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(Quasi-) momentum, (quasi) angular momentum and spin for condensed media

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Abstract. Using Noether's theorem we show on the example of the elastic medium that the quasi-momentum is a general property of condensed media. The relation of this concept to the proper momentum is discussed. Furthermore we derive balance equations for the angular momentum and quasi angular momentum and relate these concepts to the orbital (quasi) angular momentum and intrinsic angular momentum (spin).

1. Introduction

The usefulness of the knowledge of first integrals of a set of differential equations is well known. In classical mechanics the tool mostly used for searching these first integrals is Noether's theorem (e.g. for mechanical systems see Desloge and Karch [1] and for fields Kobussen [2].) This famous theorem connects the symmetry properties of a system with first integrals. It usually gives first integrals with an obvious physical meaning, such as the energy, momentum or angular momentum, corresponding to manifest symmetries of the system. However, frequently there exist also other first integrals with less obvious interpretation being related to 'hidden' symmetries.

After the discovery of a new first integral people are trying to clarify what the physical meaning of it is, since the intuition gained in this way helps to formulate or solve new problems. As this is usually done in an ad hoc way, the literature is plagued by confusive statements. An example of such a situation is the quasi- or pseudo-momentum for crystals. This first integral is born by a simple, but non-trivial, symmetry property of condensed media. Frequently the quasi momentum is interpreted in a very confusive way. Usually it is attributed to lattice structures, and believed to be directly connected with the so called Umklapp processes (U-processes) (e.g. Süssman [3]).

In this paper we will show that quasi-momentum is an inherent property of condensed media, and in a companion paper [4] that only for crystals it really is

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connected with U-processes. We shall study the set of variations generating the densities of energy, momentum and of quasi-momentum. Our formulation makes it possible to study other continuous symmetries, for example rotations, which yield the density of orbital angular momentum and of phonon spin.

For the physical identification of the introduced quantities we shall consider the scattering of a particle on a medium.

2. The model of the elastic continuum

We shall show that the quasi-momentum is a property of solids and of fluids. For this reason we shall study a model having the properties of both kinds of media. We choose the model of the elastic continuum which is a very old model for the elastic properties of solids. The Lagrangian density of an anisotropic elastic continuum consists of kinetic energy and elastic energy

$$\mathscr{L}_e(y) = \tfrac{1}{2}\rho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} - \tfrac{1}{2} C^{\alpha\beta}_{\mu\nu} \nabla_\alpha u_\mu \nabla_\beta u_\nu, \tag{2.1}$$

where ρ_0 is the mass density of the medium and a summation over Greek indices from 1 through 3 is implied. The real vector field $\mathbf{u}(\mathbf{y}, t)$ defines the actual position of the mass element with coordinate $\mathbf{y}$, at the time t. The real coefficients $C^{\alpha\beta}_{\mu\nu}$ are related to the elastic constants and have some symmetry properties

$$C^{\alpha\beta}_{\mu\nu} = C^{\beta\alpha}_{\nu\mu} = C^{\mu\beta}_{\alpha\nu}. \tag{2.2}$$

Other symmetry properties follow from symmetries of the medium. For example, from the invariance of the elastic energy with respect to all rotations one can show that the tensor $C^{\alpha\beta}_{\mu\nu}$ is built as [5]

$$C^{\alpha\beta}_{\mu\nu} = c\delta_{\alpha\mu}\delta_{\beta\nu} + a(\delta_{\alpha\beta}\delta_{\mu\nu} + \delta_{\alpha\nu}\delta_{\beta\mu}). \tag{2.3}$$

For media of lower symmetry the tensor $C^{\alpha\beta}_{\mu\nu}$ can be constructed from basic tensor-dyads and only rotations around definite axes and by definite discrete angles leave the Lagrangian unchanged [5].

In contrast with the main part of the literature, we shall interprete the independent variable $\mathbf{y}$ as a substantial (material) coordinate, i.e. $\mathbf{y}$ represents the original or equilibrium position of a material point. The dependent variable $\mathbf{u}$ is then the deviation of a material point originally at $\mathbf{y}$. The actual position is then

$$\mathbf{R}(\mathbf{y}, t) = \mathbf{y} + \mathbf{u}(\mathbf{y}, t). \tag{2.4}$$

This interpretation is analogous to the use of Lagrangian coordinates in fluid dynamics. Our formulation has the advantage that quantities as $\dot{\mathbf{u}}$ and $\frac{1}{2}\rho_0\dot{\mathbf{u}}\cdot\dot{\mathbf{u}}$ have a direct physical meaning as concepts familiar in particle dynamics such as particle velocity and kinetic energy. Only with the interpretation of $\mathbf{y}$ as a substantial coordinate, the Lagrangian density (2.1) is the difference of kinetic and potential energy densities. Using $\mathbf{y}$ as a local (Eulerian) coordinate, the same interpretation can only approximately be true. Additionally we want to stress that less restrictive approximations are needed to obtain (2.1) in terms of substantial coordinates from more realistic models, than to obtain (2.1) in terms of local (Eulerian) coordinates. One such more realistic model (cf. Ziman [6]) is a crystal built up from N atoms, each of them having a mass m. The Lagrangian of such a crystal reads

$$L = \tfrac{1}{2}m \sum_{\mathbf{l}} \dot{\mathbf{u}}_{\mathbf{l}} \cdot \dot{\mathbf{u}}_{\mathbf{l}} - \sum_{\mathbf{m} \neq \mathbf{n}} V(\mathbf{R}_{\mathbf{m}} - \mathbf{R}_{\mathbf{n}})$$

$$= \tfrac{1}{2}m \sum_{\mathbf{l}} \dot{\mathbf{u}}_{\mathbf{l}} \cdot \dot{\mathbf{u}}_{\mathbf{l}} - \tfrac{1}{2} \sum_{\mathbf{m} \neq \mathbf{n}} \nabla_\alpha \nabla_\beta V(\langle \mathbf{R}_{\mathbf{m}} \rangle - \langle \mathbf{R}_{\mathbf{n}} \rangle) \times (u^\alpha_{\mathbf{m}} - u^\alpha_{\mathbf{n}})(u^\beta_{\mathbf{m}} - u^\beta_{\mathbf{n}}) + \cdots, \tag{2.5}$$

where for simplicity we have assumed a two-body interaction. The vector $\mathbf{R}_{\mathbf{m}}$ is the actual position of the **m**th particle (atom)

$$\mathbf{R}_{\mathbf{m}} = \langle \mathbf{R}_{\mathbf{m}} \rangle + \mathbf{u}_{\mathbf{m}} \tag{2.6}$$

and $\langle \mathbf{R}_{\mathbf{m}} \rangle$ is its equilibrium position.

From (2.4) one obtains the elastic continuum model by introducing the mass density $\rho_0 = m/v$ as the mass m of an elementary cell divided by its volume v. Furthermore one introduces the fields $\mathbf{u}(\mathbf{y})$ by

$$\mathbf{u}_{\mathbf{n}} = \mathbf{u}(\langle \mathbf{R}_{\mathbf{n}} \rangle); \qquad \mathbf{u}_{\mathbf{n}} - \mathbf{u}_{\mathbf{m}} = (\langle \mathbf{R}_{\mathbf{n}} \rangle - \langle \mathbf{R}_{\mathbf{m}} \rangle)_\alpha \nabla_\alpha \mathbf{u} + \cdots. \tag{2.7}$$

In such an approximation the summation over lattice sites is replaced by an integration over y:

$$\sum_{\mathbf{n}} \to \frac{1}{v} \int d^3 y. \tag{2.8}$$

Now assuming

$$|\nabla_\alpha \mathbf{u}| \ll 1, \quad \text{and} \quad |\nabla_\alpha \nabla_\beta \mathbf{u}| \ll \frac{1}{a},$$

a being a characteristic interatomic distance, equation (2.5) may be approximated by retaining only the two first terms and equation (2.7) by retaining the first term only. Then the Lagrangian (2.5) takes the form (2.1) and we have the linear elastic continuum limit of the crystal. This continuous harmonic model of the crystal can be improved by adding non-harmonic terms H_{anh} to the Lagrangian, or one can consider quasi-continuous models with non-local interaction [7]. But let us remind that the elastic continuum model describes many experiments quite adequately, for example the light scattering in the hydrodynamical regime in crystals.

Frequently one should allow the coupling of the displacement field $\mathbf{u}(\mathbf{y})$ with other fields, such as the electric or magnetic field [8]. Our approach can also be applied to such generalized models.

Let us return to the Lagrangian density (2.1). Since it depends on time derivatives and gradients and does not depend explicitly on time and coordinates it does not change under several transformations. For our purposes it is enough to study the infinitesimal transmations. These symmetry transformations are the time shift

$$t \to t + \varepsilon, \tag{2.9}$$

space shift

$$\mathbf{y} \to \mathbf{y} + \boldsymbol{\varepsilon}, \tag{2.10}$$

and the shift of the field

$$\mathbf{u} \to \mathbf{u} + \boldsymbol{\varepsilon}. \tag{2.11}$$

Besides, for an isotropic medium, also the rotation around an arbitrary axis $\mathbf{n}$ is a symmetry transformation. Since we consider a vectorial field $\mathbf{u}$, it transforms under rotations as follows

$$u_\mu(\mathbf{y}) \to R_{\mu\nu}(\varepsilon, \mathbf{n}) u_\nu(\tilde{R}(-\varepsilon, \mathbf{n})\mathbf{y}), \tag{2.12}$$

where $\tilde{R}(\varepsilon, \mathbf{n})$ is the matrix of an infinitesimal rotation around an axis $\mathbf{n}$. Hence for such rotations we have

$$\delta u_\mu(\mathbf{y}) = \varepsilon \mathscr{E}_{\mu\rho\sigma} n_\rho u_\lambda - \varepsilon \mathscr{E}_{\gamma\rho\lambda} n_\rho y_\lambda \nabla_\gamma u_\mu, \tag{2.13}$$

or

$$\delta \mathbf{u}(\mathbf{y}) = \varepsilon(\mathbf{n} \wedge \mathbf{u} - ((\mathbf{n} \wedge \mathbf{y}) \cdot \nabla)\mathbf{u}),$$

where $\mathscr{E}_{\mu\rho\nu}$ is the totally anti-symmetric Levi-Civita symbol, and

$$R_{\mu\nu}(\varepsilon, \mathbf{n}) = \varepsilon \mathscr{E}_{\mu\rho\nu} n_\rho; \qquad n_\rho = (\mathbf{n})_\rho. \tag{2.14}$$

In the next section we will investigate the consequences of these infinitesimal symmetry transformations.

3. The conservation laws

By straightforward calculations we shall obtain two equivalent forms for the first order variation of the Lagrangian density:

$$\delta \mathscr{L}_e = \rho_0 \dot{u}_\mu \frac{d}{dt} \delta u_\mu - C^{\alpha\beta}_{\mu\nu} \nabla_\alpha u_\mu \nabla_\beta \delta u_\nu, \tag{3.1}$$

and

$$\delta \mathscr{L}_e = d_t(\rho_0 \dot{u}_\mu \delta u_\mu) - d_\alpha(C^{\alpha\beta}_{\mu\nu} \nabla_\beta u_\nu \delta u_\mu) - \delta u_\mu(\rho_0 \ddot{u}_\mu - C^{\alpha\beta}_{\mu\nu} \nabla_\alpha \nabla_\beta u_\nu), \tag{3.2}$$

where $d_t F$, $d_\alpha F$ are the total time and space derivatives. For example

$$d_t F(\mathbf{u}(\mathbf{y}, t), \mathbf{y}, t) = \left(\frac{\partial F}{\partial u_\mu}\right)_{\mathbf{y},t} \left(\frac{\partial u_\mu}{\partial t}\right)_{\mathbf{y}} + \left(\frac{\partial F}{\partial t}\right)_{\mathbf{u},\mathbf{y}} \tag{3.3}$$

Both equations (3.1) and (3.2) are identities valid for arbitrary variations δu_α. The third term of $\delta \mathscr{L}_e$ in equation (3.2) vanishes for $\mathbf{u}(\mathbf{y}, t)$ being a solution of the equation of motion

$$\rho_0 \ddot{u}_\mu - C^{\alpha\beta}_{\mu\nu} \nabla_\alpha \nabla_\beta u_\nu = 0.$$

We shall consider now the infinitesimal time shift $\delta t = \varepsilon$, or equivalently, in terms of the fields, the variations

$$\delta \mathbf{u} = \varepsilon \dot{\mathbf{u}}. \tag{3.4}$$

Then from equation (3.1) it is seen that the variation $\delta \mathscr{L}_e$ is a total derivative

$$\delta \mathscr{L}_e = \varepsilon d_t \mathscr{L}_e. \tag{3.5}$$

Therefore, the variation (3.4) is termed a Noetherian variation. In general, Noetherian variations yield equations in the form of a local conservation law (lcl.). For the present example we shall show that the variation (3.4) yields the lcl. for the energy.

Substituting (3.4) into equations (3.1) and (3.2), and afterwards subtracting them one obtains

$$d_t e + d_\beta j^E_\beta = \dot{u}_\mu(\rho_0 \ddot{u}_\mu - C^{\alpha\beta}_{\mu\nu}\nabla_\alpha\nabla_\beta u_\nu). \tag{3.6}$$

For $u_\mu(\mathbf{y}, t)$ being a solution of the equation of motion (i.e. for $\mathbf{u}$ on a phase trajectory) we obtain the differential conservation law

$$d_t e + d_\beta j^E_\beta = d_t e + \nabla \cdot \mathbf{j}^E \doteq 0. \tag{3.7}$$

Since e is clearly the density of energy

$$e(\mathbf{y}, t) = \tfrac{1}{2}\rho_0 \dot{u}_\mu \dot{u}_\mu + \tfrac{1}{2} C^{\alpha\beta}_{\mu\nu}\nabla_\alpha u_\mu \nabla_\beta u_\nu, \tag{3.8}$$

the vector $\mathbf{j}^E$ with components

$$j^E_\beta = -C^{\alpha\beta}_{\mu\nu}(\nabla_\alpha u_\mu)\dot{u}_\nu \tag{3.9}$$

is the density of the energy current.

Here we use the sign $\doteq$ for equalities which only hold for solutions of the equations of motion. Next, let us consider an infinitesimal shift of the field. From equation (3.1) it follows that $\delta\mathscr{L}_e$ vanishes for $\delta\mathbf{u} = \boldsymbol{\varepsilon}$, and we obtain the lcl.

$$d_t p_\mu + d_\alpha \pi_{\alpha\mu} \doteq 0, \tag{3.10}$$

where

$$p_\mu(\mathbf{y}) = \rho_0 \dot{u}_\mu(\mathbf{y}) \tag{3.11}$$

is the density of (proper) momentum, and

$$\pi_{\alpha\mu} = \pi_{\mu\alpha} = \frac{\partial \mathscr{L}_e}{\partial(\nabla_\alpha u_\mu)} = -C^{\alpha\beta}_{\mu\nu}\nabla_\beta u_\nu \tag{3.12}$$

is the density of momentum current.

Integrating lcl.'s with suitable boundary conditions, one obtains corresponding global conservation laws. For example, with cyclic boundary conditions [6], one obtains from (3.7) and (3.10)

$$\frac{dE}{dt} \doteq 0, \qquad \frac{d}{dt}\mathbf{P} \doteq 0. \tag{3.13}$$

As global conservation laws are less fundamental than local ones, we shall concentrate our attention mainly on lcl.'s.

For an infinitesimal space shift (2.10), the field changes as [2]

$$\delta u = -(\boldsymbol{\varepsilon}\cdot\nabla)\mathbf{u}; \qquad \delta u_\mu = -\varepsilon_\alpha \nabla_\alpha u_\mu. \tag{3.14}$$

This change, which we shall call the shift of the displacement pattern is also a Noetherian variation. Indeed, for the variation (3.14) we obtain

$$\delta\mathscr{L}_e = -\varepsilon_\gamma d_\gamma \mathscr{L}_e. \tag{3.15}$$

Thus for this variation we get the differential conservation law

$$d_t\hat{p}_\gamma + d_\alpha \hat{\pi}_{\alpha\gamma} \doteq 0, \tag{3.16}$$

where the conserved density is

$$\hat{p}_\gamma(\mathbf{y}) = -\rho_0 \dot{u}_\mu \nabla_\gamma u_\mu, \tag{3.17}$$

and

$$\hat{\pi}_{\alpha\gamma}(\mathbf{y}) = \hat{\pi}_{\gamma\alpha}(\mathbf{y}) = C^{\alpha\beta}_{\mu\nu} \nabla_\beta u_\nu \nabla_\gamma u_\mu + \delta_{\alpha\gamma}\mathscr{L}_e(y) \tag{3.18}$$

is the corresponding current density tensor.

Although the densities $\mathbf{p}(\mathbf{y})$ and $\hat{\mathbf{p}}(\mathbf{y})$ are not identical, they have the same dimension. Therefore, and for reasons which become clear later, we shall call $\hat{\mathbf{p}}(\mathbf{y})$ the density of quasi-momentum.

Comparing the derived formulae for the densities (3.11) and (3.17) we observe that the distinct feature of momentum is the linearity of the expressions for its density in the field variables. As we shall see in the companion paper [4] this is a formal reason why only quasi-momentum, and not the proper momentum, plays an outstanding role in heat transport in crystals.

Gilbert and Mollow have derived the conservation laws for momentum and quasi-momentum in a similar way [9]. They called the quantity $\hat{\mathbf{p}}(\mathbf{y})$ the tensor momentum. However as we shall show in the companion paper [4], $\hat{\mathbf{p}}$ is the well known quantity in the theory of solids. It is nothing but the quasi- or pseudo momentum.

Now consider the infinitesimal rotation (2.13). It essentially consists of two terms

$$\delta\mathbf{u} = \delta_1\mathbf{u} + \delta_2\mathbf{u} \tag{3.19}$$

$$\delta_1\mathbf{u} = \varepsilon\mathbf{n}\wedge\mathbf{u}; \qquad \delta_1 u_\mu = \varepsilon\mathscr{E}_{\mu\rho\lambda} n_\rho u_\lambda \tag{3.20}$$

$$\delta_2\mathbf{u} = -\varepsilon((\mathbf{n}\wedge\mathbf{y})\cdot\boldsymbol{\nabla})\mathbf{u}; \qquad \delta_2 u_\mu = -\varepsilon\mathscr{E}_{\gamma\rho\lambda} n_\rho y_\lambda \nabla_\gamma u_\mu. \tag{3.21}$$

Let us discuss these terms separately. For $\delta\mathbf{u} = \delta_1\mathbf{u}$, the variation (3.2) directly yields

$$\delta\mathscr{L}_e = \delta_1\mathscr{L}_e = \varepsilon n_\rho (d_t S_\rho(\mathbf{y}) + d_\beta \pi^S_{\beta\rho}(\mathbf{y})) \tag{3.22}$$

where

$$\mathbf{S} = \mathbf{u}(\mathbf{y})\wedge\rho_0\dot{\mathbf{u}}(\mathbf{y}); \qquad S_\rho(\mathbf{y}) = \mathscr{E}_{\rho\lambda\mu}\rho_0 u_\lambda \dot{u}_\mu \tag{3.23}$$

and

$$\pi^S_{\beta\rho}(\mathbf{y}) = -C^{\beta\alpha}_{\nu\mu}\mathscr{E}_{\nu\rho\lambda} u_\lambda \nabla_\alpha u_\mu = \mathscr{E}_{\nu\rho\lambda} u_\lambda \pi_{\beta\nu}. \tag{3.24}$$

On the other hand, with (3.20) and (3.1) one obtains

$$\begin{aligned}\delta_1\mathscr{L}_e &= -\varepsilon n_\rho \mathscr{E}_{\nu\rho\lambda} C^{\alpha\beta}_{\mu\nu} \nabla_\alpha u_\mu \nabla_\beta u_\lambda \\ &= \varepsilon n_\rho \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} \nabla_\beta u_\lambda\end{aligned} \tag{3.25}$$

and equations (3.22) and (3.25) together yield

$$d_t S_\rho(\mathbf{y}) + d_\beta \pi^S_{\beta\rho}(\mathbf{y}) \doteq \varepsilon_{\nu\rho\lambda}\pi_{\beta\nu}\nabla_\beta u_\lambda. \tag{3.26}$$

For $\delta\mathbf{u} = \delta_2\mathbf{u}$, the variation (3.2) yields

$$\delta\mathscr{L}_e = \delta_2\mathscr{L}_e = \varepsilon n_\rho (d_t\hat{L}_\rho(\mathbf{y}) + d_\beta(\pi^{\hat{L}}_{\beta\rho}(\mathbf{y}) - \mathscr{E}_{\beta\rho\lambda} y_\lambda \mathscr{L}_e(\mathbf{y}))), \tag{3.27}$$

where

$$\hat{L}_\rho(\mathbf{y}) = -\rho_0 \mathscr{E}_{\gamma\rho\lambda} y_\lambda \dot{u}_\mu \nabla_\gamma u_\mu; \qquad \hat{\mathbf{L}}(\mathbf{y}) = \mathbf{y} \wedge \hat{\mathbf{p}}(\mathbf{y}) \tag{3.28}$$

$$\pi^{\hat{L}}_{\beta\rho}(\mathbf{y}) = C^{\alpha\beta}_{\mu\nu} \mathscr{E}_{\gamma\rho\lambda} y_\lambda (\nabla_\alpha u_\mu) \nabla_\gamma u_\nu + \mathscr{E}_{\beta\rho\lambda} y_\lambda \mathscr{L}_e(\mathbf{y}) = \mathscr{E}_{\rho\lambda\nu} y_\lambda \hat{\pi}_{\beta\nu}, \tag{3.29}$$

On the other hand, with (3.21) and (3.1) one obtains

$$\delta_2\mathscr{L}_e = -\varepsilon n_\rho d_\beta(\mathscr{E}_{\beta\rho\lambda} y_\lambda \mathscr{L}_e(\mathbf{y})) + \varepsilon n_\rho \mathscr{E}_{\gamma\rho\beta} C^{\alpha\beta}_{\mu\nu} \nabla_\alpha u_\mu \nabla_\gamma u_\nu. \tag{3.30}$$

Together with (3.22) we now obtain

$$\begin{aligned} d_t\hat{L}_\rho(\mathbf{y}) + d_\beta \pi^{\hat{L}}_{\beta\rho}(\mathbf{y}) &\doteq \mathscr{E}_{\gamma\rho\beta} C^{\alpha\beta}_{\mu\nu} \nabla_\alpha u_\mu \nabla_\gamma u_\nu \\ &= -\mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} \nabla_\lambda u_\beta. \end{aligned} \tag{3.31}$$

Both (3.26) and (3.32) are local balance equations. Adding both together one obtains the balance equation

$$d_t(S_\rho(\mathbf{y}) + \hat{L}_\rho(\mathbf{y})) + d_\beta(\pi^S_{\beta\rho}(\mathbf{y}) + \pi^{\hat{L}}_{\beta\rho}(\mathbf{y})) \doteq \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} (\nabla_\lambda u_\beta + \nabla_\beta u_\lambda). \tag{3.32}$$

In Section 2 we have observed that the Lagrangian (2.1) can represent a continuous approximation of a discrete particle system only if the gradients $\nabla_\alpha u_\mu$ of the components of the elongation fields u are small. The densities and currents found above are all of the order $\nabla_\alpha u_\mu$. Therefore in the source term one should also keep terms of the order $\nabla_\alpha u_\mu$ and drop higher order terms. Consequently, the source term in (3.32) vanishes and we have

$$d_t(S_\rho + \hat{L}_\rho) + d_\beta(\pi^S_{\beta\rho} + \pi^{\hat{L}}_{\beta\rho}) \doteq 0. \tag{3.33}$$

For a more thorough derivation of this lcl., one has to enlarge the model Lagrangian (2.1) with higher order terms, or to discuss the full non-linear theory. In this paper we do not intend to do so.

The lcl. (3.33) is in agreement with results of Vonsovskii and Svirskii [10]. These authors interprete this equation as the lcl. for the angular momentum $\hat{\mathbf{L}}(\mathbf{y}) + \mathbf{S}(\mathbf{y})$, $\hat{\mathbf{L}}(\mathbf{y})$ being the orbital angular momentum density and $\mathbf{S}(\mathbf{y})$ the intrinsic angular momentum or spin density. They choose these names because $\hat{\mathbf{L}}(\mathbf{y})$ does depend and $\mathbf{S}(\mathbf{y})$ does not depend on the choice of the origin of the coordinate system. Nevertheless, within the interpretation of $\mathbf{y}$ as a substantial coordinate, the vector $\mathbf{S}(\mathbf{y}) + \hat{\mathbf{L}}(\mathbf{y})$ does not represent the proper angular momentum as it is defined in particle dynamics. Therefore we will call $\hat{\mathbf{L}}(\mathbf{y}) + \mathbf{S}(\mathbf{y})$ the quasi angular momentum density, $\hat{\mathbf{L}}(\mathbf{y})$ being the orbital part and $\mathbf{S}(\mathbf{y})$ the intrinsic or spin part.

In particle dynamics the true angular momentum is defined as

$$\mathbf{L}_{\text{tot}} = \sum_{\mathbf{n}} \mathbf{R}_{\mathbf{n}} \wedge m\dot{\mathbf{R}}_{\mathbf{n}},$$

$\mathbf{R}_{\mathbf{n}}$ being the position vector of the particle with number $\mathbf{n}$. With equation (2.4),

$\mathbf{R_n} = \mathbf{y_n} + \mathbf{u}(\mathbf{y_n})$, one finds

$$\mathbf{L}_{\text{tot}} = \sum_{\mathbf{n}} m(\mathbf{y_n} + \mathbf{u}(\mathbf{y_n})) \wedge \dot{\mathbf{u}}(\mathbf{y_n}).$$

Thus a reasonable expression for the true angular momentum density of the elastic medium therefore is

$$\rho_0(\mathbf{y} + \mathbf{u}) \wedge \dot{\mathbf{u}} = \mathbf{L}(\mathbf{y}) + \mathbf{S}(\mathbf{y}), \tag{3.34}$$

where $\mathbf{S}(\mathbf{y})$ is given by (3.23) and

$$\mathbf{L}(\mathbf{y}) = \mathbf{y} \wedge \mathbf{p}(\mathbf{y}) = \mathbf{y} \wedge \rho_0 \dot{\mathbf{u}}(\mathbf{y}). \tag{3.35}$$

Analogously to the derivation of equation (3.31), with

$$\delta \mathbf{u} = \varepsilon \mathbf{n} \wedge \mathbf{y}, \qquad \delta u_\mu = \varepsilon \mathscr{E}_{\mu\rho\lambda} n_\rho y_\lambda \tag{3.36}$$

one can derive

$$d_t L_\rho + d_\beta \pi^L_{\beta\rho} \doteq 0, \tag{3.37}$$

where

$$\pi^L_{\beta\rho} = -C^{\beta\alpha}_{\nu\mu} \mathscr{E}_{\nu\rho\lambda} y_\lambda \nabla_\alpha u_\mu = \mathscr{E}_{\rho\lambda\nu} y_\lambda \pi_{\beta\nu}. \tag{3.38}$$

Equations (3.37) and (3.26) together yield the local balance equation of the elastic medium for the true angular momentum

$$d_t(S_\rho(\mathbf{y}) + L_\rho(\mathbf{y})) + d_\beta(\pi^S_{\beta\rho} + \pi^L_{\beta\rho}) \doteq \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} \nabla_\beta u_\lambda. \tag{3.39}$$

This local balance equation can also be derived directly with the variation

$$\delta \mathbf{u} = \varepsilon \mathbf{n} \wedge (\mathbf{y} + \mathbf{u}); \qquad \delta u_\mu = \varepsilon \mathscr{E}_{\mu\rho\lambda} n_\rho (y_\lambda + u_\lambda). \tag{3.40}$$

Again with our approximation one should drop the source term in (3.29). Then with (3.26) and (3.31) one obtains

$$d_t S_\rho(\mathbf{y}) + d_\beta \pi^S_{\beta\rho}(\mathbf{y}) \doteq \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} \nabla_\beta u_\lambda \approx 0, \tag{3.41}$$

$$d_t \hat{L}_\rho(\mathbf{y}) + d_\beta \pi^{\hat{L}}_{\beta\rho}(\mathbf{y}) \doteq \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} \nabla_\lambda u_\beta \approx 0, \tag{3.42}$$

The local balance equation for the total quasi angular momentum (3.32) and total angular momentum then is

$$d_t(S_\rho(\mathbf{y}) + \hat{L}_\rho(\mathbf{y})) + d_\beta(\pi^S_{\beta\rho}(\mathbf{y}) + \pi^{\hat{L}}_{\beta\rho}(\mathbf{y})) \doteq \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} (\nabla_\lambda u_\beta + \nabla_\beta u_\lambda) \approx 0 \tag{3.43}$$

and

$$d_t(S_\rho(\mathbf{y}) + L_\rho(\mathbf{y})) + d_\beta(\pi^S_{\beta\rho}(\mathbf{y}) + \pi^L_{\beta\rho}(\mathbf{y})) \doteq \mathscr{E}_{\nu\rho\lambda} \pi_{\beta\nu} \nabla_\beta u_\lambda \approx 0, \tag{3.44}$$

which are valid for all systems (2.1)–(2.2) with $|\nabla \mathbf{u}| \ll 1$.

4. Identification of quasi-momentum and quasi angular-momentum; Scattering of a particle on the medium

In this section we shall study the scattering of a particle on the elastic medium. The Lagrangian of the elastic medium without a scattered particle is

$$L_e = \int d^3y \mathscr{L}_e, \tag{4.1}$$

with $\mathscr{L}_e$ given by (2.1). The Lagrangian of the composite system, elastic medium plus particle, contains L_e, the Lagrangian of a free particle L_p, i.e. its kinetic energy, and the energy of mutual interaction L_{int}.

$$L = L_e + L_p + L_{\text{int}}. \tag{4.2}$$

Assuming the mass of the particle is m_p, and its position is $\mathbf{x}$, one has

$$L_p = \tfrac{1}{2} m_p \dot{\mathbf{x}} \cdot \dot{\mathbf{x}}. \tag{4.3}$$

Let us assume that the interaction energy only depends on the mutual distance $\mathbf{R} - \mathbf{x} = \mathbf{y} + \mathbf{u}(\mathbf{y}) - \mathbf{x}$ of the particle and a material point of the medium. Then

$$L_{\text{int}} = -\int d^3y f(\mathbf{y} + \mathbf{u}(\mathbf{y}) - \mathbf{x}). \tag{4.4}$$

The Lagrangian (4.2) of the composite system then is written as

$$L = \int d^3y \mathscr{L}(\mathbf{y}), \tag{4.5}$$

$\mathscr{L}(\mathbf{y})$ being the total Lagrangian density

$$\begin{aligned}\mathscr{L}(\mathbf{y}) = {} & \tfrac{1}{2}\rho_0 \dot{\mathbf{u}}(\mathbf{y}) \cdot \dot{\mathbf{u}}(\mathbf{y}) - \tfrac{1}{2} C^{\alpha\beta}_{\mu\nu} \nabla_\alpha u_\mu \nabla_\beta u_\nu \\ & + \tfrac{1}{2} m_p \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \delta(\mathbf{y}) - f(\mathbf{y} + \mathbf{u}(\mathbf{y}) - \mathbf{x}).\end{aligned} \tag{4.6}$$

Note that $\mathscr{L}(\mathbf{y})$ depends on $\mathbf{y}$ explicitly now. Proceeding in the same way as previously, we obtain from (4.6) the equations of motion

$$\rho_0 \ddot{u}_\mu(\mathbf{y}, t) - C^{\alpha\beta}_{\mu\nu} \nabla_\alpha \nabla_\beta u_\nu + \nabla_\mu f(\mathbf{y} + \mathbf{u} - \mathbf{x}) = 0, \tag{4.7}$$

$$m_p \ddot{\mathbf{x}} - \int d^3y \boldsymbol{\nabla} f(\mathbf{y} + \mathbf{u}(\mathbf{y}) - \mathbf{x}) = 0 \tag{4.8}$$

and again two different expressions for $\delta\mathscr{L}$.

It is often said that (linear) momentum conservation is due to invariance of space shift. So, let us consider such an infinitesimal space shift. In terms of actual positions then one would write

$$\delta \mathbf{x} = \boldsymbol{\varepsilon}; \qquad \delta \mathbf{R} = \delta(\mathbf{y} + \mathbf{u}) = \boldsymbol{\varepsilon} \tag{4.9}$$

e.g.

$$\delta \mathbf{x} = \boldsymbol{\varepsilon}, \qquad \delta \mathbf{y} = 0, \qquad \delta \mathbf{u} = \boldsymbol{\varepsilon}. \tag{4.10}$$

One easily sees that this variation leaves L invariant and yields the following global conservation law

$$d/dt(\mathbf{P}_e + \mathbf{P}_p) \doteq 0, \tag{4.11}$$

where

$$\mathbf{P}_e = \int d^3y \rho_0 \dot{\mathbf{u}}(\mathbf{y}) \tag{4.12}$$

and

$$\mathbf{P}_p = m_p \dot{\mathbf{x}}. \tag{4.13}$$

Instead of (4.10) one also could study

$$\delta \mathbf{x} = \boldsymbol{\varepsilon}, \qquad \delta \mathbf{u} = 0, \qquad \delta \mathbf{y} = \boldsymbol{\varepsilon}. \tag{4.14}$$

Such a variation of the independent variable $\mathbf{y} \to \mathbf{y}' = \mathbf{y} + \boldsymbol{\varepsilon}$, is a special example of the more general variation of both field and independent variables

$$\mathbf{u}(\mathbf{y}) \to \mathbf{u}'(\mathbf{y}') = \mathbf{u}(\mathbf{y}) + \boldsymbol{\eta}(\mathbf{y}); \qquad \mathbf{y} \to \mathbf{y}' = \mathbf{y} + \boldsymbol{\varepsilon}. \tag{4.15}$$

Such variations always can be related to a variation of the field variable alone [11] (cf. also [2]).

$$\mathbf{u}(\mathbf{y}) \to \mathbf{u}'(\mathbf{y}) = \mathbf{u}(\mathbf{y}) + \delta \mathbf{u}(\mathbf{y}). \tag{4.16}$$

Then with (4.15) and (4.16)

$$\begin{aligned} \delta \mathbf{u}(\mathbf{y}) &= \mathbf{u}'(\mathbf{y}) - \mathbf{u}(\mathbf{y}) = \mathbf{u}'(\mathbf{y}' - \boldsymbol{\varepsilon}) - \mathbf{u}(\mathbf{y}) \\ &= \mathbf{u}'(\mathbf{y}') - \mathbf{u}(\mathbf{y}) - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla}')\mathbf{u}'(\mathbf{y}') = \boldsymbol{\eta}(\mathbf{y}) - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla}')\mathbf{u}'(\mathbf{y}') \\ &= \boldsymbol{\eta}(\mathbf{y}) - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla})\mathbf{u}(\mathbf{y}) + \varepsilon^2 \cdots. \end{aligned}$$

So, neglecting higher order terms in ε, we obtain

$$\delta \mathbf{u}(\mathbf{y}) = \boldsymbol{\eta}(\mathbf{y}) - (\boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla})\mathbf{u}(\mathbf{y}). \tag{4.17}$$

Thus equivalent to (4.14) is the variation

$$\delta \mathbf{x} = \boldsymbol{\varepsilon}, \qquad \delta \mathbf{u} = -(\boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla})\mathbf{u}. \tag{4.18}$$

One easily verifies that this variation is also Noetherian and yields

$$d/dt(\hat{\mathbf{P}}_e + \mathbf{P}_p) \doteq 0, \tag{4.19}$$

where

$$\hat{\mathbf{P}}_e = \int d^3y \hat{\mathbf{p}}(\mathbf{y}), \tag{4.20}$$

and $\hat{\mathbf{p}}(\mathbf{y})$ is given by (3.11).

Returning now to equation (4.11) one sees that the proper momentum of both the elastic medium and the particle together form a constant of the motion of the composite system. That means that if the momentum of the particle changes because of interaction, the momentum of the elastic medium changes with the same amount but with opposite sign. We say the momentum can be transferred additively from one part of the system into the other part. This is exactly the behaviour one expects for any constant of the motion.

Now looking at equation (4.19) we see that the quasi momentum $\hat{\mathbf{P}}_e$ of the elastic medium has exactly the same property with respect to the proper momentum $\mathbf{P}_p$ of the particle. One can express this property by saying that for a particle quasi momentum and proper momentum are identical. In order to get a more precise statement, let us substract equations (4.11) and (4.19). Then we obtain

$$d/dt\, \mathbf{D} \doteq 0; \qquad \mathbf{D} = \mathbf{P}_e - \hat{\mathbf{P}}_e. \tag{4.21}$$

Thus, also in the case of interaction with a scattering particle, the proper- and the quasi momentum of the elastic medium differ only by a constant. Consequently, by scattering experiments one cannot distinguish between proper and quasi momentum.

Let us consider infinitesimal rotations in $\mathbf{R}, \mathbf{x}$ space:

$$\mathbf{x} \to \mathbf{x}' = \mathbf{x} + \varepsilon \mathbf{n} \wedge \mathbf{x}, \tag{4.22}$$

$$\mathbf{R} = \mathbf{y} + \mathbf{u} \to \mathbf{R}' = \mathbf{y} + \mathbf{u} + \varepsilon \mathbf{n} \wedge (\mathbf{y} + \mathbf{u}). \tag{4.23}$$

Equation (4.23) can be written as either

$$\mathbf{y} \to \mathbf{y}' = \mathbf{y}; \qquad \mathbf{u} \to \mathbf{u}' = \mathbf{u} + \delta \mathbf{u}; \tag{4.24}$$

$$\delta \mathbf{u} = \varepsilon \mathbf{n} \wedge (\mathbf{y} + \mathbf{u}); \qquad \delta u_\mu = \varepsilon \mathscr{E}_{\mu\rho\nu} n_\rho (y_\nu + u_\nu), \tag{4.25}$$

or as

$$\begin{aligned} &\mathbf{y} \to \mathbf{y}' = \mathbf{y} + \varepsilon \mathbf{n} \wedge \mathbf{y}; \\ &\mathbf{u}(\mathbf{y}) \to \mathbf{u}'(\mathbf{y}') = \mathbf{u}(\mathbf{y}) + \varepsilon \mathbf{n} \wedge \mathbf{u}. \end{aligned} \tag{4.26}$$

Analogously to (4.17) and equivalently to (4.25) we may also write

$$\mathbf{y} \to \mathbf{y}' = \mathbf{y}; \qquad \mathbf{u} \to \mathbf{u}' + \delta \mathbf{u}, \tag{4.27}$$

where

$$\begin{aligned} &\delta \mathbf{u} = \varepsilon (\mathbf{n} \wedge \mathbf{u} - ((\mathbf{n} \wedge \mathbf{y}) \cdot \boldsymbol{\nabla}) \mathbf{u}); \\ &\delta u_\mu = \varepsilon n_\rho (\mathscr{E}_{\mu\rho\nu} u_\nu - \mathscr{E}_{\gamma\rho\nu} y_\nu \nabla_\gamma u_\mu). \end{aligned} \tag{4.28}$$

Analogously to the procedure of Section 3, for the system (4.6), the variation (4.22)–(4.25) yields the global balance equation or approximative global conservation laws

$$d/dt\left(\mathbf{x} \wedge m_p \dot{\mathbf{x}} + \int (\mathbf{S}(\mathbf{y}) + \mathbf{L}(\mathbf{y}))\, d^3 y\right)_\rho \doteq \int \mathscr{E}_{\nu\rho\sigma} \pi_{\beta\nu} \nabla_\beta u_\sigma \, d^3 y \approx 0 \tag{4.29}$$

and (4.22)–(4.28) yields

$$d/dt\left(\mathbf{x} \wedge m_p \dot{\mathbf{x}} + \int (\mathbf{S}(\mathbf{y}) + \hat{\mathbf{L}}(\mathbf{y}))\, d^3 y\right)_\rho \doteq \int \mathscr{E}_{\nu\rho\sigma} \pi_{\beta\nu} (\nabla_\beta u_\sigma + \nabla_\sigma u_\beta)\, d^3 y \approx 0 \tag{4.30}$$

In general the expressions (4.29) and (4.30) are valid only under the assumption $|\nabla u| \ll 1$.

The approximative global conservation laws (4.19) and (4.30) have the same structure as (4.11) and (4.29). Therefore we can repeat the discussion following on equation (4.20) and conclude: Within the context of scattering experiments, the quasi angular momentum and the true angular momentum play identical roles and cannot be distinguished.

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