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Non-regularity of the Coulomb potential in quantum electrodynamics

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Abstract. The operator $P_+(\lambda)-P_+^\circ$ is not Hilbert-Schmidt if P_+° resp. $P_+(\lambda)$ denote the spectral projections of $[0,\infty)$ for the operators $H_0=\alpha p+\beta$, resp. $H=\alpha p+\beta-\lambda/|x|$ and $\lambda\in(0,1)$. This implies that a static external Colomb field is non-regular in the sense of [1].

1. Introduction

We consider the free Dirac operator

$$H_0=\alpha p+\beta \tag{1.1}$$

in interaction with a static electric Coulomb field, i.e., the Hamiltonian

$$H(\lambda)=\alpha p+\beta-\frac{\lambda}{|x|} \qquad 0<\lambda<1 \tag{1.2}$$

In our study of the external field problem [1, 2] in quantum electrodynamics, the question arose whether a so-called strong Bogoljubow transformation exists which relates the free field to the interacting field. The mathematical problem is to prove or disprove that

$$P_+(\lambda)-P_+^\circ\in \text{H.S.} \tag{1.3}$$

is Hilbert-Schmidt (cf. [1, p. 794]). Here P_+° resp. $P_+(\lambda)$ are the spectral projections of $[0,\infty)$ for H_0 resp. $H(\lambda)$. For less singular potentials than Coulomb this question has been studied in [1] and in more detail in [3]. The latter reference contains both necessary and sufficient conditions on the potential V so that (1.3) holds for the pair H_0, H_0+V. If (1.3) holds we call the potential *regular*, if not we call it *non-regular*. It has been shown in [3] that potentials whose Fourier transform $\tilde{V}$ obeys

$$\int d^3p\frac{p^2}{1+p^{1-\varepsilon}}|\tilde{V}(p)|^2<\infty \tag{1.4}$$

for *some* $\varepsilon>0$ are *regular*, but if

$$\int d^3p\frac{p^2}{1+p^{1+\varepsilon}}|\tilde{V}(p)|^2=\infty \tag{1.5}$$

for *some* $\varepsilon>0$, V is *non-regular*. Hence the case of the Coulomb potential ($\tilde{V}\sim 1/p^2$) cannot be decided on the basis of (1.4) and (1.5). In Section II we prove that the Coulomb potential is non-regular. This fact is due to the strong singularity at the origin.

II. Non-regularity of $1/|x|$

We will prove

Theorem 1. *$P_+(\lambda)-P_+^\circ$ is not Hilbert-Schmidt for any $\lambda \in (0, 1)$.*

Remarks. (1) The phrase "by dilation" means that we perform a unitary transformation

$$(U_\sigma f)(x)=\sigma^{3/2}f(\sigma x) \tag{2.1}$$

for some $\sigma > 0$.

Notice that as $\sigma \to 0$

$$U_\sigma f \to 0 \tag{2.2}$$

weakly.

(2) We have

$$U_\sigma\left(\alpha p+\beta-\frac{\lambda}{|x|}\right)U_\sigma^* = \frac{1}{\sigma}\left(\alpha p+\sigma\beta-\frac{\lambda}{|x|}\right) \tag{2.3}$$

(3) Without going into details we mention that for $\lambda \in (\sqrt{3}/2, 1)$ we take for $H(\lambda)$ the physically distinguished self-adjoint extension of Schmincke [4], Wüst [5] and Nenciu [6]. We also know that this gives the operator of the quantum mechanics textbooks. Hence the ground state of $H(\lambda)$ is at $\sqrt{1-\lambda^2}$.

Proof of Theorem 1. Suppose $P_+(\lambda)-P_+^\circ$ were Hilbert-Schmidt. From (2.1) and the compactness of $P_+(\lambda)-P_+^\circ$ we conclude that

$$U_\sigma(P_+(\lambda)-P_+^\circ)U_\sigma^* \to 0 \tag{2.4}$$

strongly as $\sigma \to 0$.

We will show that this leads to a contradiction. From [1, p. 795]

$$P_+^\circ=\frac{1}{2}\left(1+\frac{\alpha p+\beta}{\sqrt{p^2+1}}\right) \tag{2.5}$$

so that

$$U_\sigma P_+^\circ U_\sigma^*=\frac{1}{2}\left(1+\frac{\alpha p+\beta\sigma}{\sqrt{p^2+\sigma^2}}\right)\to\frac{1}{2}\left(1-\frac{\alpha p}{p}\right)\equiv \tilde{P}_+^\circ \tag{2.6}$$

strongly as $\sigma \to 0$.

Using

$$P_+(\lambda)=\frac{1}{2}+\frac{1}{2\pi}\lim_{\rho\to\infty}\int_{-\rho}^{\rho}\frac{d\eta}{H(\lambda)-i\eta} \tag{2.7}$$

and (2.3) we get

$$(f, U_\sigma P_+(\lambda)U_\sigma^* f) = \frac{1}{2} + \frac{1}{2\pi} \lim_{\rho\to\infty} \int_{-\rho}^{\rho} \left(f, \frac{\sigma}{\alpha p + \sigma\beta - \frac{\lambda}{|x|} - i\eta\sigma} f\right) d\eta$$

$$= \frac{1}{2} + \frac{1}{2\pi} \lim_{\rho\to\infty} \int_{-\rho\sigma}^{\rho\sigma} \left(f, \frac{1}{\alpha p + \sigma\beta - \frac{\lambda}{|x|} - i\eta} f\right) d\eta$$

$$= (f, P_+^\sigma(\lambda) f) \tag{2.8}$$

so that

$$U_\sigma P_+(\lambda) U_\sigma^* = P_+^\sigma(\lambda) \tag{2.9}$$

where $P_+^\sigma(\lambda)$ is the spectral projection onto $[0, \infty)$ for $\alpha p + \sigma\beta - (\lambda/|x|)$. Obviously as $\sigma \to 0$

$$\alpha p + \sigma\beta - \frac{\lambda}{|x|} \to \alpha p - \frac{\lambda}{|x|} \tag{2.10}$$

strongly on $D(H(\lambda))$ and hence also in strong resolvent sense. Therefore, if Ker $(\alpha p - (\lambda/|x|)) = \{0\}$,

$$P_+^\sigma(\lambda) \to \tilde{P}_+(\lambda) \equiv \tilde{P}_+ \quad \text{as} \quad \sigma \to 0 \tag{2.11}$$

strongly where $\tilde{P}_+$ is the spectral projection onto $[0, \infty)$ for $\alpha p - \lambda/|x|$ [7, p. 432]. To show that $\alpha p - \lambda/|x|$ has indeed a trivial kernel one can either inspect the differential equations in the invariant subspaces of given angular momentum or one can argue as follows: If the kernel were non-trivial it would have infinite dimension for suppose Q were the projection onto the kernel and dim $Q < \infty$. Since $\alpha p - \lambda/|x|$ commutes with dilations up to a factor (see Remark 2) we have $1 = \|Q\| = \|QU_\sigma\| \to 0$ since dim $Q < \infty$. This is impossible. Hence dim $Q = \infty$ in *each* subspace of fixed angular momentum. But this is impossible since there exists at most two linearly independent solutions in each subspace [8]. By (2.6), (2.9) and (2.11)

$$U_\sigma(P_+(\lambda) - P_+^\circ)U_\sigma^* \to \tilde{P}_+ - \tilde{P}_+^\circ \tag{2.12}$$

So in view of (2.3) we need only show that $\tilde{P}_+ - \tilde{P}_+^\circ$ is non-zero. Suppose $\tilde{P}_+ = \tilde{P}_+^\circ$. Then with $\tilde{P}_-^\circ = 1 - \tilde{P}_+^\circ$, $\tilde{P}_- = 1 - \tilde{P}_+$

$$0 = \tilde{P}_-\left(\alpha p + -\frac{\lambda}{|x|}\right)\tilde{P}_+ = \tilde{P}_-^\circ\left(\alpha p - \frac{\lambda}{|x|}\right)\tilde{P}_+^\circ$$

$$= \tilde{P}_-^\circ(\alpha p)\tilde{P}_+^\circ - \lambda \tilde{P}_-^\circ \frac{1}{|x|} \tilde{P}_+^\circ \tag{2.13}$$

where these equalities hold on $D(\alpha p) \subset D(\alpha p - \lambda/|x|)$. (The latter inclusion follows from the fact that $C_0^\infty(R^3)$ is a core for αp and $1/|x|$ is (αp)-bounded.) But $\tilde{P}_-^\circ(\alpha p)\tilde{P}_+^\circ = 0$ and

$$\tilde{P}_-^\circ \frac{1}{|x|} P_+^\circ \neq 0 \tag{2.14}$$

To see (2.14) consider for instance the matrix kernel in momentum space. Hence (2.13) is not true. This finishes our proof of Theorem 1.

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