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Evanescence of states and asymptotic completeness

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Abstract. Introducing an abstract version of the geometric approach, we deduce asymptotic completeness for simple scattering systems from local decay and smallness of the interaction near infinity. In potential scattering, this gives asymptotic completeness for highly singular potentials and generalized asymptotic completeness, in the sense of Pearson, in the presence of local absorption of states. Moreover, the singularly continuous subspace that may occur due to local singularities of the potential is contained in the subspace of bound states of the Hamiltonian (defined in the geometric sense).

I. Introduction

Over the last two years a new 'geometric' method has been developed for proving existence and completeness of the wave operators in quantum scattering theory (see Refs. [1] through [8] and the review [9], all of which are based on the fundamental work of Enss [1]). In the present paper we deal with the following two points: (i) we study the mathematical structure of this approach in an abstract form, and (ii) we apply the method to the situation not previously covered where asymptotic completeness holds only in some generalized form. In particular we give a 'geometric' proof of known results about generalized asymptotic completeness in the presence of local absorption of states, and we obtain as a new result a characterization of the space-time behaviour of states in the singularly continuous subspace of Schrödinger Hamiltonians with locally strongly singular potentials.

As in previous work [10]–[12] we adopt the attitude that a study of the space-time behaviour of states should be the first step in a physical scattering theory. To define scattering states and bound states, one considers a sequence $\{F_r\}$ of self-adjoint 'localizing operators', i.e. satisfying $F_r = F_r^* \in \mathscr{B}(\mathscr{H})$, $\|F_r\| \leq \kappa$ for each $r = 1, 2, \ldots$ and

$$\operatorname*{s-lim}_{r\to\infty} F_r = I, \tag{1}$$

where I denotes the identity operator. The usual example in potential scattering is as follows: the Hilbert space is $\mathscr{H} = L^2(\mathbb{R}^n)$ and F_r is the multiplication operator by the characteristic function χ_r of the ball $B_r = \{x \in \mathbb{R}^n \mid |x| \leq r\}$, i.e.

$$(F_r f)(x) = \begin{cases} f(x) & \text{if } |x| \leq r \\ 0 & \text{if } |x| > r. \end{cases} \tag{2}$$

The *scattering states* of the Hamiltonian H for $t \to \pm\infty$ are defined as those (pure) states that are evanescent from each bounded region of configuration space as $t \to \pm\infty$ respectively. They form subspaces $\mathscr{M}_\infty^+(H)$ and $\mathscr{M}_\infty^-(H)$ respectively given by

$$f \in \mathscr{M}_\infty^\pm(H) \Leftrightarrow \lim_{t\to\pm\infty} \|F_r e^{-iHt} f\|^2 = 0^{(*)} \quad \text{for each } r = 1, 2, \ldots \tag{3}$$

The *bound states of H at positive or negative times* are defined as those states which remain essentially localized in a bounded region of configuration space at all positive or all negative times respectively. They form two subspaces $\mathscr{M}_0^\pm(H)$ defined as

$$f \in \mathscr{M}_0^\pm(H) \Leftrightarrow \lim_{r\to\infty} \sup_{t\in[0,\pm\infty)} \|(I - F_r)e^{-iHt} f\|^2 = 0. \tag{4}$$

The *bound states* $\mathscr{M}_0(H)$ are those states that remain bound at *all* times, i.e.

$$\mathscr{M}_0(H) = \mathscr{M}_0^+(H) \cap \mathscr{M}_0^-(H). \tag{5}$$

It is known [13, 10] that each vector in the closed subspace $\mathscr{H}_p(H)$ spanned by all eigenvectors of H is a bound state, but in certain cases $\mathscr{M}_0(H)$ is strictly larger than $\mathscr{H}_p(H)$.

The time evolution $\exp(-iHt)f$ of a scattering state $f \in \mathscr{M}_\infty^\pm(H)$ converges weakly to zero as $t \to \pm\infty$ respectively. On the other hand the time evolution of vectors in the singularly continuous subspace $\mathscr{H}_{sc}(H)$ of H will in general converge weakly to zero only in some averaged sense, for example $(g \in \mathscr{H}_{sc}(H))$

$$\operatorname*{w-lim}_{T\to\infty} \frac{1}{T} \int_0^{\pm T} \exp(-iHt) g \, dt = 0, \tag{6}$$

so that they cannot be scattering states in the sense of the definition (3). In order to treat such Hamiltonians, it is essential to introduce two other subspaces $\bar{\mathscr{M}}_\infty^\pm(H)$, the set of scattering states on the time average:

$$f \in \bar{\mathscr{M}}_\infty^\pm(H) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T} \int_0^{\pm T} \|F_r e^{-iHt} f\|^2 \, dt = 0 \quad \text{for each } r = 1, 2, \ldots \tag{7}$$

Since averaging over time is inherent in this approach and essential when dealing with Hamiltonians with non-empty singularly continuous spectrum, we use it in our abstract formulation, in the more general form of some invariant mean. It is then natural to replace also other time limits by limits of time

(*) An equation containing double signs is meant to hold separately for the upper and for the lower sign.

averages. On the other hand certain results will be obtained by working with convergent subsequences rather than with averages.

If scattering theory is approached as outlined above, the completeness proof should proceed as follows: (i) show that all states orthogonal to the bound states are scattering states as $t \to +\infty$ and $t \to -\infty$, (ii) prove that all scattering states become asymptotically free, i.e. lie in the range of both of the wave operators. Now for Schrödinger Hamiltonians it can be proved under suitable *local* assumptions on the potential v that $\mathcal{M}_0(H) = \mathcal{H}_p(H)$ and $\bar{\mathcal{M}}_\infty^\pm(H) = \mathcal{H}_c(H)$, i.e. that all states in the subspace $\mathcal{H}_c(H)$ of continuity of H are scattering states (at least on the time average [13, 10]). Once evanescence of all states in $\mathcal{H}_c(H)$ is known to hold, the completeness proof should involve only conditions on v near infinity. Thus, the idea of our principal theorem may be paraphrased as follows: 'Suppose that, for a single-channel scattering system, all states in $\mathcal{H}_c(H)$ are outgoing at large times and that the interaction is well-behaved at large distances. Then one has asymptotic completeness'.

The theorem just cited gives asymptotic completeness in the usual sense: the ranges $\mathcal{R}(\Omega_\pm)$ of the wave operators $\Omega_\pm$ are the orthogonal complement of the subspace $\mathcal{H}_p(H)$ spanned by the eigenvectors of H. In more general situations it is appropriate to speak of *asymptotic completeness in the geometric sense*, which is the property that

$$\mathcal{R}(\Omega_+) = \mathcal{R}(\Omega_-) = \mathcal{M}_0(H)^\perp. \tag{8}$$

This is more general, since $\mathcal{H}_p(H)$ may be a proper subspace of $\mathcal{M}_0(H)$. Some remarks on this may be found at the end of Section II. Finally, if local absorption occurs, one must introduce, in addition to $\mathcal{M}_0(H)$ and $\bar{\mathcal{M}}_\infty^\pm(H)$, the subspaces $\mathcal{M}_\Sigma^\pm(H)$ of states that are absorbed at the singularities of v (supposed to lie in a bounded subset Σ of configuration space). One finds in this case that all vectors in the singularly continuous subspace $\mathcal{H}_{sc}(H)$ of H are bound states, and that the absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of H is the (orthogonal) direct sum of the subspace of scattering states and the subspace of absorbed states (for each sign of time). This is called *generalized asymptotic completeness* and will be discussed in Section III*O*.

II. Some abstract results

We consider the scattering problem for a pair of self-adjoint operators H and H_0 in a separable complex Hilbert space $\mathcal{H}$. Among other things we want to give sufficient abstract conditions for the wave operators to exist and to be complete. These conditions involve two means $m^\pm$ and a set $\mathcal{A}$ of functions ϕ from $\mathbb{R}$ to $\mathbb{C}$. We first state our hypotheses on $\mathcal{A}$:

(A1) Each $\phi \in \mathcal{A}$ is the inverse Fourier transform of a function $\tilde{\phi}$ satisfying $\tilde{\phi} \in L^1(\mathbb{R})$ and $d/dz\,\tilde{\phi}(z) \in L^1(\mathbb{R})$.

(A2) There is a countable closed subset Γ of $\mathbb{R}$ and a sequence $\{\phi_n\} \equiv \mathcal{A}_0$ in $\mathcal{A}$ such that $\phi_n : \mathbb{R} \to \mathbb{R}$,

$$\|\phi_n\|_\infty \leq M < \infty \quad \text{for all } n \tag{9}$$

and

$$\lim_{n\to\infty} \phi_n(\lambda) = 1 \quad \text{for all } \lambda \in \mathbb{R}\setminus\Gamma. \tag{10}$$

Notice that (A1) implies for each $\phi \in \mathcal{A}$ that $\lambda \mapsto (\lambda \pm i)\phi(\lambda) \in L^\infty(\mathbb{R})$, hence that

$$(H_0 \pm i)\phi(H_0) \in \mathcal{B}(\mathcal{H}), \qquad (H \pm i)\phi(H) \in \mathcal{B}(\mathcal{H}). \tag{11}$$

Lemma 1. *Let $\{\phi_n\}$ be the sequence given by (A2) and $f \in \mathcal{H}_c(H)$. Then $\|f - \phi_n(H)f\| \to 0$ as $n \to \infty$.*

Proof. We denote by $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ the spectral family of H. Since each ϕ_n is continuous by (A1), we have

$$|1 - \phi_n(\lambda)|^2 \leq (M+1)^2 \qquad \forall \lambda \in \mathbb{R}, \forall n. \tag{12}$$

Now

$$\|f - \phi_n(H)f\|^2 = \int_{\mathbb{R}} |1 - \phi_n(\lambda)|^2 \, d(f, E_\lambda f). \tag{13}$$

By (12) the integrand in (13) is majorized, uniformly in n, by a function which is integrable with respect to the measure $d(f, E_\lambda f)$. Since $f \in \mathcal{H}_c(H)$ and Γ is countable, the measure of Γ is zero. By (10), $\phi_n(\lambda) \to 1$ a.e. with respect to the measure $d(f, E_\lambda f)$. The result now follows from (13) and the Lebesque dominated convergence theorem. ■

We next say a few words about the means $m^\pm$ that are involved in our conditions. Let $C^+(\mathbb{R})$ be the set of all bounded, continuous, non-negative functions from $\mathbb{R}$ to $\mathbb{R}$. We shall write $m^+(\psi) = 0$ whenever $\psi(x)$ converges, in a generalized sense, to zero as $x \to +\infty$. (Similarly $m^-(\psi) = 0$ for the limit $x \to -\infty$.) Any reasonable definition of this notion entails the following properties:

(M1) $m^+(\psi) = 0 \Rightarrow m^+(c\psi) = 0$ for $c \geq 0$.

(M2) $m^+(\psi_1) = 0$, $m^+(\psi_2) = 0 \Rightarrow m^+(\psi_1 + \psi_2) = 0$.

(M3) $m^+(\psi_1) = 0$ and $\psi_2 = \psi_1$ for $x > x_0 \Rightarrow m^+(\psi_2) = 0$.

(M4) $m^+(\psi_1) = 0$ and $\psi_2 \leq \psi_1 \Rightarrow m^+(\psi_2) = 0$.

(M5) $m^+(c) = 0$, $c \in [0, +\infty) \Leftrightarrow c = 0$.

(M6) The subset of $C^+(\mathbb{R})$ for which $m^+(\psi) = 0$ is closed in the uniform topology. We shall further assume that this is a non-empty subset.

For $\psi \in C(\mathbb{R})$ only (the set of all bounded, continuous functions from $\mathbb{R}$ to $\mathbb{R}$), we shall write $m^+(\psi) = 0$ whenever $m^+(|\psi|) = 0$.

Remarks. (a) An immediate consequence of the above is that $m^+(\psi) = 0$ whenever $\lim \psi(x) = 0$ as $x \to +\infty$. It also follows that, for any ψ such that $m^+(\psi) = 0$, a subsequence $\{x_n\}$ exists satisfying $x_n \to +\infty$, $\psi(x_n) \to 0$ as $n \to \infty$.

(b) It is often of interest to characterize the asymptotic evolution of states in terms of limits of *sequences* (as opposed to limits of functions). In that case one has to consider convergence of a sequence $\{c_n\}$ to zero in a generalized sense. We shall express this as $\mu^+(\{c_n\})=0$, and there is no difficulty in transcribing (M1) to (M6) into the corresponding properties for convergence of series.

It is usually convenient in applications to restrict one's attention to *invariant* means.

Definition. We shall say that m^+ is *invariant* if $m^+(\psi)=0 \Rightarrow m^+(\psi_a)=0$ for all $a \in \mathbb{R}$, where $\psi_a(x)=\psi(x+a)$. (There is a corresponding notion for sequences).

The ergodic mean

$$m_e^+(\psi)=\lim_{T\to\infty}\frac{1}{T}\int_0^T \psi(x)\,dx \tag{14}$$

is invariant in this sense, and may be used to define convergence to zero. This mean, defined on a suitable space of functions, is an example of an *invariant mean*, on which there is a considerable literature [14]. As stated in the Introduction, the ergodic mean has the property that, if H is a self-adjoint operator in a Hilbert space $\mathcal{H}$, one has for $f \in \mathcal{H}_c(H)$ and $g \in \mathcal{H}$:

$$m_e^+(\langle g, e^{-iHt}f\rangle)=0. \tag{15}$$

In other words, vectors in the continuum subspace of H converge weakly to zero in the generalized sense. In general, this property will hold, for m^+ invariant, if and only if $m^+(\langle f, e^{-iHt}f\rangle)=0$. Hence, for given f, one has to verify convergence to zero for a single function only.

In defining subspaces such as $\bar{\mathcal{M}}_\infty^\pm(H)$, we are concerned rather with strong convergence. To this end we state the following

Definition. For given $f \in \mathbb{R}$, define $\mathcal{B}^+(H,f)$ to be the set of all bounded linear operators A from $\mathcal{H}$ to $\mathcal{H}$ for which $m^+(\|Ae^{-iHt}f\|^2)=0$.

Provided m^+ is invariant, $\mathcal{B}^+(H,f)$ will have the following properties:

(B1) $\mathcal{B}^+(H,f)$ is a norm closed linear subset of $\mathcal{B}(\mathcal{H})$.

(B2) $\mathcal{B}^+(H,f)$ is left-invariant under $\mathcal{B}(\mathcal{H})$.

(B3) $\mathcal{B}^+(H,f)$ is right-invariant under $\{H\}''$, the set of all bounded measurable functions of H.

The only one of these properties which is not immediate to verify is (B3). If m^+ is invariant, then, for $A \in \mathcal{B}^+(H,f)$, $m^+(\|Ae^{-iHt}(e^{-iHs}f)\|^2)=0$, for each $s \in \mathbb{R}$. Taking limits of linear combinations for different values of s, one can approximate $\phi(H)f$ arbitrarily closely in norm by $\Sigma_k c_k e^{-iHs_k}f$ to give $m^+(\|Ae^{-iHt}\phi(H)f\|^2)=0$, which leads to the required result.

We shall use the notation $R_z=(H-z)^{-1}$ and $R_z^0=(H_0-z)^{-1}$. We denote by E_{ac}^0 the orthogonal projection with range $\mathcal{H}_{ac}(H_0)$ and by $\mathcal{B}_\infty$ the Banach space of all compact operators on $\mathcal{H}$, the norm being the operator norm. If m^+ is invariant, then

$$m^\pm(\langle f, e^{-iHt}f\rangle)=0 \Rightarrow \mathcal{B}_\infty \in \mathcal{B}^\pm(H,f). \tag{16}$$

In scattering theory it is useful to impose conditions on the difference $H-H_0$ or on $HK-KH_0$, where K is some suitable bounded operator. This motivates the following lemmas in which we derive properties of $\phi(H)K-K\phi(H_0)$ from assumptions on $HK-KH_0$.

Lemma 2. *Let K be an operator in $\mathscr{B}(\mathscr{H})$ and assume that $(H_0-i)^{-1}(H_0K-KH)(H-i)^{-1}\in\mathscr{B}^+(H,f)$, where m^+ is invariant. Then*

(a) $K\phi(H)-\phi(H_0)K\in\mathscr{B}^+(H,f)$ *for each* $\phi\in\mathscr{A}$.

(b) $\phi(H_0)[K-\exp(iH_0t)K\exp(-iHt)]\in\mathscr{B}^+(H,f)$, *for each* $\phi\in\mathscr{A}$ *and each* $t\in\mathbb{R}$.

Proof. (a) Let $\operatorname{Im} z\neq 0$. Then $R_z^0(H_0K-KH)$ is well defined on $D(H)$ if it is interpreted as $H_0R_z^0K-R_z^0KH$. From (B2), (B3) and the identity

$$R_z^0(H_0K-KH)R_z=[I+(z-i)R_z^0]R_i^0(H_0K-KH)R_i[(H-i)R_z] \tag{17}$$

one obtains that $R_z^0(H_0K-KH)R_z\in\mathscr{B}^+(H,f)$. Hence

$$KR_z-R_z^0K=R_z^0(H_0K-KH)R_z\in\mathscr{B}^+(H,f). \tag{18}$$

Furthermore one has

$$KR_z^mR_\zeta^n-R_z^{0m}R_\zeta^{0n}K=(KR_z^m-R_z^{0m}K)R_\zeta^n+R_z^{0m}(KR_\zeta^n-R_\zeta^{0n}K). \tag{19}$$

Setting in (19) $z=\zeta$, $m=1$ and $n=N-1$, one obtains by induction that $KR_z^N-R_z^{0N}K\in\mathscr{B}^+(H,f)$ for each $N=0,1,2,\dots$. Upon reinserting this into (19) one concludes that $KR_i^mR_{-i}^n-R_i^{0m}R_{-i}^{0n}K\in\mathscr{B}^+(H,f)$ for all $m,n=0,1,2,\dots$. By the complex Stone-Weierstrass theorem (applied to the one-point compactification of $\mathbb{R}$ [15]), each $\phi\in\mathscr{A}$ is the uniform limit of a sequence of polynomials in $(\lambda+i)^{-1}$ and $(\lambda-i)^{-1}$. Hence $K\phi(H)-\phi(H_0)K$ is the uniform limit of a sequence of operators in $\mathscr{B}^+(H,f)$, i.e. $K\phi(H)-\phi(H_0)K\in\mathscr{B}^+(H,f)$ by (B1).

(b) Replacing $\phi(\lambda)$ by $\exp(i\lambda t)\phi(\lambda)$ in (a), we have that $K\phi(H)\exp(iHt)-\phi(H_0)\exp(iH_0t)H\in\mathscr{B}^+(H,f)$. Upon multiplication by $\exp(-iHt)$ on the right, we obtain that $K\phi(H)-\phi(H_0)\exp(iH_0t)K\exp(-iHt)\in\mathscr{B}^+(H,f)$, and the result of (b) follows by combining this with (a). ■

Lemma 3. *Let K be an operator in $\mathscr{B}(\mathscr{H})$ and assume that $(H_0-i)^{-1}(H_0K-KH)(H-i)^{-1}\in\mathscr{B}_\infty$. Then*

(a) $K\phi(H)-\phi(H_0)K\in\mathscr{B}_\infty$ *and* $\phi(H_0)[K-\exp(iH_0t)K\exp(-iHt)]\in\mathscr{B}_\infty$ *for each* $\phi\in\mathscr{A}$ *and each* $t\in\mathbb{R}$.

(b) *If m^+ is invariant and $m^+(\langle f, e^{-iHt}f\rangle)=0$, then the conclusions of Lemma 2 follow.*

Proof. The proof of (a) is similar to that of Lemma 2. (b) follows from (16). ■

A closely related result for sequences is

Lemma 4. *Let K be an operator in $\mathscr{B}(\mathscr{H})$ and assume that $(H_0-i)^{-1}(H_0K-KH)(H-i)^{-1}\in\mathscr{B}_\infty$. Suppose generalized convergence to zero is defined for sequences and that $\mu^+(\{\langle g, e^{-iHt_n}f\rangle\})=0$ for some sequence $\{t_n\}$ and all $g\in\mathscr{H}$. (This will* *be so provided* $\mu^+(\{\langle f, e^{-iH(t_n+s)}f\rangle\})=0$; *i.e. the sequence* $\langle f, e^{-iHt_n}f\rangle$ *and its translates must converge to zero in the generalized sense.) Then the conclusions of Lemma 2 follow, where the set* $\mathcal{B}^+(H,f)$ *is now defined by the condition* $\mu^+(\{\|Ae^{-iHt_n}f\|^2\})=0$.

Theorem 1. *Let* $m^\pm$ *be invariant means and* E *an orthogonal projection commuting with* H_0 *and such that* $\mathcal{R}(E)\subseteq\mathcal{H}_c(H_0)$.

(I) *Assume there is an operator* $J\in\mathcal{B}(\mathcal{H})$ *and two self-adjoint operators* $F_\pm\in\mathcal{B}(\mathcal{H})$ *such that*

(C1) $JD(H_0)\subseteq D(H)$,

(C2) (a) $(F_+ + F_-)E=E$,

(b) $F_\mp\phi(H_0)\in\mathcal{B}^\pm(H_0,f)$ *for all* $\phi\in\mathcal{A}_0$, $f\in\mathcal{R}(E)$,

(c) $\left|\int_0^{\pm\infty} dt\|(HJ-JH_0)\phi(H_0)e^{-iH_0t}EF_\pm\|\right|<\infty, \qquad \forall\phi\in\mathcal{A}_0.$

Then the wave operators $W_\pm=\text{s-lim}\exp(iHt)J\exp(-iH_0t)E$ *as* $t\to\pm\infty$ *exist.*

(II) *Assume in addition, for some* $f\in\mathcal{H}_c(H)$ *and some* $\phi\in\mathcal{A}_0$

(C3) $(H_0-i)^{-1}(H_0J^*-J^*H)(H-i)^{-1}\in\mathcal{B}^+(H,f)$,

(C4) $(I-E)\phi(H_0)J^*\in\mathcal{B}^\pm(H,f)$.

Then

$$W_\pm^*f=0\Rightarrow J^*\phi(H)\in\mathcal{B}^\pm(H,f). \tag{20}$$

In particular, if (C3) *and* (C4) *hold for all* $\phi\in\mathcal{A}_0$ *and all* $f\in\mathcal{H}_c(H)$, *and if all such states* f *are 'evanescent with respect to* $I-J^*$', *e.g. if*

(C5) $(I-J^*)\phi(H)\in\mathcal{B}^\pm(H,f) \qquad \forall\phi\in\mathcal{A}_0, \quad \forall f\in\mathcal{H}_c(H)$,

then $\overline{\mathcal{R}(W_\pm)}=\mathcal{H}_p(H)^\perp$. *If furthermore* $\mathcal{R}(E)\subseteq\mathcal{H}_{ac}(H_0)$, *then* H *has no singularly continuous spectrum.*

(III) *Assume in addition to* (C1) *and* (C2) *that*

(C6) $(I-J)\in\mathcal{B}^\pm(H_0,h)$ *for all* $h\in\mathcal{R}(E)$.

Then $W_\pm$ *are isometric with initial set* $\mathcal{R}(E)$. *Under the stronger hypothesis*

(C6′) $\underset{t\to\pm\infty}{\text{s-lim}}\,(I-J)e^{-iH_0t}E=0$,

the wave operators $\Omega_\pm=\text{s-lim}\exp(iHt)\exp(-iH_0t)E$ *as* $t\to\pm\infty$ *also exist, and* $\Omega_\pm=W_\pm$.

Proof. (for W_+).

(i) As in Lemma 1, $\|g-\phi_n(H_0)g\|\to0$ as $n\to\infty$, for each $g\in\mathcal{R}(E)$. Thus the set $\mathcal{D}=\{\phi(H_0)g \mid g\in\mathcal{R}(E), \phi\in\mathcal{A}_0\}$ is dense in $\mathcal{R}(E)$.

We next show that the set

$$\mathcal{D}_+=\{\phi(H_0)Ee^{iH_0s}F_+e^{-iH_0s}f \mid f\in\mathcal{R}(E), s\geq0, \phi\in\mathcal{A}_0\}$$

is dense in $\mathscr{R}(E)$. For this, let $f=\phi(H_0)g\in\mathscr{D}$ and $\varepsilon>0$. We set $f_+(s)=E\exp(iH_0s)F_+\exp(-iH_0s)f$. By (C2, b) there is a sequence $\{s_m\}$ such that $s_m\to\infty$ and $\exp(iH_0s_m)F_-\exp(-iH_0s_m)f\to 0$ strongly as $m\to\infty$. By (C2, a), this implies that

$$\underset{m\to\infty}{\text{s-lim}}\, f_+(s_m)=\underset{m\to\infty}{\text{s-lim}}\, Ee^{iH_0s_m}(F_++F_-)e^{-iH_0s_m}f=f. \tag{21}$$

Now

$$\|f-\phi_n(H_0)Ee^{iH_0s_m}F_+e^{-iH_0s_m}f\|\le\|f-f_+(s_m)\|+\|f_+(s_m)-\phi_n(H_0)f_+(s_m)\|.$$

By (21), we may choose m such that $\|f-f_+(s_m)\|<\varepsilon/2$. Once m is fixed, it follows from the first part that there is n such that $\|f_+(s_m)-\phi_n(H_0)f_+(s_m)\|<\varepsilon/2$. This proves that $\mathscr{D}_+$ is dense in $\mathscr{R}(E)$.

(ii) We now show that W_+ exists. It suffices to prove existence on $\mathscr{D}_+$. So let $g=\phi(H_0)E\exp(iH_0s)F_+\exp(-iH_0s)f$ be in $\mathscr{D}_+$. Then (setting $\tau=t-s$)

$$\begin{aligned}\int_0^\infty & dt\|(HJ-JH_0)e^{-iH_0t}g\| \\ &=\int_{-s}^\infty d\tau\|(HJ-JH_0)e^{-iH_0\tau}\phi(H_0)EF_+e^{-iH_0s}f\| \\ &\le\|f\|\left\{\int_0^\infty d\tau\|(HJ-JH_0)e^{-iH_0\tau}\phi(H_0)EF_+\|\right. \\ &\qquad\qquad\left.+s\|(HJ-JH_0)\phi(H_0)\|\,\|F_+\|\right\},\end{aligned}$$

which is finite by (C2, c) and the fact that $F_+\in\mathscr{B}(\mathscr{H})$ and $(HJ-JH_0)\phi(H_0)\in\mathscr{B}(\mathscr{H})$. By the Cook criterion [16, Proposition 4.15], $\{\exp(iHt)J\exp(-iH_0t)g\}$ is strongly convergent as $t\to+\infty$, i.e. W_+ exists. Similarly one obtains the existence of W_-.

(iii) We next show that, if one assumes also (C3), one has, for the vector f satisfying (C3) and all $\phi\in\mathscr{A}_0$:

$$F_\pm E\phi(H_0)(J^*-W_\pm^*)\in\mathscr{B}^+(H,f)\cap\mathscr{B}^-(H,f). \tag{22}$$

By Lemma 2, $F_\pm E\phi(H_0)[J^*\text{-}\exp(iH_0s)J^*\exp(-iHs)]\in\mathscr{B}^+(H,f)\cap\mathscr{B}^-(H,f)$ (take $K=J^*$). As $s\to\pm\infty$, this converges weakly to the operator in (22). We must show that the convergence is even in operator norm. This follows from (C2, c) since for $f\in D(H)$,

$$\begin{aligned}\|F_\pm E\phi(H_0)&[e^{iH_0u}J^*e^{-iHu}-e^{iH_0s}J^*e^{-iHs}]f\| \\ &=\left\|\int_s^u dtF_\pm E\phi(H_0)e^{iH_0t}(H_0J^*-J^*H)e^{-iHt}f\right\| \\ &\le\left|\int_s^u dt\|(HJ-JH_0)e^{-iH_0t}\phi(H_0)EF_\pm\|\right|\,\|f\|,\end{aligned}$$

which tends to zero as $s,u\to+\infty$ or $s,u\to-\infty$ respectively.

(iv) Since $\mathscr{R}(E)\subseteq\mathscr{H}_c(H)$, the intertwining relation for $W_\pm$, together with its definition, implies $\mathscr{R}(W_\pm)\subseteq\mathscr{H}_c(H)=\mathscr{H}_p(H)^\perp$.

(v) We next prove (20). We assume $W_+^* f = 0$ and use the identity

$$\begin{aligned} J^* e^{-iHt}\phi(H)f = {} & [J^*\phi(H) - \phi(H_0)J^*]e^{-iHt}f \\ & + [I - (F_+ + F_-)E]\phi(H_0)J^* e^{-iHt}f \\ & + F_+ E\phi(H_0)(J^* - W_+^*)e^{-iHt}f \\ & + F_- E\phi(H_0)(J^* - W_-^*)e^{-iHt}f \\ & + F_+ E\phi(H_0)e^{-iH_0t}W_+^* f + F_-\phi(H_0)e^{-iH_0t}EW_-^* f. \end{aligned} \tag{23}$$

The first four terms on the r.h.s. converge to zero in the generalized sense since the operators appearing in these terms belong to $\mathcal{B}^+(H, f)$ by Lemma 2(a), (C2, a) and (C4) for the first two and by (22) for the next two. The fifth term is identically zero, whereas the last term converges to zero as $t \to +\infty$ by (C2, b). This proves (20).

(vi) Now if $f \in \mathcal{H}_c(H)$ and $W_+^* f = 0$, (C5) implies, with the result already obtained, that $\phi(H) \in \mathcal{B}^+(H, f)$ for each $\phi \in \mathcal{A}_0$. But $m^+(\|\phi(H)e^{-iHt}f\|^2) = m^+(\|\phi(H)f\|^2) = 0$ implies $\phi(H)f = 0$. Hence, if (C5) holds, we have $\phi_n(H)f = 0$ for each n, where $\{\phi_n\}$ is the sequence given by (A2). Then $f = 0$ by Lemma 1, so that there is no vector in $\mathcal{H}_c(H)$ except the zero vector satisfying $W_+^* f = 0$. Thus $\mathcal{R}(W_+) = \mathcal{H}_p(H)^\perp$.

If $\mathcal{R}(E) \subseteq \mathcal{H}_{ac}(H_0)$, then $\mathcal{R}(W_+) \subseteq \mathcal{H}_{ac}(H)$ by the intertwining relation. Hence $\mathcal{H} = \mathcal{H}_p(H) \oplus \mathcal{H}_{ac}(H)$, i.e. H has no singularly continuous spectrum.

(vii) Assume now that (C6) holds. Then, for each $f \in \mathcal{R}(E)$, there is a sequence $\{t_n\}$ such that $t_n \to \infty$ and $\|(I - J)\exp(-iH_0 t_n)f\| \to 0$ as $n \to \infty$. It follows that

$$\begin{aligned} \|W_+ f\| &= \lim_{t\to\infty} \|Je^{-iH_0t}f\| = \lim_{n\to\infty} \|Je^{-iH_0t_n}f\| \\ &= \lim_{n\to\infty} \|e^{-iH_0t_n}f\| = \|f\|, \end{aligned} \tag{24}$$

i.e. that W_+ is isometric with initial set $\mathcal{R}(E)$. The proof of the other statements in (III) is straightforward. ■

Corollary 1. *In the hypotheses of Theorem* 1, *replace* $\mathcal{B}^\pm(H, f)$ *by* $\mathcal{B}_\infty$ *in* (C3), (C4) *and* (C5). *Assume in addition that* $m^+(\langle f, e^{-iHt}f\rangle) = 0$ *for each* $f \in \mathcal{H}_c(H)$. *Then all conclusions of the theorem remain valid.*

Corollary 2. *Assume* (C1), (C2) *with* (C2,b) *replaced by* s-$\lim F_\mp\phi(H_0)\exp(-iH_0t)h = 0$ *as* $t \to \pm\infty$, *for each* $\phi \in \mathcal{A}_0$ *and each* $h \in \mathcal{R}(E)$. *Also assume* (C3) *and* (C4) *with* $\mathcal{B}^\pm(H, f)$ *replaced by* $\mathcal{B}_\infty$. *Let generalized convergence to zero be defined for sequences and let* $\mu^\pm(\{\langle g, e^{-iHt_n}f\rangle\}) = 0$ *for some sequence* $\{t_n\}$ *and for all* $g \in \mathcal{H}$. *Then*

$W_\pm^* f = 0 \Rightarrow J^*\phi(H) \in \mathcal{B}^\pm(H, f)$

($\mathcal{B}^\pm(H, f)$ *being defined as in Lemma* 4).

Proof. Use (23) and notice that now the operators appearing in the first four terms are in $\mathcal{B}_\infty$ as a consequence of Lemma 3, hence in $\mathcal{B}^\pm(H, f)$ by the analogue of (16) for sequences. The last term in (23) converges strongly to zero by hypothesis. ■

If $W_+^* f = W_-^* f = 0$, one may use a version of Corollary 2 in which the time parameter t does not appear. As an application of this we have

Corollary 3. *Assume* (C1), (C2), *as well as* (C3), (C4) *and* (C5) *with* $\mathscr{B}^{\pm}(H, f)$ *replaced by* $\mathscr{B}_\infty$. *Then each eigenvalue of* H *in* $\mathbb{R}\setminus\Gamma$ *is of finite multiplicity, and these eigenvalues accumulate at most at points of* Γ *or at* $\pm\infty$.

Proof. Let $\lambda \in \mathbb{R}\setminus\Gamma$, and assume there is an infinite orthonormal sequence $\{f_n\}$ of eigenvectors of H with $Hf_n = \lambda_n f_n$ and $\lambda_n \to \lambda$. Choose $\phi \in \mathscr{A}_0$ such that $\phi(\lambda) > 0$. One has w-lim $f_n = 0$ and $\lim \|\phi(H)f_n\| = \phi(\lambda) > 0$. Since $W_\pm^* f_n = 0$, we obtain as in (23) that

$$\phi(H)f_n = (I - J^*)\phi(H)f_n + [J^*\phi(H) - \phi(H_0)J^*]f_n$$
$$+ [I - (F_+ + F_-)E]\phi(H_0)J^* f_n + F_+ E\phi(H_0)(J^* - W_+^*)f_n + F_- E\phi(H_0)(J^* - W_-^*)f_n.$$

Since all operators appearing on the r.h.s. are compact and w-lim $f_n = 0$, we obtain $\|\phi(H)f_n\| \to 0$, a contradiction. (Note that convergence of sequences here is defined in the normal way). ■

Theorem 2. *Let* $\{F_r\}$ *be a sequence of localizing operators such that* $F_r J \in \mathscr{B}^{\pm}(H_0, f)$ *for each* $r = 1, 2, \dots$ *and each* $f \in \mathscr{R}(E)$ *and such that* $\lim \|(I - F_r)(I - J^*)\| = 0$ *as* $r \to \infty$. *Let* $\mathscr{M}_0^{\pm}(H)$ *be defined by* (4). *Assume* (C1)–(C4) *in the form stated in Corollary* 2. *Then*

(a) $\mathscr{H} = \mathscr{M}_0^+(H) \oplus \overline{\mathscr{R}(W_+)} = \mathscr{M}_0^-(H) \oplus \overline{\mathscr{R}(W_-)}$. (25)

(b) If $\mathscr{R}(E) \subseteq \mathscr{H}_{ac}(H_0)$, one has $\mathscr{H}_{sc}(H) \subseteq \mathscr{M}_0(H)$.

Proof. (i) We first show that $F_r \in \mathscr{B}^{\pm}(H, W_\pm f)$ for each $f \in \mathscr{R}(E)$. In fact

$$\|F_r e^{-iHt} W_\pm f\|^2 \leq 2\,\|F_r\|^2\, \|e^{-iHt} W_\pm f - J e^{-iH_0 t} f\|^2 + 2\,\|F_r J e^{-iH_0 t} f\|^2.$$

The first term converges to zero as $t \to \pm\infty$, hence $m^{\pm}(\|e^{-iHt}W_\pm f - Je^{-iH_0 t} f\|^2) = 0$. Also, by hypothesis, $m^{\pm}(\|F_r J e^{-iH_0 t} f\|^2) = 0$, so that our claim follows from (M1), (M2) and (M4).

(ii) Let $h \in \mathscr{M}_0^+(H)$, $f \in \mathscr{R}(E)$ and $g = W_+ f$. Then

$$|\langle h, g\rangle| = |\langle e^{-iHt}h, e^{-iHt}g\rangle|$$
$$= |\langle (I - F_r)e^{-iHt}h, e^{-iHt}g\rangle + \langle e^{-iHt}h, F_r e^{-iHt} g\rangle|$$
$$\leq \|(I - F_r)e^{-iHt}h\|\,\|g\| + \|h\|\,\|F_r e^{-iHt} g\|.$$

Given $\varepsilon > 0$, one may choose r so that the first term on the r.h.s. is less than $\varepsilon/2$ for all $t \geq 0$. Since $F_r \in \mathscr{B}^+(H, g)$, there is a $t > 0$ such that the second term is less than $\varepsilon/2$. Hence $|\langle h, g\rangle| < \varepsilon$ for each $\varepsilon > 0$, i.e. $\langle h, g\rangle = 0$. Thus $\mathscr{M}_0^+(H) \perp \mathscr{R}(W_+)$.

(iii) Now assume that f is such that $f \perp \mathscr{M}_0^+(H)$ and $f \perp \mathscr{R}(W_+)$. We must show that $f = 0$. So suppose $f \neq 0$.

Let $\phi \in \mathscr{A}_0$ be such that $g \equiv \phi(H)f \neq 0$. Since $\mathscr{M}_0^+(H)$ is invariant under e^{-iHt}, we have $g \perp \mathscr{M}_0^+(H)$. Thus, since $g \notin \mathscr{M}_0^+(H)$, there are two sequences $\{t_n\}$ and $\{r_n\}$ such that $t_n \to +\infty$, $r_n \to +\infty$ and

$$\|(I - F_{r_n})e^{-iHt_n} g\| \geq \delta > 0 \quad \text{for all } n. \tag{26}$$

Since each bounded sequence has a weakly convergent subsequence, there is a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\{\exp(-iHt_{n_k})g\}$ is weakly convergent. We relabel the subsequence $\{t_{n_k}\}$ and the corresponding subsequence $\{r_{n_k}\}$ and use the fact that compact operators map weakly convergent sequences into strongly convergent ones. We then find from (23) (cf. the proof of Corollary 2) that $J^* \exp(-iHt_n)g = J^*\phi(H)\exp(-iHt_n)f$ converges strongly to some vector in $\mathcal{H}$. Since $I - F_{r_n} \to 0$ strongly and

$$\|(I-F_{r_n})e^{-iHt_n}g\| \le \|(I-F_{r_n})(I-J^*)\|\,\|g\| + \|(I-F_{r_n})J^*e^{-iHt_n}g\|,$$

it follows that $\|(I-F_{r_n})e^{-iHt_n}g\| \to 0$ as $n\to\infty$. This contradicts (26), so that we must have $f=0$. This proves (a).

(b) If $\mathcal{R}(E) \subseteq \mathcal{H}_{ac}(H_0)$, then $\mathcal{R}(W_\pm) \subseteq \mathcal{H}_{ac}(H)$, hence by (25) $\mathcal{H}_{sc}(H) \subseteq \mathcal{M}_0^+(H) \cap \mathcal{M}_0^-(H) \equiv \mathcal{M}_0(H)$. ■

Corollary 4. *Under the assumptions of Theorem* 2, *the following two statements are equivalent:*

(α) $\mathcal{M}_0^+(H) = \mathcal{M}_0^-(H)$,

(β) *the wave operators $W_\pm$ satisfy asymptotic completeness in the geometric sense, namely* $\mathcal{R}(W_+) = \mathcal{R}(W_-) = \mathcal{M}_0(H)^\perp$.

Remark. If the wave operators $\Omega_\pm$ also exist and $W_\pm = \Omega_\pm$, then (α) or (β) implies that the scattering operator $S = \Omega_+^*\Omega_-$ is unitary in the subspace $\mathcal{R}(E)$.

Corollary 5. *Assume in Theorem* 2 *that* $\text{s-lim}\, F_r J e^{-iH_0 t} f = 0$ *for each r and each $f \in \mathcal{R}(E)$. Then*

$$\mathcal{H} = \mathcal{M}_0^+(H) \oplus \mathcal{M}_\infty^+(H) = \mathcal{M}_0^-(H) \oplus \mathcal{M}_\infty^-(H).$$

III. Remarks and examples

We collect here a few remarks about our theorems and indicate how they may be applied in non-relativistic potential scattering.

(**A**) If one replaces in (C3) and (C4) $\mathcal{B}^\pm(H,f)$ by $\mathcal{B}_\infty$, then the conditions (C1)–(C4) and (C6) in Theorem 1 are assumptions on the operators H_0, J, $HJ - JH_0$, E and $F_\pm$ which, in applications, are expected to be explicitly given. (C5) is a condition on H, called 'local evanescence', which we have purposely isolated from the other assumptions (see (**K**) for a motivation of the term 'local evanescence').

The conditions (C3) and (C4) may be replaced by the following two conditions (C3′) and (C4′):

(C3′) $E(H_0 - i)^{-1}(H_0J^* - J^*H)(H-i)^{-1} \in \mathcal{B}^\pm(H,f)$,

(C4′) $(I-E)J^* \in B^\pm(H,f)$.

Also, if $J = I$, then (C5) and (C6) clearly become redundant.

(**B**) If one knows in addition to the assumptions of Theorem 1 that $\sigma_c(H) \subseteq \sigma_c(H_0)$, one may replace in (10) the real line $\mathbb{R}$ by $\sigma_c(H_0)$.

(**C**) If one assumes in (A2) only that Γ is a set of Lebesgue measure zero and makes the assumption that $\mathcal{R}(E) \subseteq \mathcal{H}_{ac}(H_0)$, one obtains in Theorem 1 that $W_\pm$ are complete in the sense that $\overline{\mathcal{R}(W_\pm)} = \mathcal{H}_{ac}(H)$.

(**D**) Some useful possibilities for the projection E in Theorem 1 are the following:

(α) $E = E^0_{ac}$, the projection onto the absolutely continuous subspace of H_0. In this case one obtains the usual wave operators $\Omega_\pm =$ s-lim exp (iHt) exp $(-iH_0t)E^0_{ac}$ as $t \to \pm\infty$.

(β) $E = E^0_c$, the projection onto the continuous subspace of H_0. In this case the wave operators are partial isometries mapping $\mathcal{H}_c(H_0)$ onto $\mathcal{H}_c(H)$.

(γ) $E = E^0_\infty$, the orthogonal projection onto the subspace $\mathcal{M}_\infty(H_0)$ of all scattering states of H_0 [17, 16]. Here it is assumed that one has a sequence $\{F_r\}$ of localizing operators which serve to define the scattering states $\mathcal{M}^\pm_\infty(H_0)$ as in (3), and that $\mathcal{M}^+_\infty(H_0) = \mathcal{M}^-_\infty(H_0) \equiv \mathcal{M}_\infty(H_0)$.

(**E**) If $\mathcal{A}_0 \subset C^\infty_0(\mathbb{R})$, the condition (C1) may for example be weakened to (C1'): $Jf \in D(H)$ for each f in $\mathcal{H}_c(H_0)$ having compact support in the spectral representation of H_0.

(**F**) Theorem 1 has the peculiarity of not applying immediately to the trivial situation where $H = H_0$, $J = I$ and $\mathcal{H}_{sc}(H_0) = \{0\}$. In this case, the conclusions of the theorem are clearly valid. But only the assumptions (C1), (C2, c), (C3)–(C6) are trivially satisfied; it is a non-trivial fact that one may find two operators (even projections) $F_\pm$ satisfying (C2, a) and (C2, b) with $E = E^0_{ac}$ ([18], Theorem 8).

(**G**) As another illustration of the usefulness of averaging, consider the case where $J = I$, $(H - H_0)(H_0 + i)^{-2} \in \mathcal{B}_\infty$ and $D(H) = D(H_0)$. It follows [19, p. 261] that $(H - H_0)(H + i)^{-2} \in \mathcal{B}_\infty$. Hence, by (16), $(H_0 + i)^{-1}(H - H_0)(H + i)^{-1} \in \mathcal{B}^\pm(H, f)$ for all f in the dense subset $\{f = (H + i)^{-1}g \mid g \in \mathcal{H}_c(H)\}$ of $\mathcal{H}_c(H)$. Since $(H_0 + i)^{-1}(H - H_0)(H + i)^{-1}$ is bounded, this implies (C3) for all $f \in \mathcal{H}_c(H)$.

(**H**) The conditions (C1)–(C5) do not exclude the possibility that $W_\pm = 0$, i.e. $\mathcal{H}_c(H) = \{0\}$. (An example is given by the harmonic oscillator $H = -\Delta + |x|^2$, $H_0 = -\Delta$ in $L^2(\mathbb{R}^n)$, if one takes J to be multiplication by a function $j(x)$ in $C^\infty_0(\mathbb{R}^n)$). The partial isometry of $W_\pm$ is obtained only after imposing a condition of the type (C6).

(**I**) Let E_Δ and E^0_Δ be the spectral projections of H and H_0 respectively associated with the interval Δ. If one replaces in (C1) and (C2, c) the operator J by $E_\Delta J$ with $J \in \mathcal{B}(\mathcal{H})$, then part (I) of Theorem 1 gives the existence of the *local wave operators* s-lim $E_\Delta e^{iHt}Je^{-iH_0t}E^0_\Delta E$ as $t \to \pm\infty$. If furthermore Δ is compact, (C1) becomes redundant. (In (C2), E may also be replaced by $E^0_\Delta E$). If one assumes (C3)–(C5) for each $f \in E_\Delta \mathcal{H}_c(H)$, or (C6), (C6') with $E^0_\Delta E$ instead of E, one obtains the corresponding conclusions locally (on Δ).

Suppose that

$$\text{(C2, c')} \quad \left| \int_0^{\pm\infty} dt \, \|E_\Delta(HJ - JH_0)\phi(H_0)e^{-iH_0t}EF_\pm\| \right| < \infty, \qquad \forall \phi \in \mathcal{A}_0,$$

for each compact interval Δ. Then $E_\Delta e^{iHt}Je^{-iH_0t}E$ has strong limits as $t \to \pm\infty$ for each compact Δ. If one assumes in addition a weak version of (C1), e.g.

$$\text{(C1')} \quad F_\phi(H)J\phi(H_0) \in \mathcal{B}(\mathcal{H})$$

for each $\phi \in \mathcal{A}_0$, where $F_\phi : \mathbb{R} \to \mathbb{C}$ is such that $|F_\phi(\lambda)| \to \infty$ as $|\lambda| \to \infty$ and

$|F_\phi(\lambda)|\geq 1$, then $W_\pm$ exist. This follows easily from the identity

$$e^{iHt}Je^{-iH_0t}\phi(H_0)g = E_{[-M,M]}e^{iHt}Je^{-iH_0t}\phi(H_0)g + [F_\phi(H)]^{-1}(I-E_{[-M,M]})e^{iHt}F_\phi(H)J\phi(H_0)e^{-iH_0t}g$$

and the fact that $\|F_\phi(H)^{-1}(I-E_{[-M,M]})\| \to 0$ as $M\to\infty$.

Also notice that, if one assumes the existence of $W_\pm$ and (22), then (C1) and (C2, c) can be dropped altogether.

(J) *Two-space Scattering Theory.* Let H_0 act in a Hilbert space $\mathcal{H}_0$ and H in a Hilbert space $\mathcal{H}$. J is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$ and J^* maps back from $\mathcal{H}$ to $\mathcal{H}_0$. Assume (C1)–(C4) and

$$(\mathrm{C5})_2\quad m^\pm(\|(Z^*-J^*)e^{-iHt}f\|^2)=0 \qquad \forall f\in\mathcal{H}_c(H),$$

where Z is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$ satisfying $\|Z^*h\|\geq c\|h\|$ for some $c>0$ and all $h\in\mathcal{H}$. One then gets wave operators $W_\pm = \text{s-}\lim \exp(iHt)J\exp(-iH_0t)E$ as $t\to\pm\infty$. If $f\perp\mathcal{H}_p(H)$ and $f\perp\mathcal{R}(W_+)$, one obtains from (20) and $(\mathrm{C5})_2$ that $Z^*\phi(H)\exp(-iHt)f\to 0$ in the generalized sense, so that $\phi(H)\exp(-iHt)f\to 0$ in the generalized sense by the hypothesis made on Z^*. Hence $\overline{\mathcal{R}(W_\pm)}=\mathcal{H}_p(H)^\perp$ as before, and H has no singularly continuous spectrum if $\mathcal{R}(E)\subseteq\mathcal{H}_{ac}(H_0)$.

If for example

$$(\mathrm{C6})_2\quad m^\pm(\|(Z-J)e^{-iH_0t}f\|^2)=0 \qquad \forall f\in\mathcal{R}(E),$$

then, as in (24), $\|W_+f\|=\lim\|Z\exp(-iH_0t_n)f\|$ as $n\to\infty$, and W_+ is a partial isometry only if this limit is equal to $\|f\|$ for each $f\in\mathcal{R}(E)$.

(K) *Potential Scattering.* In non-relativistic scattering theory, one has $H_0=-\Delta$ in $\mathcal{H}=L^2(\mathbb{R}^n)$. J is the multiplication operator by a C^∞-function j such that $j(\mathbf{x})=0$ for $|\mathbf{x}|\leq R$ and $j(x)=1$ for $|\mathbf{x}|\geq R+1$, where R is any finite number (J satisfies (C6′)). $F_\pm$ may be taken to be the spectral projections associated with the intervals $(0,\infty)$ and $(-\infty,0)$, respectively, of the self-adjoint operator $A=\frac{1}{2}(\mathbf{P}\cdot\mathbf{Q}+\mathbf{Q}\cdot\mathbf{P})$ ($\mathbf{Q}$ = multiplication by $\mathbf{x}$, $\mathbf{P}=-i$ grad) [5]. Thus, roughly speaking, states in the subspace $F_+\mathcal{H}(F_-\mathcal{H})$ are such that the projection of the velocity onto the position vector is positive (negative). Also $E=I$.

If $v:\mathbb{R}^n\to\mathbb{R}$ is a potential such that multiplication by $v(\mathbf{x})j(\mathbf{x})$ is a H_0-bounded operator, we define H to be an arbitrary self-adjoint extension of $\hat{H}=-\Delta+v$ with $D(\hat{H})=JD(H_0)$ (notice that, unless we take $j\equiv 1$, $\hat{H}$ is not densely defined since all functions in $D(\hat{H})$ have support outside the ball $B_R=\{x \mid |x|\leq R\}$.) One has $HJ-JH_0=VJ-(\Delta J)-2i(\nabla J)\cdot\mathbf{P}$, where e.g. (ΔJ) denotes the multiplication operator by $(\Delta j)(\mathbf{x})$.

To prove (C2, c) it suffices to exhibit an invertible operator $T\in\mathcal{B}(\mathcal{H})$ such that

$$\|[VJ-(\Delta J)-2i(\nabla J)\cdot\mathbf{P}](H_0+1)^{-\alpha}T^{-1}\|<\infty \quad \text{for some} \quad \alpha\geq 0$$

and

$$\|Te^{-iH_0t}\phi(H_0)F_\pm\|\leq c(\phi)(1+|t|)^{-1-\delta} \tag{27}$$

for some $\delta>0$, all $t\gtrless 0$ respectively and all $\phi\in\mathcal{A}\equiv C_0^\infty(\mathbb{R}\setminus\{0\})$. This may be proved for instance for $T=|A+i|^{-1-2\delta}$ [5] or for $T=(I+|\mathbf{Q}|)^{-1-2\delta}$ [20] provided that

$$v(\mathbf{x})j(\mathbf{x}) = (1+|\mathbf{x}|)^{-1-2\delta}[w_\infty(\mathbf{x}) + w_p(\mathbf{x})] \tag{28}$$

with $w_\infty \in L^\infty(\mathbb{R}^n)$ and $w_p \in L^p(\mathbb{R}^n)$ for some $p > \max\{2, n/2\}$. (The terms with (ΔJ) and (∇J) give no difficulties since Δj and ∇j are in $C_0^\infty(\mathbb{R}^n)$, hence of the form of the r.h.s. of (28)).

The fact that T^* has dense range also implies (C2, b): If $g = T^*f$, then

$$\|F_\pm \phi(H_0)e^{-iH_0t}g\| \le \|Te^{+iH_0t}\phi(H_0)F_\pm\| \, \|f\| \to 0 \quad \text{as} \quad t \to \mp\infty,$$

so that (C2, b) holds, in the sense of normal convergence, on a dense set and hence on $\mathcal{H}$.

Thus, if v has the form of the r.h.s. of (28) outside some ball B_R (for some $\delta > 0$) and if all states in $\mathcal{H}_c(H)$ are evanescent in the sense of (C5), $\Omega_\pm$ exist and $\mathcal{R}(\Omega_\pm) = \mathcal{H}_p(H)^\perp$. (Local conditions on v implying evanescence can be found in [10, 13]).$^{(*)}$

A different pair of operators $F_\pm$ is used in [6].

(L) *Oscillating Potentials.* One can also treat potentials that oscillate sufficiently rapidly near infinity. Take $n = 3$, assume v to be spherically symmetric and define $w(r) = -\int_r^\infty v(s)\, ds$. If

$$|w(r)| \le (1+r)^{-1-\delta}[f_\infty(r) + f_2(r)] \quad \text{for} \quad r \ge r_0 > 0, \tag{29}$$

with $f_\infty \in L^\infty(\mathbb{R}_+)$, $rf_2 \in L^2(\mathbb{R}_+)$ and $\delta > 0$, then one can show as in Eq. (25) of [22] the existence of a function $\varphi_0 : \mathbb{R}_+ \to \mathbb{R}$ such that, as $r \to \infty$,

$$\varphi_0(r) = 1 + 0(r^{-\delta/2})$$

and

$$d\varphi_0(r)/dr = w(r)[1 + 0(r^{-\delta/2})]$$

(set $\varepsilon = \delta - 2$ in Eq. (21) of [22]). Take $\hat{j} = j\varphi_0$, with j as before. Then, as in [22],

$$(H\hat{J} - \hat{J}H_0)\varphi(H_0)e^{-iH_0t}F_\pm = \{w_0(|\mathbf{Q}|) + \mathbf{w}(|\mathbf{Q}|) \cdot \mathbf{P}\}\phi(H_0)e^{-iH_0t}F_\pm,$$

where $|w_0(r)| \le c_1(1+r)^{-m} + c_2|w(r)|$ (m any real number) and $|w_k(r)| < c_3|w(r)|$ for $r > R$ ($k = 1, 2, 3$). Hence, as in **(K)**, one has existence and strong asymptotic completeness of $\Omega_\pm$ provided v satisfies local conditions such that all states in $\mathcal{H}_p(H)^\perp$ are evanescent. The condition (29) is satisfied for instance for potentials of the form $v(r) = cr^\kappa \sin(r^\gamma)$ with $\kappa \in \mathbb{R}$ and $\gamma - \kappa > 2$. (For other results on oscillating potentials, see [4]).

(M) One sees that, for the geometric proof of completeness to work, it is not necessary to have local compactness or a condition of subordination, e.g. of the type

$$\psi(\mathbf{P})E_{(a,b)}(H) \in \mathcal{B}(\mathcal{H}) \tag{30}$$

for some ψ going to infinity at infinity and all $a, b \in \mathbb{R}$, which has been used in

$^{(*)}$ It is interesting to note that the geometric proof of asymptotic completeness, at least in potential scattering, can be carried through by purely time-dependent methods without invoking the spectral theorem [21].

previous work (e.g. [4, 7, 8]). Only evanescence is essential. However, from a condition of the type (30), evanescence can easily be inferred [10].

(N) It is an interesting feature of the geometric method that the combination of certain ergodic or similar assumptions (cf. (C2, b), (C3)–(C6)) with a Cook type hypothesis (cf. (C2, c)) may lead to the conclusion that averaging over time (for instance in the definition of the subspaces of scattering states) may be left out. In fact, if $\chi_R(\mathbf{Q})(H+i)^{-1} \in \mathcal{B}_\infty$ (χ_R is the characteristic function of the ball B_R), then s-lim $\chi_R(\mathbf{Q}) \exp(-iHt)f = 0$ for each $R<\infty$ and each $f \in \mathcal{H}_{ac}(H)$ [10], hence for each $f \in \mathcal{H}_p(H)^\perp$ under the hypotheses of Theorem 1.

(O) *Local Absorption.* Assume there is a bounded closed set Σ of Lebesgue measure zero such that $v \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \Sigma)$ for some $q > \max(2, n/2)$ and vj satisfies (28), where j is as in **(K)** and vanishes on some ball B_R containing Σ in its interior. Let H be an arbitrary self-adjoint extension of $(-\Delta + v) \restriction C_0^\infty(\mathbb{R}^n \setminus \Sigma)$. We denote by $C_\Sigma^\infty(\mathbb{R}^n)$ the set of all bounded functions in $C^\infty(\mathbb{R}^n)$ vanishing in some neighbourhood of Σ. We denote by $\mathcal{M}_\Sigma^\pm(H)$ the subspaces of all states that get absorbed at Σ as $t \to \pm\infty$ [11] and by $\bar{\mathcal{M}}_\Sigma^\pm(H)$ the subspaces of all states that get absorbed on the time average. These subspaces are defined as follows:

$$f \in \mathcal{M}_\Sigma^\pm(H) \Leftrightarrow \lim_{t\to\pm\infty} \|Ke^{-iHt}f\|^2 = 0 \qquad \forall k \in C_\Sigma^\infty(\mathbb{R}^n), \tag{31}$$

where K denotes the multiplication operator by the function k, and

$$f \in \bar{\mathcal{M}}_\Sigma^\pm(H) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_0^{\pm T} \|Ke^{-iHt}f\|^2 \, dt = 0 \qquad \forall k \in C_\Sigma^\infty(\mathbb{R}^n). \tag{32}$$

We also set $\mathcal{M}_{\Sigma,ac}^\pm(H) = \mathcal{M}_\Sigma^\pm(H) \cap \mathcal{H}_{ac}(H)$.

Theorem 3. *Let H be as above and $H_0 = -\Delta$. Then the wave operators $\Omega_\pm = \text{s-lim} \exp(iHt)\exp(-iH_0t)$ as $t \to \pm\infty$ exist and one has generalized asymptotic completeness in the following sense:*

(a) *As $t \to +\infty$, state vectors in the absolutely continuous subspace of H are either scattering states and belong to the range of Ω_+ or absorbed states, and similarly as $t \to -\infty$. More precisely:*

$$\mathcal{H}_{ac}(H) = \mathcal{M}_\infty^+(H) \oplus \mathcal{M}_{\Sigma,ac}^+(H) = \mathcal{M}_\infty^-(H) \oplus \mathcal{M}_{\Sigma,ac}^-(H) \tag{33}$$

and

$$\mathcal{R}(\Omega_\pm) = \mathcal{M}_\infty^\pm(H). \tag{34}$$

(b) *State vectors in the singularly continuous subspace of H are bound states, and as $t \to \pm\infty$, they get absorbed at Σ on the time average:*

$$\mathcal{H}_{sc}(H) \subseteq \mathcal{M}_0(H) \cap \bar{\mathcal{M}}_\Sigma^+(H) \cap \bar{\mathcal{M}}_\Sigma^-(H). \tag{35}$$

In particular, one has asymptotic completeness in the geometric sense if and only if $\mathcal{M}_{\Sigma,ac}^+(H) = \mathcal{M}_{\Sigma,ac}^-(H)$.

Proof. The domains of H and H_0 coincide locally on $\mathbb{R}^n \setminus \Sigma$ as well as outside B_R [23, 11]. This implies (C1) and the compactness of $K(I-J^*)(H+i)^{-1}$ for any $k \in C_\Sigma^\infty(\mathbb{R}^n)$. Also, by the results of **(K)**, (C2)–(C4) and (C6′) are satisfied (for each $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$), with $\mathcal{B}^\pm(H, f)$ replaced by $\mathcal{B}_\infty$, and the convergence in (C2, b) is normal convergence.

If $f \in \mathscr{H}_{ac}(H)$, then $\exp(-iHt)f$ converges weakly to zero as $t \to \pm\infty$. Hence $K(I-J^*)\exp(-iHt)(H+i)^{-1}f$ converges strongly to zero. Since the set $\{(H+i)^{-1}f \mid f \in \mathscr{H}_{ac}(H)\}$ is dense in $\mathscr{H}_{ac}(H)$, we have

$$\operatorname*{s-lim}_{t\to\pm\infty} K(I-J^*)e^{-iHt}g = 0 \qquad \forall g \in \mathscr{H}_{ac}(H). \tag{36}$$

Similarly one obtains from (6) that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^{\pm T} \|K(I-J^*)e^{-iHt}h\|^2\, dt = 0 \qquad \forall h \in \mathscr{H}_{sc}(H). \tag{37}$$

If $f \in \mathscr{H}_{ac}(H)$ and $f \perp \mathscr{R}(\Omega_+)$, then by (20) and Lemma 1, s-$\lim J^* \exp(-iHt)f = 0$ as $t \to +\infty$. Inserting this into (36), we obtain s-$\lim K \exp(-iHt)f = 0$, which proves (a). Similarly, if $h \in \mathscr{H}_{sc}(H)$, then by (6), (20) and Lemma 1:

$$\operatorname*{s-lim}_{T\to\infty} \frac{1}{T}\int_0^{\pm T} \|J^*e^{-iHt}h\|^2\, dt = 0$$

which, together with (37), implies that $h \in \bar{\mathscr{M}}_\Sigma^\pm(H)$. The fact that $h \in \mathscr{M}_0(H)$ follows from Theorem 2: the localizing operators F_r are given by (2) and satisfy the hypotheses of that theorem. ■

We expect that the result of (b) is true under weaker assumptions on v outside B_R. In that case the wave operators may not exist, and our proof will then not be applicable. The local assumptions on v could also be somewhat weakened [7, 24].

One may ask the question whether all states in $\mathscr{H}_{sc}(H)$, in addition to being bound, get absorbed at Σ in the ordinary sense, i.e. whether $\mathscr{H}_{sc}(H) \subseteq \mathscr{M}_\Sigma^\pm(H)$ (no time average!) This is so if and only if $\lim \langle h, e^{-iHt}h\rangle = 0$ as $t \to \pm\infty$ for each $h \in \mathscr{H}_{sc}(H)$. Based on results in [25], one may construct examples of operators H (e.g. of the form $H = \phi(H_0 + V)$) for which $\exp(-iHt)h$ is not weakly convergent for $h \in \mathscr{H}_{sc}(H)$. Some details on this will be given in [26]. It follows however that in general (e.g. under the hypotheses of Theorem 3):

$$\mathscr{M}_\Sigma^+(H) \cap \mathscr{H}_{sc}(H) = \mathscr{M}_\Sigma^-(H) \cap \mathscr{H}_{sc}(H). \tag{38}$$

This together with (35) shows that $\mathscr{H}_{sc}(H)$ is symmetric with respect to the behaviour at $t = -\infty$ and at $t = +\infty$.

For scattering theory (with short range potentials), one wants to be in the subspace orthogonal to the bound states. So, by Theorem 3, one is automatically in (some subspaces of) $\mathscr{H}_{ac}(H)$. Also note that $\mathscr{R}(\Omega_\pm)$ are the sets of states which evanesce to infinity in the *usual* sense of convergence.

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