6 we need only the two operators to be entire in the ordinary sense and HS for every $\varkappa$. This implies that the determinant Δ (3.37) is entire [27].

Acknowledgement

I would like to thank Professor Dr. G. Scharf for suggesting the subject of this work and for many helpful discussions.

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