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Autor: Koch, H.
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Particles exist in the low temperature φ_2^4 model

by **H. Koch**

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland

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Abstract. The existence of an upper mass gap and of a two particle bound state are shown for the $\lambda\varphi^4 - \frac{1}{4}\varphi^2$ field theory in two dimensions. The proof is based on a resummation of the low temperature cluster expansion which can be combined with Spencer's expansion in order to obtain decay properties of irreducible kernels.

I. Introduction and main results

Once the existence of some quantum field model is established (in the sense of [OSch]), it is natural to analyze physically relevant quantities such as the mass spectrum, the S-matrix etc. This has for example been done for $\lambda\mathfrak{P}(\varphi)_2 + \frac{1}{2}m_0^2\varphi^2$ models for small coupling λ [S], [SZ], [DE], [OS], [K], [GJ] after Glimm, Jaffe and Spencer [GJS I, II] had constructed these models by introducing the cluster expansion.

In the meantime much progress has been made in understanding field theories at low temperatures λ^{-1}. For the two dimensional $\lambda\varphi^4 - \frac{1}{4}\varphi^2 - \mu\varphi$ models with $|\mu| < \lambda^2 \ll \lambda$ (which is for $\mu = 0$ equivalent to a $\lambda'\varphi^4 + \frac{1}{2}\varphi^2$ model with $\lambda' \gg 1$), Glimm, Jaffe and Spencer developed a convergent expansion which establishes existence and mass gap of each pure phase associated to a minimum of the polynomial. Their methods have also been applied to other low temperature models [Br], [BF], [BG], [Su].

In this paper we study the mass spectrum in a pure phase of the Euclidean $\mathfrak{P}(\varphi) = \lambda\varphi^4 - \frac{1}{4}\varphi + (64\lambda)^{-1}$ model with $0 \le \lambda \ll 1$. Our method is a minimal synthesis of methods used in the one phase region [S], [K], with low temperature expansions [GJS III], [BG]. This involves a proof of $n+1$ particle decay for n particle irreducible kernels, $n = 0, 1, 2$. The upper gap then follows by a result of Burnap [B] which is model independent.

In the two pure phases the mean of the field φ takes a value near $\pm\xi$, where $\mathfrak{P}$ has its minimum, $\xi = (8\lambda)^{-1/2}$. In both cases the interaction is $\int_\Lambda d^2x : \mathfrak{P}(\varphi) : (x)$ in the limit $\Lambda \to \mathbb{R}^2$. We will choose the + phase by taking a large region $Y \supset \Lambda$ and imposing suitable boundary conditions on the field in the strip $Y \backslash \Lambda$.

The polynomial $\mathfrak{P}$ can be written in terms of the translated field $\phi = \varphi - \xi$,

$$\mathfrak{P}(\varphi) = \lambda\phi^4 + (2\lambda)^{1/2}\phi^3 + \tfrac{1}{2}\phi^2 = V(\phi) + \tfrac{1}{2}\phi^2.$$

Let

$$V(\Lambda)=\int_{\Lambda} d^2x : V(\phi):(x)$$

where : : denotes Wick ordering with respect to the covariance $C=(-\Delta_{\partial Y}+1)^{-1}$ with $\Delta_{\partial Y}$ being the Laplacean with zero Dirichlet data on ∂Y. Then we define the $\lambda\varphi^4-\frac{1}{4}\varphi^2-(64\lambda)^{-1}$ quantum field theory by the family of Schwinger functions

$$S(x_1,\ldots,x_n)=\langle\phi(x_1)\cdots\phi(x_n)\rangle=\frac{\int d\phi_C\phi(x_1)\cdots\phi(x_n)e^{-V(\Lambda)}}{\int d\phi_C e^{-V(\Lambda)}}$$

in the limit $\Lambda\to\mathbb{R}^2$. As usually $d\phi_C$ denotes the Gaussian process with mean zero and covariance C. A condition on the sequence Λ, Y is that $Y\supset N(\Lambda)$, where for $X\subset\mathbb{R}^2$

$$N(X)=\{x\in\mathbb{R}^2 : \text{dist}\,(x,X)\leq L\}$$

and $L\in\mathbb{N}$, $L\simeq(\log\lambda)^2$ is a given length scale.

Our main result is as follows.

Theorem 1. *Fix $\varepsilon>0$. Then there is a $\delta>0$ such that the following is true.*

(a) *The infinite volume Schwinger functions exist and are analytic in λ for $|\text{Im}\,\lambda|<\delta\ \text{Re}\,\lambda<\delta^3$.*

(b) *They are also C^∞ in $\lambda^{1/2}$ for $\lambda^{1/2}\in[0,\delta]$ and define a Wightman field theory.*

(c) *For $\lambda^{1/2}\in[0,\delta]$ the mass spectrum in the interval $[0,2(1-\varepsilon)]$ consists of two points, 0 and m, where $|m-1|<\varepsilon$, corresponding to the vacuum and the one particle state respectively. Furthermore $m=m(\lambda^{1/2})$ is C^∞ in $\lambda^{1/2}$ and*

$$m^2(\lambda^{1/2})=1-4\sqrt{3}\,\lambda+O(\lambda^2).$$

(d) *If $\lambda^{1/2}\in[0,\delta]$ then there is a two particle bound state with mass $m_B\in(2(1-\varepsilon),2m)$. Furthermore $m_B=m_B(\lambda^{1/2})$ is C^∞ in $\lambda^{1/2}$ and*

$$m_B^2(\lambda^{1/2})=4m^2(\lambda^{1/2})-144\lambda^2+O(\lambda^{5/2}).$$

Let P be the spectral projection of the mass operator associated to a subinterval of $(0,2m)$ in the complement of $\{m,m_B\}$. Then $P\varphi(f)\times(1+\varphi(g))\Omega=0$ for any pair f, g of C^∞ functions with compact support.

The first part (a) and (b) of this theorem has already been proven by [GJS III], even for small external field. Our analysis differs on this level only by the way of dividing unnormalized expectations by the partition function.

Differentiability with respect to $\lambda^{1/2}$ can be shown as usually by introducing cutoffs. The explicit formula is

$$\begin{aligned}\partial_{\lambda^{1/2}}\langle\Phi\rangle=-\int d^2x[&\langle\Phi\cdot(2\lambda^{1/2}:\phi^4:(x)+\sqrt{2}:\phi^3:(x))\rangle\\&-\langle\Phi\rangle\cdot\langle 2\lambda^{1/2}:\phi^4:(x)+\sqrt{2}:\phi^3:(x)\rangle].\end{aligned}\tag{I.1}$$

The proof of (c) and (d) is based on decay properties of the following irreducible kernels. Denote by $S^1_{k,l}$ the integral operator defined by the kernel

$$S^1_{k,l}(x_1,\dots,x_k;y_1,\dots,y_l)=S(x_1,\dots,y_l)-S(x_1,\dots,x_k)S(y_1,\dots,y_l)$$

and denote by $R_{21}(x_1,x_2;y)$ the truncated three point function. The n particle irreducible (n p.i.) kernels $S^{n+1}_{k,l}$ are inductively defined by

$$S^{n+1}_{k,l}=S^n_{k,l}(S^n_{n,n})^{-1}S^n_{n,l}. \tag{I.2}$$

Three other kernels are needed for our analysis. Namely the 1 p.i. two point function k, the 2 p.i. three point function L and the Bethe-Salpeter kernel K. They are defined by the equations

$$\begin{aligned}&S^1_{1,1}=C-CkS^1_{1,1},\\&R_{21}=S^2_{2,2}LS^1_{1,1},\\&S^2_{2,2}=2S^1_{1,1}\otimes S^1_{1,1}-2(S^1_{1,1}\otimes S^1_{1,1})KS^2_{2,2}.\end{aligned} \tag{I.3}$$

We intend to prove the following proposition.

Proposition 2. *Let* $p>1$ *and* $\varepsilon>0$ *be given. Then for* $\lambda>0$ *sufficiently small,*

(a) $S^n_{k,l}(x_1,\dots,x_k;y_1,\dots,y_l)=O_p(\exp(-(1-\varepsilon)\min_{i,j}|x^0_i-y^0_j|)), \qquad n=1,2,3$

(b) $k(x;y)=O_\infty(\exp(-2(1-\varepsilon)|x^0-y^0|))$

(c) $L(x_1,x_2;y)=O_\infty(\exp(-(1-\varepsilon)[\frac{3}{2}|2y^0-x^0_1-x^0_2|+\frac{1}{2}|x^0_1-x^0_2|]))$

(d) $K(x_1,x_2;y_1,y_2)=O_\infty(\exp(-(1-\varepsilon)[\frac{3}{2}|y^0_1+y^0_2-x^0_1-x^0_2|+\frac{1}{2}|x^0_1-x^0_2|+\frac{1}{2}|y^0_1-y^0_2|]))$

where $g(\mathbf{z})=O_p(h(\mathbf{z}))$ *means that* $|\int d\mathbf{z}g(\mathbf{z})h(\mathbf{z})^{-1}|\le \text{const.}\,\|g\|_{L_p}$ *for every continuous function* f *with support in a product of unit squares* $\square_j=[j^0,j^0+1]\times[j^1,j^1+1]$.

These bounds can now be used to prove the second half of Theorem 1. By a result of Burnap [B] the decay of $S^n_{k,l}$ for $n=1,2$ and of K imply the upper mass gap. This, together with the remaining decay properties and differentiability in $\lambda^{1/2}$, allows to analyze the bound state spectrum exactly as in the $\lambda\mathfrak{P}(\varphi)_2$ case, $\lambda\ll 1$. However, the absence of other spectrum in $(2(1-\varepsilon),2m)$ does not follow. This could probably be analyzed, as in the weak coupling region, by an n particle cluster expansion [GJS II]. It is replaced by the weaker statement in Theorem 1(d).

We shall now prepare the proof of Proposition 2. In a first step we combine the Peierls expansion with Spencer's t-expansion [S II]. Let $l\simeq|\log\lambda|^{1/4}$, $l\in\mathbb{N}$ such that $L\simeq(\log\lambda)^2$ is a multiple of l. By Δ we denote squares $l\times l$ with corners in $l\mathbb{Z}^2$, while unit squares with corners in $\mathbb{Z}^2$ will be denoted by $\square$. We suppose Λ and Y to be simply connected unions of squares Δ.

In the field theoretic version of the Peierls expansion [GJS III] the mean field $\square\mapsto\bar\varphi_\square=\int_\square d^2x\varphi(x)$ plays the role of an Ising spin variable $\square\mapsto\Sigma(\square)=\pm$ associated to unit squares $\square$ in Λ. Let $1=\chi_++\chi_-$ be the smooth partition of unity defined by

$$\chi_+(x)=\chi_-(-x)=\pi^{-1/2}\int_0^\infty d^2ye^{-(x-y)^2}.$$

Then we split the measure $e^{-V}\,d\varphi$ into $2^{|\Lambda|}$ parts by inserting

$$1=\prod_{\square\subset\Lambda}[\chi_+(\bar\varphi_\square)+\chi_-(\bar\varphi_\square)]=\sum_{\Sigma\in\{+,-\}^{|\Lambda|}}\chi_\Sigma \tag{I.4}$$

where

$$\chi_\Sigma=\prod_{\square\subset\Lambda}\chi_{\Sigma(\square)}(\bar\varphi_\square).$$

Each term in the sum (I.4) represents a partition of Λ into pure phases, i.e. regions where Σ has a definite sign. By setting $\Sigma(\square)=+$ for $\square\subset Y\backslash\Lambda$ this sign is uniquely determined by the phase boundaries, which we will also denote by Σ.

Notice that it is very improbable for the field φ to take values near zero, or to change from $\pm\xi$ to $\mp\xi$. This will make the Σ expansion converge. On the other hand because of the factor χ_Σ the effective potential in a pure phase is near to $\frac{1}{2}(\varphi\pm\xi)^2$. This will be used to perform a cluster expansion in the region $Y\backslash N(\Sigma)$. For this purpose we translate the field $\phi=\varphi-\xi$ by a C^∞ function $\xi-g$ satisfying

$$g(x)=\Sigma(\square)\cdot\xi \quad\text{for}\quad x\in\square \quad\text{with}\quad \operatorname{dist}(\square,\Sigma)>\tfrac{1}{2}L.$$

For the explicit form of g see [GJS III]. The expansion in phase boundaries for $\langle\Phi\rangle$ is then as follows.

$$\langle\Phi\rangle=\frac{\sum_\Sigma\int d\psi_C e^{-F(\Lambda,\Sigma)}\Phi\chi_\Sigma e^{-V(\Lambda)}}{\sum_\Sigma\int d\psi_C\chi_\Sigma e^{-V(\Lambda)-F(\Lambda,\Sigma)}} \tag{I.5}$$

where $\psi=\varphi-g$ and

$$\begin{aligned}F(\Lambda,\Sigma)&=\log\frac{d\psi_C}{d\phi_C}\\&=\tfrac{1}{2}\langle(\xi-g),(-\Delta+1)(\xi-g)\rangle+\langle\phi,(-\Delta+1)(\xi-g)\rangle.\end{aligned} \tag{I.6}$$

This is the starting point for the cluster expansion which will be defined later and which allows to take the limit $\Lambda\to\mathbb{R}^2$.

We shall now illustrate Spencer's t-expansion. It was designed to prove Proposition 2 in the one phase region. Let $\Delta_\mathfrak{L}$ be the Laplacean with zero Dirichlet data on a closed curve $\mathfrak{L}$ in $\mathbb{R}^2\cup\{\infty\}$, and for $t\in[0,1]$ let

$$C(t)=(1-t)(-\Delta_\mathfrak{L}+1)^{-1}+t\cdot(-\Delta+1)^{-1}.$$

Notice that $C(t;x,y)=O(e^{-(1-\varepsilon)|x-y|})$, and $C(t;x,y)=0$ if x and y are separated by $\mathfrak{L}$. Then for every $G_1,G_2\subset\mathbb{R}^2$ separated by $\mathfrak{L}$ and for each finite set A of pairs $a=(x,y)$ with $x,y\in G_1\cup G_2$ (for simplicity we assume $|x-y|\geq$ const.) let us define $K_t=\prod_{a\in A}C(t;a)$. It is easy to see that

$$\partial_t^\alpha K_t|_{t=0}=0 \quad\text{for}\quad \alpha<n \tag{I.7}$$

implies

$$|K_t|\leq O(1)^{|A|}\exp(-n(1-\varepsilon)\operatorname{dist}(G_1,G_2)). \tag{I.8}$$

But if we take A such that K_t is a graph of a $\mathfrak{P}(\varphi)_2$ theory, then (I.7) $\Rightarrow$ (I.8) is in general not true. In this case irreducibility should be checked at each line $\mathfrak{L}$ separating G_1 from G_2. However if the vertices produce small contributions, then it is sufficient to consider a certain discrete set of lines $\mathfrak{L}_i$, $i \in I$, depending on G_1 and G_2. For $t = (t_i)_{i\in I}$ the corresponding covariance is defined by

$$C(t) = \prod_{i\in I} [(1-t_i)\theta_{\mathfrak{L}_i} + t_i](-\Delta_{\partial y} + 1)^{-1}$$

where $\theta_{\mathfrak{L}}(-\Delta_\Gamma + 1)^{-1} = (\Delta_{\Gamma\cup\mathfrak{L}} + 1)^{-1}$. t-derivatives are indexed by functions $\alpha : I \to \mathbb{Z}_+$,

$$\partial_t^\alpha K_t = \prod_{i\in I} \partial_{t_i}^{\alpha(i)} K_t.$$

It was a goal of Spencer [S] to establish a relation analogous to (I.7) $\Rightarrow$ (I.8) for kernels K_t of a weakly coupled $\lambda\mathfrak{P}(\varphi)_2 + \frac{1}{2}m_0^2\varphi^2$ theory.

In trying to apply the same method to our problem, the following difficulty arises. In order to show (I.7) using an explicit formula for t derivatives and a factorization property (see (I.13), (I.14)), that t dependent covariance has to be introduced before doing the translation $\phi \to \psi$ (notice that g(x) depends on $\Sigma(\square)$ if dist $(x, \square) < \frac{1}{2}L$). But if we do so it is impossible to translate the field in the case where $\xi - g \neq 0$ on $\mathfrak{L}_i$ and $t_i \neq 1$.

This problem can be solved by taking advantage of the small factors associated to nonempty intersections $\Sigma \cap N(\mathfrak{L}_i)$, so that we need not to consider $t_i \neq 1$. We perform a resummation of the Peierls expansion (r-expansion) by specifying phase boundary free region ($r_i = 0$), where Spencer's expansion can be applied. In the complement of these pure phases, small factos (r-derivatives) associated to phase boundaries will compensate the missing convergence factors from t derivatives.

Let $\mathfrak{I}$ be a finite subset of $l\mathbb{Z}$, to be determined later. To each $i \in \mathfrak{I}$ we associate the Dirichlet lines $\mathfrak{L}_i = \{(x^0, x^1) \in \mathbb{R}^2 : x^0 = i\}$ and an expansion in two variables

$$F(r_i = 1, t_i = 1) = F(0, 0) + \int_0^1 dr_i \partial_{r_i} F(r_i, 1) + \int_0^1 dt_i \partial_{t_i} F(0, t_i) \tag{I.9}$$

applied to the expectation

$$\langle\Phi\rangle_{r,t} = \frac{\sum_\Sigma \prod_{i\in\mathfrak{I}} r_i^{|\Sigma\cap N(\mathfrak{L}_i)|} \int d\phi_{C(t)} \Phi \chi_\Sigma e^{-V(\Lambda)}}{\sum_\Sigma \prod_{i\in\mathfrak{I}} r_i^{|\Sigma\cap N(\mathfrak{L}_i)|} \int d\phi_{C(t)} \chi_\Sigma e^{-V(\Lambda)}} \tag{I.10}$$

where $0^0 = 1$, $r = r^{\mathfrak{I}} = (r_i)_{i\in\mathfrak{I}}$ and $t = t^{\mathfrak{I}} = (t_i)_{i\in\mathfrak{I}}$. Notice that setting $r_i = 0$ eliminates all terms in the sum over phase boundaries for which $\xi - g$ is not zero in a neighbourhood of $\mathfrak{L}_i$. Thus in the range of parameters r, t used in the expansion

(I.9) the field ϕ can be translated by $\xi - g$ as before, with

$$\log \frac{d\psi_{C(t)}}{d\phi_{C(t)}} = F(\Lambda, \Sigma)$$

given by (I.6) independent of t.

We now define the r, t-dependent irreducible kernels by the same Neumann series as in [K] in terms of C(t) and partially amputated Schwinger functions

$$S(\mathbf{x}_1, \ldots, \mathbf{x}_m, y_1, \ldots, y_n)_{r,t} = \frac{\sum_{\Sigma} \prod_{i \in \mathfrak{I}} r_i^{|\Sigma \cap N(\mathfrak{L}_i)|} \int d\phi_{C(t)} \prod_{i=1}^{m} \frac{d}{d\phi(x_i)} \prod_{j=1}^{n} \phi(y_j) \chi_\Sigma e^{-V(\Lambda)}}{\sum_{\Sigma} \prod_{i \in \mathfrak{I}} r_i^{|\Sigma \cap N(\mathfrak{L}_i)|} \int d\phi_{C(t)} \chi_\Sigma e^{-V(\Lambda)}} \tag{I.12}$$

They have the factorization property

$$\begin{aligned} &S(\mathbf{x}_1, \ldots, \mathbf{x}_{m+m'}, y_1, \ldots, y_{n+n'})_{r,t} \\ &\quad = S(\mathbf{x}_1, \ldots, \mathbf{x}_m, y_1, \ldots, y_n)_{r,t} S(\mathbf{x}_{m+1}, \ldots, \mathbf{x}_{m+m'}, y_{n+1}, \ldots, y_{n+n'})_{r,t} \end{aligned} \tag{I.13}$$

if $\mathfrak{L}_i$ separates $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ from $\{x_{m+1}, \ldots, x_{m+m'}, y_{n+1}, \ldots, y_{n+n'}\}$ and $r_i = t_i = 0$. To calculate t-derivatives we use the following formula

$$\begin{aligned} &\partial_{t_i} S(\mathbf{x}_1, \ldots, \mathbf{x}_m, y_1, \ldots, y_n)_{r,t} \\ &\quad = \tfrac{1}{2} \int d^4z S(\mathbf{z}_1, \mathbf{z}_2; \mathbf{x}_1, \ldots, \mathbf{x}_m, y_1, \ldots, y_n)_{r,t} \partial_{t_i} C(t; z_1, z_2) \end{aligned} \tag{I.14}$$

where; denotes truncation. This formula is proved as usually by introducing cutoffs. It is now possible to repeat the same calculations as in [K, Section III]. This shows formally

Lemma 3. *Let* $x_1^0 < x_2^0 < y_1^0 < y_2^0$ *and let* $\alpha : l\mathbb{Z} \to \{0, 1, 2\}$ *be given such that* $\alpha(i) \leq 1$ *if* $x_1^0 < i < x_2^0$ *or* $y_1^0 < i < y_2^0$. *Then*

$$\prod_{x_1^0 < i < x_2^0} \partial_{t_i}^{\alpha(i)} k(x_1; x_2)_{r,t} \big|_{t_i = 0} = 0,$$

$$\prod_{x_1^0 < i < y_1^0} \partial_{t_i}^{\alpha(i)} L(x_1, x_2; y_1)_{r,t} \big|_{t_i = 0} = 0$$

and

$$\prod_{x_1^0 < i < y_2^0} \partial_{t_i}^{\alpha(i)} K(x_1, x_2; y_1, y_2)_{r,t} \big|_{t_i = 0} = 0,$$

where $r_i = 0$ *whenever* $t_i \neq 1$. *Under the same condition on* (r, t) *and for* $n = 1, 2, 3$ *we have*

$$\prod_{x_k^0 < i < y_1^0} \partial_{t_i}^{\alpha(i)} S_{k,l}^n(x_1, \ldots, x_k; y_1, \ldots, y_l)_{r,t} \big|_{t_i = 0} = 0$$

if $\alpha(i) < n$ *and* $x_1^0 < \cdots < x_k^0 < y_1^0 < \cdots < y_l^0$.

Suppose now that for a certain kernel $K_{r,t}(X; Y)$ we have shown that

$$\partial_t^\alpha K_{r,t}(X; Y)\,|_{r^I=t^I=0} = 0$$

for every $I \subset \mathfrak{J}$ and every $\alpha : I \to \mathbb{Z}_+$ which is pointwise smaller than some $\beta = \beta(K)$. The r-t-expansion for $K(X; Y) = K_{1,1}(X; Y)$ is then

$$K(X, Y) = \sum_{\substack{I \cup J = \mathfrak{J} \\ I \cap J = \emptyset}} \int_{\{0\}}^{\{1\}} dr^J \int_{\{0\}}^{\{1\}} dt^I \prod_{i \in I} \frac{(1-t_i)^{\beta(i)-1}}{(\beta(i)-1)!} \partial_r^J \partial_t^\beta K_{r,t}(X; Y)\,\Big|_{\substack{r^I=0 \\ t^J=1}} \tag{I.15}$$

For the kernels we are interested in, $\mathfrak{J}$ is given by the set of i's considered in the previous lemma, $\mathfrak{J} = l\mathbb{Z} \cap (x^0, y^0)$ for some interval $(x^0, y^0) \subset \mathbb{R}$. Then since .

$$\sum_{\substack{I \cup J = \mathfrak{J} \\ I \cap J = \emptyset}} 1 = 2^{|\mathfrak{J}|} \leq e^{l^{-1}|x^0 - y^0|}$$

the proof of Proposition 2 is reduced to the proof of the following lemma.

Lemma 4. *Let $\varepsilon < 0$ be given and let $(K(X; Y), O, \mathfrak{J})$ denote one of the kernels considered in Proposition 2 and Lemma 3, with the corresponding order symbol O and index sets $\mathfrak{J} = \mathfrak{J}(K, X, Y)$. Then we have*

$$\partial_t^\alpha \partial_r^J K_{r,t}(X; Y)\,|_{r^I=0} = O(1) \exp\left(-(1-4\varepsilon) \sum_{i \in I} \alpha(i) - L\,|J|\right) \tag{I.16}$$

for $r = r^{\mathfrak{J}}$, $t = t^I$, $I, J \subset \mathfrak{J}$ and $\alpha : I \to \{0, 1, 2, 3\}$ depending on X, Y (taking constant values on products of unit lattice squares) such that $I \cap J = \emptyset$ and $r_j, t_i \in [0, 1]$.

The proof of this lemma will be given in the next section. It is based on a bound for modified expectations $\langle \Phi \rangle_{r,t,h}$ which will be shown in Section IV to be analytic in the variables r_i and $h(\alpha)$ in a large domain, and on a connection between t_i-derivatives and derivatives with respect to the new variables $h(\alpha)$. Corresponding irreducible kernels $K_{r,t,h}$ with $K_{r,t,0} = K_{r,t}$ are defined by convergent Neumann series in terms of expectations $\langle \Phi \rangle_{r,t,h}$ and thus Cauchy's formula can be applied to calculate $\partial_r^J \partial_t^\beta K_{r,t}$.

Remark. Recently J. Imbrie [I] has proven Theorem 1 by using a method which also works for small external field.

II. The expansion

In this section we define expectations $\langle \Phi \rangle_{r,t,h}$ and their expansion. We fix two sets I, J with $\mathfrak{J} = I \cup J \subset l\mathbb{Z} \cap \{x^0 : x \in \Lambda\}$, $I \cap J = \emptyset$. Then the α's which occur in (I.16) are elements of $I^{(4)}$, where $I^{(m)}$ is the set of maps from I to $\{0, 1, \ldots, m-1\}$. Let

$$\partial_{t_i} C(t) \cdot \Delta_\phi = \frac{1}{2} \int d^2x\, d^2y\, \partial_{t_i} C(t; x, y) \frac{d^2}{d\phi(x)\, d\phi(y)}.$$

Then multiple t-derivatives can be written as

$$\partial_t^\beta \int d\phi_{C(t)} Q = \sum_{\substack{\alpha_1+\cdots+\alpha_l=\beta \\ \alpha_j \le 1}} \int d\phi_{C(t)} \prod_{j=1}^{l} [\partial_t^{\alpha_j} C(t) \cdot \Delta_\phi] Q \tag{II.1}$$

or, by introducing variables $h(\alpha_j)$

$$\begin{aligned} &\partial_t^\beta \int d\phi_{C(t)} Q \\ &= \sum_{\substack{\alpha_1+\cdots+\alpha_l=\beta \\ \alpha_j \le 1}} \prod_{i=1}^{l} \partial_{h(\alpha_i)} \int d\phi_{C(t)} \prod_{\alpha\in\pi_0} \prod_{j\in l\mathbb{Z}^4} [1+h(\alpha)\partial_t^\alpha C_j(t)\cdot\Delta_\phi] Q \,|_{h=0} \end{aligned} \tag{II.2}$$

where $C_j(t; x, y) = \Delta_{j1}(x) C(t; x, y) \Delta_{j2}(y)$ for $j = (j_1, j_2) \in l\mathbb{Z}^4$, and Δ_i denotes the square $[i^0, i^0+1] \times [i^1, i^1+1]$, respectively its characteristic function. π_0 is a subset of $P(I^{(4)})$, $P(I^{(m)}) = \{\pi \subset I^{(m)} : \sum_{\alpha\in\pi} \alpha \in I^{(m)}, \emptyset \notin \pi\}$, which contains $\alpha_1, \ldots, \alpha_l$.

We now define

$$\langle\Phi\rangle_{r,t,h} = \frac{\sum_\Sigma \prod_{i\in\mathfrak{I}} r_i^{|\Sigma\cap N(\mathfrak{L}_i)|} \int d\phi_{C(t)} \prod_{\alpha\in\pi_0} \prod_{j\in l\mathbb{Z}^4} [1+h(\alpha)\partial_t^\alpha C_j(t)\cdot\Delta_\phi] \Phi \chi_\Sigma e^{-V(\Lambda)}}{\sum_\Sigma \prod_{i\in\mathfrak{I}} r_i^{|\Sigma\cap N(\mathfrak{L}_i)|} \int d\phi_{C(t)} \prod_{\alpha\in\pi_0} \prod_{j\in l\mathbb{Z}^4} [1+h(\alpha)\partial_t^\alpha C_j(t)\cdot\Delta_\phi] \chi_\Sigma e^{-V(\Lambda)}} \tag{II.3}$$

The set $\pi_0 \in P(I^{(4)})$ is always supposed to be chosen according to the derivatives that we want to compute, and the domain of parameters (r, h) which we consider is (with ε given by Lemma 4)

$$\begin{aligned} \mathfrak{R} = \{(r, h) : r \in \mathbb{C}^{|\mathfrak{I}|}, h \in \mathbb{C}^{|\pi_0|}, r_i = 0 \text{ for } i \in I, \\ |r_i| < e^{3L} \text{ for } i \in \mathfrak{I}, |h(\alpha)| < \exp((1-2\varepsilon)(d(\alpha)+l)) \text{ for } \alpha \in \pi_0\} \end{aligned}$$

where $d(\alpha) = \max\{|i-j| : \alpha(i), \alpha(j) \neq 0\}$ for $\alpha \in I^{(2)}$.

To construct our kernels $K_{r,t,h}$ we need fields of the form

$$\Phi = \prod_{i=1}^{M} \square_{j_i}(x_i) \frac{d}{d\phi(x_i)} \prod_{i=M+1}^{N} \square_{j_i}(x_i) : \phi^{\nu_i} : (x_i) \tag{II.4}$$

where $\square_j$ denotes the unit square $[j^0, j^0+1] \times [j^1, j^1+1]$, respectively its characteristic function. The support $\Omega = \square_{j1} \cup \cdots \cup \square_{j_N}$ is supposed to be in Λ. We define two functions m_Φ and n_Φ by

$$m_\Phi(\Omega') = \sum_{i=1}^{M} |\Omega' \cap \square_{j_i}|,$$

$$n_\Phi(\Omega') = \sum_{i=M+1}^{N} |\Omega' \cap \square_{j_i}| \cdot \nu_i,$$

for $\Omega' \subset \mathbb{R}^2$. The expectation is considered as a function of the variables $x_1, \ldots, x_N$. Since $d/d\phi(x_i)$ can contract to some $:\phi^{\nu_i}:(x_i)$ or to a vertex V together with some other $d/d\phi(x_i)$, this function has the form

$$f(x_1, \ldots, x_N) = \sum_{\pi \in P(\{1,\ldots,N\})} \int \prod_{j=1}^{|\pi|} d^2y_j \prod_{i\in\sigma_j} \delta(x_i - y_i) \cdot f_\pi(y_1, \ldots, y_{|\pi|})$$

where the sum runs over partitions $\pi = \{\sigma_1, \ldots, \sigma_{|\pi|}\}$ of $\{1, \ldots, N\}$. Let π denote a fixed partition of $\{1, \ldots, N_0\}$, $N_0 \geq N$. We define

$$|f|_q = \|f_{\pi_1}\|_{L^q}$$

if $\pi = \pi_1 \cup \pi_2$ for some partition π_1 of $\{1, \ldots, N\}$. Otherwise let $|f|_q = 0$. Notice that $|f|_q = |f^{(1)}|_q \cdot |f^{(2)}|_q$ if $f = f_1 \otimes f_2$.

Lemma 5. *Fix* $q \geq 1$ *and* $\varepsilon < 0$. *Then there are positive constants* β *and* ρ *such that for* $(r, h) \in \mathfrak{R}$, $t \in [0, 1]^I$ *and for* $\lambda > 0$ *sufficiently small* $\langle\Phi\rangle_{r,t,h}$ *is analytic in* r *and* h. *Furthermore for* $m_\Phi + n_\Phi \neq 0$

$$|\langle\Phi\rangle_{r,t,h}|_q \leq C_1(m_\Phi, q) C_2(n_\Phi, q)$$

independently of Λ, *where* $C_1(0, q) = 1$, $C_2(0, q) = \lambda^{1/2-\varepsilon}$ *and for* $m, n \neq 0$

$$C_1(m, q) = e^{\rho m(\Omega)^{4/3}}$$

$$C_2(n, q) = (1 + \lambda^{-n(\Omega)/2} e^{-\beta\lambda^{-1/2}}) e^{ln(\Omega)} \prod_{\Box \subset \Lambda} n(\Box)!.$$

This lemma will be proven in the next section, together with the following one. Let

$$\Phi_{\{i\}} = \Box_{j_i}(x_i) \frac{d}{d\phi(x_i)} \qquad i = 1, \ldots, M,$$

$$\Phi_{\{i\}} = \Box_{j_i}(x_i) : \phi^{\nu_i} : (x_i) \qquad i = M+1, \ldots, N$$

be given with $\Omega = \Box_{j_1} \cup \cdots \cup \Box_{j_N} \subset \Lambda$. To subsets $\Omega' \subset \Omega$ and $R' \subset R = \{1, \ldots, N\}$ we associate products

$$\Phi_{\Omega'} = \prod_{\substack{i \leq M \\ \Box_{j_i} \subset \Omega'}} \Phi_{\{i\}} \prod_{\substack{i > M \\ \Box_{j_i} \subset \Omega'}} \Phi_{\{i\}}$$
$$\Phi(R') = \prod_{\substack{i \leq M \\ i \in R'}} \Phi_{\{i\}} \prod_{\substack{i > M \\ i \in R'}} \Phi_{\{i\}}. \tag{II.5}$$

Then for partitions $\{R_1, \ldots, R_s\}$ of R we define truncated expectations as usually by

$$\langle\Phi(R_1); \Phi(R_2); \ldots; \Phi(R_s)\rangle = -\sum_{p=1}^{s} \frac{(-1)^p}{p} \sum_{\sigma_1 \cup \cdots \cup \sigma_p = \{1,\ldots,s\}} \prod_{j=1}^{p} \left\langle \Phi\left(\bigcup_{i\in\sigma_j} R_i\right)\right\rangle \tag{II.6}$$

Lemma 6. *For given* $q \geq 1$ *there are constants* K, $\delta > 0$ *such that for* $(r, h) \in \mathfrak{R}$, $t \in [0, 1]^I$ *and for* $\lambda > 0$ *sufficiently small*

$$|\langle\Phi(R_1); \ldots; \Phi(R_s)\rangle_{r,t,h}|_q \leq (Ks)^s C_1(m_{\Phi(R)}, q) C_2(n_{\Phi(R)}, q) e^{-\delta d(\Omega_1, \ldots, \Omega_s)}$$

where Ω_k *is the support of* $\Phi(R_k)$ *and* $d(\Omega_1, \ldots, \Omega_s) = \sup\{|x - y| : x \in \Omega_i, y \in \Omega_j, 1 \leq i < j \leq s\}$.

Proof of Lemma 3 *and* 4. With (I.14) and Lemma 5 and 6 as input we can exactly follows [K] in order to show that irreducible kernels $K_{r,t,h}$ are well defined (i.e. that Proposition 2 holds for $1-\varepsilon>0$ sufficiently small), that they are analytic in r and h for $(r,h)\in\mathfrak{R}$, and that they verify the statements of Lemma 3.

Let now K denote one of the kernels k, L, K or $S^n_{k,l}$. Then from (II.2) it follows [S III] that

$$\partial_t^\alpha K_{r,t}=\sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha\\ \alpha_i\le 1}}\prod_{i=1}^{l}\partial_{h(\alpha_i)}K_{r,t,h}\big|_{h=0} \tag{II.7}$$

and thus

$$\begin{aligned}&\partial_t^\alpha\partial_r^J K(X;Y)_{r,t}\\ &=\sum_{\substack{\alpha_1+\cdots+\alpha_l=\alpha\\ \alpha_j\le 1}}\oint\prod_{\alpha\in\{\alpha_1,\ldots,\alpha_l\}}\frac{h(\alpha)!\,dh(\alpha)}{2\pi i h(\alpha)^{n(\alpha)+1}}\oint\prod_{j\in J}\frac{dz_j}{2\pi i(z_j-r_j)^2}K(X;Y)_{z,t,h}\end{aligned} \tag{II.8}$$

where $n(\alpha)$ is the number of copies of α in $(\alpha_1,\ldots,\alpha_l)$ and where the integrations are over circles

$$|h(\alpha)|=\exp((1-3\varepsilon)(d(\alpha)+l)),$$
$$|z_j|=e^{2L}.$$

We suppose r_i, $t_i\in[0,1]$, $r^I=0$. Since $n(\alpha)\le 3$ and since the number of sets $(\alpha_1,\ldots,\alpha_k)$ with $0\ne\alpha_j\le 1$, $\sum_{j=1}^k\alpha_j=\alpha$ and $\sum_{j=1}^k(d(\alpha_j)+l)=ml$ is zero for $m<|\alpha|=\sum_{i\in I}\alpha(i)$ and bounded by 32^m for $m\ge|\alpha|$, we obtain

$$\begin{aligned}\partial_t^\alpha\partial_r^J K(X;Y)_{r,t}&=O(1)\cdot\sum_{m\ge|\alpha|}32^m\cdot 6^m\exp(-(1-3\varepsilon)ml-L|J|)\\ &=O(1)\exp(-(1-4\varepsilon)l|\alpha|-L|J|).\end{aligned}$$

Following [GJS III] we now define a cluster expansion for expectations $F_{\Omega\cap X,\mathbf{X},\Sigma}(r,t,h,s=1)$, which will be defined below so that for $r^I=0$

$$\langle\Phi\rangle_{r,t,h}=\frac{S_\Omega}{S_\emptyset}=\frac{\sum_\Sigma F_{\Omega,\mathbf{Y},\Sigma}(r,t,h,s=1)}{\sum_\Sigma F_{\emptyset,\mathbf{Y},\Sigma}(r,t,h,s=1)} \tag{II.9}$$

with $\mathbf{Y}=(Y,\partial Y,\emptyset)$. Φ denotes a fixed field of the form (II.4).

Let X be a subset of Y built up from lattice squares Δ and let $\partial X=\partial X^+\cup\partial X^-$ be a partition of its boundary. We suppose the triple $\mathbf{X}=(X,\partial X^+,\partial X^-)$ to be regular, i.e.

$$N(\partial X^-)\cap(N(\partial X^+)\cup(Y\backslash\Lambda))=\emptyset.$$

For each $X'\subset X$ let $b(X')$ denote the set of $(lZ)^2$ lattice bonds b in X' with $|b\cap\partial X'|=0$. Furthermore let $A(\mathbf{X})$ denote the set of spin configurations Σ on X such that $\Sigma(\square)=\pm$ for $\square\subset N(\partial X^\pm)$. Given a configuration $\Sigma\in A(\mathbf{X})$ we introduce a cluster expansion which only involves bounds b far from phase boundaries, i.e. $b\in b=b(X\backslash N(\Sigma))$.

Let C(t, s) be the covariance

$$C(t,s)=\prod_{b\in b}[(1-s_b)\theta_b+s_b]\prod_{i\in I}[(1+t_i)\theta_{\mathfrak{L}_i}+t_i](-\Delta_{\partial x}+1)^{-1} \tag{II.10}$$

for $t\in[0,1]^I$, $s\in[0,1]^b$, where $\theta_{\Gamma_2}(-\Delta_{\Gamma_1}+1)^{-1}=(-\Delta_{\Gamma_1\cup\Gamma_2}+1)^{-1}$, and where Δ_Γ denotes the Laplacean with zero Dirichlet data on each $b\in\Gamma$. For $\Gamma\subset b$ and for $F=F(s)$ we define $\Gamma^c=b\setminus\Gamma$, $\partial_s^\Gamma=\prod_{b\in\Gamma}\partial_{s_b}$,

$$s(\Gamma^c)_b=\begin{cases}0 & \text{if } b\in\Gamma^c\\ s_b & \text{if } b\in\Gamma\end{cases}$$

and

$$\delta_s^\Gamma F=\int_{\{0\}}^{\{1\}}ds\,\partial_s^\Gamma F(s(\Gamma^c)).$$

The cluster expansion is then the identity

$$F(1)=\sum_{\Gamma\subset b}\delta_s^\Gamma F, \tag{II.11}$$

applied to expectations ($\Omega'\subset X$)

$$\begin{aligned}F_{\Omega',\mathbf{X},\Sigma}(r,t,h,s)&=\prod_{i\in\mathfrak{J}}r_i^{|\Sigma\cap N(\mathfrak{L}_i)|}\int d\Psi_{C(t,s)}e^{-F(\Lambda,\Sigma)}\\&\times\prod_{\alpha\in\pi_0}\prod_{j\in l\mathbb{Z}^4}[1+h(\alpha)\partial_t^\alpha C_j(t,s)\cdot\Delta_\phi]\Phi_{\Omega'}\chi_\Sigma e^{-V(\Lambda)}.\end{aligned} \tag{II.12}$$

We continue by considering $\mathbf{X}=\mathbf{Y}$. Equation (II.9) is obtained from (II.3) through the translation $\phi\mapsto\psi=\phi+\xi-g$ (see (I.6)) by using that

$$\frac{d\psi_{C(t,s)}}{d\phi_{C(t,s)}}=\frac{d\psi_{C(1,1)}}{d\phi_{C(1,1)}}=e^{-F(\Lambda,\Sigma)}$$

for our particular choice of Dirichlet lines.

In the usual way we order the sum (II.11) such that subsums are indexed by a covering of Y with closed, nonempty connected sets $X_1,\ldots,X_n$ having pairwise disjoint interior. The set of such coverings will be denoted by C(Y). For given $\{X_1,\ldots,X_n\}\in C(Y)$ we sum first over those Γ's satisfying

$$Y\setminus\Big(\partial Y\bigcup_{b\in\Gamma^c}b\Big)=\bigcup_{j=1}^{n}Y_j \tag{II.13}$$

with Y_j open, connected and closure $(Y_j)=X_j$ for $j=1,\ldots,n$. Of course the X_j must be compatible with Σ, i.e. $\partial X_j\subset b(Y\setminus N(\Sigma))$. If we orient their boundaries by defining

$$\partial X_j^\pm=\partial X_j\cap\{\square:\Sigma(\square)=\pm\},$$

then they are pairwise compatible, i.e.

$$N(\partial X_i^-)\cap((N(\partial X_j^+)\cup(Y\setminus\Lambda))=\emptyset. \tag{II.14}$$

The restrictions of Σ and Γ to X_j will be denoted by Σ_j and Γ_j respectively. Notice that $\Sigma_j \in A(\mathbf{X}_j)$ and $\Gamma_j \in B(X_j, \Sigma_j)$, where

$$B(X, \Sigma) = \left\{\Gamma \in b(X \backslash N(\Sigma)) : X \backslash \bigcup_{b \in \Gamma^c} b \text{ is connected}\right\}.$$

Furthermore we have the factorization property

$$F_{\Omega,\mathbf{Y},\Sigma}(r, t, h, s(\Gamma^c)) = \prod_{j=1}^{n} F_{\Omega \cap X_j, \mathbf{X}_j, \Sigma_j}(r, t, h, s(\Gamma_j^c)). \tag{II.15}$$

The expansion for S_Ω and $S_\emptyset$ can now be written as follows. Let $\Omega' \subset \Omega$ be given and let $C(\mathbf{Y})$ denote the set of sets $\{\mathbf{X}_1, \ldots, \mathbf{X}_n\}$ with pairwise compatible elements, such that $\{X_1, \ldots, X_n\} \in C(Y)$. Then

$$\begin{aligned} S_{\Omega'} &= \sum_{\Sigma} \sum_{\Gamma \in b(Y)} \delta_s^{\Gamma} F_{\Omega', \mathbf{Y}, \Sigma} \\ &= \sum_{\Sigma} \sum_{\Gamma \in b(Y \backslash N(\Sigma))} \prod_j \delta_s^{\Gamma_j} F_{\Omega' \cap X_j, \mathbf{X}_j, \Sigma_j} \\ &= \sum_{\{\mathbf{X}_j\} \in C(\mathbf{Y})} \prod_j \sum_{\Sigma \in A(\mathbf{X}_j)} \sum_{\Gamma \in B(X_j, \Sigma)} \delta_s^{\Gamma} F_{\Omega' \cap X_j, \mathbf{X}_j, \Sigma}. \end{aligned} \tag{II.16}$$

The proof of the convergence of the expansion for $\langle \Phi \rangle = S_\Omega S_\emptyset^{-1}$ can now be based on the following estimate.

Lemma 7. *Fix* $q \geq 1$ *and* $\varepsilon > 0$. *Then there are constants* $a, b, c > 0$ *such that for regular* $\mathbf{X}, \Omega' \subset \Omega \cap X$, $(r, h) \in \mathfrak{R}$, $t \in [0, 1]^I$ *and for* $\lambda > 0$ *sufficiently small*

$$|\delta_s^{\Gamma} F_{\Omega', \mathbf{X}, \Sigma}(r, t, h, s)|_q \leq C_1(m_{\Phi_{\Omega'}}, q) C_3(n_\Phi, \Omega', q) \times \exp\left(-c|\Gamma| - b\lambda^{-1/2} |N(\Sigma) \cap X| + a l^{-2} |X|\right)$$

where

$$C_3(n, \Omega', q) = C(\mathbf{X}, \Sigma)^{n(\Omega')} \exp\left(\tfrac{1}{2} + 8c\right) \ln(\Omega') \prod_{\square \subset \Omega'} n(\square)!$$

if $n \neq 0$, $\Omega' \neq \emptyset$,

$$C(\mathbf{X}, \Sigma) = \begin{cases} 1 & \textit{if} \quad |\Sigma| = |\partial X^-| = 0 \\ 2\xi & \textit{otherwise} \end{cases},$$

$$C_3(0, \Omega', q) = \begin{cases} 1 & \textit{if} \quad \Omega' = \emptyset \\ \lambda^{1/2 - \varepsilon} & \textit{otherwise} \end{cases}.$$

This lemma will be proved in the last section.

III. Convergence of the expansion

In this section we prove the convergence of the low temperature cluster expansion for the modified expectations

$$\langle\Phi\rangle_{r,t,h}=\frac{\sum_{\{\mathbf{Y}_j\}\in C(\mathbf{Y})}\prod_j\sum_{\Sigma\in A(\mathbf{Y}_j)}\sum_{\Gamma\in B(\mathbf{Y}_j,\Sigma)}\delta_s^{\Gamma}F_{\Omega'\cap Y_j,\mathbf{Y},\Sigma}(r,t,h,s)}{\sum_{\{\mathbf{Y}_j\}\in C(\mathbf{Y})}\prod_j\sum_{\Sigma\in A(\mathbf{Y}_j)}\sum_{\Gamma\in B(\mathbf{Y}_j,\Sigma)}\delta_s^{\Gamma}F_{\emptyset,\mathbf{Y}_j,\Sigma}(r,t,h,s)} \tag{III.1}$$

and the exponential clustering of the corresponding truncated expectations. We use a method due to H. Kunz, B. Souillard, T. Balaban and K. Gawedski [BG].

First the numerator and denominator in (III.1) are divided by a product $\prod_{\Delta\subset\Lambda}Z_\Delta$ of $l\mathbb{Z}^2$ square partition functions

$$Z_\Delta=\sum_{\Gamma\in B(\Delta,\emptyset)}\delta_s^{\Gamma}F_{\emptyset,\mathbf{\Delta},\emptyset}(0,0,0,s)$$

which do not depend on the sign of the boundary $\partial\Delta$. The expansion can then be written in terms of the following quantities.

$$S_{\Omega',\mathbf{X}}(r,t,h)=Z_\Delta^{-l^{-2}|X\cap\Lambda|}\sum_{\Sigma\in A(\mathbf{X})}\sum_{\Gamma\in B(X,\Sigma)}\delta_s^{\Gamma}F_{\Omega',\mathbf{X},\Sigma}(r,t,h,s) \tag{III.2}$$

if $\Omega'\subset\Omega\cap X$, $|X|>l^2$, X is connected and $\mathbf{X}$ regular; otherwise let $S_{\Omega',\mathbf{X}}=0$. The new expression for $\langle\Phi\rangle_{r,t,h}$ is

$$\langle\Phi\rangle_{r,t,h}=\frac{\sum_{\{\mathbf{Y}_j\}\text{ comp.}}\prod_j S_{\Omega\cap Y_j,\mathbf{Y}_j}}{\sum_{\{\mathbf{Y}_j\}\text{ comp.}}\prod_j S_{\Phi,\mathbf{Y}_j}} \tag{III.3}$$

The sum ranges over sets $M=\{\mathbf{Y}_1,\ldots,\mathbf{Y}_m\}$ such that $M\cup N$ is a partition of $\mathbf{Y}$ for some $N=\{\mathbf{X}_1,\ldots,\mathbf{X}_n\}$ with $|X_j|=l^2$. Such sets M will be called compatible.

Lemma 8. *Fix* $q\geq 1$. *Then there is a positive constant* δ *such that for* $X\subset Y$, $\Omega'\subset\Omega\cap X$, $(r,h)\in\mathfrak{R}$, $t_i\in[0,1]$ *and* $\lambda>0$ *sufficiently small*

$$|S_{\emptyset,\mathbf{X}}(r,t,h)|\leq e^{-2\delta|X|l^{-1}}$$
$$|S_{\Omega',\mathbf{X}}(r,t,h)|_q\leq C_1(m_{\Phi_{\Omega'}},q)C_2(n_{\Phi_{\Omega'}},q)e^{-2\delta|X|l^{-1}}.$$

Proof. The proof is as in [GJS III]: One part of the factor $e^{-b\lambda^{-1/2}|N(\Sigma)|}$ in Lemma 7 is used for the first factor in C_2 (see Lemma 5). The other part controls the first sum in (III.2).

$$\sum_{\Sigma\in A(\mathbf{X})}\exp(-b'\lambda^{-1/2}|N(\Sigma)|)\leq\sum_{\Sigma\in A(\mathbf{X})}\exp(-\tfrac{1}{2}b'\lambda^{-1/2}|\Sigma|)$$
$$\leq(1+\exp(-\tfrac{1}{2}b'\lambda^{-1/2}))^{2|X|}\leq e^{\lambda|X|}. \tag{III.5}$$

To bound the second sum in (III.2) notice that for $\Gamma \in B(X, \Sigma)$ we have $2l^{-1}|\Gamma| \geq |X|\, l^{-2} - 1$. Thus

$$\sum_{\Gamma \in B(X,\Sigma)} e^{-c|\Gamma|} \leq \exp\left(-\frac{c}{2}\, l^{-1}(|X| - l^2)\right) \sum_{\Gamma \in B(X,\Sigma)} 1$$
$$\leq \exp\left(-\frac{c}{2}\, l^{-1}(|X| - l^2)\right) 2^{2l^{-2}|X| - (1/2)l^{-1}|\partial X|} \leq \exp\left(-\frac{c}{4}\, l^{-1}(|X| - l^2)\right), \tag{III.6}$$

and we can take $2\delta = c/10$ if we put a factor $e^{\lambda l^2 + 2\delta l}$ into C_2 for the case $|X| = l^2$. Finally the factor $Z_\Delta^{-l^{-2}|X \cap \Lambda|}$ is bounded by using [GJS III]

$$|Z_\Delta^{\pm 1}| \leq e^{O(1)\lambda^{1/2} l^2}. \tag{III.7}$$

We continue by expanding the ratio on the left hand side of (III.3). Notice that a compatible set $\{\mathbf{Y}_1, \ldots, \mathbf{Y}_m\}$ is already determined by $\{Y_1, \ldots, Y_m\}$ and the relative signs of the boundary loops (since the outermost loop must have the + sign), i.e. by $\{\tilde{\mathbf{Y}}_1, \ldots, \tilde{\mathbf{Y}}_m\}$, where $\tilde{\mathbf{X}} = \{(X, \partial X^+, \partial X^-), (X, \partial X^-, \partial X^+)\}$. The starting point for the expansion of [BG] is the observation that the factor

$$\varphi = \begin{cases} \prod_{j=1}^{m} S_{\Omega_j, \mathbf{Y}_j} & \text{if } \{\mathbf{Y}_1, \ldots, \mathbf{X}_m\} \text{ are compatible} \\ 0 & \text{otherwise} \end{cases}$$

can be expressed in terms of an 'interaction' $U(\alpha)$ between pairs $\alpha = ((\Omega_i, \tilde{\mathbf{Y}}_i), (\Omega_j, \tilde{\mathbf{Y}}_j))$.

Let us call $\mathbf{X}$ positive or negative according to the sign of its external boundary loop. A negative (positive) boundary loop of a positive (negative) $\mathbf{X}$ will be called inner. Then by regarding

$$S_m((\Omega_1, \mathbf{Y}_1), \ldots, (\Omega_m, \mathbf{Y}_m)) = S_{\Omega_1, \mathbf{Y}_1} \cdots S_{\Omega_m, \mathbf{Y}_m}$$

for fixed $\Omega_i, \tilde{\mathbf{Y}}_i$, $i = 1, \ldots, m$ as a function of m signs we define

$$[U((\Omega_1, \tilde{\mathbf{Y}}_1), (\Omega_2, \tilde{\mathbf{Y}}_2))S_m]((\Omega_1, \mathbf{Y}_1), (\Omega_2, \mathbf{Y}_2), \ldots)$$
$$= \begin{cases} 0 \quad \text{if} \quad Y_1 \cap Y_2 \neq \emptyset. \\ S_m((\Omega_1, (Y_1, \partial Y_1^-, \partial Y_1^+)), (\Omega_2, \mathbf{Y}_2), \ldots) \quad \text{if} \quad \Omega_1 \neq \emptyset \\ \text{and if } Y_1 \text{ is surrounded by an inner boundary} \\ \text{loop of } \mathbf{Y}_2. \\ S_m((\Omega_1, \mathbf{Y}_1), (\Omega_2, (Y_2, \partial Y_2^-, \partial Y_2^+)), \ldots) \quad \text{if} \quad \Omega_2 \neq \emptyset \\ \text{and if } Y_2 \text{ is surrounded by an inner boundary} \\ \text{loop of } \mathbf{Y}_1. \\ S_m((\Omega_1, \mathbf{Y}_1), (\Omega_2, \mathbf{Y}_2), \ldots) \quad \text{otherwise.} \end{cases} \tag{III.8}$$

The remaining $U(\alpha)$ for $\alpha \in \mathfrak{A} = \{((\Omega_i, \tilde{\mathbf{Y}}_i), (\Omega_j, \tilde{\mathbf{Y}}_j)) \colon 1 \leq i < j \leq m\}$ are defined analogously. By identifying $\tilde{\mathbf{X}}$ with its positive element, φ can be written as follows

$$\varphi = \varphi((\Omega_1, \tilde{\mathbf{Y}}_1), \ldots, (\Omega_m, \tilde{\mathbf{Y}}_m))$$
$$= \prod_{\alpha \in \mathfrak{A}} U(\alpha) S_m((\Omega_1, \tilde{\mathbf{Y}}_1), \ldots, (\Omega_m, \tilde{\mathbf{Y}}_m)). \tag{III.9}$$

Here we have used that by the $\varphi \mapsto -\varphi$ symmetry $S_{\emptyset,(X,\partial X^+,\partial X^-)} = S_{\emptyset,(X,\partial X^-,\partial X^+)}$. Next we write a graphical expansion for $\prod_{\alpha\in\mathfrak{A}} U(\alpha)$, namely

$$\prod_{\alpha\in\mathfrak{A}} U(\alpha) = \prod_{\Omega_i,\Omega_j\neq 0} U((\Omega_i, \tilde{\mathbf{Y}}_i), (\Omega_j, \tilde{\mathbf{Y}}_j)) \sum_{G_c} \prod_{\alpha\in G_c} A(\alpha) \prod_{\alpha\in\mathfrak{A}'} U(\alpha) \tag{III.10}$$

where $A(\alpha) = U(\alpha) - 1$. The sum ranges over connected graphs G_c whose lines are a subset of those $\alpha = ((\Omega_i, \tilde{\mathbf{Y}}_i), (\Omega_j, \tilde{\mathbf{Y}}_j)) \in \mathfrak{A}$ with $\Omega_i = \emptyset$ or $\Omega_j = \emptyset$, and whose connected components contain at least one $(\Omega_j, \tilde{\mathbf{Y}}_j)$ with $\Omega_j \neq \emptyset$. $\mathfrak{A}'$ contains the pairs of those points $\{(\emptyset, \mathbf{Y}_i)\}_{i\in I}$ which are not involved in G_c.

With a sum over the $\Omega_j \in \{\Omega \cap Y_j, \emptyset\}$ such that $\bigcup_j \Omega_j = \Omega$, the numerator in (III.3) can then be written as

$$\sum_m \frac{1}{m!} \sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_m \text{ pos.} \\ \Omega_1,\ldots,\Omega_m}} \varphi((\Omega_1, \mathbf{Y}_1), \ldots, (\Omega_m, \mathbf{Y}_m))$$

$$= \sum_m \frac{1}{m!} \sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_m \text{ pos.} \\ \Omega_1,\ldots,\Omega_m}} \prod_{\Omega_i,\Omega_j\neq\emptyset} U((\Omega_i, \tilde{\mathbf{Y}}_i), (\Omega_j, \tilde{\mathbf{Y}}_j)) \tag{III.11}$$

$$\times \sum_{G_c} \prod_{\alpha\in G_c} A(\alpha) S_{m-|I|}(\{(\Omega_j, \mathbf{Y}_j)\}_{j\notin I}) \prod_{\alpha\in\mathfrak{A}'} U(\alpha) S_{|I|}(\{(\emptyset, \mathbf{Y}_i)\}_{i\in I})$$

or, by summing first the terms with fixed $\{(\Omega_j, \mathbf{Y}_j)\}_{j\notin I} = \{(\Omega_1', \mathbf{X}_1), \ldots, (\Omega_k', \mathbf{X}_k)\}$, $k = m - |I|$,

$$\sum_m \frac{1}{m!} \sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_m \text{ pos.} \\ \Omega_1,\ldots,\Omega_m}} \varphi((\Omega_1, \mathbf{Y}_1), \ldots, (\Omega_m, \mathbf{Y}_m))$$

$$= \sum_k \frac{1}{k!} \sum_{\substack{\mathbf{X}_1,\ldots\mathbf{X}_k \text{ pos.} \\ \Omega_1',\ldots,\Omega_k'}} \varphi_C(\{(\Omega_n', \mathbf{X}_n) : \Omega_n' \neq \emptyset\}; \{(\emptyset, \mathbf{X}_n) : \Omega_n' = \emptyset\}) \tag{III.12}$$

$$\times \sum_l \frac{1}{l!} \sum_{\mathbf{Y}_1,\ldots,\mathbf{Y}_e \text{ pos.}} \varphi((\Omega, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_l))$$

with

$$\varphi_C((\Omega_1, \mathbf{X}_1), \ldots, (\Omega_i, \mathbf{Y}_i); (\emptyset, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_j))$$

$$= \prod_{1\leq r<s\leq i} U((\Omega_r, \tilde{\mathbf{X}}_r), (\Omega_s, \tilde{\mathbf{X}}_s)) \sum_{G_c} \prod_{\alpha\in G_c} A(\alpha) S_{i+j}((\Omega_1, \mathbf{X}_1), \ldots, (\emptyset, \mathbf{Y}_j)). \tag{III.13}$$

The sum $\sum_{G_c}$ is over all graphs whose lines α contain at least one $(\emptyset, \mathbf{Y}_r)$ and in which every $(\emptyset, \mathbf{Y}_r)$ is contained in a connected component containing some $(\Omega_s, \mathbf{X}_s)$. The second factor in (III.12) is now equal to the denominator in (III.3) and thus

$$\langle\Phi\rangle = \sum_{\substack{\{\mathbf{X}_1,\ldots,\mathbf{X}_i\} \text{ pos.} \\ \Omega\cap X_n\neq 0,\, \Omega\subset X_1\cup\cdots\cup X_i}} \sum_{\mathbf{Y}_1,\ldots,\mathbf{Y}_j \text{ pos.}} \frac{1}{j!} \tag{III.14}$$

$$\times \varphi_C((\Omega\cap X_1, \mathbf{X}_1), \ldots, (\Omega\cap X_i, \mathbf{X}_i); (\emptyset, \mathbf{X}_1), \ldots, (\emptyset, \mathbf{Y}_j)).$$

Lemma 9. *Given* $q \geq 1$ *there is an* $\varepsilon > 0$ *such that for* $\lambda > 0$ *sufficiently small* *and* $i, j \geq 1$

$$\sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_j \\ \sum_s |y_s| = l^2 N}} |\varphi_C((\Omega \cap X_1, \mathbf{X}_1), \ldots, (\Omega \cap X_i, \mathbf{X}_i); (\emptyset, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_j))|_q \leq j! \prod_{r=1}^{i} C(\Omega \cap X_r, \mathbf{X}_r) \exp\left(2l^{-2} \sum_{r=1}^{i} |X_r| - \varepsilon l N\right) \tag{III.15}$$

where $C(\Omega \cap X, \mathbf{X})$ *is the bound on* $S_{\Omega \cap X, \mathbf{X}}$ *obtained in Lemma* 8.

Proof. First we sum in (III.13) over graphs G_c with $\{s : ((\Omega_1, \tilde{\mathbf{X}}_1), (\emptyset, \tilde{\mathbf{Y}}_s)) \in G_c\} = I$ fixed. For such graphs vertices $(\emptyset, \mathbf{Y}_s)$, $s \notin I$ must be connected directly or indirectly to $\{(\Omega_r, \mathbf{X}_r) : r = 2, \ldots, i\} \cup \{(\emptyset, \mathbf{Y}_s) : s \in I\}$, but there is no restriction on lines $((\emptyset, \tilde{\mathbf{Y}}_s), (\emptyset, \tilde{\mathbf{Y}}_s))$ and $((\Omega_r, \tilde{\mathbf{X}}_r), (\emptyset, \tilde{\mathbf{Y}}_s))$, $s, s' \in I$, $r = 2, \ldots, i$. Then by summing over $I \subset \{1, \ldots, j\}$ we obtain the recursion relation

$$\begin{aligned}
&\varphi_C((\Omega_1, \mathbf{X}_1), \ldots, (\Omega_i, \mathbf{X}_i); (\emptyset, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_j)) \\
&\quad = \sum_{I \subset \{1,\ldots,j\}} \prod_{r=2}^{i} U((\Omega_1, \tilde{\mathbf{X}}_1), (\Omega_r, \tilde{\mathbf{X}}_r)) \prod_{s \in I} A((\Omega_1, \tilde{\mathbf{X}}_1), (\emptyset, \tilde{\mathbf{Y}}_s)) \\
&\qquad \times (S_1 \times \varphi_C)((\Omega_1, \mathbf{X}_1), \ldots, (\Omega_i, \mathbf{X}_i), \{(\emptyset, \mathbf{Y}_s)\}_{s \in I}; \{(\emptyset, \mathbf{Y}_s)\}_{s \notin I}).
\end{aligned} \tag{III.16}$$

To prove (III.15) we proceed by induction over $i + j$. By setting $\varphi_C(\emptyset; (\emptyset, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_j)) = 0$ we can start with $i + j = 1$, and clearly $\varphi_C((\Omega_1, \mathbf{X}_1); \emptyset) = S_{\Omega_1, \mathbf{X}_1}$ satisfies (III.15). Notice that $A((\Omega_1, \mathbf{X}_1), (\emptyset, \mathbf{Y}_s))f \neq 0$ only if Y_s overlaps or surrounds X_1. We say Y_s surrounds X_1 if X_1 is contained in a hole of Y_s and we will write Y_s os X_1. The number of clusters $\mathbf{Y}_s$ satisfying this condition with X_1 and $|Y_s| = l^2 M$ fixed is bounded by $l^{-2} |X_1| \exp O(1)M$.

Define $f(\tilde{\mathbf{X}}) = \sup\{|f(\mathbf{X})|_q : \mathbf{X} \in \tilde{\mathbf{X}}\}$. Then by using Lemma 8 (take $3\varepsilon \leq 2\delta$) and the induction hypothesis we obtain

$$\begin{aligned}
&\sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_j \\ \sum_s |Y_s| = l^2 N}} |\varphi_C((\Omega_1, \mathbf{X}_1), \ldots, (\Omega_i, \mathbf{X}_i); (\emptyset, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_j))|_q \\
&\quad \leq \sum_{I \subset \{1,\ldots,j\}} \sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_j \\ \sum_s |Y_s| = l^2 N \\ S \in I \Rightarrow Y_s \text{ os } X_1}} 2^{|I|} S_1((\Omega_1, \tilde{\mathbf{X}}_1)) \\
&\qquad \times \varphi_C(\{(\Omega_n, \tilde{\mathbf{X}}_n)\}_{n=2,\ldots,i}, \{(\emptyset, \tilde{\mathbf{Y}}_s)\}_{s \in I}; \{(\emptyset, \mathbf{Y}_s)\}_{s \notin I}) \\
&\quad \leq S_1((\Omega_1, \tilde{\mathbf{X}}_1)) \sum_{\substack{\mathbf{Y}_1,\ldots,\mathbf{Y}_j \\ \sum_s |Y_s| = l^2 N}} [\varphi_C(\{(\Omega_n, \tilde{\mathbf{X}}_n)\}_{n=2,\ldots,i}; \{(\emptyset, \mathbf{Y}_m)\}_{m=1,\ldots,j}) \\
&\qquad + 2^j \varphi_C(\{(\Omega_n, \tilde{\mathbf{X}}_n)\}_{n=2,\ldots,i}, \{(\emptyset, \mathbf{Y}_m)\}_{m=1,\ldots,j}; \emptyset) \\
&\qquad + \sum_{\substack{I \subset \{1,\ldots,j\} \\ 0 < |I| < j}} 2^{|I|} S_1((\Omega_1, \tilde{\mathbf{X}}_1)) \sum_{k=1}^{n-1} \sum_{\substack{(\mathbf{Y}_s)_{s \in I} \\ \sum_s |Y_s| = l^2 k \\ Y_s \text{ os } X_1}} \sum_{\substack{(Y_s)_{s \notin I} \\ \sum_s |Y_s| = l^2 (N-k)}} \\
&\qquad \times \varphi_C(\{(\Omega_n, \tilde{\mathbf{X}}_n)\}_{n=2,\ldots,i}, \{(\emptyset, \mathbf{Y}_s)\}_{s \in I}; \{(\emptyset, \mathbf{Y}_s)\}_{s \notin I})
\end{aligned} \tag{III.17}$$

$$\leq j! \prod_{r=1}^{i} C(\Omega_r, \mathbf{X}_r) \exp\left(2l^{-2} \sum_{r=2}^{i} |X_r|\right)\left[e^{-\varepsilon l N}\right.$$

$$\left. + \sum_{1<|I|} \frac{1}{|I|!} \sum_{k=1}^{N} \sum_{\substack{\sigma_s \geq 1 \\ \sum_{1\leq s \leq |I|} \sigma_s = k}} (2l^{-2} |X_1|)^{|I|} e^{(O(1)+2-3\varepsilon l)k} e^{-\varepsilon l(N-k)}\right]$$

$$\leq j! \prod_{r=1}^{i} C(\Omega_r, \mathbf{X}_r) \exp\left(2l^{-2} \sum_{r=2}^{i} |X_r| - \varepsilon l N\right)$$

$$\times \left[1 + \sum_{1<|I|} \frac{1}{|I|!} \left((2l^{-2} |X_1|)^{|I|} \sum_{\sigma_1, \ldots, \sigma_{|I|} \geq 1} \exp\left(-2\varepsilon l \sum_s \sigma_s\right)\right)\right]$$

$$\leq j! \prod_{r=1}^{i} C(\Omega_r, \mathbf{X}_r) \exp\left(2l^{-2} \sum_{r=1}^{i} |X_r| - \varepsilon l N\right).$$

Proof of Lemma 5. By inserting the bounds (III.15) into (III.14) we obtain

$$|\langle\Phi\rangle|_q \leq \sum_{\substack{\{\mathbf{X}_1, \ldots, \mathbf{X}_i\} \\ \Omega \cap X_n \neq \emptyset,\ \Omega \subset X_1 \cup \cdots \cup X_i}} \sum_{\mathbf{Y}_1, \ldots, \mathbf{Y}_j} \frac{1}{j!} |\varphi_C((\Omega \cap X_1, \mathbf{X}_1), \ldots$$

$$\ldots, (\Omega \cap X_i, \mathbf{X}_i); (\emptyset, \mathbf{Y}_1), \ldots, (\emptyset, \mathbf{Y}_j))|_q$$

$$\leq O(1) \sum_{\substack{\{\mathbf{X}_1, \ldots, \mathbf{X}_i\} \\ X_m \cap X_n = \emptyset,\ m \neq n \\ \Omega \cap X_n \neq \emptyset,\ \Omega \subset X_1 \cup \cdots \cup X_i}} \prod_{r=1}^{i} C(\Omega \cap X_r, \mathbf{X}_r) \exp\left(2l^{-2} \sum_{r=1}^{i} |X_r|\right)$$

$$\leq C_1(m_\Phi, q) C_2(n_\Phi, q) \sum_{k=1}^{\infty} \binom{m_\Phi(\Omega) + n_\Phi(\Omega)}{k} \left(\sum_{\sigma=1}^{\infty} \exp\left(O(1) + 2 - 3\varepsilon l)\sigma\right)\right)^k$$

$$\leq C_1(m_\Phi, q) C_2(n_\Phi, q) 2^{m_\Phi(\Omega) + n_\Phi(\Omega)}. \tag{III.17}$$

The third inequality follows since the number of clusters $\mathbf{X}$ with $|X| = l^2\sigma$ and containing a fixed square $\Delta \subset \Omega$ is bounded by $e^{O(1)\sigma}$, while the number of choices of k squares in Ω is bounded by $\binom{m_\Phi(\Omega) + n_\Phi(\Omega)}{k}$. The factor $2^{m_\Phi(\Omega)+n_\Phi(\Omega)}$ can be absorbed into C_1C_2. This proves Lemma 5.

Proof of Lemma 6. We insert (III.14) into (II.6) and obtain the expansion

$$\langle\Phi(R_1); \ldots; \Phi(R_s)\rangle = -\sum_{p=1}^{s} \frac{(-1)^p}{p} \sum_{\substack{\sigma_1 \cup \cdots \cup \sigma_p = \{1, \ldots, s\} \\ \sigma_j \neq \emptyset}} \prod_{k=1}^{p}$$

$$\times \sum_{\substack{\{\mathbf{X}_1^k, \ldots, \mathbf{X}_{i_k}^k\} \text{pos.} \\ [\Omega^k \cap X_j^k = \phi, \Omega^k \subset X_1^k \cup \ldots \cup X_{i_k}^k}} \sum_{\{\mathbf{Y}_1^k, \ldots, \mathbf{Y}_{j_k}^k\} \text{ pos.}}$$

$$\times \varphi_C^k((\Omega^k \cap X_1^k, \mathbf{X}_1^k), \ldots, (\Omega^k \cap X_{i_k}^k, \mathbf{X}_{i_k}^k); (\emptyset, \mathbf{Y}_1^k), \ldots, (\emptyset, \mathbf{Y}_{j_k}^k))]$$

where for the definition of φ_C^k the fields $\Phi_{\Omega'}$ in (II.12) are replaced by $\Phi(\bigcup_{i\in\sigma_k} R_k)_{\Omega_k'}$. By the proofs of the Lemmas 5 and 9 we know that this sum is absolutely convergent, also if the φ_C^k are expressed as a sum of products of functions S_1. We may thus sum up first all contributions from sets $C = (\mathbf{X}_1^1, \ldots, \mathbf{X}_{i_p}^p, \mathbf{Y}_1^1, \ldots, \mathbf{Y}_{j_p}^p)$ which factorize, i.e. for which we have two regions

$G_1, G_2 \subset \mathbb{R}^2$ without hole separated by a positive distance, and a partition (C_1, C_2) of C such that $Z \subset G_1$ for $\mathbf{Z} \in C_1$ and $Z \subset G_2$ for $\mathbf{Z} \in C_2$. Notice that if the argument A of a φ_C^k only contains clusters of a factorizing set C, then $\varphi_C^k(A)$ factorizes accordingly to C into a product $\varphi_C^k(A_1) \cdot \varphi_C^k(A_2)$. From a standard argument using formal power series [R], [EMS] we conclude that this first sum is zero. From the remaining terms we can extract a factor $e^{-\delta d(\Omega_1, \ldots, \Omega_s)}$ by using Lemma 8. The assertion now follows since $C_1(\cdot, q)$, $C_2(\cdot, q)$ are concave and since $|f_1 \otimes f_2|_q = |f_1|_q \cdot |f_2|_q$ and

$$\sum_{\sigma_1 \cup \cdots \cup \sigma_p = \{1, \ldots, s\}} \frac{1}{p} \leq s^s.$$

IV. Proof of Lemma 7

We assume that $|\Sigma \cap N(\mathfrak{L}_i)| = 0$ for all $i \in I$ (otherwise $F_{\Omega', \mathbf{X}, \Sigma} = 0$) and omit the r-factors since they are bounded by

$$\left|\prod_{i \in \mathfrak{I}} r_i^{|\Sigma \cap N(\mathfrak{L}_i)|}\right| \leq ((e^{3L})^{(2L+1)l^{-1}})^{|\Sigma|}$$
$$\leq e^{7l^{15}|\Sigma|} e^{o(\lambda^{-1/2})|\Sigma|},$$

and will be dominated by a factor $e^{-b\lambda^{-1/2}|N(\Sigma)|}$, $b > 0$. Let $B_0 = \{(j, \alpha) : j \in l\mathbb{Z}^4 \cap X, \alpha \in \pi_0, \alpha(i) = 0 \text{ if } \mathfrak{L}_i \cap X = \emptyset\}$. By expanding the product over $\beta \in B_0$ we obtain the following expression for $\delta_s^\Gamma F_{\Omega', \mathbf{X}, \Sigma}(r, t, h, s)$

$$\delta_s^\Gamma F_{\Omega', \mathbf{X}, \Sigma}$$
$$= \sum_{B \subset B_0} \delta_s^\Gamma \int d\psi_{C(t,s)} e^{-F(\Lambda \cap X, \Sigma)} \prod_{\beta \in B} [h(\alpha) \partial_t^\alpha C_j(t, s) \cdot \Delta_\phi] \Phi_{\Omega'} \chi_\Sigma e^{-V(\Lambda \cap X)}.$$

The derivatives are computed as in (II.1), leading to

$$\delta_s^\Gamma F_{\Omega', \mathbf{X}, \Sigma} = \sum_{\Gamma_1 \cup \Gamma_2 = \Gamma} \sum_{B \subset B_0} \int_{\{0\}}^{\{1\}} ds \int d\psi_{C(t, s(\Gamma))} \left[\sum_{\pi \in P(\Gamma_2)} \prod_{\gamma \in \pi} \partial_s^\gamma C(t, s(\Gamma)) \cdot \Delta_\psi\right]$$
$$\times e^{-F(\Lambda \cap X, \Sigma)} \left[\sum_{\cup_{\beta \in B} \gamma_\beta = \Gamma_2} \prod_{\beta \in B} h(\alpha) \partial_s^{\gamma_\beta} \partial_t^\alpha C_j(t, s(\Gamma)) \cdot \Delta_\phi\right] \Phi_{\Omega'} \chi_\Sigma e^{-V(\Lambda \cap X)}$$

where $P(\Gamma_2)$ is the set of partitions of Γ_2. We also write the kernels $\partial_s^\gamma C$ as a sum over localizations in the $l\mathbb{Z}^2$ lattice

$$\delta_s^\Gamma F_{\Omega', \mathbf{X}, \Sigma} = \Sigma' \int d\psi_{C(t, s(\Gamma))} \left[\prod_{\gamma \in \pi} \partial_s^\gamma C_{j_\gamma}(t, s(\Gamma)) \cdot \Delta\psi\right] e^{-F(\Lambda \cap X, \Sigma)}$$
$$\times \left[\prod_{\beta \in B} h(\alpha) \partial_s^{\gamma_\beta} \partial_t^\alpha C_j(t, s(\Gamma)) \cdot \Delta_\phi\right] \Phi_{\Omega'} \chi_\Sigma e^{-V(\Lambda \cap X)} \qquad \text{(IV.1)}$$

where

$$\Sigma' = \sum_{\Gamma_1 \cup \Gamma_2 = \Gamma} \sum_{B \subset B_0} \sum_{\cup_{\beta \in B} \gamma_\beta = \Gamma_1} \sum_{\pi \in P(\Gamma_2)} \sum_{\{j_\gamma\}_{\gamma \in \pi}}.$$

Each term in this sum is of the form

$$\Sigma'' \int d\psi_{C(t,s(\Gamma))} W\Phi' \chi'_\Sigma e^{-V(\Lambda\cap X)-F(\Lambda\cap X,\Sigma)}. \tag{IV.2}$$

Σ'' is obtained and interpreted as follows. By definition $d/d\phi : \phi^n := n : \phi^{n-1}:$ and this is rewritten as $\Sigma : \phi^{n-1}:$. This convention is inductively used for all derivatives $d/d\phi$ acting on $\Phi_{\Omega'}\chi_\Sigma e^{-V}$ or on derivatives of this expression. Every field ϕ is then rewritten as a function of ψ, $\phi(x) = \psi(x) + (\xi - g(x))$, producing again a sum of terms. Now we perform the derivatives $d/d\psi$ in the same way.

In order to bound the norm $|\delta_s^\Gamma F_{\Omega',\mathbf{X},\Sigma}|_q$, $q>1$, associated to a given partition π_1 of $\{1, \ldots, N\}$ we bound the product

$$\delta_s^\Gamma F_{\Omega',\mathbf{X},\Sigma}[\omega] = \langle (\delta_s^\Gamma F_{\Omega',\mathbf{X},\Sigma})_{\pi_1}, \omega \rangle$$

for functions $\omega \in L^p$, $p^{-1} + q^{-1} = 1$.

By Hölder's inequality

$$|\delta_s^\Gamma F_{\Omega',\mathbf{X},\Sigma}[\omega]| \le \Sigma'\Sigma'' \|W\Phi'[\omega]\|_r \|\chi'_\Sigma \exp(-V(\Lambda\cap X) - F(\Lambda\cap X, \Sigma))\|_{r'} \tag{IV.3}$$

where r and r′ are dual Hölder indices and r is some sufficiently large even integer ($r \simeq 35\,000$). The last factor in (IV.3) is bounded as in [GJS III] by

$$\|\chi'_\Sigma e^{-V(\Lambda\cap X)-F(\Lambda\cap X,\Sigma)}\|_{r'}$$
$$\le \Big(\prod_{\square\subset X\cap\Lambda} \nu(\square)!\Big) \exp(a'\lambda^{1/2}|X\cap\Lambda|) \exp(-3b\lambda^{-1/2}(|N(\Sigma)| + |X'|)) \tag{IV.4}$$

for some positive constants a', b, where $\nu(\square)$ is the number of times $\chi_{\Sigma(\square)}$ has been derived, and where

$$X' = \bigcup_{\nu(\square)>0} \square.$$

Each covariance is now written as a sum of terms localized (in both variables) in unit lattice squares. This induces an expansion of $W\Phi'[\omega]$ into a sum

$$W\Phi'[\omega] = \sum_u W_u \Phi'[\omega] \tag{IV.5}$$

of Wick monomials localized in unit squares. The resulting number of terms is bounded by

$$\sum_u 1 \le \prod_{\gamma\in\pi} l^4 \prod_{\beta\in B} l^4 \tag{IV.6}$$

and the monomials in W_u are smeared out with localized products of functions $(\xi - g)$ and $\partial_s^\gamma \partial_t^\alpha C$.

In order to bound the resulting graphs and the sum Σ' we use the decay properties of kernels $\partial_s^\gamma \partial_t^\alpha C_j$ established in [S II], adapted to the case of unit mass and large length scale. Define

$$i(\alpha) = \min\{i \in l\mathbb{Z} : \alpha(i) \neq 0\},$$
$$d(\beta) = d(j, \alpha) = \mathrm{dist}(\Delta_{j_1}, \mathfrak{L}_{i(\alpha)}) + \mathrm{dist}(\Delta_{j_2}, \mathfrak{L}_{i(\alpha)}) + \mathrm{dist}(\Delta_{j_1}, \Delta_{j_2}),$$

$d(j,\gamma)=\max\{\text{dist}(\Delta_{j_1},b)+\text{dist}(\Delta_{j_2},b):b\in\gamma\}$,
$d(\gamma,\alpha)=\min\{\text{dist}(b,\mathfrak{L}_{i(\alpha)}):b\in\gamma\}$,
$d(\gamma)$: the length of the shortest path in $\mathbb{R}^2$ connecting all $b\in\gamma$.

Lemma 10. *Let* $p\geq 1$ *and* $\varepsilon>0$ *be given. Then for* l_0 *sufficiently large there are positive constants* $M(\gamma,\beta)$, M_3 *and* c *such that for* $l\geq l_0$.

$$\|\partial_s^\gamma\partial_t^\alpha C_j(t,s)\|_p\leq l^{4/p}M_2(\gamma,\beta)e^{-(1-\varepsilon)d(\alpha)} \times e^{-6cd(j,\alpha)}e^{-5c(2d(\gamma)+3d(j,\gamma))} \tag{IV.7}$$

and (see also [GJS I])

$$\sum_{\pi\in P(\Gamma_2)}\prod_{\gamma\in\pi}M_2(\gamma,(\alpha=0,j))\leq e^{M_3|\Gamma_2|l^{-1}}, \tag{IV.8}$$

$$\sum_{\bigcup_{\beta\in B}\gamma_\beta=\Gamma_1}M_2(\gamma_\beta,\beta)\leq e^{M_3|X|l^{-2}}. \tag{IV.9}$$

Furthermore one has

$$\sum_{\{j\gamma\}_{\gamma\in\pi}}\prod_{\gamma\in\pi}e^{-cd(j_\gamma,\gamma)}\leq e^{M_7|\Gamma_2|l^{-1}}, \tag{IV.10}$$

$$\sum_{B\subset B_0}\prod_{\beta\in B}e^{-cd(j,\alpha)}\leq e^{M_0|X|l^{-2}} \tag{IV.11}$$

for some constants $M_0, M_7>0$. (IV.7) *remains true if the kernel is regarded as a function of a single variable* $\partial_s^\gamma\partial_t^\alpha C_j(t,s)(x,x)$, $\gamma\neq 0$ *or* $\alpha\neq 0$.

By using Lemma 10 we bound $\|W\Phi'[\omega]\|_r$ times the first factor of (IV.4) as follows. Let

$P(\Delta)=\{(j,\alpha)\in B:\Delta\subset\Delta_{j_1}\cup\Delta_{j_2}\}\cup\{(j_\gamma,\gamma):\gamma\in\pi,\ \Delta\subset\Delta_{j_{\gamma,1}}\cup\Delta_{j_{\gamma,2}}\}$,
$M(\Delta)=\text{card}\,P(\Delta)$,
$M_-(\Delta)=\text{card}\{(j,\alpha)\in B:\Delta\subset\Delta_{j_1}\cup\Delta_{j_2},\ (\xi-g)\restriction\Delta\neq 0\}$,
$M_+(\Delta)=M(\Delta)+m_{\Phi_{\Omega'}}(\Delta)$,

and let $m=m_{\Phi_{\Omega'}}$, $n=n_{\Phi_{\Omega'}}$. Then

$$\begin{aligned}\prod_{\square\subset X\cap\Lambda}\nu(\square)!\,\|W\Phi'[\omega]\|_r&\leq\|\omega\|_p C_3(0,\Omega',q)e^{26l^4|X'|}\\ &\times C(\mathbf{X},\Sigma)^{n(\Omega')}e^{(1/2+6c)ln(\Omega')}\prod_{\square\subset X\cap\Lambda}\nu(\square)!\,[(rn(\square))!]^{1/r}\\ &\times\prod_{\Delta\subset X}\Big[(4r(M_+(\Delta)-\sum_{\square\subset\Delta}\nu(\square)))!\Big]^{1/r}M_1(p,r)^{M_+(\Delta)}C(\mathbf{X},\Sigma)^{3M_-(\Delta)}\\ &\times\prod_{\gamma\in\pi}M_2(\gamma,(0,j_\gamma))e^{-5cd(j_\gamma,\gamma)}e^{-2cl|\gamma|}\\ &\times\prod_{\beta\in B}M_2(\gamma_\beta,\beta)e^{-5cd(\beta)}e^{-2cl|\gamma_\beta|}\end{aligned} \tag{IV.12}$$

where we have made the substitution

$$h(\alpha)e^{-(1-\varepsilon)d(\alpha)}e^{-cd(j,\alpha)}$$
$$\times\prod_{\gamma\in\pi} l^8 e^{-10[d(\gamma)+d(j_\gamma,\gamma)]}\prod_{\beta\in B} l^8 e^{-10c[d(\gamma_\beta)+d(j,\gamma_\beta)]} \tag{IV.13}$$
$$\to e^{26l^4|X'|}e^{(1/2+6c)\ln(\Omega')}e^{-2cl|\Gamma|}C_3(0,\Omega',q).$$

Except for this substitution and for the factors $C(\mathbf{X},\Sigma)$ arising when $\phi\neq\psi$, (IV.13) is a standard estimate (see [GJS I], but with separate copies of unit squares for the monomials of W_u and $\Phi'[\omega]$). The convergence factors allowing for (IV.13) are obtained as follows. We use

(1) for terms from contractions $\partial_s^\gamma C_{j_\gamma}\cdot\Delta_\psi$

$$l^8 e^{-10c[d(\gamma)+d(j_\gamma,\gamma)]}\le e^{-2cl|\gamma|}\quad\text{if}\quad d(\gamma)+d(j_\gamma,\gamma)>0.$$

If $d(\gamma)+d(j_\gamma,\gamma)=0$ then $|\gamma|\le 4$ and $F(\Delta_{j_{\gamma,1}}\cup\Delta_{j_{\gamma,2}},\Sigma)=0$, and we use

$\lambda^{\varepsilon/4}l^4<e^{-cl|\gamma|}$ when the contraction is to V.

$l^4<e^{-cl|\gamma|}e^{l^4|\square\cap X'|}$ when the contraction is to $\chi_{\Sigma(\square)}$.

$l^4<e^{-cl|\gamma|}e^{5cl}$ when the contraction is to $\Phi_{\Omega'}$.

The same holds if (j_γ,γ) is replaced by (j,γ_β).

(2) for terms from contractions $\partial_t^\alpha C_j\cdot\Delta_\phi$

$(|h(\alpha)|\le e^{(1-2\varepsilon)(d(\alpha)+l)})$

$l^8e^{-3\varepsilon l}<1$ if $d(\alpha)\ge\varepsilon^{-1}l$.

$l^8e^le^{-cd(j,\alpha)}<1$ if $d(j,\alpha)\ge 2c^{-1}l$.

If $d(\alpha)<\varepsilon^{-1}l$ and $d(j,\alpha)<2c^{-1}l$ then we use

$\lambda^{\varepsilon/4}l^4e^{l/2}<1$ when the contraction is to V.

$l^4e^{l/2}<e^{l(1/2+c)}$ when the contraction is to $\Phi_{\Omega'}$.

$l^4e^{l/2}<(e^{l^4|\square\cap X'|})^{1/2Kl^2}$ for $K=24/c\varepsilon$, when the contraction is to $\chi_{\Sigma(\square)}$. This is sufficient since the number of β's in B such that $d(\alpha)<\varepsilon^{-1}l$ and $d(j,\alpha)<2c^{-1}l$ and $\square\subset\Delta_{j_1}\cup\Delta_{j_2}$ is bounded by Kl^2.

(3) If a contraction $d/d\phi$ from $\Phi_{\Omega'}$ is to V then we get a factor $\lambda^{1/2-\varepsilon}$, and if it is to $\chi_{\Sigma(\square)}$ we use that

$$1<\lambda^{1/2-\varepsilon}e^{l^4|\square\cap X'|}.$$

Next we use a similar argument to cancel the factor $C(\mathbf{X},\Sigma)^{3M_-(\Delta)}$. Namely for each contraction $\partial_s^{\gamma_\beta}\partial_t^\alpha C_j\cdot\Delta_\phi$ with $(\xi-g)\restriction(\Delta_{j_1}\cup\Delta_{j_2})\neq 0$ we have $d(\beta)>L$ and thus

$$C(\mathbf{X},\Sigma)^6e^{-cd(\beta)}<1$$

since $C(\mathbf{X},\Sigma)=O(\lambda^{-1/2})$ and $L\simeq(\log\lambda)^2$.

We continue by estimating (IV.3). The sum Σ' is controlled by (IV.8), ..., (IV.11) and a factor $e^{-cl|\Gamma|}$ (notice that $\sum_{\Gamma_1\cup\Gamma_2=\Gamma}1=2^{|\Gamma|}$). There remains a factor

$$e^{-cl|\Gamma|}e^{-b\lambda^{-1/2}(|N(\Sigma)|+|X'|)}e^{a''l^{-2}|X\cap\Lambda|}\|\omega\|_p C_3(0,\Omega',q) \tag{IV.14}$$

times the maximum over admissible Γ_i, B, γ_β, π and j_γ of

$$\Sigma''\prod_{\square}\nu(\square)!\prod_{\Delta}\left[\left(4r\left(M_+(\Delta)-\sum_{\square\subset\Delta}\nu(\square)\right)\right)!\right]^{1/r}M_1^{M_+(\Delta)} \\ \times\prod_{\beta\in B}e^{-4cd(\beta)}\prod_{\gamma\in\pi}e^{-4cd(j_\gamma,\gamma)}, \tag{IV.15}$$

where we have put the factors

$$C(\mathbf{X},\Sigma)^{n(\Omega')}e^{(1/2+6c)\ln(\Omega')}\prod_{\square}[(rn(\square))!]^{1/r}\cdot C_3(0,\Omega',q)$$

into $C_3(n,\Omega',q)$ by using $(ab)!\leq a^{ab}(b!)^a$. This inequality together with the facts that $a!\,b!\leq(a+b)!$, $a!\leq a^a$ and $(a+b)!\leq(2a)!\,(2b)!$ implies that

$$\prod_{\square}\nu(\square)!\prod_{\Delta}\left[\left(4r\left(M_+(\Delta)-\sum_{\square\subset\Delta}\nu(\square)\right)\right)!\right]^{1/r}M_1^{M_+(\Delta)} \\ \leq\prod_{\Delta}(4^4r^4M_1)^{M(\Delta)+m(\Delta)}((2M(\Delta))!)^4((2m(\Delta))!)^4 \\ \leq e^{O(1)m(\Omega')^{4/3}}\prod_{\Delta}e^{O(1)M(\Delta)^{4/3}} \tag{IV.16}$$

In a similar way we bound the number of terms coming from differentiations and from expressing V′ and $\Phi_{\Omega'}$ in terms of ψ:

$$\Sigma''1\leq\prod_{\square}2^{n(\square)}\prod_{\Delta}K^{M(\Delta)} \\ \times(n(\Delta)+|\Delta\cap X'|+4)(n(\Delta)+|\Delta\cap X'|+8)\cdots(n(\Delta)+|\Delta\cap X'|+4M_+(\Delta)). \tag{IV.17}$$

By using

$$(N+4)(N+8)\cdots(N+4M)\leq(N+4M)^M \\ \leq(4M)^M\left(1+\frac{N}{4M}\right)^M \\ \leq(4M)^Me^{N/4}$$

the right side of (IV.17) can be bounded as follows.

$$\Sigma''1\leq e^{n(\Omega')+1/4|X'|}e^{O(1)m(\Omega')^{4/3}}\prod_{\Delta}e^{O(1)M(\Delta)^{4/3}} \tag{IV.18}$$

Now by putting together (IV.14), ..., (IV.18) we arrive at the bound

$$|\delta_s^\Gamma F_{\Omega',\mathbf{X},\Sigma}[\omega]|\leq\|\omega\|_p\,C_1(m,q)C_3(n,\Omega',q) \\ \times\exp\left(-cl\,|\Gamma|-b\lambda^{-1/2}\,|N(\Sigma)|+a''l^{-2}\,|X\cap\Lambda|\right) \\ \times\sup_{B,\{\gamma_\beta\},\{\gamma\},\{j_\gamma\}}\prod_{\Delta}\left[e^{\rho M(\Delta)^{4/3}}\prod_{p\in P(\Delta)}e^{-2cd(p)}\right] \tag{IV.19}$$

The product $\prod_\Delta[\cdots]$ in (IV.19) is bounded by using the 'pushout principle' as in [S II], cf. also [GJS I]. This goes as follows. There are not more than $8(a+3)^2$

choices of $(j_\gamma, \gamma) \in P(\Delta)$ such that $d(j_\gamma, \gamma) \leq al$, and there are less than $24(a+3)^3$ choices of $\beta \in P(\Delta)$ such that $d(\beta) \leq al$. Thus by ordering the elements p_i of $P(\Delta)$ we can achieve

$$l^{-1}\, d(p_k) > \tfrac{1}{6} k^{1/3} - 3.$$

Or, by summing over k

$$l^{-1} \sum_{p \in P(\Delta)} d(p) > \tfrac{1}{8} M(\Delta)^{4/3} - 3M(\Delta)$$

and thus

$$\rho M(\Delta)^{4/3} \leq O(1) + \sum_{p \in P(\Delta)} cd(p)$$

This proves Lemma 7.

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