

Zeitschrift: Helvetica Physica Acta

Band: 53 (1980)

Heft: 1

Artikel: Non-regularity of the Coulomb potential in quantum electrodynamics

Autor: Klaus, Martin

DOI: <https://doi.org/10.5169/seals-115107>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 21.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Non-regularity of the Coulomb potential in quantum electrodynamics

by **Martin Klaus**

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22903, USA

(21. I. 1980)

Abstract. The operator $P_+(\lambda) - P_+^\circ$ is not Hilbert-Schmidt if P_+° resp. $P_+(\lambda)$ denote the spectral projections of $[0, \infty)$ for the operators $H_0 = \alpha p + \beta$, resp. $H = \alpha p + \beta - \lambda/|x|$ and $\lambda \in (0, 1)$. This implies that a static external Coulomb field is non-regular in the sense of [1].

1. Introduction

We consider the free Dirac operator

$$H_0 = \alpha p + \beta \tag{1.1}$$

in interaction with a static electric Coulomb field, i.e., the Hamiltonian

$$H(\lambda) = \alpha p + \beta - \frac{\lambda}{|x|} \quad 0 < \lambda < 1 \tag{1.2}$$

In our study of the external field problem [1, 2] in quantum electrodynamics, the question arose whether a so-called strong Bogoljubow transformation exists which relates the free field to the interacting field. The mathematical problem is to prove or disprove that

$$P_+(\lambda) - P_+^\circ \in \text{H.S.} \tag{1.3}$$

is Hilbert-Schmidt (cf. [1, p. 794]). Here P_+° resp. $P_+(\lambda)$ are the spectral projections of $[0, \infty)$ for H_0 resp. $H(\lambda)$. For less singular potentials than Coulomb this question has been studied in [1] and in more detail in [3]. The latter reference contains both necessary and sufficient conditions on the potential V so that (1.3) holds for the pair $H_0, H_0 + V$. If (1.3) holds we call the potential *regular*, if not we call it *non-regular*. It has been shown in [3] that potentials whose Fourier transform \tilde{V} obeys

$$\int d^3p \frac{p^2}{1+p^{1-\varepsilon}} |\tilde{V}(p)|^2 < \infty \tag{1.4}$$

for some $\varepsilon > 0$ are *regular*, but if

$$\int d^3p \frac{p^2}{1+p^{1+\varepsilon}} |\tilde{V}(p)|^2 = \infty \tag{1.5}$$

for some $\varepsilon > 0$, V is *non-regular*. Hence the case of the Coulomb potential ($\tilde{V} \sim 1/p^2$) cannot be decided on the basis of (1.4) and (1.5). In Section II we

prove that the Coulomb potential is non-regular. This fact is due to the strong singularity at the origin.

II. Non-regularity of $1/|x|$

We will prove

Theorem 1. $P_+(\lambda) - P_+^\circ$ is not Hilbert-Schmidt for any $\lambda \in (0, 1)$.

Remarks. (1) The phrase "by dilation" means that we perform a unitary transformation

$$(U_\sigma f)(x) = \sigma^{3/2} f(\sigma x) \tag{2.1}$$

for some $\sigma > 0$.

Notice that as $\sigma \rightarrow 0$

$$U_\sigma f \rightarrow 0 \tag{2.2}$$

weakly.

(2) We have

$$U_\sigma \left(\alpha p + \beta - \frac{\lambda}{|x|} \right) U_\sigma^* = \frac{1}{\sigma} \left(\alpha p + \sigma \beta - \frac{\lambda}{|x|} \right) \tag{2.3}$$

(3) Without going into details we mention that for $\lambda \in (\sqrt{3}/2, 1)$ we take for $H(\lambda)$ the physically distinguished self-adjoint extension of Schmincke [4], Wüst [5] and Nenciu [6]. We also know that this gives the operator of the quantum mechanics textbooks. Hence the ground state of $H(\lambda)$ is at $\sqrt{1-\lambda^2}$.

Proof of Theorem 1. Suppose $P_+(\lambda) - P_+^\circ$ were Hilbert-Schmidt. From (2.1) and the compactness of $P_+(\lambda) - P_+^\circ$ we conclude that

$$U_\sigma (P_+(\lambda) - P_+^\circ) U_\sigma^* \rightarrow 0 \tag{2.4}$$

strongly as $\sigma \rightarrow 0$.

We will show that this leads to a contradiction. From [1, p. 795]

$$P_+^\circ = \frac{1}{2} \left(1 + \frac{\alpha p + \beta}{\sqrt{p^2 + 1}} \right) \tag{2.5}$$

so that

$$U_\sigma P_+^\circ U_\sigma^* = \frac{1}{2} \left(1 + \frac{\alpha p + \beta \sigma}{\sqrt{p^2 + \sigma^2}} \right) \rightarrow \frac{1}{2} \left(1 - \frac{\alpha p}{p} \right) \equiv \tilde{P}_+^\circ \tag{2.6}$$

strongly as $\sigma \rightarrow 0$.

Using

$$P_+(\lambda) = \frac{1}{2} + \frac{1}{2\pi} \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \frac{d\eta}{H(\lambda) - i\eta} \tag{2.7}$$

and (2.3) we get

$$\begin{aligned}
(f, U_\sigma P_+(\lambda) U_\sigma^* f) &= \frac{1}{2} + \frac{1}{2\pi} \lim_{\rho \rightarrow \infty} \int_{-\rho}^{\rho} \left(f, \frac{\sigma}{\alpha p + \sigma \beta - \frac{\lambda}{|x|} - i\eta \sigma} f \right) d\eta \\
&= \frac{1}{2} + \frac{1}{2\pi} \lim_{\rho \rightarrow \infty} \int_{-\rho\sigma}^{\rho\sigma} \left(f, \frac{1}{\alpha p + \sigma \beta - \frac{\lambda}{|x|} - i\eta} f \right) d\eta \\
&= (f, P_+^\sigma(\lambda) f)
\end{aligned} \tag{2.8}$$

so that

$$U_\sigma P_+(\lambda) U_\sigma^* = P_+^\sigma(\lambda) \tag{2.9}$$

where $P_+^\sigma(\lambda)$ is the spectral projection onto $[0, \infty)$ for $\alpha p + \sigma \beta - (\lambda/|x|)$. Obviously as $\sigma \rightarrow 0$

$$\alpha p + \sigma \beta - \frac{\lambda}{|x|} \rightarrow \alpha p - \frac{\lambda}{|x|} \tag{2.10}$$

strongly on $D(H(\lambda))$ and hence also in strong resolvent sense. Therefore, if $\text{Ker}(\alpha p - (\lambda/|x|)) = \{0\}$,

$$P_+^\sigma(\lambda) \rightarrow \tilde{P}_+(\lambda) \equiv \tilde{P}_+ \quad \text{as } \sigma \rightarrow 0 \tag{2.11}$$

strongly where \tilde{P}_+ is the spectral projection onto $[0, \infty)$ for $\alpha p - \lambda/|x|$ [7, p. 432]. To show that $\alpha p - \lambda/|x|$ has indeed a trivial kernel one can either inspect the differential equations in the invariant subspaces of given angular momentum or one can argue as follows: If the kernel were non-trivial it would have infinite dimension for suppose Q were the projection onto the kernel and $\dim Q < \infty$. Since $\alpha p - \lambda/|x|$ commutes with dilations up to a factor (see Remark 2) we have $1 = \|Q\| = \|QU_\sigma\| \rightarrow 0$ since $\dim Q < \infty$. This is impossible. Hence $\dim Q = \infty$ in *each* subspace of fixed angular momentum. But this is impossible since there exists at most two linearly independent solutions in each subspace [8]. By (2.6), (2.9) and (2.11)

$$U_\sigma (P_+(\lambda) - P_+^\circ) U_\sigma^* \rightarrow \tilde{P}_+ - \tilde{P}_+^\circ \tag{2.12}$$

So in view of (2.3) we need only show that $\tilde{P}_+ - \tilde{P}_+^\circ$ is non-zero. Suppose $\tilde{P}_+ = \tilde{P}_+^\circ$. Then with $\tilde{P}_-^\circ = 1 - \tilde{P}_+^\circ$, $\tilde{P}_- = 1 - \tilde{P}_+$

$$\begin{aligned}
0 &= \tilde{P}_- \left(\alpha p + -\frac{\lambda}{|x|} \right) \tilde{P}_+ = \tilde{P}_-^\circ \left(\alpha p - \frac{\lambda}{|x|} \right) \tilde{P}_+^\circ \\
&= \tilde{P}_-^\circ (\alpha p) \tilde{P}_+^\circ - \lambda \tilde{P}_-^\circ \frac{1}{|x|} \tilde{P}_+^\circ
\end{aligned} \tag{2.13}$$

where these equalities hold on $D(\alpha p) \subset D(\alpha p - \lambda/|x|)$. (The latter inclusion follows from the fact that $C_0^\infty(\mathbb{R}^3)$ is a core for αp and $1/|x|$ is (αp) -bounded.) But $\tilde{P}_-^\circ (\alpha p) \tilde{P}_+^\circ = 0$ and

$$\tilde{P}_-^\circ \frac{1}{|x|} \tilde{P}_+^\circ \neq 0 \tag{2.14}$$

To see (2.14) consider for instance the matrix kernel in momentum space. Hence (2.13) is not true. This finishes our proof of Theorem 1.

Acknowledgements

It is a pleasure to thank Prof. G. Scharf for some useful remarks about this paper.

REFERENCES

- [1] M. KLAUS, G. SCHARF, *The regular external field problem in quantum electrodynamics*, *Helv. Phys. Acta*, 50, 779 (1977).
- [2] M. KLAUS, G. SCHARF, *Vacuum polarization in Fock space*, *Helv. Phys. Acta*, 50, 803 (1977).
- [3] G. NENCIU, G. SCHARF, *On Regular External Fields in Quantum Electrodynamics*, *Helv. Phys. Acta*, 51, 413 (1978).
- [4] U. W. SCHMINCKE, *Distinguished self-adjoint extensions of Dirac operators*, *Math. Z.* 129, 335 (1972).
- [5] R. WÜST, *Distinguished self-adjoint extensions of Dirac operators constructed by means of cut-off potentials*. *Math. Z.* 131, 339 (1973).
- [6] G. NENCIU, *Self-adjointness and invariance of the essential spectrum for Dirac operators defined as quadratic forms*. *Comm. Math. Phys.* 48, 235 (1976).
- [7] T. KATO, *Perturbation theory for linear operators*, Berlin, Heidelberg, New York: Springer 1976.
- [8] P. A. REJTO, *Israel J. Math.* 9, 111 (1971).