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A comment on relativistic classical mechanics

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Abstract. The canonical formalism for relativistic classical mechanics described by Horwitz and Piron, Piron and Reuse, and Reuse is studied from a different point of view.

Recently Horwitz and Piron [1], Piron and Reuse [2], and Reuse [3] described a canonical formalism for the relativistic classical mechanics of many particles. Within the given approach the authors discussed the evolution equations for a charged particle in an electromagnetic field and the relativistic two-body problem.

In the present paper we describe how the evolution equations can be obtained from a different approach. We compare both approaches. For the sake of completeness we also consider non-relativistic classical mechanics.

First of all let us briefly describe the approach due to Horwitz and Piron [1] (compare also Reuse [3]). In order to obtain the evolution equations, the authors consider the differential 1-form

$$\alpha = \sum_{i=1}^{N} \sum_{j=1}^{3} p_{ij}\, dq_{ij} - \sum_{i=1}^{N} E_i\, dt_i - K(p_{11}, \ldots, t_N)\, d\lambda, \tag{1}$$

where N is the number of particles. Given a closed curve C in $\Gamma = \{p_{11}, \ldots, p_{N3}, q_{11}, \ldots, q_{N3}, E_1, \ldots, E_N, t_1, \ldots, t_N, \lambda\}$ the requirement that the integral

$$\oint_C \alpha \tag{2}$$

is invariant for any continuous deformations of C obtained by arbitrary displacements of its points along the trajectories leads to the canonical equations

$$\begin{aligned} \frac{dp_{ij}}{d\lambda} &= -\frac{\partial K}{\partial q_{ij}} \\ \frac{dE_i}{d\lambda} &= \frac{\partial K}{\partial t_i} \\ \frac{dq_{ij}}{d\lambda} &= \frac{\partial K}{\partial p_{ij}} \\ \frac{dt_i}{d\lambda} &= -\frac{\partial K}{\partial E_i}. \end{aligned} \tag{3}$$

λ is called the historical time.

We mention that in non-relativistic classical mechanics the differential 1-form under consideration is

$$\alpha = \sum_{i=1}^{3} \sum_{i=1}^{3} p_{ij}\, dq_{ij} - H(p_{11}, \ldots, q_{N3})\, dt. \tag{4}$$

The equations of motion are

$$\begin{aligned} \frac{dp_{ij}}{dt} &= -\frac{\partial H}{\partial q_{ij}} \\ \frac{dq_{ij}}{dt} &= \frac{\partial H}{\partial p_{ij}}. \end{aligned} \tag{5}$$

In the following we deal with differential forms and vector fields and with the concept of the Lie derivative of a differential form (or a vector field) with respect to a vector field [4]. Throughout invariance conditions are formulated with the help of the Lie derivative.

Consider now a different approach for obtaining the canonical equations in relativistic classical mechanics. For the sake of simplicity we consider first of all the case with one particle. Later on we give the extension to more than one particle.

The states of a particle are described by four independent variables, namely (p, q, E, t). Let

$$\omega = dp \wedge dq - dE \wedge dt. \tag{6}$$

$\wedge$ denotes the exterior product. ω is a differential 2-form (in short 2-form). Let

$$V = V_1(p, q, E, t)\frac{\partial}{\partial p} + V_2(p, q, E, t)\frac{\partial}{\partial q} + V_3(p, q, E, t)\frac{\partial}{\partial E} + V_4(p, q, E, t)\frac{\partial}{\partial t} \tag{7}$$

be an analytic vector field defined on $\mathbf{R}^4$. In physics V is called an infinitesimal generator. The vector field V generates via the Lie series

$$\exp(\lambda V)\begin{pmatrix} p \\ q \\ E \\ t \end{pmatrix} \tag{8}$$

(parameter $\lambda \in I \subset \mathbf{R}$) the solution (local group of local transformations) of the autonomous system of differential equations

$$\begin{aligned} \frac{dp}{d\lambda} &= V_1(p, q, E, t), & \frac{dE}{d\lambda} &= V_3(p, q, E, t) \\ \frac{dq}{d\lambda} &= V_2(p, q, E, t), & \frac{dt}{d\lambda} &= V_4(p, q, E, t). \end{aligned} \tag{9}$$

We emphasize that the parameter λ which has the dimension of time is the parameter of evolution for the entire system.

Let us assume that

$$L_V \omega = 0. \tag{10}$$

$L_V\omega$ denotes the Lie derivative of the differential form ω with respect to the vector field V. Condition (10) means that the differential form ω is invariant with respect to V and therefore does not change as it propagates down the trajectories of V [4]. In the following we use the identity $L_V\omega \equiv d(V \lrcorner \omega) + V \lrcorner d\omega$, where $d\omega$ denotes the exterior derivative of the differential form ω and $V \lrcorner \omega$ is the contraction of ω by V. Using the invariance requirement of the differential form ω with respect to V we derive the equations of motion. Since $d\omega = 0$, it follows from equation (10) that

$$d(V \lrcorner \omega) = 0. \tag{11}$$

The converse of the Poincaré lemma tells us that there exists at least locally a function K such that

$$V \lrcorner \omega = -dK \tag{12}$$

We mention that the exterior derivative is a linear operation and for an arbitrary differential form, say γ, we have $dd\gamma = 0$.

Let

$$X = ct\frac{\partial}{\partial q} + q\frac{\partial}{c\,\partial t} \tag{13}$$

and

$$Y = cp\frac{\partial}{\partial E} + E\frac{\partial}{c\,\partial p}. \tag{14}$$

X and Y are the infinitesimal generators of the Lorentz transformation. We mention that $L_{(X+Y)}\omega = 0$. The requirement that K is invariant under the Lorentz transformation can be expressed as $L_{(X+Y)}K = 0$.

Now the equation $V \lrcorner \omega = -dK$ yields

$$\begin{aligned} V_1 &= -\frac{\partial K}{\partial q}, & V_2 &= \frac{\partial K}{\partial p} \\ V_3 &= \frac{\partial K}{\partial t}, & V_4 &= -\frac{\partial K}{\partial E}. \end{aligned} \tag{15}$$

Inserting equation (15) into equation (9) we obtain the equations of motion

$$\begin{aligned} \frac{dp}{d\lambda} &= -\frac{\partial K}{\partial q}, & \frac{dq}{d\lambda} &= \frac{\partial K}{\partial p} \\ \frac{dE}{d\lambda} &= \frac{\partial K}{\partial t}, & \frac{dt}{d\lambda} &= -\frac{\partial K}{\partial E}. \end{aligned} \tag{16}$$

As a consequence we find that $L_VK \equiv V(K) = 0$. In other words

$$\frac{d}{d\lambda}K = 0. \tag{17}$$

This means K is a constant of motion.
Moreover,

$$L_{(X+Y)}V \equiv [X+Y, V] = 0, \tag{18}$$

where [,] denotes the Lie bracket (commutator). $L_{(X+Y)}V=0$ means that the equations of motion are invariant under the Lorentz transformation as it must be.

The extension of the described approach to more than one coordinate (in q and p) and more than one particle is straightforward. Let N be the number of particles. The states of a particle are described by eight independent variables $(p_{n1}, p_{n2}, p_{n3}, q_{n1}, q_{n2}, q_{n3}, t_n, E_n)$, where $n=1,\ldots,N$. Hence the phase space under consideration is $\mathbf{R}^{8N}$. We now consider the 2-form

$$\omega=\sum_{i=1}^{N}\sum_{j=1}^{3} dp_{ij}\wedge dq_{ij}-\sum_{i=1}^{N} dE_i\wedge dt_i \tag{19}$$

and the vector field

$$V=\sum_{i=1}^{N}\sum_{j=1}^{3}\left(V_{ij}\frac{\partial}{\partial p_{ij}}+W_{ij}\frac{\partial}{\partial q_{ij}}\right)+\sum_{i=1}^{N}\left(R_i\frac{\partial}{\partial E_i}+S_i\frac{\partial}{\partial t_i}\right). \tag{20}$$

Again we require that $L_V\omega=0$. As a consequence we obtain $d(V\rfloor\omega)=0$. It follows that there is at least locally a 0-form K such that $V\rfloor\omega=-dK$. We require Lorentz invariance of K. Again we have $L_VK\equiv V(K)=0$. Moreover, let

$$\Omega=\prod_{i=1}^{N}\left(\left(\prod_{j=1}^{N} dp_{ij}\wedge dq_{ij}\right)\wedge dE_i\wedge dt_i\right) \tag{21}$$

be the volume element of $\mathbf{R}^{8N}$. Then $L_V\Omega=0$. This is Liouville's theorem.

In summary we state that in the approach due to Horwitz and Piron [1] the equations of motion are determined by

$$Z\rfloor d\alpha=0, \tag{22}$$

where α is given by equation (1) and Z takes the form

$$Z=\sum_{i=1}^{N}\sum_{j=1}^{3}\left(V_{ij}\frac{\partial}{\partial p_{ij}}+W_{ij}\frac{\partial}{\partial q_{ij}}\right)+\sum_{i=1}^{N}\left(R_i\frac{\partial}{\partial E_i}+S_i\frac{\partial}{\partial t_i}\right)+\frac{\partial}{\partial\lambda}. \tag{23}$$

The condition (22) means that the differential form α is a relative integral invariant of Z [4]. In the present approach the invariance condition $L_V\omega=0$ is the starting point, where V is given by equation (20) and ω is given by equation (19). As a consequence we find $V\rfloor\omega=-dK$. In the approach due to Horwitz and Piron [1] the starting point is the differential form α which contains the Lorentz scalar $K(p, q, E, t)$. In the present approach the Lorentz scalar $K(p, q, E, t)$ is a consequence of the invariance condition $L_V\omega=0$ and that the differential form ω is invariant under Lorentz transformation. We mention that in non-relativistic classical mechanics both approaches are well known [5].

In the appendix we briefly mention the non-relativistic case.

Appendix

For the sake of completeness let us briefly describe the non-relativistic case. Here we consider the phase space $\mathbf{R}^{6N}=(p_{11},\ldots,p_{N3}, q_{11},\ldots,q_{N3})$ and the two form

$$\omega = \sum_{i=1}^{N} \sum_{j=1}^{3} dp_{ij} \wedge dq_{ij},$$

where N denotes the number of particles. ω is called a symplectic form and $(\mathbf{R}^{6N}, \omega)$ is called a symplectic manifold. Let

$$V = \sum_{i=1}^{N} \sum_{j=1}^{3} \left(V_{ij} \frac{\partial}{\partial p_{ij}} + W_{ij} \frac{\partial}{\partial q_{ij}} \right).$$

The requirement that the two-form is invariant under V, i.e. $L_V \omega = 0$, leads to $d(V \lrcorner \omega) = 0$. Again we apply Poincaré's lemma. Consequently, there is a 0-form H such that $V \lrcorner \omega = -dH$. The function H is determined up to an additive constant and is called the Hamiltonian. The condition $V \lrcorner \omega = -dH$ leads to the Hamilton equations.

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