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The scattering cross section and its Born approximation at high energies

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Abstract. For $H=-\Delta+V$ in $L^2(\mathbf{R}^3)$ with $V(x)=(1+x^2)^{-\beta}W(x)$, $\beta>1$, $W\in L^q(R^3)$, $\frac{3}{2}\leq q\leq\infty$, we prove

$$\bar{\sigma}(\lambda)=\bar{\sigma}_1(\lambda)+O(\lambda^{-1-(1/2)+4/3q})$$

as $\lambda\to\infty$, where $\bar{\sigma}(\lambda)$ is the total scattering cross section, averaged over all incident directions, and $\bar{\sigma}_1(\lambda)$ its (first) Born approximation.

I. Introduction

We consider scattering theory in $L^2(\mathbf{R}^3)$ for $H=-\Delta+V$ with V multiplication by a real-valued function $V(x)$ satisfying the condition

$$(1+x^2)^{\beta}V(x)\in L^q(\mathbf{R}^3) \tag{1.1}$$

for some $\beta>1, \frac{3}{2}\leq q\leq\infty$. For such potentials the total cross section averaged over all incident directions, $\bar{\sigma}(\lambda)$, and its (first) Born approximation, $\bar{\sigma}_1(\lambda)$, are known to exist and be finite for all energies $\lambda\in(0,\infty)\backslash e$. Here e is a bounded closed set of measure zero, and in most cases $e=\sigma_p(H)\cap(0,\infty)$, $\sigma_p(H)$ the point spectrum of H. See [2, 4] for definition and discussion of $\bar{\sigma}(\lambda)$ and $\bar{\sigma}_1(\lambda)$.

It is well known that $\bar{\sigma}(\lambda)=O(\lambda^{-1})$ and $\bar{\sigma}_1(\lambda)=O(\lambda^{-1})$ as $\lambda\to\infty$. One also has $\bar{\sigma}(\lambda)-\bar{\sigma}_1(\lambda)=o(\lambda^{-1})$ as $\lambda\to\infty$. This is probably well known, although we have not been able to find the result in the literature. In any case the proof is given below. Here we consider the question of how good an approximation $\bar{\sigma}_1(\lambda)$ is to $\bar{\sigma}(\lambda)$ as $\lambda\to\infty$. There seems to be no rigorous remainder estimates, even for Yukawa potentials.

Our result is the following: If V satisfies (1.1) for some $q, \frac{3}{2}\leq q\leq\infty$, then

$$\bar{\sigma}(\lambda)-\bar{\sigma}_1(\lambda)=O(\lambda^{-\frac{3}{2}(1-1/2q)}) \tag{1.2}$$

as $\lambda\to\infty$. In particular, for a Yukawa potential $V(x)=c\,|x|^{-1}\exp(-a\,|x|)$, $a>0$, one has $\bar{\sigma}(\lambda)-\bar{\sigma}_1(\lambda)=O(\lambda^{-5/4+\delta})$, for any $\delta>0$. The proof of (1.2) is based on the expression for the scattering matrix obtained in Kato–Kuroda scattering theory, a resolvent estimate due to Agmon, and complex interpolation.

2. The remainder estimate

Let V satisfy (1.1). Let $H_0=-\Delta$ and define $H=H_0+V$ as the quadratic form sum. Write $V(x)=\rho(x)^2W(x)$, $\rho(x)=(1+x^2)^{-\beta/2}$, $W\in L^q(\mathbf{R}^3)$. By Hölder's inequality $V\in L^1(\mathbf{R}^3)\cap L^{3/2}(\mathbf{R}^3)$ for $\frac{3}{2}\leq q<2$ and $V\in L^2(\mathbf{R}^3)\cap L^q(\mathbf{R}^3)$ for $2\leq q\leq\infty$, so all the results in the scattering theory for (H_0, H) are well known, and can be obtained by the Kato-Kuroda method, see e.g. [2, 4, 5, 6, 9, 10].

Write $V=AB$ with $A(x)=\rho(x)\,|W(x)|^{1/2}$ and $B(x)=\rho(x)\,|W(x)|^{1/2}\times \operatorname{sgn}(W(x))$. Let $R_0(\zeta)=(H_0-\zeta)^{-1}$ be the free resolvent. It is given by

$$R_0(\zeta):\frac{\exp(i\sqrt{\zeta}\,|x-y|)}{4\pi\,|x-y|},\qquad \operatorname{Im}(\sqrt{\zeta})>0, \tag{2.1}$$

where $T: k(x, y)$ means that the operator T has the integral kernel $k(x, y)$. The operator $Q(\zeta)=BR_0(\zeta)A$, $\operatorname{Im}(\zeta)\neq 0$, is a Hilbert-Schmidt operator by Sobolev's inequality. The boundary values $(\lambda>0)$

$$Q_\pm(\lambda)=\lim_{\varepsilon\downarrow 0} Q(\lambda\pm i\varepsilon)$$

exist in the Hilbert-Schmidt norm and are given by

$$Q_\pm(\lambda): B(x)\frac{\exp(\pm i\sqrt{\lambda}\,|x-y|)}{4\pi\,|x-y|}A(y). \tag{2.2}$$

Note that the Hilbert-Schmidt norm of $Q_\pm(\lambda)$ is independent of λ. Let us also mention the Zemach–Klein result [10, 12]

$$\|Q_\pm(\lambda)\|_{B(L^2(\mathbf{R}^3))}\to 0\quad\text{as}\quad \lambda\to\infty. \tag{2.3}$$

$B(L^3(\mathbf{R}^3))$ denotes the bounded operators on $L^2(\mathbf{R}^3)$, and $B_2(L^2(\mathbf{R}^3))$ $(B_1(L^2(\mathbf{R}^3)))$ the Hilbert–Schmidt (trace class) operators. $L^2(S^2)$ is the space of square integrable functions on the unit sphere S^2.

The operators $T(\lambda; A)$, $T(\lambda; B)$ are the bounded operators from $L^2(\mathbf{R}^3)$ to $L^2(S^2)$ given by

$$T(\lambda; A): (2\pi)^{-3/2}2^{-1/2}\lambda^{1/4}\exp(-i\lambda^{1/2}\omega\cdot x)A(x), \tag{2.4}$$

$$T(\lambda; B): (2\pi)^{-3/2}2^{-1/2}\lambda^{1/4}\exp(-i\lambda^{1/2}\omega\cdot x)B(x). \tag{2.5}$$

We have the relation (see [5])

$$Q_+(\lambda)-Q_-(\lambda)=2\pi i T(\lambda; B)^* T(\lambda; A). \tag{2.6}$$

$(1+Q_\pm(\lambda))^{-1}$ exists as a bounded operator for $\lambda\in(0,\infty)\backslash e$, where e is a closed bounded set of measure zero. We can find $\lambda_0>0$ such that $(0,\infty)\backslash e\supset(\lambda_0,\infty)$, and in the sequel *we always assume* $\lambda>\lambda_0$.

The scattering matrix $S(\lambda)$ for (H_0, H) is the unitary operator on $L^2(S^2)$ given by

$$S(\lambda)=1-2\pi i T(\lambda; A)(1+Q_+(\lambda))^{-1}T(\lambda; B)^*. \tag{2.7}$$

One has

$$S(\lambda)^*=1+2\pi i T(\lambda; A)(1+Q_-(\lambda))^{-1}T(\lambda; B)^*. \tag{2.8}$$

See [2, 4, 5, 6, 9, 10], where these results are given (for different classes of V which together cover the class considered here).

Under assumption (1.1) $S(\lambda)-1$ is a Hilbert-Schmidt operator. The cross section averaged over all incident directions is given by

$$\bar{\sigma}(\lambda)=\frac{\pi}{\lambda}\|S(\lambda)-1\|^2_{B_2(L^2(S^2))}.$$

See [2, 4], from which the result can be obtained for the class of V considered here. Unitarity of $S(\lambda)$ implies $(S(\lambda)-1)^*(S(\lambda)-1)=2-S(\lambda)-S(\lambda)^*\in B_1(L^2(S^2))$ and hence the convenient expression

$$\bar{\sigma}(\lambda)=-\frac{\pi}{\lambda}\operatorname{tr}(S(\lambda)+S(\lambda)^*-2). \tag{2.9}$$

The proof of (1.2) is based on the following Lemma:

Lemma 2.1. *Let V satisfy* (1.1) *for some* $q, \frac{3}{2}\le q\le\infty$. *Then we have*

$$\|Q_\pm(\lambda)\|_{B(L^2(\mathbf{R}^3))}\le c\lambda^{-\frac{1}{2}(1-3/2q)} \tag{2.10}$$

for $\lambda\ge 1$.

Proof. The proof is by complex interpolation and is similar to the proof of Theorem 4.1 in [11]. Let us prove the result for $Q_+(\lambda)$. First note that the norm of $Q_+(\lambda)$ equals the norm of $\tilde{Q}_+(\lambda)$, where

$$\tilde{Q}_+(\lambda):|W(x)|^{1/2}\rho(x)\frac{\exp(i\sqrt{\lambda}\,|x-y|)}{4\pi\,|x-y|}\rho(y)\,|W(y)|^{1/2}.$$

$\rho\in L^3(\mathbf{R}^3)\cap L^\infty(\mathbf{R}^3)$ and $W\in L^q(\mathbf{R}^3)$, $\frac{3}{2}\le q\le\infty$, imply $\rho\,|W|^{1/2}\in L^3(\mathbf{R}^3)$ and $\|\rho\,|W|^{1/2}\|_{L^3}\le c\,\|W\|^{1/2}_{L^q}$. Estimating the operator norm by the Hilbert-Schmidt norm one obtains from Sobolev's inequality

$$\|\tilde{Q}_+(\lambda)\|\le c\,\|\rho\,|W|^{1/2}\|^2_{L^3}\le c'\,\|W\|_{L^q}.$$

A result due to Agmon [1] implies that the operator

$$K_+(\lambda):\rho(x)\frac{\exp(i\sqrt{\lambda}\,|x-y|)}{4\pi\,|x-y|}\rho(y)$$

is bounded on $L^2(\mathbf{R}^3)$ and $\|K_+(\lambda)\|\le c\lambda^{-1/2}$, $\lambda\ge 1$. Agmon states the result without proof. A complete proof can be found in [6]. Rauch [7] has proved that the power $\lambda^{-1/2}$ cannot be improved.

Assume $W\in L^\infty(\mathbf{R}^3)$ with compact support. Fix q, $\frac{3}{2}\le q\le\infty$. Consider for z satisfying $0\le\operatorname{Re} z\le 1$ the family of operators

$$F(z):|W(x)|^{qz/3}\rho(x)\frac{\exp(i\sqrt{\lambda}\,|x-y|)}{4\pi\,|x-y|}\rho(y)\,|W(y)|^{qz/3}.$$

Let $f, g\in L^2(\mathbf{R}^3)$. $z\to(f, F(z)g)$ is analytic for $0<\operatorname{Re} z<1$ and continuous for

$0 \le \text{Re } z \le 1$. The estimates given above imply

$$|(f, F(iy)g)| \le c\lambda^{-1/2} \|f\| \|g\|$$

and

$$|(f, F(1+iy)g)| \le c(\| |W|^{q/3}\|_{L^3})^2 \|f\| \|g\|$$

for all $y \in \mathbf{R}$. The three line theorem (see e.g. [8]) implies

$$\left|\left(f, F\left(\frac{3}{2q}\right)g\right)\right| \le c\lambda^{-\frac{1}{2}(1-3/2q)} \|W\|_{L^q} \|f\| \|g\|.$$

The result now follows for general $W \in L^q(\mathbf{R}^3)$, $\frac{3}{2} \le q < \infty$, by an approximation argument. For $q = \infty$ the result follows directly from the estimate for $K_+(\lambda)$ given above.

Theorem 2.2. *Let V satisfy* (1.1) *for some* $q, \frac{3}{2} \le q \le \infty$. *We then have*

$$\bar{\sigma}(\lambda) = \bar{\sigma}_1(\lambda) + O(\lambda^{-\frac{3}{2}(1-1/2q)}) \tag{2.11}$$

as $\lambda \to \infty$, *where*

$$\bar{\sigma}_1(\lambda) = \frac{1}{\lambda}\frac{1}{4\pi} \iint \frac{V(x)V(y)}{|x-y|^2} (\sin(\sqrt{\lambda}\,|x-y|))^2 \, dx \, dy \tag{2.12}$$

is the (*first*) *Born approximation to* $\bar{\sigma}(\lambda)$.

Remark 2.3.

(i) It is well known that (2.12) is the first Born approximation to $\bar{\sigma}(\lambda)$, as defined in [2].

(ii) A change of variables, $(\sin u)^2 = \frac{1}{2}(1 - \cos(2u))$, and the Riemann-Lebesgue lemma imply

$$\bar{\sigma}_1(\lambda) = \frac{1}{\lambda}\frac{1}{8\pi} \iint \frac{V(x)V(y)}{|x-y|^2} \, dx \, dy + o(\lambda^{-1})$$

as $\lambda \to \infty$, and hence

$$\lim_{\lambda\to\infty} \lambda\bar{\sigma}(\lambda) = \lim_{\lambda\to\infty} \lambda\bar{\sigma}_1(\lambda) = \frac{1}{8\pi} \int\int \frac{V(x)V(y)}{|x-y|^2} \, dx \, dy$$

(iii) The integrals in (2.12) and (ii) above are non-negative. This is well known for V a Rollnik potential ([10]) for the integral in (ii), and will also follow from the proof given below.

Proof. The proof is based on the expression (2.9) for $\bar{\sigma}(\lambda)$. (2.7) and (2.8) imply

$$S(\lambda) + S(\lambda)^* - 2 = -2\pi i T(\lambda; A)((1+Q_+(\lambda))^{-1} - (1+Q_-(\lambda))^{-1})T(\lambda; B)^*.$$

The Born expansion consists in this case of an expansion of $(1+Q_\pm(\lambda))^{-1}$. We use only the finite form of the expansion, so there are no convergence problems.

$$(1+Q_+(\lambda))^{-1}-(1+Q_-(\lambda))^{-1}$$
$$=-(Q_+(\lambda)-Q_-(\lambda))+Q_+(\lambda)^2(1+Q_+(\lambda))^{-1}-Q_-(\lambda)^2(1+Q_-(\lambda))^{-1}.$$

Hence we have $S(\lambda)+S(\lambda)^*-2=\alpha+\beta_++\beta_-$ with an obvious notation. (2.6) implies

$$\alpha=2\pi i T(\lambda;A)(Q_+(\lambda)-Q_-(\lambda))T(\lambda;B)^*$$
$$=-4\pi^2T(\lambda;A)T(\lambda;B)^*T(\lambda;A)T(\lambda;B)^*.$$
$$\operatorname{tr}\alpha=-4\pi^2\operatorname{tr}((T(\lambda;B)^*T(\lambda;A))^2).$$

(2.2) and (2.6) imply that $T(\lambda;B)^*T(\lambda;A)$ is a Hilbert–Schmidt operator given by

$$T(\lambda;B)^*T(\lambda;A):\frac{1}{4\pi^2}B(x)\frac{\sin(\sqrt{\lambda}\,|x-y|)}{|x-y|}A(y).$$

Using $V=AB$ one obtains

$$\operatorname{tr}\alpha=-\frac{1}{4\pi^2}\iint\frac{V(x)V(y)}{|x-y|^2}(\sin(\sqrt{\lambda}\,|x-y|)^2\,dx\,dy.$$

This gives the term $\bar{\sigma}_1(\lambda)$ in (2.11), and (2.12). The integral in (2.12) is non-negative, being the trace of a non-negative operator, c.f. Remark 2.3 (iii). One easily verifies that this term is the first Born approximation as defined in [2].

Let us estimate $\operatorname{tr}(\beta_+)$. The estimate for $\operatorname{tr}(\beta_-)$ is similar and is omitted. Using (2.6) we get

$$\operatorname{tr}(\beta_+)=-2\pi i\operatorname{tr}(T(\lambda;A)Q_+(\lambda)^2(1+Q_+(\lambda))^{-1}T(\lambda;B)^*)$$
$$=-2\pi i\operatorname{tr}(T(\lambda;B)^*T(\lambda;A)Q_+(\lambda)^2(1+Q_+(\lambda))^{-1})$$
$$=-\operatorname{tr}((Q_+(\lambda)-Q(\lambda))Q_+(\lambda)^2(1+Q_+(\lambda))^{-1}).$$

$$|\operatorname{tr}(\beta_+)|\le(\|Q_+(\lambda)\|_B+\|Q(\lambda)\|_B)\,\|Q_+(\lambda)\|_{B_2}^2\,\|(1+Q_+(\lambda))^{-1}\|_B \tag{2.13}$$

where $B=B(L^2(\mathbf{R}^3))$ and $B_2=B_2(L^2(\mathbf{R}^3))$. Lemma 2.1 gives under assumption (1.1)

$$\|Q_+(\lambda)\|_B+\|Q_-(\lambda)\|_B\le c\lambda^{-\frac{1}{2}+3/4q}.$$

As noted above $\|Q_+(\lambda)\|_{B_2}$ is independent of λ. (2.3) gives $\|(1+Q_+(\lambda))^{-1}\|_B\le c$ for $\lambda\ge\lambda_0$. This gives the remainder estimate in (2.11).

Corollary 2.4. *Let V satisfy* (1.1) *with $q=\frac{3}{2}$. When then have*

$$\bar{\sigma}(\lambda)-\bar{\sigma}_1(\lambda)=\sigma(\lambda^{-1})$$

as $\lambda\to+\infty$.

Proof. This follows from (2.3) and (2.13).

Remark 2.5. The above results can easily be extended to cover potentials of the form

$$V(x)=(1+x^2)^{-\beta}(V_1(x)+V_2(x))$$

with $\beta > 1$, $V_1 \in L^q(\mathbf{R}^3)$, $\frac{3}{2} \leq q < \infty$, $V_2 \in L^\infty(\mathbf{R}^3)$. This follows by Kato's remark [3; Remark 1.10] and an easy extension of Lemma 2.1. Thus Theorem 2.2 is valid for this class of potentials.

Remark 2.6. One can prove the estimate (2.10) for a larger class of potentials. We shall indicate one such extension. Assume

$$V(x) = (1+x^2)^{-(\alpha/2)(1-3/2q)} W(x) \tag{2.14}$$

for $\alpha > 1$, $W \in L^q(\mathbf{R}^3)$, $\frac{3}{2} \leq q \leq \infty$. Factorize $V = AB$ as above with $\rho(x)$ replaced by $(1+x^2)^{-\alpha(1-3/2q)/4}$. Let $Q_\pm(\lambda)$ be given by (2.2). Then the estimate (2.10) is valid for $Q_\pm(\lambda)$. The proof is similar to the proof of Lemma 2.1, if one considers instead of $F(z)$ the family $G(z)$ defined as follows. Let $\tilde{\rho}(x) = (1+x^2)^{-\alpha/4}$. Fix $q, \frac{3}{2} \leq q < \infty$, and define

$$G(z) : |W(x)|^{qz/3} \, \tilde{\rho}(x)^{1-z} \frac{e^{i\sqrt{\lambda}\,|x-y|}}{4\pi\,|x-y|} \tilde{\rho}(y)^{1-z} \, |W(y)|^{qz/3}.$$

The condition (2.14) on V is not sufficient to ensure the finiteness of $\bar{\sigma}(\lambda)$, and the boundedness of the exceptional set e. But if one imposes further conditions on V satisfying (2.14), one can prove our result for a slightly larger class of V. For instance one could assume (2.14) with $\alpha > 2$.

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Note added in proof: G. Hagedorn (Comm. Math. Phys. *66*, 77–94 (1979)) has given a remainder estimate for the Born approximation to the scattering amplitude.

REFERENCES

[1] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa, Ser. IV *2*, 151–218 (1975).
[2] W. Amrein, J. M. Jauch and K. Sinha, *Scattering theory in quantum mechanics*, W. A. Benjamin, Reading, 1977.
[3] T. Kato, *Wave operators and similarity for non-selfadjoint operators*, Math. Ann. *162*, 258–279 (1966).
[4] T. Kato, *Scattering theory, Studies in Mathematics*, Math. Soc. Amer., Vol. 7, 1971, pp. 90–115.
[5] S. T. Kuroda, *Scattering theory for differential operators. I: Operator theory; II: Selfadjoint elliptic operators*. J. Math. Soc. Japan *25* (1973), pp. 75–104; 222–234.
[6] S. T. Kuroda, *An introduction to scattering theory*, Lecture notes series #51, Mathematisk Institut, Aarhus Universitet, 1978.
[7] J. Rauch, *Local decay of scattering solutions to Schrödinger's equation*, Comm. Math. Phys. *61*, 149–168 (1978).
[8] M. Reed and B. Simon, *Methods of modern mathematical physics, II: Fourier Analysis, Self-adjointness*, Academic Press, New York, 1975.
[9] M. Reed and B. Simon, *Methods of modern mathematical physics, III: Scattering theory*, Academic Press, New York, 1979.
[10] B. Simon, *Quantum mechanics for Hamiltonians defined as quadratic forms*, Princeton University Press, Princeton, 1971.
[11] B. Simon, *Trace ideals and their applications*, Cambridge University Press, Cambridge, 1979.
[12] C. Zemach and A. Klein, *The Born expansion in non-relativistic quantum theory*, Nuovo Cimento *10*, 1078–1087 (1958).