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Analyticity and Borel-summability of the perturbation expansion for correlation functions of continuous spin systems

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Abstract. Systems of continuous spins with single spin distribution $\exp(-\mu\phi^2 - \lambda P(\phi))$ on a lattice are studied. The pair interaction between the spins may have infinite range. It is proved that the correlation functions are analytic in λ for small $|\lambda|$, $\operatorname{Re}\lambda \geqslant 0$. The perturbation series in λ is Borel-summable in the ordinary sense if the degree of $P(\phi)$ is $\leqslant 4$, and in the generalized sense if it is >4.

1. Introduction

We study a system of continuous spins $\phi \in \mathbb{R}$ with single spin distribution $\exp[-\mu\phi^2 - \lambda P(\phi)]$ on a lattice of arbitrary dimension. The function P is assumed to be bounded from below and polynomially bounded from above. Such systems are of considerable interest because they arise as lattice approximations of $P(\phi)$-Euclidean quantum field theories [1]. In this case the spins are coupled by a nearest neighbour interaction

$$\mathfrak{J}(\mathbf{k}, \mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}, \ |\mathbf{k}-\mathbf{l}| = 1.$$

In this paper we show analyticity and Borel-summability in λ of correlation functions and free energy. Our method of proof is the cluster expansion used by Eckmann, Magnen and Sénéor [2] in order to study these questions for $(\phi^4)_2$. Since our interest comes from statistical mechanics, we allow the interaction to be of arbitrary range. Then, however, the bounds on the derivatives necessary to prove Borel-summability can not be obtained by the technique of studying truncated correlation functions. This method has been worked out only in the nearest neighbour case. Instead we use term-by-term estimation which, besides of being more general, greatly simplifies the proof. Nevertheless, the method of truncated correlation functions can be adapted to the lattice problem with finite range interaction. It is then possible to derive cluster properties [2]. Strong cluster properties have been obtained by Malyshev [3]. For the analysis of the perturbation expansions it is not necessary to use these complicated methods.

For nearest neighbour interactions results similar to ours have been published (analyticity) respectively announced (Borel-summability) recently by Constantinescu [4]. We apply his method of analytic continuation in Section 8.

2. Description of the system

Let the pair-interaction

$$\mathfrak{I}:\mathbb{Z}^\nu\times\mathbb{Z}^\nu\to\mathbb{C}$$
$$(\mathbf{k},\mathbf{l})\mapsto\mathfrak{I}(\mathbf{k},\mathbf{l})$$

be subject to the following conditions:

$$\begin{aligned}&\text{(i) } \mathfrak{I}(\mathbf{k},\mathbf{k})=0\ \forall\mathbf{k}\in\mathbb{Z}^\nu\\ &\text{(ii) } \mathfrak{I}(\mathbf{k},\mathbf{l})=\mathfrak{I}(\mathbf{l},\mathbf{k})\ \forall\mathbf{k},\mathbf{l}\in\mathbb{Z}^\nu\\ &\text{(iii) } \mathfrak{I}(\mathbf{k}+\mathbf{m},\mathbf{l}+\mathbf{m})=\mathfrak{I}(\mathbf{k},\mathbf{l})\ \forall\mathbf{k},\mathbf{l},\mathbf{m}\in\mathbb{Z}^\nu\\ &\text{(iv) } \sum_{\mathbf{k}\in\mathbb{Z}^\nu}|\mathfrak{I}(\mathbf{o},\mathbf{k})|^{1/2}=\mathfrak{I}<\infty\end{aligned}\tag{1}$$

Let the function

$$P:\mathbb{R}\to\mathbb{R}$$

satisfy

$$\begin{aligned}&\text{(i) } P(\phi)\geqslant -A\ \forall\phi\in\mathbb{R} \text{ and some constant } A\geqslant 0.\\ &\text{(ii) } P(\phi)\leqslant B_1+B_2\phi^{2d}\ \forall\phi \text{ and constants } B_1,\ B_2\geqslant 0,\ d\in\mathbb{N}.\end{aligned}\tag{2}$$

In Sections 8 and 9 we shall assume P to be a polynomial. As the objects we shall investigate are only trivially changed by the addition of a constant to $P(\phi)$, we shall assume $A=0$.

Let now Λ, the volume of the spin system, be a finite subset of $\mathbb{Z}^\nu$. A spin $\phi_\mathbf{k}$ at lattice site $\mathbf{k}\in\Lambda$ may assume any real value. We define a finite measure on the configuration space $\mathbb{R}^\Lambda$ of our system by

$$d\mu_\Lambda^\lambda\Big(\mathsf{X}_{\mathbf{k}\in\Lambda}\ \phi_\mathbf{k}\Big)=\exp\Big[-\frac{1}{2}\sum_{\mathbf{k},\mathbf{l}\in\Lambda}\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_\mathbf{k}\phi_\mathbf{l}\Big]\prod_{\mathbf{k}\in\Lambda}\exp\left[-\mu\phi_\mathbf{k}^2-\lambda P(\phi_\mathbf{k})\right]d\phi_\mathbf{k}\tag{3}$$

for Re μ sufficiently large, Re $\lambda\geqslant 0$. Now, for $\Omega\subseteq\Lambda$ and abbreviating $\mathsf{X}_{\mathbf{k}\in\Omega}\ \phi_\mathbf{k}$ by ϕ_Ω we define

$$S_\Lambda^\lambda(\phi_\Omega)=\left(\int d\mu_\Lambda^\lambda\right)^{-1}\int e^{-\frac{1}{2}\sum_{\mathbf{k},\mathbf{l}\in\Lambda}\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_\mathbf{k}\phi_\mathbf{l}}\prod_{\mathbf{k}\in\Lambda}e^{-\mu\phi_\mathbf{k}^2-\lambda P(\phi_\mathbf{k})}\prod_{\mathbf{k}\in\Lambda\backslash\Omega}d\phi_\mathbf{k}.\tag{4}$$

$$S_\Lambda^\lambda(\phi_\Omega)\prod_{\mathbf{k}\in\Omega}d\phi_\mathbf{k}$$

is a normed complex-valued measure on $\mathbb{R}^\Omega$, a probability measure, if μ, λ and $\mathfrak{I}(\mathbf{k},\mathbf{l})\ \forall\mathbf{k},\mathbf{l}$ are real.

3. Cluster expansion

We choose some translation-invariant ordering on $\mathbb{Z}^\nu$ and write the pair-interaction term

$$\prod_{\substack{\mathbf{k},\mathbf{l}\in\Lambda\\ \mathbf{k}<\mathbf{l}}}\exp\left[-\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_\mathbf{k}\phi_\mathbf{l}\right]$$

as

$$\prod_{\substack{\mathbf{k},\mathbf{l}\in\Lambda\\ \mathbf{k}<\mathbf{l}}} [(\exp[-\Im(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}]-1)+1] = \sum_{\gamma} \prod_{(\mathbf{k},\mathbf{l})\in\gamma} (\exp[-\Im(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}]-1)$$

where the sum runs over the subsets γ of the set of ordered pairs $(\mathbf{k},\mathbf{l})$, $\mathbf{k}<\mathbf{l}$, $\mathbf{k}$, $\mathbf{l}\in\Lambda$. Obviously, the γ's can be represented by graphs on Λ which allows us to apply the graph-theoretic notion of connectedness to them.

Let $\Omega\subset\mathbb{Z}^\nu$ be finite, non-empty, $X\supseteq\Omega$. Then, with $\Gamma_\varnothing(X)$ the set of all connected graphs on X, $\Gamma_\Omega(X)$ the set of all the graphs on X which connect every $\mathbf{l}\in X\backslash\Omega$ to some $\mathbf{k}\in\Omega$ and do not contain any direct connections between elements of Ω, we define

$$\sigma^{\varnothing}(\phi_X) = \sum_{\gamma\in\Gamma_\varnothing(X)} \prod_{(\mathbf{k},\mathbf{l})\in\gamma} (e^{-\Im(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}}-1) \prod_{\mathbf{k}\in X} e^{-\mu\phi_{\mathbf{k}}^2-\lambda P(\phi_{\mathbf{k}})}\, d\phi_{\mathbf{k}} \tag{5}$$

$$\sigma^{\Omega}(\phi_X) = \prod_{\substack{\mathbf{k},\mathbf{l}\in\Omega\\ \mathbf{k}<\mathbf{l}}} e^{-\Im(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}} \prod_{\mathbf{k}\in\Omega} e^{-\mu\phi_{\mathbf{k}}^2-\lambda P(\phi_{\mathbf{k}})}$$
$$\times \sum_{\gamma\in\Gamma_\Omega(X)} \prod_{(\mathbf{k},\mathbf{l})\in\gamma} (e^{-\Im(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}}-1) \prod_{\mathbf{l}\in X\backslash\Omega} e^{-\mu\phi_{\mathbf{l}}^2-\lambda P(\phi_{\mathbf{l}})} \tag{6}$$

If $\Omega = X$, the sum is put equal to one.

Now $S_\Lambda^\lambda(\phi_\Omega)$ can be written in terms of the σ^Ω's and $\sigma^\varnothing$'s:

$$S_\Lambda^\lambda(\phi_\Omega) = \sum_{X:\Omega\subseteq X\subseteq\Lambda} \int \sigma^\Omega(\phi_X) \prod_{\mathbf{k}\in X\backslash\Omega} d\phi_{\mathbf{k}} \sum_{[W_1,\dots,W_r]\in\mathscr{P}(\Lambda)} \prod_{i=1}^{r} \int \sigma^\varnothing(\phi_{W_i}) \prod_{\mathbf{l}\in W_i} d\phi_{\mathbf{l}}$$
$$\times \left[\sum_{[W_1,\dots,W_r]\in\mathscr{P}(\Lambda\backslash X)} \prod_{i=1}^{r} \int \sigma^\varnothing(\phi_{W_i}) \prod_{\mathbf{k}\in W_i} d\phi_{\mathbf{k}} \right]^{-1} \tag{7}$$

where, for $X\subset\mathbb{Z}^\nu$, $\mathscr{P}(X)$ is the set of all partitions of X into disjoint subsets.

With the definitions

$$Z(\lambda) = \int e^{-\mu\phi^2-\lambda P(\phi)}\, d\phi, \tag{8}$$

$$s_X^\varnothing = \begin{cases} 0 & \text{if } |X|\leq 1 \\ Z^{-|X|} \int \sigma^\varnothing(\phi_X) \prod_{\mathbf{k}\in X} d\phi_{\mathbf{k}} & \text{otherwise} \end{cases} \tag{9}$$

$$s_X^\Omega(\phi_\Omega) = \begin{cases} Z^{-|X|} \int \sigma^\Omega(\phi_X) \prod_{\mathbf{k}\in X\backslash\Omega} d\phi_{\mathbf{k}} & \text{if } X\supseteq\Omega \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$(\exp s^\varnothing)_X = \begin{cases} 1 & \text{if } X=\varnothing \\ \sum_{[W_1,\dots,W_r]\in\mathscr{P}(X)} \prod_{i=1}^{r} s_{W_i}^\varnothing & \text{otherwise} \end{cases} \tag{11}$$

we get

$$S_\Lambda^\lambda(\phi_\Omega) = \frac{\sum_{X\subseteq\Lambda} s_X^\Omega(\phi_\Omega) \sum_{W\subseteq\Lambda\backslash X} (\exp s^\varnothing)_W}{\sum_{W\subseteq\Lambda} (\exp s^\varnothing)_W} \tag{12}$$

Expansion of the denominator leads to

$$S_\Lambda^\lambda(\phi_\Omega)=\sum_{X\subseteq\Lambda} s_X^\Omega(\phi_\Omega)\sum_{W_0\subseteq\Lambda\setminus X}(\exp s^\varnothing)_{W_0}$$
$$\times\sum_{n=0}^{\infty}(-1)^n\sum_{\substack{W_1,\dots,W_n\\ W_i\neq\varnothing,1\leq i\leq n}}\prod_{i=1}^{n}(\exp s^\varnothing)_{W_i} \tag{13}$$

This expression can be reordered in the following manner:

Let $\mathcal{N}(\Lambda)$ be the set of functions

$$N:\mathbb{Z}^\nu\to\mathbb{Z}_+$$
$$\mathbf{k}\mapsto N(\mathbf{k})$$

the support $\tilde{N}$ of which is contained in Λ.

For a collection $[W_0,\dots,W_n]$ of finite subsets of $\mathbb{Z}^\nu$ and N as above, we write $[W_0,\dots,W_n]\sim N$ if the number of sets in $[W_0,\dots,W_n]$ containing $\mathbf{k}$ is $N(\mathbf{k})\ \forall\mathbf{k}\in\mathbb{Z}^\nu$.

Introducing

$$q_N^X=\sum_{\substack{[W_0,\dots,W_n]\sim N\\ W_0\cap X=\varnothing\\ W_i\neq\varnothing,1\leq i\leq n}}(-1)^n\prod_{i=0}^{n}(\exp s^\varnothing)_{W_i} \tag{14}$$

we can write (13) as

$$S_\Lambda^\lambda(\phi_\Omega)=\sum_{X:\Omega\subseteq X\subseteq\Lambda}s_X^\Omega(\phi_\Omega)\sum_{N\in\mathcal{N}(\Lambda)}q_N^X \tag{15}$$

For the q_N^X a recurrence relation can be derived, which, with

$$|N|=\sum_{\mathbf{k}\in\mathbb{Z}^\nu}N(\mathbf{k}) \tag{16}$$

allows induction on $|X|+|N|$:

Proposition 3.1

$$q_N^\varnothing=\begin{cases}1 & \text{if } N=0\\ 0 & \text{otherwise}\end{cases} \tag{17}$$

$$q_N^X=q_N^{X\setminus\{\mathbf{k}\}}-\sum_{V\subseteq\widetilde{N-\chi_{\{\mathbf{k}\}}}}s_{V\cup\{\mathbf{k}\}}^\varnothing q_{N-\chi_{V\cup\{\mathbf{k}\}}}^{X\cup V}\quad \text{for}\quad \mathbf{k}\in X, \tag{18}$$

where χ denotes the characteristic function. The sum in the above expression is put to zero if $\mathbf{k}\notin\tilde{N}$.

Proof

(17):

$$q_0^{\varnothing} = (\exp s^{\varnothing})_{\varnothing} = 1$$

$$\begin{aligned}
N \neq 0 : q_N^{\varnothing} &= \sum_{\substack{[W_0, W_1, \ldots, W_n] \sim N \\ W_i \neq \varnothing, 1 \leq i \leq n}} (-1)^n \prod_{i=0}^{n} (\exp s^{\varnothing})_{W_i} \\
&= \sum_{\substack{[W_0, W_1, \ldots, W_n] \sim N \\ W_i \neq \varnothing, 1 \leq i \leq n \\ W_0 = \varnothing}} (-1)^n \prod_{i=0}^{n} (\exp s^{\varnothing})_{W_i} + \sum_{\substack{[W_0, W_1, \ldots, W_n] \sim N \\ W_i \neq \varnothing, 0 \leq i \leq n}} (-1)^n \prod_{i=0}^{n} (\exp s^{\varnothing})_{W_i}. \\
&= \sum_{\substack{[W_1, \ldots, W_n] \sim N \\ W_i \neq \varnothing, 1 \leq i \leq n}} (-1)^n \prod_{i=1}^{n} (\exp s^{\varnothing})_{W_i} + \sum_{\substack{[W_1, \ldots, W_n] \sim N \\ W_i \neq \varnothing, 1 \leq i \leq n}} (-1)^{n-1} \prod_{i=1}^{n} (\exp s^{\varnothing})_{W_i} \\
&= 0.
\end{aligned}$$

(18):

$$\begin{aligned}
& q_N^{X \setminus \{\mathbf{k}\}} - q_N^X \\
&= \sum_{\substack{W_0 \subseteq \widetilde{N - X_{\{\mathbf{k}\}}} \\ W_0 \cap X = \varnothing}} (\exp s^{\varnothing})_{W_0 \cup \{\mathbf{k}\}} \left[\sum_{n=0}^{\infty} (-1)^n \sum_{[W_1, \ldots, W_n] \sim N - X_{W_0 \cup \{\mathbf{k}\}}} \prod_{i=1}^{n} (\exp s^{\varnothing})_{W_i} \right] \\
&= \sum_{\substack{V \subseteq \widetilde{N - X_{\{\mathbf{k}\}}} \\ V \cap X = \varnothing}} s^{\varnothing}_{V \cup \{\mathbf{k}\}} \sum_{\substack{W_0 \subseteq \widetilde{N - X_{V \cup \{\mathbf{k}\}}} \\ W_0 \cap (V \cup X) = \varnothing}} (\exp s^{\varnothing})_{W_0} \\
&\quad \times \left[\sum_{n=0}^{\infty} (-1)^n \sum_{[W_1, \ldots, W_n] \sim N - X_{V \cup \{\mathbf{k}\}} - X_{W_0}} \prod_{i=1}^{n} (\exp s^{\varnothing})_{W_i} \right] \\
&= \sum_{\substack{V \subseteq \widetilde{N - X_{\{\mathbf{k}\}}} \\ V \cap X = \varnothing}} s^{\varnothing}_{V \cup \{\mathbf{k}\}} q^{X \cup V}_{N - X_{V \cup \{\mathbf{k}\}}} \qquad \square
\end{aligned}$$

It is not difficult to see, that every q_N^X can be calculated by repeated application of (18) starting from $q_N^{\varnothing}$'s.

4. Estimates

In this section we derive various estimates which are needed later on to control the cluster expansion of the correlation functions. By

$$0 \leq (\phi_{\mathbf{k}} \pm \phi_{\mathbf{l}})^2 = \phi_{\mathbf{k}}^2 \pm 2\phi_{\mathbf{k}}\phi_{\mathbf{l}} + \phi_{\mathbf{l}}^2$$

we have

$$-\frac{|\mathfrak{I}(\mathbf{k}, \mathbf{l})|}{2} (\phi_{\mathbf{k}}^2 + \phi_{\mathbf{l}}^2) \leq |\mathfrak{I}(\mathbf{k}, \mathbf{l})| \, \phi_{\mathbf{k}}\phi_{\mathbf{l}} \leq \frac{|\mathfrak{I}(\mathbf{k}, \mathbf{l})|}{2} (\phi_{\mathbf{k}}^2 + \phi_{\mathbf{l}}^2)$$

and therefore

$$|\mathfrak{I}(\mathbf{k}, \mathbf{l}) \phi_{\mathbf{k}}\phi_{\mathbf{l}}| \leq \frac{|\mathfrak{I}(\mathbf{k}, \mathbf{l})|}{2} (\phi_{\mathbf{k}}^2 + \phi_{\mathbf{l}}^2) \tag{19}$$

This implies

$$|e^{-\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}}| \leq e^{|\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}|} \leq e^{[|\mathfrak{I}(\mathbf{k},\mathbf{l})|/2](\phi_{\mathbf{k}}^2+\phi_{\mathbf{l}}^2)} \tag{20}$$

$$|e^{-\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}} - 1| \leq e^{|\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}|} - 1 \leq e^{[|\mathfrak{I}(\mathbf{k},\mathbf{l})|/2](\phi_{\mathbf{k}}^2+\phi_{\mathbf{l}}^2)} - 1. \tag{21}$$

From now on we shall assume

$$\operatorname{Re}\mu > \mathfrak{I} = \sum_{\mathbf{k}} |\mathfrak{I}(\mathbf{o},\mathbf{k})|^{1/2} \tag{22}$$

and

$$|\mathfrak{I}(\mathbf{k},\mathbf{l})| \leq 1, \tag{23}$$

a restriction, which can be easily removed by scaling of the $\phi_{\mathbf{k}}$'s. Let

$$K(\mathbf{k},\mathbf{l}) = |\mathfrak{I}(\mathbf{k},\mathbf{l})|^{1/2} \frac{\operatorname{Re}\mu}{2\mathfrak{I}} > \frac{|\mathfrak{I}(\mathbf{k},\mathbf{l})|}{2} \tag{24}$$

and $\mu_{\mathbf{k}}$, $\mu_{\mathbf{l}}$ arbitrary complex constants. Then we have

Proposition 4.1

$$|e^{-\mu_{\mathbf{k}}\phi_{\mathbf{k}}^2} e^{-\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}} e^{-\mu_{\mathbf{l}}\phi_{\mathbf{l}}^2}| \leq e^{-[\operatorname{Re}\mu_{\mathbf{k}} - K(\mathbf{k},\mathbf{l})]\phi_{\mathbf{k}}^2} e^{-[\operatorname{Re}\mu_{\mathbf{l}} - K(\mathbf{k},\mathbf{l})]\phi_{\mathbf{l}}^2} \tag{25}$$

$$|e^{-\mu_{\mathbf{k}}\phi_{\mathbf{k}}^2}(e^{-\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}} - 1) e^{-\mu_{\mathbf{l}}\phi_{\mathbf{l}}^2}| \leq e^{-[\operatorname{Re}\mu_{\mathbf{k}} - K(\mathbf{k},\mathbf{l})]\phi_{\mathbf{k}}^2} e^{-[\operatorname{Re}\mu_{\mathbf{l}} - K(\mathbf{k},\mathbf{l})]\phi_{\mathbf{l}}^2} \frac{\mathfrak{I}}{\operatorname{Re}\mu - \mathfrak{I}} |\mathfrak{I}(\mathbf{k},\mathbf{l})|^{1/2}. \tag{26}$$

Proof

(25) is obvious from (20), (24).

(26):

$$\begin{aligned} |e^{-\mu_{\mathbf{k}}\phi_{\mathbf{k}}^2}(e^{-\mathfrak{I}(\mathbf{k},\mathbf{l})\phi_{\mathbf{k}}\phi_{\mathbf{l}}} - 1) e^{-\mu_{\mathbf{l}}\phi_{\mathbf{l}}^2}| &\leq e^{-\operatorname{Re}\mu_{\mathbf{k}}\phi_{\mathbf{k}}^2} e^{-\operatorname{Re}\mu_{\mathbf{l}}\phi_{\mathbf{l}}^2} (e^{[|\mathfrak{I}(\mathbf{k},\mathbf{l})|/2](\phi_{\mathbf{k}}^2+\phi_{\mathbf{l}}^2)} - 1) \\ &\leq e^{-[\operatorname{Re}\mu_{\mathbf{k}} - K(\mathbf{k},\mathbf{l})]\phi_{\mathbf{k}}^2} e^{-[\operatorname{Re}\mu_{\mathbf{l}} - K(\mathbf{k},\mathbf{l})]\phi_{\mathbf{l}}^2} (e^{-[K(\mathbf{k},\mathbf{l}) - |\mathfrak{I}(\mathbf{k},\mathbf{l})|/2](\phi_{\mathbf{k}}^2+\phi_{\mathbf{l}}^2)} - e^{-K(\mathbf{k},\mathbf{l})(\phi_{\mathbf{k}}^2+\phi_{\mathbf{l}}^2)}) \end{aligned}$$

The last factor is bounded by

$$\begin{aligned} &\sup_{x\geq 0} (e^{-[K(\mathbf{k},\mathbf{l}) - |\mathfrak{I}(\mathbf{k},\mathbf{l})|/2]x} - e^{-K(\mathbf{k},\mathbf{l})x}) \\ &\leq \frac{|\mathfrak{I}(\mathbf{k},\mathbf{l})|/2}{K(\mathbf{k},\mathbf{l}) - |\mathfrak{I}(\mathbf{k},\mathbf{l})|/2} = \frac{|\mathfrak{I}(\mathbf{k},\mathbf{l})|}{|\mathfrak{I}(\mathbf{k},\mathbf{l})|^{1/2}(\operatorname{Re}\mu)/\mathfrak{I} - |\mathfrak{I}(\mathbf{k},\mathbf{l})|} \\ &= \frac{\mathfrak{I}\,|\mathfrak{I}(\mathbf{k},\mathbf{l})|^{1/2}}{\operatorname{Re}\mu - \mathfrak{I}\,|\mathfrak{I}(\mathbf{k},\mathbf{l})|^{1/2}} \end{aligned}$$

which is smaller than $\dfrac{\mathfrak{I}}{\operatorname{Re}\mu - \mathfrak{I}} |\mathfrak{I}(\mathbf{k},\mathbf{l})|^{1/2}$ by (22), (23).

Here we have used

$$\sup_{x\geq 0}(e^{-\alpha x}-e^{-\beta x})=\left(1-\frac{\alpha}{\beta}\right)e^{-\alpha x_0}\leq\frac{\beta-\alpha}{\beta}$$

for $\beta\geq\alpha>0$, where $x_0>0$ solves

$$\frac{d}{dx}(e^{-\alpha x}-e^{-\beta x})=0. \quad \square$$

For Ω, X finite subsets of $\mathbb{Z}^\nu$, $\Omega\subseteq X$, we define

$$T_X^\Omega=\sum_\gamma\prod_{(\mathbf{k},\mathbf{l})\in\gamma}|\Im(\mathbf{k},\mathbf{l})|^{1/2} \tag{27}$$

where the sum runs over the trees connecting each $\mathbf{l}\in X\backslash\Omega$ with exactly one $\mathbf{k}\in\Omega$.

Lemma 4.2

$$|s_X^\varnothing|\leq\frac{\sqrt{2\pi}}{|Z|\sqrt{\operatorname{Re}\mu}}\left(\frac{\sqrt{2\pi}\Im}{|Z|\sqrt{\operatorname{Re}\mu}(\operatorname{Re}\mu-\Im)}\right)^{|X|-1}T_X^{\{\mathbf{k}\}}, \tag{28}$$

$\mathbf{k}$ *some element of* X.

$$|s_X^\Omega(\phi_\Omega)|\leq\left(\frac{\sqrt{2\pi}\Im}{|Z|\sqrt{\operatorname{Re}\mu}(\operatorname{Re}\mu-\Im)}\right)^{|X\backslash\Omega|}\prod_{\mathbf{k}\in\Omega}e^{-(\operatorname{Re}\mu/2)\phi_\mathbf{k}^2}T_X^\Omega \tag{29}$$

Proof. For $\Omega\subseteq X$, $\mathbf{k}_1\in\Omega$, $\Omega'=\Omega\backslash\{\mathbf{k}_1\}$, $X'=X\backslash\{\mathbf{k}_1\}$ we have

$$\sigma^\Omega(\phi_X)=e^{-\mu\phi_{\mathbf{k}1}^2-\lambda P(\phi_{\mathbf{k}1})}\prod_{\mathbf{k}\in\Omega'}e^{-\Im(\mathbf{k}_1,\mathbf{k})\phi_{\mathbf{k}1}\phi_\mathbf{k}}e^{-\mu\phi_\mathbf{k}^2-\lambda P(\phi_\mathbf{k})}$$
$$\times\sum_{Y\subseteq X\backslash\Omega}\prod_{\mathbf{l}\in Y}(e^{-\Im(\mathbf{k}_1,\mathbf{l})\phi_{\mathbf{k}1}\phi_\mathbf{l}}-1)\sigma^{\Omega'\cup Y}(\phi_{X'}). \tag{30}$$

This recurrence relation, together with

$$\sigma^{\{\mathbf{k}\}}(\phi_\mathbf{k})=e^{-\mu\phi_\mathbf{k}^2-\lambda P(\phi_\mathbf{k})} \tag{31}$$

as a starting point, allows for induction on $|X|$:

Let us assume that for any $\Psi\subseteq X'$ the inequality

$$|\sigma^\Psi(\phi_{X'})|\leq\left(\frac{\Im}{\operatorname{Re}\mu-\Im}\right)^{|X'\backslash\Psi|}\prod_{\mathbf{k}\in X'}\exp\left\{-\left[\operatorname{Re}\mu-\sum_{\substack{\mathbf{l}\in X'\\ \mathbf{l}>\mathbf{k}}}K(\mathbf{k},\mathbf{l})\right]\phi_\mathbf{k}^2\right\}T_{X'}^\Psi$$

holds. Then for $\Omega \subseteq X$ follows

$$|\sigma^{\Omega}(\phi_X)| \leqslant e^{-\operatorname{Re}\mu\phi_{\mathbf{k}_1}^2} \prod_{\mathbf{k}\in\Omega'} |e^{-\Im(\mathbf{k}_1,\mathbf{k})\phi_{\mathbf{k}_1}\phi_{\mathbf{k}}}|$$

$$\times \sum_{Y\subseteq X\setminus\Omega} \left[\prod_{\mathbf{l}\in Y} |e^{-\Im(\mathbf{k}_1,\mathbf{l})\phi_{\mathbf{k}_1}\phi_{\mathbf{l}}} - 1| \times \left(\frac{\Im}{\operatorname{Re}\mu - \Im}\right)^{|X\setminus(\Omega\cup Y)|}\right.$$

$$\left. \times \prod_{\mathbf{k}\in X'} \exp\left\{-\left[\operatorname{Re}\mu - \sum_{\substack{\mathbf{l}\in X' \\ \mathbf{l}>\mathbf{k}}} K(\mathbf{k},\mathbf{l})\right]\phi_{\mathbf{k}}^2\right\} T_{X'}^{\Omega'\cup Y}\right]$$

$$\leqslant \left(\frac{\Im}{\operatorname{Re}\mu - \Im}\right)^{|X\setminus\Omega|} \prod_{\mathbf{k}\in X} \exp\left\{-\left[\operatorname{Re}\mu - \sum_{\substack{\mathbf{l}\in X \\ \mathbf{l}>\mathbf{k}}} K(\mathbf{k},\mathbf{l})\right]\phi_{\mathbf{k}}^2\right\}$$

$$\times \sum_{Y\subseteq X\setminus\Omega} \prod_{\mathbf{l}\in Y} |\Im(\mathbf{k}_1,\mathbf{l})|^{1/2} T_{X'}^{\Omega'\cup Y}$$

by (25), (26). This yields

$$|\sigma^{\Omega}(\phi_X)| \leqslant \left(\frac{\Im}{\operatorname{Re}\mu - \Im}\right)^{|X\setminus\Omega|} \prod_{\mathbf{k}\in X} \exp\left\{-\left[\operatorname{Re}\mu - \sum_{\substack{\mathbf{l}\in X \\ \mathbf{l}>\mathbf{k}}} K(\mathbf{k},\mathbf{l})\right]\phi_{\mathbf{k}}^2\right\} T_X^{\Omega},$$

which completes the induction.

Now we have, by (24)

$$|\sigma^{\Omega}(\phi_X)| \leqslant \left(\frac{\Im}{\operatorname{Re}\mu - \Im}\right)^{|X\setminus\Omega|} \prod_{\mathbf{k}\in X} e^{-(\operatorname{Re}\mu/2)\phi_{\mathbf{k}}^2} T_X^{\Omega}. \tag{32}$$

The estimates (28), (29) then follow from (9), (10), (32) and

$$s_X^{\varnothing} = \int s_X^{\{\mathbf{k}\}}(\phi_{\mathbf{k}})\, d\phi_{\mathbf{k}} \quad \text{for some} \quad \mathbf{k}\in X. \quad \square$$

Proposition 4.3. *Let* $|\arg\mu| \leqslant \varphi < \pi/2$ *and* λ *be in a semicircle* $S_R = \{\lambda \mid |\lambda| \leqslant R, \operatorname{Re}\lambda \geqslant 0\}$ *with radius* $R < B_1^{-1}(\cos\varphi)^{1/2}$. *Then, for every* κ *with*

$$0 < \kappa < \frac{B_1^{-1}(\cos\varphi)^{1/2} - R}{B_1^{-1}(\cos\varphi)^{1/2}}$$

there is a constant μ_0 *such that the integral* $Z(\lambda)$ *defined by* (8) *satisfies*

$$\kappa\sqrt{\frac{\pi}{|\mu|}} \leqslant |Z(\lambda)| \leqslant (2-\kappa)\sqrt{\frac{\pi}{|\mu|}} \tag{33}$$

for $|\mu| \geqslant \mu_0$.

Proof. For B_1, B_2 introduced in (2ii)

$$\sup_{\operatorname{Re}\lambda\geqslant 0}\left|\frac{dZ(\lambda)}{d\lambda}\right|\leqslant\int P(\phi)e^{-\operatorname{Re}\mu\phi^2}\,d\phi$$

$$\leqslant\sqrt{\frac{\pi}{\operatorname{Re}\mu}}\,[B_1+B_2\,d!\,(\operatorname{Re}\mu)^{-d}] \tag{34}$$

$$=\sqrt{\frac{\pi}{|\mu|}}\,(\cos\varphi)^{-1/2}\left[B_1+\frac{B_2\,d!}{(\cos\varphi)^d}\,|\mu|^{-d}\right]$$

It follows, that

$$\sqrt{\frac{\pi}{|\mu|}}\left(1-R(\cos\varphi)^{-1/2}\left[B_1+\frac{B_2\,d!}{(\cos\varphi)^d}\,|\mu|^{-d}\right]\right)\leqslant|Z(\lambda)|$$

$$\leqslant\sqrt{\frac{\pi}{|\mu|}}\left(1+R(\cos\varphi)^{-1/2}\left[B_1+\frac{B_2\,d!}{(\cos\varphi)^d}\,|\mu|^{-d}\right]\right)\quad\square \tag{35}$$

Proposition 4.4 *For* arg μ, λ *as in Proposition* 4.3 *we can, for every* $\varepsilon>0$ *find a* μ_0 *such that for* $|\mu|\geqslant\mu_0$

$$\frac{\sqrt{2\pi}\,\mathfrak{J}}{|Z|\sqrt{\operatorname{Re}\mu}\,(\operatorname{Re}\mu-\mathfrak{J})}\leqslant\varepsilon \tag{36}$$

Proof

$$\frac{\sqrt{2\pi}\,\mathfrak{J}}{|Z|\sqrt{\operatorname{Re}\mu}\,(\operatorname{Re}\mu-\mathfrak{J})}=\frac{\sqrt{2}\,\mathfrak{J}}{|Z|\sqrt{|\mu|/\pi}\,(\cos\varphi)^{1/2}(|\mu|\cos\varphi-\mathfrak{J})}$$

$$\leqslant\frac{\sqrt{2}\,\mathfrak{J}}{\kappa(\cos\varphi)^{1/2}(|\mu|\cos\varphi-\mathfrak{J})} \tag{37}$$

for $|\mu|$ large enough, which implies that

$$\frac{\sqrt{2\pi}\,\mathfrak{J}}{|Z|\sqrt{\operatorname{Re}\mu}\,(\operatorname{Re}\mu-\mathfrak{J})}$$

goes to zero for $|\mu|\to\infty$.

Proposition 4.5

$$\sum_{\substack{X:\Omega\subseteq X\\|X\setminus\Omega|=n}}T_X^{\Omega}\leqslant e^{\mathfrak{J}(|\Omega|+n-1)}\quad\text{for}\quad\Omega\neq\varnothing. \tag{38}$$

Proof. We proceed again by induction on $|X|$:

$$\sum_{\substack{X:\Omega\subseteq X\\|X\setminus\Omega|=n}} T_X^\Omega = n!^{-1} \sum_{\mathbf{k}_1,\dots,\mathbf{k}_n} T^\Omega_{\Omega\cup\{\mathbf{k}_1,\dots,\mathbf{k}_n\}}, \tag{39}$$

with $T^\Omega_{\Omega\cup\{\mathbf{k}_1,\dots,\mathbf{k}_n\}}=0$, if $\mathbf{k}_i=\mathbf{k}_j$ for some i, j, $i\neq j$ or if $\mathbf{k}_i\in\Omega$ for some i.

For $\mathbf{l}_1\in\Omega$, $\Omega'=\Omega\setminus\{\mathbf{l}_1\}$ we have

$$\begin{aligned}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n} T^\Omega_{\Omega\cup\{\mathbf{k}_1,\dots,\mathbf{k}_n\}} &= \sum_{\mathbf{k}_1,\dots,\mathbf{k}_n}\ \sum_{Y\subseteq\{\mathbf{k}_1,\dots,\mathbf{k}_n\}}\ \prod_{\mathbf{l}\in Y}|\mathfrak{I}(\mathbf{l}_1,\mathbf{l})|^{1/2}T^{\Omega'\cup Y}_{\Omega'\cup\{\mathbf{k}_1,\dots,\mathbf{k}_n\}}\\ &\leqslant \sum_{m=0}^{n}\binom{n}{m}\prod_{i=1}^{m}\left(\sum_{\mathbf{k}_i}|\mathfrak{I}(\mathbf{l}_1,\mathbf{k}_i)|^{1/2}\right)\sum_{\mathbf{k}_{m+1},\dots,\mathbf{k}_n} T^{\Omega'\cup\{\mathbf{k}_1,\dots,\mathbf{k}_m\}}_{\Omega'\cup\{\mathbf{k}_1,\dots,\mathbf{k}_n\}}\end{aligned} \tag{40}$$

which is smaller than

$$n!\sum_{m=0}^{\infty}\frac{\mathfrak{I}^m}{m!}e^{\mathfrak{I}(|\Omega'|+n-1)}=n!\,e^{\mathfrak{I}(|\Omega|+n-1)}$$

if we assume, that the assertion of the proposition is true for $|X|-1$. We complete the proof by observing that we can start induction with

$$T^{\{\mathbf{k}\}}_{\{\mathbf{k}\}}=1 \quad \square$$

Now, let $[\mathbf{k}_1,\dots,\mathbf{k}_n]$ be a collection of points in $\mathbb{Z}^\nu$, We write $[\mathbf{k}_1,\dots,\mathbf{k}_n]\sim N$, if $[\{\mathbf{k}_1\},\dots,\{\mathbf{k}_n\}]\sim N$, with N, $\sim$ as defined in Section 3, that is, $[\mathbf{k}_1,\dots,\mathbf{k}_n]\sim N$, if every $\mathbf{k}\in\mathbb{Z}^\nu$ is contained $N(\mathbf{k})$ times in $[\mathbf{k}_1,\dots,\mathbf{k}_n]$.

If $[\mathbf{k}_1,\dots,\mathbf{k}_n]$ contains r different points occurring $n_1,\dots,n_r$ times $(n_1+\dots+n_r=n)$, we define

$$[\mathbf{k}_1,\dots,\mathbf{k}_n]!=\frac{n!}{n_1!,\dots,n_r!},\quad\text{and 1,}\quad\text{if}\quad n=0. \tag{41}$$

Introducing $q^X_{[\mathbf{k}_1,\dots,\mathbf{k}_n]}=q^X_N$, if $[\mathbf{k}_1,\dots,\mathbf{k}_n]\sim N$ we get

$$\sum_{N\in\mathcal{N}(\Lambda)}q^X_N=\sum_{n=0}^{\infty}\sum_{\substack{N\in\mathcal{N}(\Lambda)\\|N|=n}}q^X_N=\sum_{n=0}^{\infty}\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n\in\Lambda}q^X_{[\mathbf{k}_1,\dots,\mathbf{k}_n]}[\mathbf{k}_1,\dots,\mathbf{k}_n]!^{-1} \tag{42}$$

Now, with $\mathbf{l}\in X$, the recurrence relation (18) yields:

$$\begin{aligned}&\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n\in\Lambda}q^X_{[\mathbf{k}_1,\dots,\mathbf{k}_n]}[\mathbf{k}_1,\dots,\mathbf{k}_n]!^{-1}\\ &\quad=\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n\in\Lambda}q^{X\setminus\{\mathbf{l}\}}_{[\mathbf{k}_1,\dots,\mathbf{k}_n]}[\mathbf{k}_1,\dots,\mathbf{k}_n]!^{-1}-\sum_{\mathbf{k}_1,\dots,\mathbf{k}_n\in\Lambda}[\mathbf{k}_1,\dots,\mathbf{k}_n]!^{-1}\\ &\qquad\times\sum_{m=1}^{n}s^{\varnothing}_{[\mathbf{l},\mathbf{k}_1,\dots,\mathbf{k}_m]}q^{X\cup\{\mathbf{k}_1,\dots,\mathbf{k}_m\}}_{[\mathbf{k}_{m+1},\dots,\hat{\mathbf{l}},\dots,\mathbf{k}_n]}\frac{[\mathbf{k}_1,\dots,\mathbf{k}_n]!}{[\mathbf{k}_1,\dots,\mathbf{k}_m]![\mathbf{k}_{m+1},\dots,\mathbf{k}_n]!},\end{aligned} \tag{43}$$

where $s^{\varnothing}_{[\mathbf{l},\mathbf{k}_1,\dots,\mathbf{k}_m]}$ is put equal to zero if two of the arguments are equal and $q^{X\cup\{\mathbf{k}_1,\dots,\mathbf{k}_m\}}_{[\mathbf{k}_{m+1},\dots,\hat{\mathbf{l}},\dots,\mathbf{k}_n]}$ is put equal to zero if $\mathbf{l}$ is not contained in $[\mathbf{k}_{m+1},\dots,\mathbf{k}_n]$, otherwise one of the $\mathbf{k}$'s equal to $\mathbf{l}$ is omitted.

Lemma 4.6. *For $|\arg \mu| \leqslant \varphi < \pi/2$, $\lambda \in S_R$ with $R < B_1^{-1}(\cos \varphi)^{1/2}$, $C > 1$, $c > 0$ we can find μ_0 such that*

$$\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_1, \ldots, \mathbf{k}_n]!^{-1} \leqslant C^{|X|} c^n, \tag{44}$$

if $|\mu| \geqslant \mu_0$.

Proof. By (41)

$$\begin{aligned}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_1, \ldots, \mathbf{k}_n]!^{-1} &\leqslant \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^{X\setminus\{\mathbf{l}\}}_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_1, \ldots, \mathbf{k}_n]!^{-1} \\ &\quad + \sum_{m=1}^{n} \Bigg[\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m} |s^{\varnothing}_{[\mathbf{l},\mathbf{k}_1,\ldots,\mathbf{k}_m]}| \, m!^{-1} \\ &\quad \times \sum_{\substack{\mathbf{k}_{m+1},\ldots,\mathbf{k}_n \\ \exists i, m+1 \leqslant i \leqslant n \\ \text{with } \mathbf{l} = \mathbf{k}_i}} |q^{X\cup\{\mathbf{k}_1,\ldots,\mathbf{k}_m\}}_{[\mathbf{k}_{m+1},\ldots,\hat{\mathbf{i}},\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_{m+1}, \ldots, \mathbf{k}_n]!^{-1} \Bigg] \\ &= \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^{X\setminus\{\mathbf{l}\}}_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_1, \ldots, \mathbf{k}_n]!^{-1} \\ &\quad + \sum_{m=1}^{n} \Bigg[\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m} |s^{\varnothing}_{[\mathbf{l},\mathbf{k}_1,\ldots,\mathbf{k}_m]}| \, m!^{-1} \\ &\quad \times \sum_{\mathbf{k}_{m+2},\ldots,\mathbf{k}_n} |q^{X\cup\{\mathbf{k}_1,\ldots,\mathbf{k}_m\}}_{[\mathbf{k}_{m+2},\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_{m+2}, \ldots, \mathbf{k}_n]!^{-1} \Bigg]\end{aligned} \tag{45}$$

By (28), (38) we have

$$\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m} |s^{\varnothing}_{[\mathbf{l},\mathbf{k}_1,\ldots,\mathbf{k}_m]}| \, m!^{-1} \leqslant \frac{\sqrt{2\pi}}{|Z| \sqrt{\operatorname{Re} \mu}} \left(\frac{\sqrt{2\pi} \Im e^{\Im}}{|Z| \sqrt{\operatorname{Re} \mu} (\operatorname{Re} \mu - \Im)} \right)^m \tag{46}$$

Now we can show, again by induction, that, for $C > 1$, $c' > 0$

$$\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}| \, [\mathbf{k}_1, \ldots, \mathbf{k}_n]!^{-1} \leqslant C^{|X|+n} c'^n \tag{47}$$

for $|\arg \mu| \leqslant \varphi < \pi/2$, $\lambda \in S_R$, $R < B_1^{-1}(\cos \varphi)^{1/2}$, $|\mu|$ sufficiently large: Abbreviating

$$\frac{\sqrt{2\pi}}{|Z| \sqrt{\operatorname{Re} \mu}} \quad \text{as} \quad a$$

and

$$\frac{\sqrt{2\pi} \Im e^{\Im}}{|Z| \sqrt{\operatorname{Re} \mu} (\operatorname{Re} \mu - \Im)} \quad \text{as} \quad b$$

we have by (42), (43)

$$\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_1,\ldots,\mathbf{k}_n]!^{-1} \leqslant C^{|X|+n-1}c'^n + aC^{|X|+n-1}\sum_{m=1}^{n} b^m c'^{n-1-m}$$
$$= C^{|X|+n-1}c'^n\left[1+\frac{a}{c'}\sum_{m=1}^{n}\left(\frac{b}{c'}\right)^m\right]$$

for $b<c'$ this implies

$$\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_1,\ldots,\mathbf{k}_n]!^{-1} \leqslant C^{|X|+n-1}c'^n\left(1+\frac{a}{c'}\frac{b/c'}{1-b/c'}\right).$$

Choosing

$$C \geqslant \left(1+\frac{a}{c'}\frac{b/c'}{1-b/c'}\right)$$

we obtain (47). With $c = Cc'$ the assertion of the lemma follows by Propositions 4.3. and 4.4. □

5. Convergence of the cluster expansion, thermodynamic limit and analyticity

Let F be a polynomially bounded function on $\mathbb{R}^\Omega$ and

$$\int |F(\phi_\Omega)| \prod_{\mathbf{k}\in\Omega} e^{-(\mathrm{Re}\,\mu/2)\phi_\mathbf{k}^2}\,d\phi_\mathbf{k} \leqslant K_F. \tag{48}$$

Defining for finite Λ, $\Lambda \supseteq \Omega$

$$\langle F\rangle^\lambda_\Lambda = \int F(\phi_\Omega) S^\lambda_\Lambda(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_\mathbf{k} \tag{49}$$

we have

Theorem 5.1. *For* $|\arg\mu| \leqslant \varphi < \pi/2$, $\lambda \in S_R$ *with* $R < B_1^{-1}(\cos\varphi)^{1/2}$ *we can find a* μ_0 *such that for* $|\mu| \geqslant \mu_0$
(a) *the series*

$$\sum_{X:\Omega\subseteq X\subseteq\Lambda}\ \sum_{N\in\mathcal{N}(\Lambda)} \int F(\phi_\Omega) s^\Omega_X(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_\mathbf{k} \cdot q^X_N \tag{50}$$

is absolutely convergent to a limit $\langle F\rangle^\lambda_\Lambda$ *uniformly in* Λ *and uniformly in* $\lambda \in S_R$.
(b) *for any increasing sequence* $\{\Lambda_n\}_{n\in\mathbb{N}}$ *of finite subsets of* $\mathbb{Z}^\nu$ *with* $\bigcup_{n\in\mathbb{N}}\Lambda_n = \mathbb{Z}^\nu$ *and* $\lambda \in S_R$ *the thermodynamic limit*

$$\langle F\rangle^\lambda = \lim_{n\to\infty}\langle F\rangle^\lambda_{\Lambda_n} \tag{51}$$

exists and is independent of the choice of the sequence.
(c) *the series*

$$\sum_{X:\Omega\subseteq X}\ \sum_{N} \int F(\phi_\Omega) s^\Omega_X(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_\mathbf{k} \cdot q^X_N \tag{52}$$

is absolutely convergent to $\langle F\rangle^{\lambda}$ *uniformly in* $\lambda \in S_R$.

Proof. The expressions (50), (52) can be estimated by

$$\sum_{X:\Omega\subseteq X} \int |F(\phi_\Omega)|\, |s_X^\Omega(\phi_\Omega)| \prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}} \cdot \sum_N |q_N^X| \leqslant K_F \sum_{X:\Omega\subseteq X} b'^{|X\setminus\Omega|} T_X^\Omega \cdot C^{|X|} \sum_{n=0}^{\infty} c^n$$

$$\leqslant K_F \sum_{m=0}^{\infty} b'^m e^{\Im(|\Omega|+m-1)} C^{|\Omega|+m} \frac{1}{1-c}$$

$$= K_F e^{\Im(|\Omega|-1)} \frac{C^{|\Omega|}}{(1-bC)(1-c)}, \tag{53}$$

where we have used Lemmas 4.2 and 4.6, Propositions 4.5 and 4.4 and

$$b' = \frac{\sqrt{2\pi}\Im}{|Z|\sqrt{\operatorname{Re}\mu}(\operatorname{Re}\mu - \Im)}, \qquad b = b' e^{\Im}$$

This shows (a) and the convergence of (52). That $\langle F\rangle^{\lambda}$ exists and equals (52) follows from the fact that the right hand side of (51) is just a reordering of the absolutely convergent series (52). □

Corollary 5.2. *Under the conditions of Theorem* 5.1 *on* μ *and* λ, $\langle F\rangle^{\lambda}_{\Lambda}$ *and* $\langle F\rangle^{\lambda}$ *are analytic in* λ *for* λ *in the interior of* S_R.

Proof. The assertion follows from analyticity of the q_N^X's and the integrals $\int F(\phi_\Omega) s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}}$ and uniform convergence of (50), (52) by a standard theorem of complex analysis.

6. Bounds on the derivatives of correlation functions

Proposition 6.1. *For* $|\arg\mu| \leqslant \varphi < \pi/2$ *there are constants* μ_0, R, $0<R<B_1^{-1}(\cos\varphi)^{1/2}$ *and* K_0, K_1 *such that for* $|\mu| \geqslant \mu_0$, $\lambda \in S_R$

$$|D_\lambda^p(Z(\lambda)^{-1})| \leqslant |\mu|^{1/2} K_0 K_1^p p!^d \tag{54}$$

Proof. With

$$\Delta(\lambda) = Z(0)^{-1}(Z(\lambda) - Z(0)) \tag{55}$$

we have

$$Z(\lambda)^{-1} = Z(0)^{-1}(1+\Delta(\lambda))^{-1} \tag{56}$$

By (34), with $R_1 < B_1^{-1}(\cos\varphi)^{1/2}$ we have

$$|\Delta(\lambda)| \leqslant |\lambda| \sup_{\lambda\in S_{R_1}} |D_\lambda \Delta(\lambda)| \leqslant R\,(\cos\varphi)^{-1/2}\left[B_1 + \frac{B_2\, d!}{(\cos\varphi)^d}|\mu|^{-d}\right] \tag{57}$$

for $\lambda \in S_R$, $R \leqslant R_1$ which shows that we can find a μ_0 ensuring that

$$|\Delta(\lambda)| \leqslant c_0 < 1 \tag{58}$$

for $|\mu| \geqslant \mu_0$, $\lambda \in S_{R_1}$. Consequently

$$Z(\lambda)^{-1} = Z(0)^{-1} \sum_{n=0}^{\infty} [-\Delta(\lambda)]^n \tag{59}$$

converges absolutely and uniformly in λ and therefore

$$D_\lambda^p(Z(\lambda)^{-1}) = Z(0)^{-1} \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{p_1,\ldots,p_k \geqslant 0 \\ \sum p_i = p}} \frac{p!}{p_1! \cdot \ldots \cdot p_k!} \prod_{i=1}^{k} D_\lambda^{p_i} \Delta(\lambda) \tag{60}$$

for $|\mu| \geqslant \mu_0$, $\lambda \in S_{R_1}$.

For $p \geqslant 1$ we have

$$\begin{aligned} |D_\lambda^p \Delta(\lambda)| &\leqslant \sqrt{\frac{|\mu|}{\pi}} \int |P(\phi)|^p e^{-\mathrm{Re}\, \mu\phi^2}\, d\phi \\ &\leqslant \sqrt{\frac{|\mu|}{\pi}} \sum_{q=0}^{p} \binom{p}{q} B_1^{p-q} B_2^q (\mathrm{Re}\, \mu)^{-(dq+1/2)} \Gamma(dq + \tfrac{1}{2}) \\ &\leqslant (\cos\varphi)^{-1/2} \sum_{q=0}^{p} \binom{p}{q} B_1^{p-q} B_2^q (|\mu| \cos\varphi)^{-dq} (dq)! \\ &\leqslant (\cos\varphi)^{-1/2} \left[B_1 + \frac{B_2\, d^d}{(\cos\varphi)^d} |\mu|^{-d} \right]^p p!^d \end{aligned} \tag{61}$$

Therefore, choosing C_{μ_0} such that

$$C_{\mu_0} \geqslant (\cos\varphi)^{-1/2} \left[B_1 + \frac{B_2\, d^d}{(\cos\varphi)^d} \mu_0^{-d} \right], \tag{62}$$

we can, using (57), (61), estimate (60) for $R \leqslant R_1$, $|\mu| \geqslant \mu_0$, $p \geqslant 1$ by

$$\begin{aligned} &\sqrt{\frac{|\mu|}{\pi}} \sum_{k=0}^{\infty} \sum_{l=1}^{\min(p,k)} \binom{k}{l} (RC_{\mu_0})^{k-l} \sum_{\substack{p_1,\ldots,p_k \geqslant 1 \\ \sum p_i = p}} \frac{p!}{p_1! \cdot \ldots \cdot p_k!} \prod_{i=1}^{l} C_{\mu_0}^{p_i} p_i!^d \\ &\qquad \leqslant \sqrt{\frac{|\mu|}{\pi}}\, p!\, C_{\mu_0}^p \sum_{k=0}^{\infty} \sum_{l=1}^{\min(k,p)} \binom{k}{l} (RC_{\mu_0})^{k-l} \sum_{\substack{p_1,\ldots,p_k \geqslant 1 \\ \sum p_i = p}} \prod_{i=1}^{l} p_i!^{d-1} \end{aligned} \tag{63}$$

for $\lambda \in S_R$. By

$$\sum_{\substack{p_1,\ldots,p_k \geqslant 1 \\ \sum p_i = p}} 1 = \binom{p-1}{k-1} \leqslant 2^p \tag{64}$$

we have now

$$|D_\lambda^p Z(\lambda)^{-1}| \leqslant \sqrt{\frac{|\mu|}{\pi}}\, p!^d (2C_{\mu_0})^p \sum_{k=0}^{\infty} \sum_{l=1}^{\min(k,p)} \binom{k}{l} (RC_{\mu_0})^{k-l}. \tag{65}$$

Let us consider the two cases
$k \leqslant p$:

$$(2C_{\mu_0})^p \sum_{l=1}^{\min(p,k)} \binom{k}{l}(RC_{\mu_0})^{k-l} \leqslant (2C_{\mu_0})^p(1+RC_{\mu_0})^k \leqslant [2C_{\mu_0}(1+RC_{\mu_0})]^p, \tag{66}$$

$k > p$:
For $R < 1/C_{\mu_0}$

$$\begin{aligned}(2C_{\mu_0})^p \sum_{l=1}^{\min(k,p)} \binom{k}{l}(RC_{\mu_0})^{k-l} &\leqslant (2C_{\mu_0})^p(RC_{\mu_0})^{k-p} \sum_{l=0}^{k} \binom{k}{l}\\ &\leqslant \left(\frac{2}{R}\right)^p (2RC_{\mu_0})^k.\end{aligned} \tag{67}$$

Choosing $C_{\mu_0} \geqslant \frac{1}{2}$ and fulfilling condition (62), R with $R \leqslant R_1$ and $R < 1/2C_{\mu_0}$ we get by (65), (66), (67)

$$\begin{aligned}|D_\lambda^p(Z(\lambda)^{-1})| &\leqslant \sqrt{\frac{|\mu|}{\pi}}\, p!^d\left([2C_{\mu_0}(1+RC_{\mu_0})]^p(p+1) + \left(\frac{2}{R}\right)^p \sum_{k=p+1}^{\infty} (2RC_{\mu_0})^k\right)\\ &\leqslant \sqrt{\frac{|\mu|}{\pi}}\, p!^d\left([4C_{\mu_0}(1+RC_{\mu_0})]^p + 2RC_{\mu_0}(4C_{\mu_0})^p \frac{1}{1-2RC_{\mu_0}}\right)\end{aligned} \tag{68}$$

Therefore

$$|D_\lambda^p(Z(\lambda)^{-1})| \leqslant \sqrt{\frac{|\mu|}{\pi}} \frac{1}{1-2RC_{\mu_0}} [4C_{\mu_0}(1+RC_{\mu_0})]^p p!^d \tag{69}$$

for $\lambda \in S_R$. As

$$|Z(\lambda)^{-1}| \leqslant \sqrt{\frac{|\mu|}{\pi}} \frac{1}{1-RC_{\mu_0}} \tag{70}$$

by Proposition 4.3, (35), (62), the estimate (54) is valid for

$$K_0 = \frac{1}{\sqrt{\pi}(1-2RC_{\mu_0})}, \qquad K_1 = [4C_{\mu_0}(1+RC_{\mu_0})] \quad \square \tag{71}$$

Proposition 6.2. *With* $|\arg \mu| \leqslant \varphi$, φ, μ_0, R *as in Proposition* 6.1 *we have for* $|\mu| \geqslant \mu_0$, $\lambda \in S_R$, $m \in \mathbb{N}$

$$|D_\lambda^p(Z(\lambda)^{-m})| \leqslant |\mu|^{m/2}(2K_0)^m(2K_1)^p p!^d \tag{72}$$

Proof. By Proposition 6.1

$$|D_\lambda^p(Z(\lambda)^{-m})| \leqslant (|\mu|^{1/2}K_0)^m \sum_{\substack{p_1,\ldots,p_m \geqslant 0 \\ \sum p_i = p}} \frac{p!}{p_1! \cdot \ldots \cdot p_m!} \prod_{i=1}^m K_1^{p_i} p_i!^d$$
$$\leqslant (|\mu|^{1/2}K_0)^m K_1^p p! \sum_{l=1}^m \binom{m}{l} \sum_{\substack{p_1,\ldots,p_l \geqslant 1 \\ \sum p_i = p}} \prod_{i=1}^l p_i!^{d-1}$$
$$\leqslant |\mu|^{m/2}(2K_0)^m (2K_1)^p p!^d \tag{73}$$

by (64). □

Proposition 6.3. *For* $|\arg \mu| \leqslant \varphi < \pi/2$ *there are constants* μ_0, K_2, K_3 *such that for* $|\mu| \geqslant \mu_0$, $\operatorname{Re} \lambda \geqslant 0$

(a) $$|D_\lambda^p(Z(\lambda)^{|X|} s_X^{\varnothing})| \leqslant K_2^{|X|} |\mu|^{-|X|/2} \left(\frac{\mathfrak{I}}{|\mu| \cos \varphi - \mathfrak{I}}\right)^{|X|-1} K_3^p p!^d \, T_X^{\{\mathbf{k}\}} \tag{74}$$

for some $\mathbf{k} \in X$ where T_X^Ω is given by (27),

(b) *for any* F *on* $\mathbb{R}^\Omega$, $\Omega \subset \mathbb{Z}^\nu$ *finite,*

$$|F(\phi_\Omega)| \leqslant K_F \prod_{\mathbf{k} \in \Omega} |\phi_\mathbf{k}|^{l_\mathbf{k}}, \tag{75}$$

$$\left| D_\lambda^p(Z(\lambda)^{|X|} \int F(\phi_\Omega) s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k} \in \Omega} d\phi_\mathbf{k} \right| \leqslant K_F \prod_{\mathbf{k} \in \Omega} \left(\frac{l_\mathbf{k}!}{(2\,|\mu| \cos \varphi)^{l_\mathbf{k}}}\right)^{1/2} K_2^{|X|} |\mu|^{-|X|/2}$$
$$\times \left(\frac{\mathfrak{I}}{|\mu| \cos \varphi - \mathfrak{I}}\right)^{|X \setminus \Omega|} K_3^p p!^d \, T_X^\Omega \tag{76}$$

Proof.

$$D_\lambda^p\left(Z(\lambda)^{|X|} \int F(\phi_\Omega) s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k} \in \Omega} d\phi_\mathbf{k}\right) = D_\lambda^p \int F(\phi_\Omega) \sigma_X^\Omega(\phi_X) \prod_{\mathbf{k} \in X} d\phi_\mathbf{k} \tag{77}$$

by (10). Then, by (32),

$$\left| D_\lambda^p\left(Z(\lambda)^{|X|} \int F(\phi_\Omega) s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k} \in \Omega} d\phi_\mathbf{k}\right)\right|$$
$$\leqslant \sum_{\substack{p_\mathbf{k} \geqslant 0, \mathbf{k} \in X \\ \sum p_\mathbf{k} = p}} \frac{p!}{\prod_{\mathbf{k} \in X} p_\mathbf{k}!} \int |F(\phi_\Omega)|\, |\sigma_X^\Omega(\phi_X)| \prod_{\mathbf{k} \in X} |P(\phi)|^{p_\mathbf{k}} \, d\phi_\mathbf{k}$$
$$\leqslant p! \left(\frac{\mathfrak{I}}{\operatorname{Re} \mu - \mathfrak{I}}\right)^{|X \setminus \Omega|} K_F T_X^\Omega$$
$$\times \sum_{\substack{p_\mathbf{k} \geqslant 0, \mathbf{k} \in X \\ \sum p_\mathbf{k} = p}} \prod_{\mathbf{k} \in X} p_\mathbf{k}!^{-1} \int |\phi_\mathbf{k}|^{l_\mathbf{k}} (B_1 + B_2 \phi^{2d})^{p_\mathbf{k}} e^{-(\operatorname{Re} \mu/2)\phi_\mathbf{k}^2} \, d\phi_\mathbf{k} \tag{78}$$

where, of course, $l_{\mathbf{k}}$ is put equal to zero if $\mathbf{k} \notin \Omega$. For integrals of the type occurring in the last expression, we have bounds

$$\int |\phi|^l (B_1 + B_2 \phi^{2d})^p e^{-(\mathrm{Re}\,\mu/2)\phi^2}\, d\phi$$

$$\leqslant \sum_{q=0}^{p} \binom{p}{q} B_1^{p-q} B_2^q \left(\frac{\mathrm{Re}\,\mu}{2}\right)^{-[dq+(l+1)/2]} \Gamma\left(dq + \frac{l+1}{2}\right)$$

$$\leqslant \sqrt{\frac{\pi}{|\mu|}} \frac{l!^{1/2}}{(2\cos\varphi)^{(l+1)/2}} |\mu|^{-l/2} \left[B_1 + \frac{B_2\, d^d}{(2\cos\varphi)^d} |\mu|^{-d}\right]^p p!^d. \tag{79}$$

This leads to

$$\left| D_\lambda^p (Z(\lambda)^{|X|} \int F(\phi_\Omega) s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}} \right|$$

$$\leqslant K_F \prod_{\mathbf{k}\in\Omega} \left(\frac{l_{\mathbf{k}}!}{(2|\mu|\cos\varphi)^{l_{\mathbf{k}}}}\right)^{1/2} \left[(2\cos\varphi)^{-1/2} \sqrt{\frac{\pi}{|\mu|}}\right]^{|X|} T_X^\Omega$$

$$\times \left[B_1 + \frac{B_2\, d^d}{(2|\mu|\cos\varphi)^d}\right]^p p! \sum_{\substack{p_{\mathbf{k}}\geqslant 0, \mathbf{k}\in X \\ \sum p_{\mathbf{k}} = p}} \prod_{\mathbf{k}\in X} p_{\mathbf{k}}!^{d-1}$$

$$\leqslant K_F \prod_{\mathbf{k}\in\Omega} \left(\frac{l_{\mathbf{k}}!}{(2|\mu|\cos\varphi)^d}\right)^{1/2} \left(\sqrt{\frac{2\pi}{\cos\varphi}}\right)^{|X|} |\mu|^{-|X|/2}$$

$$\times \left(2\left[B_1 + \frac{B_2\, d^d}{(2|\mu|\cos\varphi)^d}\right]\right)^p p!^d T_X^\Omega, \tag{80}$$

where we have used again

$$\sum_{\substack{p_1,\ldots,p_m \geqslant 0 \\ \sum p_i = p}} 1 \leqslant \sum_{l=1}^{m} \binom{m}{l} \sum_{\substack{p_1,\ldots,p_l \geqslant 1 \\ \sum p_i = p}} 1 \leqslant 2^{p+m}.$$

(80) implies (76) and, with $\Omega = \mathbf{k}$, $F(\phi_{\mathbf{k}}) \equiv 1$, $\mathbf{k} \in X$, (74) follows.

Lemma 6.4. *For* $|\arg\mu| \leqslant \varphi < \pi/2$ *there are constants* $R > 0$, μ_0, K_4, K_5 *such that for* $|\mu| \geqslant \mu_0$, $\lambda \in S_R$

(a) $$|D_\lambda^p s_X^{\varnothing}| \leqslant K_4^{|X|} \left(\frac{\mathfrak{F}}{|\mu|\cos\varphi - \mathfrak{F}}\right)^{|X|-1} K_5^p p!^d\, T_X^{\{\mathbf{k}\}} \tag{81}$$

for some $\mathbf{k} \in X$.

(b) *for* F *on* $\mathbb{R}^\Omega$ *satisfying* (75)

$$\left| D_\lambda^p \left(\int F(\phi_\Omega)\, s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}} \right) \right| \leqslant C_F K_4^{|X|} \left(\frac{\mathfrak{F}}{|\mu|\cos\varphi - \mathfrak{F}}\right)^{|X\setminus\Omega|} K_5^p p!^d\, T_X^\Omega, \tag{82}$$

where

$$C_F = K_F \prod_{\mathbf{k}\in\Omega} \left(\frac{l_{\mathbf{k}}!}{(2\,|\mu|\cos\varphi)^{l_{\mathbf{k}}}}\right)^{1/2}. \tag{83}$$

Proof.

$$D_\lambda^p\left(\int F(\phi_\Omega)s_X^\Omega(\phi_\Omega)\prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}}\right)$$
$$= \sum_{q=0}^{p}\binom{p}{q} D_\lambda^{p-q}(Z(\lambda)^{-|X|})D_\lambda^q\left(Z(\lambda)^{|X|}\int F(\phi_\Omega)s_X^\Omega(\phi_\Omega)\prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}}\right) \tag{84}$$

Propositions 6.2, 6.3 yield

$$\left|D_\lambda^p\left(\int F(\phi_\Omega)s_X^\Omega(\phi_\Omega)\prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}}\right)\right|$$
$$\leqslant C_F(2K_0K_2)^{|X|}\left(\frac{\mathfrak{I}}{|\mu|\cos\varphi-\mathfrak{I}}\right)^{|X\backslash\Omega|} T_X^\Omega$$
$$\times \sum_{q=0}^{p}\binom{p}{q}(2K_1)^{p-q}K_3^q(p-q)!^d\, q!^d$$
$$\leqslant C_F(2K_0K_2)^{|X|}\left(\frac{\mathfrak{I}}{|\mu|\cos\varphi-\mathfrak{I}}\right)^{|X\backslash\Omega|}(2K_1+K_3)^p p!^d\, T_X^\Omega \tag{85}$$

This proves (82) and (81). □

Lemma 6.5. *For $|\arg\mu|\leqslant\varphi$, φ, R, μ_0, K_5, F as in Lemma 6.4 there is a constant K_6 such that for $|\mu|\geqslant\mu_0$, $\lambda\in S_R$*

$$\text{(a)}\quad \sum_{\substack{X:\Omega\subseteq X\\ |X|=m}}\left|D_\lambda^p\int F(\phi_\Omega)s_X^\Omega(\phi_\Omega)\prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}}\right| \leqslant C_F K_6^m\left(\frac{\mathfrak{I}}{|\mu|\cos\varphi-\mathfrak{I}}\right)^{m-|\Omega|}K_5^p p!^d \tag{86}$$

$$\text{(b)}\quad \sum_{\substack{X:\mathbf{l}\in X\\ |X|=m}}|D_\lambda^p s_X^\varnothing| \leqslant K_6^m\left(\frac{\mathfrak{I}}{|\mu|\cos\varphi-\mathfrak{I}}\right)^{m-1}K_5^p p!^d \tag{87}$$

where $\mathbf{l}\in\mathbb{Z}^\nu$ is fixed.

Proof. By Proposition 4.5 we need only replace K_4 by $K_6=e^{\mathfrak{I}}K_4$.

Lemma 6.6. *For $|\arg\mu|\leqslant\varphi<\pi/2$ and given constants $C_D>1$, $c_D>0$, $K_7>K_5$ we can find $R>0$, μ_0 such that for $|\mu|\geqslant\mu_0$, $\lambda\in S_R$*

$$\sum_{N:|N|=n}|D_\lambda^p q_N^X| \leqslant C_D^{|X|}c_D^n K_7^p p!^d \tag{88}$$

Proof. Up to a point, we follow the proof of Lemma 4.6 closely. By (43) we have

$$\begin{aligned}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} &|D_\lambda^p q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_1,\ldots,\mathbf{k}_n]!^{-1}\\ &\leq \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n} |D_\lambda^p q^{X\setminus\{\mathbf{l}\}}_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_1,\ldots,\mathbf{k}_n]!^{-1}\\ &\quad+\sum_{m=1}^{n}\sum_{q=0}^{p}\binom{p}{q}\Bigg[\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_m}|D_\lambda^q s^{\varnothing}_{[\mathbf{l},\mathbf{k}_1,\ldots,\mathbf{k}_m]}|\,m!^{-1}\\ &\quad\times\sum_{\mathbf{k}_{m+2},\ldots,\mathbf{k}_n}|D_\lambda^{p-q}q^{X\cup\{\mathbf{k}_1,\ldots,\mathbf{k}_m\}}_{[\mathbf{k}_{m+2},\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_{m+2},\ldots,\mathbf{k}_n]!^{-1}\Bigg] \end{aligned}\tag{89}$$

Now we can show by induction on $|X|+n$, that

$$\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}|D_\lambda^p q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_1,\ldots,\mathbf{k}_n]!^{-1}\leq C_D^{|X|+n}c_D'^{n}K_7^p p!^d \tag{90}$$

for any $C_D>1$, $c_D'>0$, $K_7>K_5$, $\lambda\in S_R$ for some $R>0$, $|\arg\mu|\leq\varphi$, provided that $|\mu|$ is greater than some μ_0 depending on C_D, c_D', K_7. With

$$a=\frac{\mathfrak{J}}{|\mu|\cos\varphi-\mathfrak{J}} \tag{91}$$

we have

$$\begin{aligned}\sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n}&|D_\lambda^p q^X_{[\mathbf{k}_1,\ldots,\mathbf{k}_n]}|\,[\mathbf{k}_1,\ldots,\mathbf{k}_n]!^{-1}\\ &\leq C_D^{|X|+n-1}c_D'^{n}K_7^p p!^d+\sum_{m=1}^{n}\sum_{q=0}^{p}\binom{p}{q}K_6^{m+1}a^mK_5^q q!^d\\ &\quad\times C_D^{|X|+n-1}c_D'^{n-m-1}K_7^{p-q}(p-q)!^d\\ &=C_D^{|X|+n-1}c_D'^{n}\left[K_7^p p!^d+\frac{K_6}{c_D'}\sum_{m=1}^{n}\left(\frac{K_6a}{c_D'}\right)^m\sum_{q=0}^{p}\binom{p}{q}K_5^qK_7^{p-q}q!^d(p-q)!^d\right]\\ &=C_D^{|X|+n-1}c_D'^{n}\left[K_7^p p!^d+\left(\frac{K_6}{c_D'}\right)^2a\sum_{m=0}^{n-1}\left(\frac{K_6a}{c_D'}\right)^mK_7^p p!^d\sum_{q=0}^{p}\left(\frac{K_5}{K_7}\right)^q\right]\\ &\leq C_D^{|X|+n-1}c_D'^{n}\left[1+a\left(\frac{K_6}{c_D'}\right)^2\frac{1}{1-\dfrac{K_6a}{c_D'}}\,\frac{1}{1-\dfrac{K_5}{K_7}}\right]K_7^p p!^d \end{aligned}\tag{92}$$

where we have assumed $a<c_D'/K_6$ and used Lemma 6.5b. As we can make a arbitrarily small by making $|\mu|$ large, (92) proves the lemma. □

Now we are ready to prove the main theorem of this section.

Theorem 6.7. *Let F satisfy* (75). *Then, for $|\arg\mu|\leq\varphi<\pi/2$ there are constants $R>0$, μ_0, K such that for $|\mu|\geq\mu_0$, $\lambda\in S_R$*

$$|D_\lambda^p\langle F\rangle_\Lambda^\lambda|\leq K^p p!^d \tag{93}$$

uniformly in Λ *and*

$$|D_\lambda^p\langle F\rangle^\lambda| \leqslant K^p p!^d \tag{94}$$

Proof. By Theorem 5.1

$$\begin{aligned}|D_\lambda^p\langle F\rangle^\lambda| &\leqslant \sum_{q=0} \binom{p}{q} \sum_{m=0}^{\infty} \sum_{\substack{X:\Omega\subseteq X\\ |X|=m}} \left| D_\lambda^{p-q} \int F(\phi_\Omega) s_X^\Omega(\phi_\Omega) \prod_{\mathbf{k}\in\Omega} d\phi_{\mathbf{k}} \right| \\ &\quad \times \sum_{n=0}^{\infty} \sum_{|N|=n} |D_\lambda^q q_N^X| \\ &\leqslant C_F(K_6 C_D)^{|\Omega|} \sum_{q=0}^{p} \binom{p}{q} K_5^{p-q} K_7^q (p-q)!^d\, q!^d \\ &\quad \times \sum_{m=0}^{\infty} \left(\frac{K_6 C_D \mathfrak{F}}{|\mu| \cos\varphi - \mathfrak{F}} \right)^m \sum_{n=0}^{\infty} c_D^n \end{aligned} \tag{95}$$

by Lemmas 6.5a, 6.6. Obviously, this estimate is valid for the finite volume correlation functions, too.

For $c_D < 1$, $|\mu|$ sufficiently large, the series in (95) converge and for some constant C $|D_\lambda^p\langle F\rangle_\Lambda^\lambda|$, $|D_\lambda^p\langle F\rangle^\lambda|$ are bounded by $C(K_5+K_7)^p p!^d$. □

Corollary 6.8. *Let* F, φ, R, μ_0 *be as in Theorem* 6.7, $\{\Lambda_n\}_{n\in\mathbb{N}}$ *as in Theorem* 5.1b. *Then, for* $|\mu| \geqslant \mu_0$, $\langle F\rangle_{\Lambda_n}^\lambda$ *converges to* $\langle F\rangle^\lambda$ *uniformly in* λ *for* $\lambda \in S_R$.

Proof. By Theorem 6.7, $|D_\lambda\langle F\rangle_\Lambda^\lambda|$ is bounded uniformly in Λ, which implies that the $\langle F\rangle_\Lambda^\lambda$ are equicontinuous. Together with pointwise convergence (Theorem 5.1b) this proves the assertion. □

7. The free energy

Concerning the thermodynamic function

$$f_\Lambda(\lambda) = \frac{1}{|\Lambda|} \log \int d\mu_\Lambda^\lambda \tag{96}$$

we shall content ourselves with a few remarks. It follows from (3), that

$$D_\lambda f_\Lambda(\lambda) = \frac{1}{|\Lambda|} \sum_{\mathbf{k}\in\Lambda} \langle P(\phi_{\mathbf{k}})\rangle_\Lambda^\lambda \tag{97}$$

It is not difficult to see, that the thermodynamic limit of (97) exists and equals $\langle P(\phi_{\mathbf{k}})\rangle^\lambda$ for any $\mathbf{k}$, if the cluster expansion converges. The conditions on the sequence $\{\Lambda_n\}_{n\in\mathbb{N}}$ shall have to be chosen somewhat more restrictive (e.g. tending to infinity in the sense of Fisher). We have then

$$f(\lambda) = \lim_{\Lambda\to\infty} f_\Lambda(\lambda) = f(0) + \int_0^\lambda d\lambda \langle P(\phi_{\mathbf{k}})\rangle^\lambda \tag{98}$$

showing that for $f(\lambda)$ analyticity and bounds on derivatives hold as well as for the infinite volume correlation functions.

8. Extension of the domain of analyticity

In this section we shall show, using a scaling trick due to Constantinescu [4], that, with P a polynomial of degree $2d$, μ real the correlation functions can be analytically continued to a sector $\{\lambda \mid |\lambda| \leq R, \ |\arg \lambda| \leq \psi\}$ with $\psi < (d+1)\pi/2$, where ψ can be chosen arbitrarily close to $(d+1)\pi/2$ provided that R and μ, depending on ψ, are small respectively large enough. Throughout the preceding sections we have assumed P real-valued and subject to conditions (2i), (2ii). In this section P's occur which are complex-valued and satisfy for any $\theta < \pi/2$

$$\operatorname{Re}(\lambda P(\phi)) \geq -A|\lambda| \tag{99}$$

for λ in the sector $\{\lambda \mid |\arg| \leq \theta\}$ and some constant A and

$$|P(\phi)| \leq B_1 + B_2\phi^{2d} \tag{100}$$

for some constants B_1, B_2. It is not difficult to convince oneself that the results of the previous sections hold in that case with S_R replaced by $S_R^\theta = \{\lambda \mid |\lambda| \leq R, |\arg \lambda| \leq \theta\}$.

Theorem 8.1. *For μ real*

$$P(\phi) = \phi^{2d} + \sum_{n=1}^{2d-1} \beta_n \phi^n, \qquad \beta_n \in \mathbb{C}, \qquad 1 \leq n \leq 2d-1, \tag{101}$$

$$F(\phi_\Omega) = \prod_{\mathbf{k} \in \Omega} \phi_{\mathbf{k}}^{l_{\mathbf{k}}} \tag{102}$$

$$\psi < (d+1)\frac{\pi}{2} \tag{103}$$

we can find constants R, μ_0 such that for $\mu \geq \mu_0$

(a) *analytic continuations $\widetilde{\langle F \rangle_\Lambda^\lambda}$, $\widetilde{\langle F \rangle^\lambda}$ of $\langle F \rangle_\Lambda^\lambda$, $\langle F \rangle^\lambda$ to S_R^ψ exist,*
(b) *there is a constant K such that for $\lambda \in S_R^\psi$*

$$|D_\lambda^p \widetilde{\langle F \rangle_\Lambda^\lambda}| \leq K^p p!^d \tag{104}$$

$$|D_\lambda^p \widetilde{\langle F \rangle^\lambda}| \leq K^p p!^d \tag{105}$$

Proof. For $\alpha \in \mathbb{C}$, $|\alpha| = 1$ we define

$$P_\alpha(\phi) = \phi^{2d} + \sum_{n=1}^{2d-1} \alpha^{n/2-d} \beta_n \phi^n \tag{106}$$

$$g_\Lambda(\lambda, \alpha) = \frac{\alpha^{l/2} \displaystyle\int F(\phi_\Omega) \exp\left[-\frac{\alpha}{2} \sum_{\mathbf{k},\mathbf{l}} \mathfrak{I}(\mathbf{k}, \mathbf{l}) \phi_{\mathbf{k}} \phi_{\mathbf{l}}\right] \prod_{\mathbf{k} \in \Lambda} e^{-\alpha\mu\phi_{\mathbf{k}}^2 - \lambda P_\alpha(\phi_{\mathbf{k}})}\, d\phi_{\mathbf{k}}}{\displaystyle\int \exp\left[-\frac{\alpha}{2} \sum_{\mathbf{k},\mathbf{l}} \mathfrak{I}(\mathbf{k}, \mathbf{l}) \phi_{\mathbf{k}} \phi_{\mathbf{l}}\right] \prod_{\mathbf{k} \in \Lambda} e^{-\alpha\mu\phi_{\mathbf{k}}^2 - \lambda P_\alpha(\phi_{\mathbf{k}})}\, d\phi_{\mathbf{k}}} \tag{107}$$

where

$$l = \sum_{\mathbf{k}\in\Omega} l_{\mathbf{k}}$$

For $g_\Lambda(\lambda, \alpha)$ we get by the substitution $\alpha^{1/2}\phi \to \phi$ the relation

$$g_\Lambda(\lambda, \alpha) = g_\Lambda(\alpha^{-d}\lambda, 1) \tag{108}$$

$P_\alpha(\phi)$ fulfills condition (99) and (100) for any sector S_R^θ with $\theta < \pi/2$ where A, B_1, B_2 can be chosen so that the inequalities hold uniformly in α for $|\alpha| = 1$. Now, by Corollary 5.2, Theorem 6.7 and the remarks preceding Theorem 8.1 we see that for $|\arg \alpha| \leqslant \theta < \pi/2$ we can find R, μ_0 such that for $\lambda \in S_R^\theta$, $\mu \geqslant \mu_0$ $g_\Lambda(\lambda, \alpha)$ and its thermodynamic limit in the sense of Theorem 5.1b, $g(\lambda, \alpha)$ are analytic in λ and their derivatives have bounds of the form (104), (105). As (108) carries over to $g(\lambda, \alpha)$ this proves the theorem, if we choose $\theta = \psi/(d+1)$ and take into account, that

$$g_\Lambda(\lambda, 1) = \langle F(\phi_\Omega)\rangle_\Lambda^\lambda \quad \text{for} \quad \lambda \in S_R^\theta$$

We obtain constants R, μ_0, K in (104), (105) independent of α because the estimates used to prove Corollary 5.2, Theorem 6.7 depend on $|\mathfrak{I}(\mathbf{k}, \mathbf{l})|$, A, B_1, B_2 only. (We could also use the fact, that we need only a finite set of α's for our analytic continuation.)

9. Borel-summability

Theorem 9.1. *For P, F as in Theorem* 8.1 *there are constants $R > 0$, μ_0 such that for μ, λ real, $\mu \geqslant \mu_0$, $0 < \lambda < R$ $\langle F\rangle_\Lambda^\lambda$, $\langle F\rangle^\lambda$, $f_\Lambda(\lambda)$, $f(\lambda)$ are Borel-summable for $d = 2$, Borel-summable in the generalized sense for $d > 2$.*

Proof. The theorem follows from Theorem 8.1 and a generalized version of Watson's theorem (see [5], chapter XII.4).

Remark. If P is real and satisfies conditions (2i), (2ii) with $d = 2$, analytic continuation is not necessary to prove Borel-summability and P need not be a polynomial. One has only to apply Nevanlinna's theorem [6], which is stronger than Watson's theorem. It yields convergence of the Borel-sum in a circle lying in the right half plane and tangent to the imaginary axis at the origin. This theorem seems to have been rediscovered only recently [7].

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