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A relativistic two-body model for hydrogen-like and positronium-like systems I

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Abstract. A new theory for the description of relativistic particles is applied to a two-body model for hydrogen-like and positronium-like systems. In this paper, we consider a simplified approach where the interactions due to the spins are not taken into account. It is shown that the spectrum predicted by this model is in agreement with the one obtained from the Klein–Gordon equation with the mass replaced by the reduced mass.

Introduction

The present paper is devoted to a study of a relativistic two-body model for the hydrogen atom, considered as two interacting spin $\frac{1}{2}$ particles of opposite electric charges. The model is based on a relativistic theory of spin $\frac{1}{2}$ particles previously developed [1] and already successfully applied in a single particle approach to the hydrogen atom (i.e. to a charged spin $\frac{1}{2}$ particle in an external electromagnetic Coulomb potential) [2]. Our model is constructed in the framework of a relativistic dynamics developed first by L. P. Horwitz and C. Piron [3], and for which a survey has been given in [4]. Briefly, these authors suggest a (many particl) relativistic canonical formalism based on the following ideas: Particles are not identified with trajectories in space-time, as is usual, but with points (events) in space-time. In order to describe the 'true' evolution of the system, one further postulates the existence of another time parameter τ, called the 'historical time'. Such a parameter τ has been introduced by several authors to develop a relativistically invariant classical mechanics. Recently, in this way, D. Dominici, J. Gomis, G. Longhi have proposed a Lagrangian formalism applied to two interacting relativistic particles [5], [6]. We also refer to the relativistically invariant classical Hamiltonian mechanics presented by P. M. Pearle [7].

The state of each particle is then characterized by eight independent numbers

$$q^\mu = (q^1, q^2, q^3, q^4) = (\vec{q}, t)$$

the position in space-time and

$$p_\mu = (p_1, p_2, p_3, p_4) = (\vec{p}, -E)$$

the momentum-energy of the particle.

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The evolution itself is governed by the canonical equations

$$\frac{dq^\mu_{(i)}}{d\tau}=\frac{\partial K}{\partial p_{(i)\mu}} \quad \text{and} \quad \frac{dp_{(i)\mu}}{d\tau}=-\frac{\partial K}{\partial q^\mu_{(i)}} \tag{1}$$

where K depends on the state of each particle (i). The covariance is guaranteed if K transforms like a scalar field under the action of the Poincaré group, the historical time τ being invariant by assumption. The metric tensor is given by $g_{\mu\nu}=(1, 1, 1, -c^2)$, where c denotes the velocity of light in vacuum.

For example, if we neglect the radiation phenomena, the evolution of a charged particle in an external electromagnetic field corresponding to a 4-vector potential $A_\mu(x)=(\vec{A}(x), -V(x))$, can be obtained from the following function K:

$$K=\frac{1}{2M}g^{\mu\nu}(p_\mu-eA_\mu(q))(p_\nu-eA_\nu(q)) \tag{2}$$

where e denotes the electric charge and M the mass of the particle. In this formalism, M is a dynamical constant characteristic of the particle. In this example, the canonical equations (1) read

$$\frac{dq^\mu}{d\tau}=\frac{p^\mu-eA^\mu(q)}{M} \quad \text{and} \quad \frac{dp_\mu}{d\tau}=e\frac{p^\nu-eA^\nu(q)}{M}\partial_\mu A_\nu(q)$$

where $\partial_\mu A_\nu$ denotes the partial derivative of A_ν with respect to x^μ.

Hence

$$g_{\mu\nu}\frac{dq^\mu}{d\tau}\frac{dq^\nu}{d\tau}=\frac{2K}{M}$$

and since K is a constant of the motion, the proper time takes the same values as the historical time for $K=-Mc^2/2$.

$$g^{\mu\nu}(p_\mu-eA_\mu(q))(p_\nu-eA_\nu(q))=-M^2c^2 \tag{3}$$

is satisfied for any τ.

The above relativistic formalism sketched for classical particles can be quantized in a way that avoids the usual difficulties associated with the Klein–Gordon equation and a new relativistic model of quantum spinless particle has then been obtained. We refer to [3] for further details. For the spin $\frac{1}{2}$ particle, a model has then been developed [1] that is based on the following physical interpretation of the spin $\frac{1}{2}$ in Relativity: Let us consider the Stern–Gerlach apparatus that measures the spin orientation and more precisely, let us consider the symmetry of the present magnetic field. Such a field is characterized by a strong gradient. However, the latter defines not only a direction for the spin but also a unique time-like direction, the direction of the time in the frame where the electromagnetic field is purely magnetic. Then the spin state of the particle is characterized by a direction in space (the spin) and a time-like 4-vector n^μ with $n^4>0$ and which we choose to be normalized to $n_\mu n^\mu=-c^2$. This 4-vector n^μ is, by assumption, a superselection rule [8]. Accordingly, the state space of the spin $\frac{1}{2}$ particle is given by a family of Hilbert spaces H_n indexed by n^μ. More precisely, to every time-like unit 4-vector n^μ, we associate a Hilbert space H_n isomorphic to

$\mathbb{C}^2 \otimes L^2(\mathbb{R}^4, d^4x)$ supplied with the scalar product

$$\langle \psi, \varphi \rangle = \int_{\mathbb{R}^4} d^4x \sum_{i=1}^{2} \psi_i^*(x)\varphi_i(x) = \int_{\mathbb{R}^4} d^4x\, \psi^+(x)\varphi(x)$$

In other words, each spin $\frac{1}{2}$ particle state is characterized by a 4-vector n^μ and a vector $\psi \in H_n$.

As regards to the observables, the position (in space-time) corresponds to the self-adjoint operators:

$$(q^\mu \psi)(x) = x^\mu \psi(x), \qquad \psi \in H_n \tag{4}$$

and the momentum-energy to the self-adjoint operators:

$$(p_\mu \psi)(x) = -i\hbar\, \partial_\mu \psi(x), \qquad \psi \in H_n \tag{5}$$

whose commutation relations are given by

$$i[p_\mu, q^\nu] = \hbar\, \delta_\mu^\nu$$

The spin observable corresponds in each H_n to a set of four matrices W_n^μ. For $n^\mu = n_0^\mu \equiv (0, 0, 0, 1)$ these matrices are given by

$$W_{n_0}^i = \tfrac{1}{2}\sigma^i, \qquad i = 1, 2, 3 \quad \text{and} \quad W_{n_0}^4 = 0$$

where σ^i are the Pauli matrices. For an arbitrary n^μ, one sets

$$W_n^\mu = L(n)_\nu^\mu W_{n_0}^\nu \tag{6}$$

where $L(n)$ are Lorentz boosts satisfying $L(n)_\nu^\mu n_0^\nu = n^\mu$.

This definition is justified by the following considerations: if we consider a space-like 4-vector s^μ with

$$s_\mu s^\mu = 1 \quad \text{and} \quad n_\mu s^\mu = 0$$

and the observable defined by the operator $s_\mu W_n^\mu$, then the corresponding two eigenstates associated with a measurement of the spin with a Stern–Gerlach apparatus are such that

(i) the time direction is given by n^μ
(ii) the direction of the magnetic field gradient is given by s^μ, i.e. $(\vec{s}_0, 0) = L^{-1}(n)_\nu^\mu s^\nu$ in the reference frame of the apparatus.

This follows immediately from the relation

$$s_\mu W_n^\mu = s_\mu L(n)_\nu^\mu W_{n_0}^\nu = \tfrac{1}{2}\vec{s}_0 \vec{\sigma}$$

From the definition (6) follows immediately the relation

$$W_n^\mu n_\mu = 0$$

the commutation relations

$$[W_{n\mu}, W_{n\nu}] = i\varepsilon_{\mu\nu\rho\lambda} W_n^\rho n^\lambda$$

where $\varepsilon_{\mu\nu\rho\lambda}$ denotes the totally antisymmetric tensor with $\varepsilon_{1234} = 1$, and the anticommutation relations

$$\{W_{n\mu}, W_{n\nu}\} = \tfrac{1}{2}(g_{\mu\nu} + n_\mu n_\nu / c^2)$$

(7)

What concerns the covariance, the Lorentz group acts on the state space by means of operators $U(\Lambda)$ in the following way

$$(U(\Lambda)\psi)_{\Lambda n}(x) = D(L(\Lambda n)^{-1}\Lambda L(n))\psi_n(\Lambda^{-1}x) \tag{8}$$

where D denotes the usual spin $\frac{1}{2}$ unitary ray representation of the rotation group. A Lorentz transformation Λ thus maps the Hilbert space H_n onto $H_{\Lambda n}$.

Furthermore, one easily verifies the covariance of the observables, that is

$$\begin{aligned} U(\Lambda)^{-1}q^\mu U(\Lambda) &= \Lambda^\mu_\nu q^\nu \\ U(\Lambda)^{-1}p_\mu U(\Lambda) &= \Lambda^\nu_\mu p_\nu \\ U(\Lambda)^{-1}W^\mu_n U(\Lambda) &= \Lambda^\mu_\nu W^\nu_n \end{aligned} \tag{9}$$

As in the classical case, the evolution of the particle is parametrized by the historical time τ. As we are in a Schrödinger picture this evolution is governed by a Schrödinger-like equation:

$$i\hbar \frac{d}{d\tau}\psi_\tau = K_{n_\tau}\psi_\tau \tag{10}$$

In addition, because of the existence of the superselection rule n^μ, this equation is connected with a second equation of the form

$$\frac{dn^\mu_\tau}{d\tau} = f^\mu(n_\tau, \psi_\tau) \tag{11}$$

where the f^μ fulfil the condition $n_\mu f^\mu(n, \psi) = 0$. Obviously, in (10), K_n is a self-adjoint operator, scalar, covariant under the action of the Poincaré group.

It is important to note at this point that the dependence on ψ_τ in (11) may involve irreversible processes. On the other hand the comparison of this model with the one by Dirac [1] suggests that the evolution is such that n^μ_τ tends to be parallel to $\langle p^\mu \rangle_{\psi_\tau}$, the mean value of the momentum-energy observable and in many cases both directions can simply be assumed to be parallel.

Finally, it is important for what follows, to remember the explicit form of the operator K_n that corresponds to a spin $\frac{1}{2}$ particle in an external electromagnetic field described by the 4-vector potential $A_\mu(x) = (\vec{A}(x), -V(x))$. When radiation phenomena are neglected, the operator K_n is that of the spin 0 case, modified by terms due to the interaction of the spin with the electromagnetic field in particular, we have proposed in [1] the following operator

$$\begin{aligned} K_n = {} & \frac{1}{2M} g^{\mu\nu}(p_\mu - eA_\mu(q))(p_\nu - eA_\nu(q)) - \frac{g_1\mu_0}{Mc^2}(p^\mu - eA^\mu(q))\tilde{F}_{\mu\nu}(q)W^\nu_n \\ & + \frac{g_2^2\mu_0^2}{8Mc^4} F_{\mu\nu}(q)n^\nu F^\mu{}_\rho(q)n^\rho - \frac{g_3\mu_0}{c^2} n^\mu \tilde{F}_{\mu\nu}(q)W^\nu_n \end{aligned} \tag{12}$$

where e denotes the electric charge of the particle, M its mass, $\mu_0 = e\hbar/2M$ is the Bohr magneton, g_1, g_2, g_3 are dimensionless phenomenological constants and

$$\tilde{F}_{\mu\nu}(x) = -\frac{c^2}{2}\varepsilon_{\mu\nu}{}^{\rho\lambda}F_{\rho\lambda}(x)$$

is the dual tensor of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This expression for K_n has been applied, in particular it can be seen that, in the semi-classical approximation, it leads to the BMT equation [9].

1. The system of two relativistic spin $\frac{1}{2}$ particles

In the framework of quantum mechanics the states of a system of two particles are generally assumed to be described by rays in the tensor product of both Hilbert spaces corresponding to these particles. Characteristics of such a description are the quantum correlations between both particles.

In the above context, in accordance with our interpretation of the spin, the states of two distinguishable particles (1) and (2) having quantum correlations, are assumed to correspond to common 4-vectors $n^\mu = n^\mu_{(1)} = n^\mu_{(2)}$ for the superselection rules of both particles and to be given by the rays of the Hilbert spaces $H_n \otimes H_n$: i.e. by a 4-vector n^μ and a four-component wave function in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^4_{(1)} \times \mathbb{R}^4_{(2)}, d^4x_{(1)}\, d^4x_{(2)})$ supplied with the scalar product:

$$\langle \psi, \varphi \rangle = \int_{\mathbb{R}^4_{(1)} \times \mathbb{R}^4_{(2)}} d^4x_{(1)}\, d^4x_{(2)} \psi^+(x_{(1)}, x_{(2)}) \varphi(x_{(1)}, x_{(2)}) \tag{13}$$

On this state space, the Lorentz group acts by means of operators $U(\Lambda)$, in the following way:

$$(U(\Lambda)\psi)_{\Lambda n}(x_{(1)}, x_{(2)}) = (D(n, \Lambda) \otimes D(n, \Lambda))\psi_n(\Lambda^{-1}x_{(1)}, \Lambda^{-1}x_{(2)}) \tag{14}$$

where the $D(n, \Lambda)$ denote the same 2×2 matrices $D(L(\Lambda n)^{-1}\Lambda L(n))$ as in (8).

The observables position in space-time, momentum-energy and spin for the particles (1) and (2), respectively, correspond to the following families of self-adjoint operators

$$\begin{aligned} q^\mu_{(1)}\psi(x_{(1)}, x_{(2)}) &= x^\mu_{(1)}\psi(x_{(1)}, x_{(2)}), \qquad \psi \in H_n \otimes H_n \\ p_{(1)\mu}\psi(x_{(1)}, x_{(2)}) &= -i\hbar\, \partial^{(1)}_\mu \psi(x_{(1)}, x_{(2)}) \\ W^\mu_{(1)} &= W^\mu_n \otimes \mathbb{1}_{\mathbb{C}^2} \end{aligned}$$

and (15)

$$\begin{aligned} q^\mu_{(2)}\psi(x_{(1)}, x_{(2)}) &= x^\mu_{(2)}\psi(x_{(1)}, x_{(2)}) \\ p_{(2)\mu}\psi(x_{(1)}, x_{(2)}) &= -i\hbar\, \partial^{(2)}_\mu \psi(x_{(1)}, x_{(2)}) \\ W^\mu_{(2)} &= \mathbb{1}_{\mathbb{C}^2} \otimes W^\mu_n \end{aligned}$$

where $\partial^{(1)}_\mu$ and $\partial^{(2)}_\mu$ denote partial derivative with respect to $x^\mu_{(1)}$ and $x^\mu_{(2)}$, respectively. It is straightforward to verify that these operators have transformation properties analogous to those indicated in (9) for one particle.

The evolution of such a system is again parametrized by the 'historical time' τ and is governed by a Schrödinger-like equation

$$i\hbar \frac{d}{d\tau} \psi_\tau = K_{n_\tau}\psi_\tau \tag{16}$$

which is (generally) connected with an evolution equation for n^μ. Again K_n is assumed to be a self-adjoint operator, scalar, covariant under the action of the Poincaré group.

2. A two body model of hydrogen-like system: a first approach without spin interaction

Our aim is now to propose and to discuss a model for two interacting particles of opposite electric charges where the interactions due to the spins are first neglected. More precisely we are interested in the bound states of such a system, i.e., roughly speaking, in those states where both particles 'move together' in space time in a 'stable' way. That is, in view of 1, and of the above remark concerning the evolution of the superselection rule n^μ, we restrict ourselves (the consistency of this assumption will become clear later on) to the case $n^\mu_{(1)} = n^\mu_{(2)} = n^\mu$, for which we propose the following self-adjoint operator:

$$K_n = \frac{g^{\mu\nu}}{2M_{(1)}}(p_{(1)\mu} - \lambda_{(1)}eA_{(1)\mu}(q))(p_{(1)\nu} - \lambda_{(1)}eA_{(1)\nu}(q)) + \frac{g^{\mu\nu}}{2M_{(2)}}(p_{(2)\mu} - \lambda_{(2)}eA_{(2)\mu}(q))(p_{(2)\nu} - \lambda_{(2)}eA_{(2)\nu}(q)) \tag{17}$$

where q^μ stands for $q^\mu_{(2)} - q^\mu_{(1)}$, $\lambda_{(1)}$ and $\lambda_{(2)}$ are dimensionless constants and

$$A_{(2)\mu}(x) = -A_{(1)\mu}(-x) = \frac{e}{4\pi\varepsilon_0}\frac{n_\mu/c^2}{d(n, x)} \tag{18}$$

with

$$d(n, x) = \sqrt{x_\mu x^\mu + (n_\mu x^\mu)^2/c^2}$$

In these expressions e, $M_{(1)}$ and $-e$, $M_{(2)}$ respectively denote the charges and the masses of the particles (1) and (2) and ε_0 is the vacuum dielectric constant.

The 4-vector fields $A_{(2)\mu}(x)$ ($A_{(1)\mu}(-x)$) in (18) are simply the Liénard–Wiechert 4-potentials corresponding to a charge e ($-e$) whose motion is uniform and in the direction $n^\mu_{(1)} = n^\mu$ ($n^\mu_{(2)} = n^\mu$). In addition $A_{(i)\mu}(x) =$ verifies

$$\partial_\mu A^\mu_{(i)}(x) \equiv 0, \qquad i = 1, 2 \tag{19}$$

and also fulfil the Coulomb gauge condition relatively to n^μ.

Hence in this model each particle interacts with the other one via a 4-vector potential that looks by assumption like in the external approximation, except for the dimensionless constants $\lambda_{(1)}$ and $\lambda_{(2)}$. The role of these constants is to balance with respect to the masses the 'potential momentum-energy' due to the electromagnetic interaction. These constants are assumed to depend only on the masses and their values will be discussed later on. The introduction of such constants in the problem of two interacting particles has already been suggested by L. Brillouin and applied by L. de Broglie in the Galilean case [10] for interactions described by potentials.

It is convenient to introduce at this point the following observables:

(i) total momentum-energy

$$P_\mu = p_{(1)\mu} + p_{(2)\mu} \tag{20}$$

(ii) 'center of mass' position

$$Q^\mu = \frac{M_{(1)}}{M} q^\mu_{(1)} + \frac{M_{(2)}}{M} q^\mu_{(2)}, \qquad M = M_{(1)} + M_{(2)} \tag{21}$$

(iii) relative momentum-energy

$$p_\mu = \frac{M_{(1)}}{M} p_{(2)\mu} - \frac{M_{(2)}}{M} p_{(1)\mu} \tag{22}$$

Correspondingly, we change the coordinates as follows:

$$X^\mu = \frac{M_{(1)}}{M} x^\mu_{(1)} + \frac{M_{(2)}}{M} x^\mu_{(2)} \quad \text{and} \quad x^\mu = x^\mu_{(2)} - x^\mu_{(1)} \tag{23}$$

The states are then described by functions $\psi(X, x)$ in

$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(\mathbb{R}^4_X \times \mathbb{R}^4_x, d^4X\, d^4x)$

with the scalar product

$$\langle \psi, \varphi \rangle = \int_{\mathbb{R}^4_X \times \mathbb{R}^4_x} d^4X\, d^4x\, \psi^+(X, x)\varphi(X, x) \tag{24}$$

Furthermore, we now have

$$\begin{aligned} P_\mu \psi(X, x) &= -i\hbar \frac{\partial}{\partial X^\mu} \psi(X, x) \\ Q^\mu \psi(X, x) &= X^\mu \psi(X, x) \\ p_\mu \psi(X, x) &= -i\hbar \frac{\partial}{\partial x^\mu} \psi(X, x) \\ q^\mu \psi(X, x) &= x^\mu \psi(X, x) \end{aligned} \tag{25}$$

These operators satisfy the commutation relations

$$i[P_\mu, Q^\nu] = \hbar \mathbb{1} \delta^\nu_\mu \quad \text{and} \quad i[p_\mu, q^\nu] = \hbar \mathbb{1} \delta^\nu_\mu \tag{26}$$

Together with the explicit form (18) of the potentials, the operators K_n then read

$$\begin{aligned} K_n = {} & \frac{1}{2M} g^{\mu\nu} \left(P_\mu + \Lambda \frac{e^2}{4\pi\varepsilon_0} \frac{n_\mu/c^2}{d(n, q)} \right) \left(P_\nu + \Lambda \frac{e^2}{4\pi\varepsilon_0} \frac{n_\nu/c^2}{d(n, q)} \right) \\ & + \frac{1}{2m} g^{\mu\nu} \left(p_\mu + \lambda \frac{e^2}{4\pi\varepsilon_0} \frac{n_\mu/c^2}{d(n, q)} \right) \left(p_\nu + \lambda \frac{e^2}{4\pi\varepsilon_0} \frac{n_\nu/c^2}{d(n, q)} \right) \end{aligned} \tag{27}$$

where m denotes the 'reduced mass'

$$m = \frac{M_{(1)} M_{(2)}}{M_{(1)} + M_{(2)}} \tag{28}$$

and

$$\Lambda = \lambda_{(1)} + \lambda_{(2)}, \qquad \lambda = \frac{M_{(1)}\lambda_{(2)} - M_{(2)}\lambda_{(1)}}{M_{(1)} + M_{(2)}} \tag{29}$$

Since K_n does not depend on Q^μ, we have

$$[K_n, P_\mu] = 0 \tag{30}$$

i.e. the total momentum-energy operator corresponds to a constant of the motion.

Hence for the eigenvalue-equation

$$K_n\psi_K(X,x)=K\psi_K(X,x), \qquad K\in\mathbb{R} \tag{31}$$

we first consider solutions of the form:

$$\psi_K(X,x)=\exp\left(iP_\mu X^\mu/\hbar\right)\phi_{K,P}(x) \tag{32}$$

where P_μ now denotes a real 4-vector.

Since we are interested in bound states that satisfy the evolution condition that $\langle p^\mu_{(1)}\rangle$ and $\langle p^\mu_{(2)}\rangle$ are along n^μ for any τ, we now consider solutions of (31) of the form (32) satisfying the condition

$$P_\mu = Wn_\mu/c^2 \tag{33}$$

which fix n_μ from a given P_μ. In this relation W denotes the energy of the system 'at rest'. It also follows that, because of (30), n^μ is constant in τ. The relative wave function $\phi_{K,P}(x)$ in (32) is then a solution of the following equation, which is the restriction of (31) to the spectral subspace of P_μ corresponding to (33).

$$\left\{\frac{-1}{2Mc^2}\left(W+\Lambda\frac{e^2}{4\pi\varepsilon_0}\frac{1}{d(n,q)}\right)^2+\frac{1}{2m}g^{\mu\nu}\left(p_\mu+\lambda\frac{e^2}{4\pi\varepsilon_0}\frac{n_\mu/c^2}{d(n,q)}\right)\right.$$
$$\left.\times\left(p_\nu+\lambda\frac{e^2}{4\pi\varepsilon_0}\frac{n_\nu/c^2}{d(n,q)}\right)\right\}\phi_{K,P}(x)=K\phi_{K,P}(x) \tag{34}$$

For convenience and without loss of generality we can choose $n^\mu=n_0^\mu=(0,0,0,1)$. In this case $d(n_0,x)=|\vec{x}|\equiv r$ and, obviously, $n_0^\mu p^\mu\equiv -i\hbar\,\partial/\partial x^4$ commutes with $d(n_0,q)$. Hence, $[n_0^\mu p_\mu, K_{n_0}]=0$ since $n_0^\mu p_\mu$ also commutes with P_μ. Moreover the relative angular momentum operators $\vec{L}=\vec{q}\wedge\vec{p}$ commute with K_{n_0}, P_μ and $n_0^\mu p_\mu$. As a consequence, we can now find solutions of (31) for $n^\mu=n_0^\mu$ satisfying (32) where $\phi_{K,P}(x)$ (in spherical coordinates r, θ, ϕ) is of the form

$$\phi_{K,P}(x)=\phi_0\exp\left(-iwx^4/\hbar\right)Y_l^m(\theta,\phi)R(r) \tag{35}$$

Here ϕ_0 denotes an element of $\mathbb{C}^4$, $Y_l^m(\theta,\phi)$ are the spherical harmonics, $R(r)$ is a radial wave function and $-w$ is the eigenvalue of the equation

$$n_0^\mu p_\mu\phi_{K,P}(x)=-w\phi_{K,P}(x), \qquad w\in\mathbb{R} \tag{36}$$

The corresponding radial equation for $R(r)$ is obtained from (34) and is given by

$$\left\{\frac{-1}{2Mc^2}\left(W+\frac{e^2}{4\pi\varepsilon_0}\frac{\Lambda}{r}\right)^2-\frac{1}{2mc^2}\left(w+\frac{e^2}{4\pi\varepsilon_0}\frac{\lambda}{r}\right)^2\right.$$
$$\left.-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right\}R(r)=KR(r) \tag{37}$$

Collecting similar terms, this radial equation, formally looks like the non-relativistic one. Indeed we have

$$\left\{\frac{-\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}+\frac{\hbar^2}{2m}\frac{\gamma(\gamma+1)}{r^2}-\chi\alpha^2mc^2\frac{a_0}{r}\right\}R(r)=\varepsilon R(r) \tag{38}$$

with

$$\gamma(\gamma+1)=l(l+1)-\alpha^2\left(\frac{m}{M}\Lambda^2+\lambda^2\right)$$

$$\chi=\Lambda\frac{W}{Mc^2}+\lambda\frac{w}{mc^2} \tag{39}$$

$$\varepsilon=\frac{1}{2}\left(2K+\frac{W^2}{Mc^2}+\frac{w^2}{mc^2}\right)$$

Here $\alpha=e^2/4\pi\varepsilon_0\hbar c$ is the fine structure constant and $a_0=\hbar/\alpha mc$ the Bohr radius (involving the reduced mass).

The solutions $\phi^m_{K,P,w,l}(x)$ susceptible of describing bound states of the two particles, i.e. states where, in particular, both particles remain spatially close to each other, necessarily correspond to the discrete part of the spectrum in (38). From the analogy of (38) with the radial equation in the non-relativistic Coulomb interaction, such solutions do exist in $L^2(\mathbb{R}_+, r^2\,dr)$ (we assume $\gamma>-\frac{1}{2}$) if and only if $\chi>0$ and

$$\varepsilon=-\tfrac{1}{2}mc^2\frac{\alpha^2\chi^2}{(\gamma+n')^2} \tag{40}$$

where n' takes all positive integer values. The conditions (40) imply from (39) that $\Lambda W/M+\lambda w/m>0$ and

$$K=\frac{-1}{2}\left\{\frac{W^2}{Mc^2}+\frac{w^2}{mc^2}+\frac{\alpha^2mc^2}{(\gamma+n')^2}\left(\Lambda\frac{W}{Mc^2}+\lambda\frac{w}{mc^2}\right)^2\right\} \tag{41}$$

with

$$\gamma=-\tfrac{1}{2}+\sqrt{(l+\tfrac{1}{2})^2-\alpha^2\left(\frac{m}{M}\Lambda^2+\lambda^2\right)}.$$

We denote $R_{n',l}(r)$ the corresponding solutions of (38) in $L^2(\mathbb{R}_+, r^2\,dr)$ and $\phi^m_{P,w,n',l}(x)$ the corresponding relative wave function in (35) (remember that K in (41) depends on P, w, n' and l).

For bound states, both particles remain not only 'spatially close to each other' but also they are assumed to move 'together' in space-time. Let us thus consider the 'relative velocity' operators given, for $n^\mu=n_0^\mu$, by:

$$\dot{q}^\mu\equiv\frac{i}{\hbar}[K_{n_0}, q^\mu]=\frac{1}{m}\left(p^\mu+\lambda\frac{e^2}{4\pi\varepsilon_0}\frac{n_0^\mu/c^2}{d(n_0,q)}\right) \tag{42}$$

and 'relative' states $\Phi^m_{P,n',l}(x)$ obtained from a set of functions $\phi^m_{P,w,n',l}(x)$ having fixed quantum numbers n', l, m, fixed P and which are sharply defined in w.

$$\Phi^m_{P,n',l}(x)=\int dw\sigma(w)\phi^m_{P,w,n',l}(x) \tag{43}$$

Such states are susceptible of describing bound states only if $\langle\dot{q}^\mu\rangle_\Phi\cong 0$. For symmetry reasons, we however necessarily have that $\langle\dot{\vec{q}}\rangle_\Phi=0$. The last condition

$\langle \dot{q}^4 \rangle_\Phi \cong 0$ implies

$$\langle p^4 \rangle_\Phi \cong -\lambda \frac{e^2}{4\pi\varepsilon_0 c^2} \left\langle \frac{1}{d(n_0, q)} \right\rangle_\Phi = -\lambda\alpha^2 m \left\langle \frac{a_0}{d(n_0, q)} \right\rangle_\Phi \tag{44}$$

Observing that (see e.g. [11])

$$\int_0^\infty \frac{a_0}{r} |R_{n',l}(r)|^2 r^2 \, dr = \frac{\chi}{(\gamma+n')^2} \tag{45}$$

for normalized $R_{n',l}(r)$, we have (assuming an appropriate normalization of the functions $\phi^m_{P,w,n',l}(x)$)

$$\begin{aligned} \left\langle \frac{a_0}{d(n_0, q)} \right\rangle_\Phi &= \int_{\mathbb{R}^4} d^4x \frac{a_0}{r} |\Phi^m_{P,n',l}(x)|^2 \\ &= \int dw \, |\sigma(w)|^2 \frac{\chi(w)}{(\gamma+n')^2} = \frac{1}{(\gamma+n')^2} \left(\Lambda \frac{W}{Mc^2} + \lambda \frac{\bar{w}}{mc^2} \right) \end{aligned} \tag{46}$$

where $\bar{w} = \int dww \, |\sigma(w)|^2$. In addition

$$\langle p^4 \rangle_\Phi = \bar{w}/c^2 \tag{47}$$

and consequently equation (44) implies

$$\frac{\bar{w}}{c^2} \cong -\frac{\lambda\alpha^2 m}{(\gamma+n')^2} \left(\Lambda \frac{W}{Mc^2} + \lambda \frac{\bar{w}}{mc^2} \right) \tag{48}$$

giving the following value for $\bar{w}$:

$$\bar{w} \cong \frac{-\lambda\Lambda\alpha^2 mc^2}{(\gamma+n')^2 + \lambda^2\alpha^2} \frac{W}{Mc^2} \tag{49}$$

It follows from this last equation that

$$\chi \cong \Lambda \frac{W}{Mc^2} \frac{(\gamma+n')^2}{(\gamma+n')^2 + \lambda^2\alpha^2}$$

when $w \cong \bar{w}$. Hence, the condition $\chi > 0$ in (40) is satisfied whenever $W > 0$ (assuming $\Lambda > 0$).

Finally, the states

$$\Psi^m_{n',l}(X, x)_\tau = \int d^4P \int dw\sigma(P, w) \psi^m_{P,w,n',l}(X, x) \exp(-iK\tau/\hbar) \tag{50}$$

built up from

$$\psi^m_{P,w,n',l}(X, x) = \exp(iP_\mu X^\mu/\hbar) \phi^m_{P,w,n',l}(x) \tag{51}$$

for a fixed set of quantum numbers n', l, m, and sharply defined in w around $\bar{w}$ and in P^μ around Wn_0^μ/c^2, describe particles moving together in space-time, the system globally moving like a free particle. Moreover, such states fulfil the conditions that $\langle p^\mu_{(1)} \rangle$ and $\langle p^\mu_{(2)} \rangle$ are parallel to n_0^μ since $\langle \vec{p} \rangle \cong 0$.

Considering now the operators

$$\dot{q}^{\mu}_{(1)} \equiv \frac{\hbar}{i}[K_n, q^{\mu}_{(1)}] = \frac{1}{M_{(1)}}(p^{\mu}_{(1)} - \lambda_{(1)} e A^{\mu}_{(1)}(q))$$

and

$$\dot{q}^{\mu}_{(2)} \equiv \frac{\hbar}{i}[K_n, q^{\mu}_{(2)}] = \frac{1}{M_{(2)}}(p^{\mu}_{(2)} + \lambda_{(2)} e A^{\mu}_{(2)}(q))$$

and noting that the expectation values for $g_{\mu\nu}\dot{q}^{\mu}_{(1)}\dot{q}^{\nu}_{(1)}$ and $g_{\mu\nu}\dot{q}^{\mu}_{(2)}\dot{q}^{\nu}_{(2)}$ are around $-c^2$, and that

$$K_n = \frac{M_{(1)}}{2} g_{\mu\nu}\dot{q}^{\mu}_{(1)}\dot{q}^{\nu}_{(1)} + \frac{M_{(2)}}{2} g_{\mu\nu}\dot{q}^{\mu}_{(2)}\dot{q}^{\nu}_{(2)} \tag{53}$$

we have to consider values of K around $-Mc^2/2$. From (41) and taking (49) into account we find that for $w \cong \bar{w}$

$$K \cong -\frac{1}{2}\frac{W^2}{Mc^2}\left(1 + \frac{m}{M}\frac{\Lambda^2\alpha^2}{(\gamma+n')^2+\lambda^2\alpha^2}\right) \tag{54}$$

The values of W such that the right-hand side of (54) is equal to $-Mc^2/2$ are now interpreted as the energy spectrum. They are given by

$$W = Mc^2\left[1 + \frac{m}{M}\frac{\Lambda^2\alpha^2}{(\gamma+n')^2+\lambda^2\alpha^2}\right]^{-1/2} \tag{55}$$

(only the positive root is compatible with the condition $\chi > 0$ in (40) whenever $\Lambda > 0$).

In order to discuss the results, let us consider the power expansion of (55) with respect to α^2. From (55) and from the expression of γ given in (41) we obtain the following expansion

$$W = Mc^2 - mc^2\left\{\frac{\Lambda^2\alpha^2}{2n^2} + \frac{\Lambda^4\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}}\left(\frac{m}{M}+\frac{\lambda^2}{\Lambda^2}\right) - \frac{3}{4}\frac{m}{M} - \frac{\lambda^2}{\Lambda^2}\right) + O(\alpha^6)\right\} \tag{56}$$

where $n = l + n'$ denotes the principal quantum number.

Comparing this result with the non-relativistic one we assume that $\Lambda = 1$.

To choose $\lambda_{(1)}$ and $\lambda_{(2)}$ we have to consider the fine structure contribution due to the term,

$$-\frac{\alpha^4 mc^2}{2n^4}\frac{n}{l+\frac{1}{2}}\left(\frac{m}{M}+\lambda^2\right)$$

in (56). When one of both masses is larger than the other, the previous expression is expected to be similar to the corresponding one from the Klein–Gordon spectrum expansion [12]. In other words, we expect that $\lambda^2 \cong 1$.

Furthermore, there is also theoretical evidence from the Breit equation [11], [13] and experimental support from the fine structure measurements for the positronium [14] to assume more generally that

$$\frac{m}{M} + \lambda^2 = 1 \tag{57}$$

for any mass ratio. Then, up to a term mc^2, with m the reduced mass, the fine structure term is in every case the same as in the Klein–Gordon expansion. Let us recall the expansion of the Klein–Gordon energy spectrum for the hydrogen atom (i.e. in the external Coulomb field approximation):

$$W = mc^2\left\{1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - \frac{3}{4}\right) + O(\alpha^6)\right\}$$

The resulting fine structure term of order α^4 is:

$$-\frac{\alpha^4 mc^2}{2n^3(l+\frac{1}{2})}$$

The above assumptions

$$\Lambda = 1 \quad \text{and} \quad \lambda^2 = 1 - m/M \tag{58}$$

lead, from (56), to the following expansion for the spectrum of our model

$$W = Mc^2 - mc^2\left\{\frac{\alpha^2}{2n^2} + \frac{\alpha^4}{2n^4}\left(\frac{n}{l+\frac{1}{2}} - 1 + \frac{m}{M}\right) + O(\alpha^6)\right\} \tag{59}$$

Finally, because of the definitions (29), the values of $\lambda_{(1)}$ and $\lambda_{(2)}$ according to (58) are:

$$\lambda_{(1)} = \frac{M_{(1)}}{M} - \lambda \quad \text{and} \quad \lambda_{(2)} = \frac{M_{(2)}}{M} + \lambda \tag{60}$$

with

$$\lambda = \pm\sqrt{1 - m/M}$$

We can decide to choose the sign of λ in such a way that $\lambda_{(2)} \to 1$ (and then $\lambda_{(1)} \to 0$) if $M_{(1)} \to \infty$, i.e. the external field approximation. Consequently $\lambda > 0$ whenever $M_{(2)} \le M_{(1)}$.

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