Zeitschrift: Helvetica Physica Acta

Band: 52 (1979)

Heft: 4

Artikel: On the group-theoretical foundations of classical and quantum physics : kinematics and state spaces

Autor: Giovannini, N. / Piron, C.

DOI: https://doi.org/10.5169/seals-115038

Helvetica Physica Acta, Vol. 52 (1979), Birkhäuser Verlag, Basel

On the group-theoretical foundations of classical and quantum physics: kinematics and state spaces

by **N. Giovannini** and **C. Piron**

Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland

(20. VII. 1979)

Abstract. Elementary physical systems are analyzed from the point of view of the kinematical symmetry groups which are defined by their action on a set of corresponding observables. The properties and structure of the corresponding possible state spaces are investigated in detail and we exhibit how, by a necessary generalization of the usual frame of quantum mechanics we obtain a framework allowing a complete classification. The theory is applied to the relativistic and non relativistic particles and it is shown how our theory leads in both cases to the quantal as well as to the classical particles as solutions in a unified way, exhibiting thus a new scheme of quantization.

Introduction

Poincaré and Galilei invariance and their relationship to the group theoretical concept of elementary particles [1, 2] are so often emphasized and so well accepted by most physicists that it might seem obsolete to come back to some aspects of the very bases of this approach. Nevertheless, as is well known, beside great successes, the corresponding theories describe in fact only free particles and are even then not completely lacking of difficulties and ambiguities. In view also of the role of cornerstone they play in the dynamics and in field theories, we find it not useless to try to formulate the corresponding basic principles under a slightly different light.

For this reason we shall begin with an elementary discussion of the essential features of a physical system with respect to the ones of its mathematical description. The basic point of view thereby is that a physical system, e.g. an elementary particle, has to be described in terms of its properties, that implies in terms of some chosen set of observables.

Physically this means that the type of the system is characterized by the existence of a particular set of observables. The results of the corresponding measurements will be assumed to be plotted in some subset of a n-dimensional real space (as is true for real numbers, or intervals, but also for matrices or other objects). On the set Γ of these spaces, the usual physical equivalence principles will then naturally correspond to the action of a group, the *kinematical symmetry group G*. Any set of such principles will thus lead to such a group which we call kinematical in order to emphasize that up to now no dynamics enter the play and that the introduction of interactions does not change the representation nor the interpretation of the chosen observables.

Mathematically, we want to find realizations of the above physical properties, this means that we have to represent a physical system by triples (K, U, P) where K is some state space, U is a representation of G in K and P is a mapping from each possible outcome of the measurements to a projection in K, that is to an operator on K having as range the subset of K that contains the states having the corresponding property *in actuality*. These three kinds of objects and their interrelations are just the ones that we want to precise and study in the present paper. This paper is thus a contribution to a field whose essential developments, starting from the paper of Birkhoff and von Neumann [3], can be found e.g. in [4] and [5].

The paper will be organized as follows. In the first part we shall discuss the structure of the space Γ and specify it as an example both in the contexts of relativistic and of non relativistic elementary particles. In particular we shall then be led to symmetry groups which, although based on usual physical equivalence principles, will slightly differ from the Galilei and Poincaré groups, thus giving rise to slightly different representations as well as to a better clear-cut distinction between the kinematical and the dynamical aspects of the theory. In part two we shall briefly discuss the realizations (K, U, P) within the usual frame of quantum mechanics, in the spirit of the approach of Mackey [4]. This frame not only excludes classical mechanics (whereas the above ideas are independent of the particularities of both theories) but it is also shown to be in a way too restrictive even for quantum mechanics. In the third part, which is the main one, we therefore analyse in detail an extended frame and in part four we apply our results to the (spinless) relativistic and non relativistic particles. In both cases we are led to two distinct solutions, a classical one and a quantal one. The unified way in which we treat and characterize elementary systems thus provide us a new scheme of quantization. Let us finally mention here that parts of the results of the present paper have been announced elsewhere [6].

1. Spaces of observables and kinematical symmetry groups

In usual classical physics states are represented by points in the phase space Ω or, better, by point valued functions on Ω. Observables are themselves functions on these states with values in subsets of Ω. Hence Ω can be identified with a set $\Gamma=\{\Gamma_A\}$ of values of a collection of observables $\{A\}$. In quantum physics, if states are rays in Hilbert spaces, their structures can be characterized by the same set of physical observables, via a collection of spectral families defined on the same Γ.

Our point of view is thus to start from this (real) space Γ, which is naturally equipped with a Borel structure. We then define on Γ a kinematical symmetry group G by usual physical equivalence postulates, i.e. as a set of (continuous) transformations relating co-ordinations of Γ that are equivalent in the sense that they are all as good as each other for the description of the system. The space Γ has thus the structure of a G-space, i.e. of a Borel space with an action σ of the group G in the sense of an (anti)-homomorphic mapping σ from the group G into the automorphisms of Γ [10]. The problem then is to search for the various possible realizations of the physical system in the following sense: an elementary system is assumed to be an irreducible representation of G in a state space K that admits covariant projections defined for each Borel subset E of each Γ_A, corresponding to the subset of states having as property a value within E for the observable A. The subject of this paper will just be to make all these notions more precise.

As an illustration, we shall consider throughout the paper the following two examples:

Example 1. The non relativistic particle: the space Γ consists of the values of the observables position momentum and time. Its elements are denoted $\{\vec{q}, \vec{p}, t\}$ and so Γ is isomorphic to $\mathbb{R}^7$. The group G then follows from the following postulates: there exists no privileged frame for position, momentum nor time, and the group is generated by the following operations:

$$\begin{aligned} &\text{(i)} \quad \{\vec{q}, \vec{p}, t\} \mapsto \{\vec{q}+\vec{a}, \vec{p}, t\}, && \vec{a} \in \mathbb{R}^3 \\ &\text{(ii)} \quad \{\vec{q}, \vec{p}, t\} \mapsto \{\vec{q}, \vec{p}+\vec{w}, t\}, && \vec{w} \in \mathbb{R}^3 \\ &\text{(iii)} \quad \{\vec{q}, \vec{p}, t\} \mapsto \{\vec{q}, \vec{p}, t+a^0\}, && a^0 \in \mathbb{R} \\ &\text{(iv)} \quad \{\vec{q}, \vec{p}, t\} \mapsto \{\alpha\vec{q}, \alpha\vec{p}, t\}, && \alpha \in SO(3) \end{aligned} \tag{1.1}$$

The group so generated will be termed the *Newton group* and it will be denoted by $\mathbb{N}$.

Example 2. The relativistic particle: the observables are the 4-position q^μ and the 4-momentum p^μ and the group G will be generated similarly as above, by

(i) the space-time translations: $\{q^\mu, p^\mu\} \mapsto \{q^\mu + a^\mu, p^\mu\}\, a \in \mathbb{R}^4$;
(ii) the momentum translations: $\{q^\mu, p^\mu\} \mapsto \{q^\mu; p^\mu + w^\mu\}, \quad w \in \mathbb{R}^4$; (1.2)
(iii) the Lorentz transformations (there exists no absolute Lorentz frame): $\{q^\mu, p^\mu\} \mapsto \{\Lambda^\mu_\nu q^\nu, \Lambda^\mu_\nu p^\nu\}$, $\Lambda \in SO(3, 1)$.

This group will be termed the *Einstein group* and will be denoted by $\mathbb{E}$.

It is worthwhile to note here that these groups are not isomorphic to the usual Galilei and Poincaré groups even though they contain essentially the same covariance postulates.

The groups we just defined are purely kinematical, as they do not depend on the dynamics. They thus do correspond to the properties of the state spaces and not to the possible evolutions in these spaces. Let us insist in this respect on the fact that the physical time has two different meanings that need to be carefully distinguished. In the above discussion it is an *observable*, and as such it corresponds to a property (the *date*), which can be measured by a clock. This meaning is different from the one of an *evolution parameter*, the latter being, by definition, not even an observable: indeed an observable must correspond to potential or actual properties and an evolution parameter that labels the *changes* of the state does not correspond to a property.

Let us now briefly discuss the structure of the above two fundamental groups.

(i) *The Newton group*

As follows from its definition (1.1) this group can be parametrized by

$$(\vec{w}, a^0, \vec{a}, \alpha) \in \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \times SO(3) \tag{1.3}$$

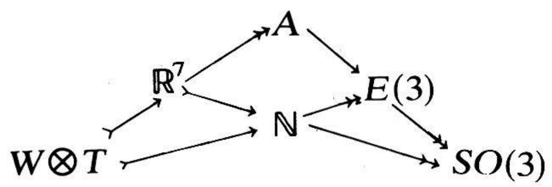
with product

$$(\vec{w}, a^0, \vec{a}, \alpha)(\vec{w}', a^{0\prime}, \vec{a}', \alpha') = (\vec{w} + \alpha\vec{w}', a^0 + a^{0\prime}, \vec{a} + \alpha\vec{a}', \alpha\alpha') \tag{1.4}$$

The unit element being $(\vec{0}, 0, \vec{0}, 1)$, the inverse is obtained as

$$(\vec{w}, a^0, \vec{a}, \alpha)^{-1} = (-\alpha^{-1}\vec{w}, -a^0, -\alpha^{-1}\vec{a}, \alpha^{-1}) \tag{1.5}$$

It is useful for visualization and for the sequel to illustrate its structure as follows. Let A, T and W denote the normal subgroups of space, time and momentum translations respectively, then $\mathbb{N}$ appears in the following complex of group extensions[1)][2)]



(1.6)

with $E(3)$ the Euclidean group. All exact sequences in (1.6) can be seen to be split (i.e. correspond to semi-direct products). The corresponding actions are all directly obtained from the following two ones:

$$\begin{aligned} &\varphi_1\colon SO(3) \to \mathrm{Aut}\,(A), && \varphi_1(\alpha)\cdot\vec{a} = \alpha\vec{a} \\ &\varphi_2\colon E(3) \to \mathrm{Aut}\,(W\otimes T), && \varphi_2(\vec{a}, \alpha)(\vec{w}, a^0) = (\alpha\vec{w}, a^0) \end{aligned} \tag{1.7}$$

The Lie-algebra n of $\mathbb{N}$ is then easily obtained: denoting the generators, in the same order as in (1.3) by V_i, P_0, P_i, J_i, $i = 1, 2, 3$, respectively, the commutators of the algebra are given by

$$\begin{aligned} &[V_i, J_j] = \varepsilon_{ij}^{k} V_k \\ &[P_i, J_j] = \varepsilon_{ij}^{k} P_k \\ &[J_i, J_j] = \varepsilon_{ij}^{k} J_k \end{aligned} \tag{1.8}$$

all other commutators vanishing, including $[V_i, P_0]$, in contradistinction to the usual Galilei group commutators.

Other properties of this group that we shall need in the sequel are given in Appendix A (central extensions) and in Appendix B (irreducible (projective) unitary representations).

(ii) *The Einstein group*

From the definition (1.2) the elements of the group can be parametrized by

$$(w^\mu, a^\mu, \Lambda^\mu_\nu) \in \mathbb{R}^4 \times \mathbb{R}^4 \times SO(3, 1) \tag{1.9}$$

with product

$$(w, a, \Lambda)(w', a', \Lambda') = (w + \Lambda w', a + \Lambda a', \Lambda\Lambda') \tag{1.10}$$

---

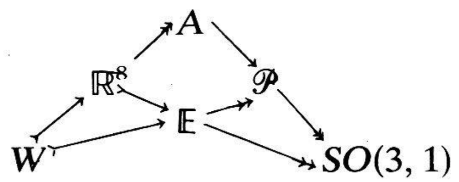
1) $\rightarrowtail$ denotes a monomorphism $\twoheadrightarrow$ an epimorphism.

2) For a simple introduction to this useful extension language we refer to [7] and for more details to [8].

hence with unit (0, 0, 1) and with inverse

$$(w, a, \Lambda)^{-1} = (-\Lambda^{-1}w, -\Lambda^{-1}a, \Lambda^{-1}) \tag{1.11}$$

Similarly as in (1.6), and denoting by A and W the (normal) subgroups generated by space-time and momentum translations respectively, $\mathbb{E}$ can be decomposed as follows



(1.12)

with $\mathcal{P}$ the Poincaré group and where all sequences are again split and completely characterized by the two following actions

$$\begin{aligned} &\psi_1\colon SO(3,1) \to \mathrm{Aut}\,(A), && (\psi_1(\Lambda)a)^\mu = \Lambda^\mu_\nu a^\nu \\ &\psi_2\colon \mathcal{P} \to \mathrm{Aut}\,(W), && (\psi_2(a,\Lambda)w)^\mu = \Lambda^\mu_\nu w^\nu \end{aligned} \tag{1.13}$$

Analogously as for $\mathbb{N}$, we also give here the commutators of the Lie-algebra e of $\mathbb{E}$: denoting by V_μ, P_μ and $M_{\mu\nu}$ the generators of momentum and space translations and of Lorentz rotations respectively, these commutators are given by

$$\begin{aligned} &[V_\sigma, M_{\mu\nu}] = -g_{\mu\sigma}V_\nu + g_{\nu\sigma}V_\mu \\ &[P_\sigma, M_{\mu\nu}] = -g_{\mu\sigma}P_\nu + g_{\nu\sigma}P_\mu \\ &[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\rho}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma} - g_{\mu\sigma}M_{\nu\rho} \end{aligned} \tag{1.14}$$

all the other commutators being zero. The central extensions of this group and its (projective) unitary representations, needed for the purpose of this paper, are also given in the Appendix A and in the Appendix B, respectively.

We now come back to the general discussion of our approach.

2. Observables and systems of imprimitivity

In the state space, that we denote by K, we first ask the covariance postulates to be represented, that is K will be some carrier space of a representation U of the kinematical symmetry group G. Before specifying K further, let us ask which other conditions it should fulfill, in view of the philosophy of our approach.

By definition we have to require the existence on K of a set of operators that correspond to the set of observables we have chosen to characterize the physical system. This means that for each observable A, there should exist a set of operators $\{P^A_\Delta \mid \Delta \in \mathcal{B}(\Gamma_A)$, the Borel sets of $\Gamma_A\}$ that project onto the states in K having for A a value within Δ. In other words we ask for a mapping in the projections of K[3])

$$P^A : \Delta \mapsto P^A_\Delta \in \mathcal{P}(K), \qquad \forall \Delta \in \mathcal{B}(\Gamma_A) \tag{2.1}$$

[3]) For a justification of this notion and a precise definition of observables see [5].

where the operators P_Δ^A are required to have the following properties, for all observables A

$$\begin{aligned} &\text{(i) } P_\varnothing^A = 0_K, \ \varnothing \text{ the void set, and} \\ &\quad\ P_{\Gamma_A}^A = \mathbb{1}_K \\ &\text{(ii) } P_{\Delta_1 \cap \Delta_2}^A = P_{\Delta_1}^A \cdot P_{\Delta_2}^A \\ &\text{(iii) } P_{\bigcup_i \Delta_i}^A = \sum_i P_{\Delta_i}^A, \ \textit{for } \Delta_i \cap \Delta_j = \varnothing, \text{ and } i \in I \text{ a countable set.} \end{aligned} \tag{2.2}$$

These mappings naturally extend to Γ, with $\mathscr{B}(\Gamma)$ the Borel structure generated by the direct union topology of the Γ_A. In the sequel we shall thus only consider these extended mappings.

In addition, we have to require that these mappings should be covariant under G, i.e. the following diagrams

$$\begin{array}{ccc} \mathscr{B}(\Gamma) & \xrightarrow{P^A} & \mathscr{P}(K) \\ \Big\downarrow {\scriptstyle \sigma(g)} & & \Big\downarrow {\scriptstyle U(g)} \\ \mathscr{B}(\Gamma) & \xrightarrow{P^A} & \mathscr{P}(K) \end{array} \tag{2.3}$$

should be commutative, for all $g \in G$, with $\sigma(g)$ the defining action of G on Γ. In equations, (2.3) reads:

$$U(g) P_\Delta^A U(g)^{-1} = P_{\sigma(g)\Delta}^A \tag{2.3$'$}$$

In the *particular case* where K is a separable Hilbert space $\mathscr{H}$, these sets are just the well known *systems of imprimitivity* (short s.o.i.) introduced by Mackey [9].

Definition. An *ordinary* (*or quantal*) *observable* is a set of projectors in a separable Hilbert space $\mathscr{H}$ that forms a system of imprimitivity as above for the corresponding kinematical symmetry group G.

Our problem is however slightly more general, as can be seen in the following

Definition. An *elementary G-particle* is a triple (K, U, P) with U an irreducible representation of G in a state space K such that there exists a set P of covariant systems of projections satisfying (2.2) and (2.3), one for each observable A.

Beside the precisions that it needs and that will be studied in the sequel, this definition shows what is the problem: how to determine all possible solutions (K, U, P). This is quite a general and difficult problem as long as nothing more is said about K and we shall discuss under which assumptions it effectively can be solved. We first want to exhibit that the usual frame of quantum mechanics is not sufficient for this purpose, while enlightening and useful for the sequel.

Suppose thus first that $K = \mathscr{H}$ is a separable Hilbert space. Then Mackey's theory of induced representations (see e.g. [10]) provides useful tools: Let $H \subseteq G$ be some closed subgroup of G, L a representation of H, $s \in G/H$, sg the natural action of G on the coset space, $\{g_s\}$ a set of (fixed) right coset representatives, $f \in \mathcal{L}^2(G/H, \mathcal{H}(L))$ with $\mathcal{H}(L)$ the carrier space of L, then the induced representation[4])

$$(U^L(g)f)(s) = (L(h)f)(s \cdot g) \tag{2.4}$$

where $h = g_s \cdot g \cdot g_{s \cdot g}^{-1} \in H$, gives, via the characteristic functions on G/H a system of imprimitivity for U^L based on the Borel sets of G/H. Such a s.o.i. will be termed *canonical*.

The important result is the converse of that construction as follows from the fundamental theorem of imprimitivity of Mackey that we briefly quote here.

Theorem 2.1 [7]. *Let U be a unitary representation of a separable locally compact group G, S a transitive G-space (i.e. a Borel space with a transitive associative action of G) and Q an s.o.i. for U based on S then*

(i) *there exists a closed subgroup H of G and a Borel isomorphism $\phi : G/H \to S$ such that*

$$(\phi(H \cdot g)) \cdot g' = \phi(H \cdot gg') \qquad \forall g, g' \in G \tag{2.5}$$

(ii) *there exists a representation L of H, unique up to equivalence, and a linear isometry V such that*

$$VUV^{-1} = U^L$$
$$VQ_\Delta V^{-1} = P_{\phi^{-1}(\Delta)} \tag{2.6}$$

with $\Delta \in \mathcal{B}(S)$, U^L as in (2.4) and P the corresponding canonical s.o.i.

(iii) *The commuting algebra of L is isomorphic to the algebra of all operators in $\mathcal{H}$ that commute with both the range of P and the range of U.*

If the action of G on S is not transitive, then (up to regularity conditions) this theorem can easily be extended, the correspondence being now not with one subgroup H but with a set $\{H_i\}$ corresponding to the decomposition of the space S into orbits.

As follows from this theorem we thus have a 1 to 1 correspondence between the observables of the space Γ and a set of closed subgroups of G. This correspondence can be explicited as follows: Let $\Delta \subseteq \Gamma_A$ an arbitrary set corresponding to a measure within Δ for the observable A and arbitrary values for the other observables. Let $\gamma_A \in \Delta$ and $H_{\gamma_A} \subseteq G$ the stability group of this arbitrary point. Then H_{γ_A} corresponds to the observable A. Note that this group is thus only determined up to conjugation.

Let us illustrate this relationship at the hand of our two examples

(i) For the Newton group

$$\begin{aligned} &\text{Observable } \vec{q} \leftrightarrow H_{\vec{q}} = \{(\vec{w}, a^0, \vec{0}, \alpha)\} \\ &\text{Observable } \vec{p} \leftrightarrow H_{\vec{p}} = \{(\vec{0}, a^0, \vec{a}, \alpha)\} \\ &\text{Observable } t \leftrightarrow H_t = \{(\vec{w}, 0, \vec{a}, \alpha)\} \end{aligned} \tag{2.7}$$

[4]) In view of our problem and to enlight notation we shall assume G to be unimodular.

(ii) For the Einstein group

$$\begin{aligned} &\text{Observable } q \leftrightarrow H_q = \{(w, 0, \Lambda)\} \\ &\text{Observable } p \leftrightarrow H_p = \{(0, a, \Lambda)\} \end{aligned} \tag{2.8}$$

The problem thus gets a first answer: the question 'which observables are compatible with a representation U in $\mathcal{H}$' reduces to: 'from which set of subgroups $\{H_i\}$ is the representation U induced, up to equivalence'.

As follows from the inducing in steps theorem of Mackey [10] this set of subgroups can be identified with a connected upper piece of the lattice of all closed subgroups of G; this piece is closed under union.

Under these general results, we now want to examine the possible solutions in our two examples. For this we first exhibit a simple but important condition on the existence of observables that we prove for our particular case:

Proposition 2.2. *Let A (resp. $\mathcal{W}$) be the subgroup of space (-time) (resp. momentum) translations and U an irreducible unitary representation of $\mathbb{N}$ or $\mathbb{E}$. Then there exist an s.o.i. for the position (resp. momentum) observable only if the restriction $U(A)$ (resp. $U(\mathcal{W})$) is faithful.*

Proof. Suppose U(A) is not faithful, i.e. there exist a nonzero translation $\vec{b}$ such that

$$U(\vec{b}) = \mathbb{1}_{\mathcal{H}}$$

Then the condition (2.3) implies

$$P_\Delta = P_{\Delta+\vec{b}} = P_{\Delta+n\vec{b}}, \qquad \forall n \in \mathbb{Z} \tag{2.9}$$

Choosing Δ such that $(\Delta+\vec{b}) \cap \Delta = \varnothing$ this implies with (2.2) (i) and (ii), multiplying this equation (2.9) by P_Δ and $P_{\Delta+n\vec{b}}$ respectively

$$P_\Delta = P_{\Delta+n\vec{b}} = 0_{\mathcal{H}}, \qquad \forall n \in \mathbb{Z} \tag{2.10}$$

As Δ can be chosen such that $\bigcup_n (\Delta+n\vec{b}) = \Gamma$, this implies that the corresponding selfadjoint operator vanishes.

We now can state the following important

Proposition 2.3. *Let U be an irreducible ordinary i.e. vector or non-projective unitary representation of $\mathbb{N}$, then U does not admit systems of imprimitivity simultaneously for the observables position and momentum.*

Proof. Suppose U admits an observable position. This implies, by the Theorem 2.1 that U can be brought in the following form: $U \sim U^L$ with

$$(U^L(g)f)(\vec{x}) = L(h)f(g^{-1}\vec{x}), \qquad \vec{x} \in \mathbb{R}^3 \tag{2.11}$$

with $f \in \mathcal{L}^2(G/H_{\bar{q}}, \mathcal{H}(L))$ and L a representation of the subgroup $H_{\bar{q}}$ given in (2.7). From the Theorem 2.1 this representation L needs to be irreducible, hence, as can be computed via the induction technique, it is necessarily of one of the two classes given by

$$(L_1(\vec{w}, a^0, \alpha)f)(s) = e^{i\beta(s)^{-1}\vec{k}\cdot\vec{w}} e^{ip_0 a^0} D_z(\beta(s)\alpha\beta(s')^{-1}) f(s') \tag{2.12}$$

with $p_0 \in \mathbb{R}$, $s \in SO(3)/SO(2)$, $\beta(s)$ a coset representative, $\beta(s')$ determined by the condition $\beta(s)\cdot\alpha\cdot\beta(s')^{-1}\in H_{\vec{k}}\cong SO(2)$, D_z an irreducible representation of $SO(2)$ and $f\in\mathscr{L}^2(SO(3)/SO(2),\mathscr{H}(D_z))$. The other class is given by

$$L_2(\vec{w},a^0,\alpha)f_0=e^{ip_0a^0}D_{\sigma\sigma'}(\alpha)f_{\sigma'} \tag{2.13}$$

with $D_{\sigma\sigma'}$ an irreducible representation of SO(3), and $f\in\mathbb{C}^{2\sigma+1}$.

As now the coset representatives of $SO(3)/SO(2)$ can be chosen in such a way that the set $\{\beta(s)\vec{k}\}$ lies in a plane containing the 3-vector $\vec{k}$, any momentum translation in a direction perpendicular to this plane is mapped under U onto $\mathbb{1}$, so that, with the Proposition 2.2, there cannot exist an s.o.i. The same is of course true in case (ii) where all momentum translations are in the kernel of the representation.

From the explicit calculation of the projective representations of $\mathbb{N}$ (see Appendix B), we now can state the following result

Proposition 2.4. *Let $K=\mathscr{H}$ be a separable Hilbert space, then there exists only one class of elementary (spinless) $\mathbb{N}$-particles, when only $\vec{p}$ and $\vec{q}$ are requested to be observables.*

This solution follows from the results of Appendix B (see (B.7)) and is given by

$$(U(\vec{w},a^0,\vec{a},\alpha)f)(\vec{x})=e^{-i\eta\lambda\vec{w}\vec{x}}e^{ip_0a^0}f(\alpha^{-1}(\vec{x}-\vec{a})) \tag{2.14}$$

with $\eta,\lambda,p_0\in\mathbb{R}$ and $f\in\mathscr{L}^2(\mathbb{R}^3)$. The observables $\vec{p}$ and $\vec{q}$ are correspondingly given by the following selfadjoint operators

$$\begin{aligned}\underline{\vec{q}}\,f(\vec{x})&=\vec{x}\,f(\vec{x})\\ \underline{\vec{p}}\,f(\vec{x})&=\left(\frac{i}{\eta\lambda}\right)\vec{\partial}f(\vec{x})\end{aligned} \tag{2.15}$$

obtained from the s.o.i. given by the characteristic functions in $\vec{x}$ for $\vec{q}$ and in $\vec{k}$ (after Fourier transformation) for $\vec{p}$.

We thus recover the usual quantal solution and as far as projective representations are concerned, the uniqueness of the solution is simply a reflexion of the well known Stone–von Neumann theorem.

The situation is perfectly analogous for the Einstein group, so that we just quote the result without entering again into details.

Proposition 2.5. *Let $K=\mathscr{H}$ be a separable Hilbert space, then there exists only one class of elementary (spinless) $\mathbb{E}$-particles (admitting thus systems of imprimitivity for the observables p^μ and q^μ).*

This solution also follows from the results of Appendix B. We however postpone the results up to part four of the present paper where we shall discuss this solution into more details.

For the moment, let us come back to the non relativistic case. Not only the unique solution found in (2.14) only describes the quantal case (whereas the spirit of our approach was principally independent of this particular theory), but the fact that only $\vec{p}$ and $\vec{q}$ are observable implies that the only measurable quantities are necessarily functions of $\vec{p}$ and $\vec{q}$ only and this excludes e.g. all time dependent problems. In the above frame, there is however no possible solution for time to be an observable as a consequence of the following

Proposition 2.6. *Let $\mathcal{H}$ be a separable Hilbert space and $t, (\vec{p}, \vec{q})$ be (all) observable for a representation U of $\mathbb{N}$ in $\mathcal{H}$. Then U is necessarily reducible.*

Proof. If time is observable, then we may decompose $\mathcal{H}$ along the corresponding spectral projectors

$$H \cong \int^{\oplus} \mathcal{H}_t \, dt, \qquad t \in \mathbb{R} \tag{2.16}$$

where isomorphism is meant as $\mathbb{N}$-module. As however time translations are in the center of $\mathbb{N}$, the characteristic functions on the corresponding dual space provide non-trivial projectors that commute with $U(\mathbb{N})$, hence U is reducible.

By these results we see that we have to relax some assumption to find $\mathbb{N}$-particles. Relax irreducibility would introduce new observables compatible with the desired ones and thus would no longer correspond to our goal. Hence we have to change K.

A possible solution is suggested from a theorem by Piron [5], which says that any propositional system (except for some exceptional cases) can be realized within the set of projectors of a *family of Hilbert spaces.*

In the remainder of this paper we shall study such objects taken as state spaces, first in general and then, in section four, as applied to our two examples.

3. State spaces and generalized imprimitivity systems

Let thus S be some index set that we assume to be equipped with a Borel structure and let

$$K = \bigvee_{s \in S} \mathcal{H}_s \tag{3.1}$$

a state space obtained as (topological) direct union of copies of some separable (complex) Hilbert space $\mathcal{H}$. An element $\Psi \in K$ thus consists of an index $s \in S$ and a ray $\Psi_s \in \mathcal{H}_s$, i.e. S plays the role of a possibly continuous set of *superselection variables.*

Let us first quote the following generalized version of the theorem of Wigner.

Theorem 3.1 [11]. *Every symmetry of a proposition system defined by a family $\{\mathcal{H}_s, s \in S\}$ is given by a permutation f of the index set S and a family of (anti-) unitary transformations $U_s : \mathcal{H}_s \to \mathcal{H}_{f(s)}$. Each U_s is defined up to a phase.*

That is if the symmetries form a group G, we have

$$U_{f_2(s)}(g_1) U_s(g_2) = \omega_s(g_1, g_2) U_s(g_1 g_2) \tag{3.2}$$

where ω is a phase depending on g_1, g_2 *and* s, satisfying thus

$$|\omega_s(g_1, g_2)| = 1, \qquad \forall s \in S, \qquad g_1, g_2 \in G \tag{3.3}$$

For our purposes we may omit the dependence on these phase factors, in the sequel.

Let now G be some separable locally compact group and K as in (3.1). As a consequence of this theorem we may define a representation U of G in K as a set of mappings $\{U(g) \mid g \in G\}$

$$U(g): K \to K \quad \text{(onto)} \tag{3.4}$$

satisfying the following properties

(i) $U(e)$ is the identity mapping
(ii) $U(g_1)U(g_2) = U(g_1 g_2)$ (3.5)
(iii) $(U(g)\phi)_s = (L(s, g)U)_{s'}$, with $L(s, g) \in \mathcal{U}(\mathcal{H}_s)$ the group of unitary operators in $\mathcal{H}_s$ and with $s' \equiv s \cdot \rho(g)$ depending only on s and g
(iv) $U(g)$ is continuous in the product topology of S and $\mathcal{H}$ (with the strong topology for $\mathcal{H}$).

There are two important consequences of the above theorem and definition. First a representation U induces, from (3.2) or (3.5)(iii) a set of mappings $\rho(g)$ that satisfy the following two conditions

$$\begin{aligned} &\text{(i)} \quad s \cdot \rho(e) = s \qquad \forall s \in S \\ &\text{(ii)} \quad (s \cdot \rho(g_1)) \cdot \rho(g_2) = s \cdot \rho(g_1 g_2), \qquad \forall s, g_1, g_2 \end{aligned} \tag{3.6}$$

so that S is promoted to a G-space [10], i.e. to a Borel space with an action ρ of G satisfying precisely (3.6).

Second, it also follows from (3.2) or (3.5) that the set $\{L(s, g)\}$ of unitary operators, satisfies the relations

$$\begin{aligned} &\text{(i)} \quad L(s, e) = \mathbb{1}_{\mathcal{H}_s}, \forall s \\ &\text{(ii)} \quad L(s, g_1 g_2) = L(s \cdot \rho(g_2), g_1) L(s, g_2), \qquad \forall s, g_1, g_2 \end{aligned} \tag{3.7}$$

hence the set $\{L(s, g)\}$ forms a so-called *normalized* $(G, S, \mathcal{U}(\mathcal{H}))$*-cocycle* (see e.g. [12]). This implies that these equations (3.7) can be tackled by usual cocycle techniques and also, because they are the same as in the usual Mackey induction theory, their solutions can be obtained in the same way: denoting by H_s the stabilizer under ρ of an arbitrary but fixed point $s \in S$, we see from (3.7) that the set $\{L(s, h) \mid h \in H\}$ is a unitary representation of H_s. Hence it is sufficient to find all solutions of (3.7) for normalized (G, S, H_s)*-cocycles*, that is for mappings $\nu(s, g)$ satisfying

$$\begin{aligned} &\text{(i)} \quad \nu(s, g) \in H_s \\ &\text{(ii)} \quad \nu(s, e) = e, \qquad \forall s \\ &\text{(iii)} \quad \nu(s \cdot \rho(g_2), g_1) \nu(s, g_2) = \omega(s, g_1 g_2) \end{aligned} \tag{3.8}$$

and the general solution of (3.7) is then obtained by taking an arbitrary unitary representation D of H_s in the carrier space $\mathcal{H}$ and by setting

$$L^{\nu}(s, g) \overset{\text{def}}{=} \mathrm{D}(\nu(s, g)) \tag{3.9}$$

We are naturally led to the following important generalizations of the usual concepts.

Definition. A representation U of G in K is termed *irreducible* if and only if

(i) the action $\rho(G)$ is transitive on S (3.10)

(ii) the set $\{L(s_0, h)\}$, for s_0 arbitrary on S, forms an irreducible representation of H_{s_0} with carrier space $\mathcal{H}_{s_0}$.

Defining the commutant as the set of commuting families $A=\{A_s\}$, this corresponds to the following

Proposition 3.2. *U is irreducible if and only if its commutant is trivial.*

Proof. From (3.10)(ii) and the Schur lemma an element of the commutant is necessarily of the form

$$(A)_s=\lambda(s)\mathbb{1}_{\mathcal{H}_s}$$

Furthermore $[A, U(g)]=0_K$ implies $U(g)\lambda(s)=\lambda(s\cdot\rho(g))U(g)=\lambda(s)U(g)$ hence by transitivity λ is constant on S.

Definition. Two representations U_1 and U_2 of G on K_1 and K_2 are termed *equivalent* if there exist a Borel isomorphism $\tau: S_1\to S_2$ and a linear isometry $V:\mathcal{H}_1\to\mathcal{H}_2$ such that

$$VL_1(\tau^{-1}(s_2), g)V^{-1}=L_2(s_2, g), \qquad \forall s_2\in S_2, \qquad \forall g\in G \tag{3.11}$$

For irreducible representations we then have

Proposition 3.3. *$U_1\sim U_2$ if and only if the corresponding $(G, S, \mathcal{U}(\mathcal{H}))$-cocycles are in the same cohomology class i.e. differ by a coboundary.*

The proof follows from direct computation.

As a consequence the classes of representations of a group G can be completely classified by the classes of the representations of all closed subgroups H and of the cohomological properties of G with respect to these subgroups (see (3.8)). We do not go into the details of this problem as we shall, as for the Hilbert space case, give more restrictions on the allowable representations. Let us just quote here an important result concerning continuity.

Theorem 3.4. *Let G be a separable locally compact group and K a direct union of copies of some Hilbert space $\mathcal{H}$ over a G-space S. Then a representation U of G on K satisfying* (3.5)(i) *to* (iii) *is continuous if and only if L in* (3.5) *is continuous and the ν-cocycles* (3.8) *are Borel functions.*

The result is here the same as in Mackey's theory and the proof follows from the same argument (see [10] and also [12] and [13] for an analogous extension of Mackey's theorem).

Let us now make a further step: suppose we have given G, K, S and U as above and suppose in addition, in order to avoid pathologies, that the ρ-action of G is regular (i.e. that there exist Borel cross-sections to the orbits). We may thus decompose S into a direct union of disjoint Borel spaces on which the action is transitive and hence focus our attention to the latter case.

Let thus S be a transitive G-space, i.e. $\forall s, s'\in S\ \exists g\in G$ such that $s\cdot\rho(g)=s'$

and let $s_0 \in S$ be arbitrary (but fixed) in S. Then it follows from Theorem 3.10(i) of [10] that

$$\begin{aligned}&\text{(i) } H_{s_0} \subseteq G, \text{ the stabilizer of } s_0, \text{ is a closed subgroup}\\&\text{(ii) } S \text{ is Borel isomorphic with } G/H_{s_0}\end{aligned} \tag{3.12}$$

Hence the set of possible K spaces can be classified using the following immediate generalization of the induction procedure:

Let $H \subseteq G$ be some closed subgroup, and let L be some unitary representation of H with carrier space $\mathcal{H}(L)$. Let K denote the direct union of copies of $\mathcal{H}(L)$ over the coset space G/H

$$K = \bigvee_{s \in G/H} (\mathcal{H}(L))_s \tag{3.13}$$

The generalized induced representation U^L is then defined as

$$(U^L(g)\varphi)_s = (L(h)\varphi)_{s \cdot \rho(g)} \tag{3.14}$$

with, as in the usual case $\rho(g)$ the (right) action of G on an (arbitrary but fixed) set of (right) coset representatives $\{s\}$ of G/H and where h is determined by the condition

$$s \cdot \rho(g) \cdot (s \cdot g)^{-1} \in H \tag{3.15}$$

Denoting $L(s, g)$ the transformations $L(h)$ of (3.14) with h depending on s via (3.15), these representations of course fulfill the conditions (3.7), hence U^L is a representation of G in the sense of (3.5), with as corresponding cocycle (3.8) obtained from (3.15) as given by $\nu(s, g) = s \cdot \rho(g) \cdot (s \cdot g)^{-1}$. Conversely and as in the usual case, each normalized cocycle $\nu(s, g)$ can be brought in this standard form with as normalized coset representatives

$$g_s = (\nu(s_0, g))^{-1} \cdot g, \qquad g_e = e \tag{3.16}$$

where $g \in H \cdot g_s$, H the stabilizer of s_0. This definition does not depend on the choice of g as can be verified using the cocycle equations and as is of course necessary for consistency. As these algebraic equations are analogous to the usual case we drop the details.

We are now in position to extend the notion of systems of imprimitivity. For this purpose let us first extend the usual notion of projectors.

Definition. A *system of projectors* on K, or a *projection in* K, is a family $\{P_s, s \in S\}$ of projectors defined on the corresponding family $\{\mathcal{H}_s\}$ of Hilbert spaces.

The following operators on K

$$P_E \stackrel{\text{def}}{=} \{\mathbb{1}_{\mathcal{H}_s} \cdot \chi_E \mid \chi_E \text{ the characteristic function of } E \in \mathcal{B}(S)\} \tag{3.17}$$

thus form projections in K, which with S, U^L as above satisfy by construction

$$U^L(g)^{-1} P_E U^L(g) = P_{E \cdot \rho(g)}. \tag{3.18}$$

This is formally the same covariance condition as for the usual systems of imprimitivity. Furthermore the projections (3.17) satisfy the properties (2.2):

$$\begin{aligned}&\text{(i)}\ P_{\varnothing}=0_K \qquad P_S=\mathbb{1}_K\\ &\text{(ii)}\ P_{E_i}P_{E_j}=P_{E_i\cap E_j}\\ &\text{(iii)}\ P_{\bigcup_i E_i}=\sum_i P_{E_i}, \quad \text{for} \quad E_i\cap E_j=\varnothing, \quad i\neq j\end{aligned} \tag{3.19}$$

We shall call *supersystems of imprimitivity* sets of projections satisfying (3.17) to (3.19) and *superselection parameters or classical observables* the corresponding observables (see §2). The reasons for this terminology will be even more explicit in the Section 4. Of course the same construction extends to the case where $\mathcal{H}_s$ is itself assumed to be of the more general form K_s with K as in (3.1), this corresponding to the inducing in steps procedure [10].

We thus see from the above construction that, with a careful extension of the usual definitions, the whole problem is set in a similar context as in Section 2 for the separable Hilbert space case. In fact we have even more: although the frame, because of the state space K, is a proper generalization of the case considered by Mackey, the method and the results follow, with the appropriately generalized concepts, as a particular case of the ones of Mackey. Indeed, if the local and algebraic problems turn out to be exactly the same, as we have shown, the global conditions of the usual theory (e.g. square integrability) now become trivial.

Let us thus reformulate and complete the essential results of the present analysis and combine them with the ones of Mackey in the following

Theorem 3.5. *Let $K=\bigvee_{s\in S}\mathcal{H}_s$ as in (3.1), U a representation (3.5) of a separable locally compact group G, transitive on S and let Q be a transitive set of covariant projections in K (satisfying (3.18) and (3.19)) and defined on some G-space T i.e. $Q:\mathcal{B}(T)\to\mathcal{P}(K)$. Then there exist closed subgroups $H_1\subseteq H_2\subseteq H_3\subseteq G$ and unitary representations L_i of H_i $(i=1,2)$ such that*

(i) *$S\cong G/H_2$, and $Q\sim P\cdot R$ (i.e. $Q=\{Q_s\}$ and $Q_s\cong P_s\cdot R_s\ \forall s\in S$) where P is of the canonical form (3.17) for G/H_3 and R_s is itself an ordinary s.o.i. for H_2/H_1 on $\mathcal{H}_s$. In other terms K can be written as $K=V_{S'}(V_{S/S'}\mathcal{H}_s)$, with $S'\cong G/H_3$ and P_s depending only on S'. Furthermore, either $T\cong G/H_1$ and $S'=S$, or $T\cong G/H_3$ and R_3 is then trivial (i.e. $H_1=H_2$).*

(ii) *$U\sim U^{L_2}$ as in (3.14) on S and $L_2\sim U^{L_1}$ on H_2/H_1.*

From the Definition (3.10) it also immediately follows that U is irreducible if and only if L_2 is irreducible.

Hence, and analogously to the case of Section 2, *all supersystems of imprimitivity are equivalent to canonical ones* and the results of this section provide us with a general method of classification of elementary G-particles, our primitive goal, method that we shall apply in the next section and whose essential steps can be resumed as follows:

1. As the set of observables is given *a priori*, we may choose as basic G-space, and without loss of generality, the space Γ discussed in Section 1. Also without loss of generality the corresponding action of G may be

assumed to be transitive. Let $\Gamma_0 \subseteq \Gamma$ correspond to a subset $\{B\}$ of the observables $\{A\}$ defining Γ and let H_0 be the corresponding subgroup of G.
2. Determine, for each choice, which irreducible unitary representations U_0 of H_0 give rise to ordinary s.o.i. corresponding to each observable B, and no other.
3. Use the generalized induction as in (3.14) on $K = V_S\mathcal{H}(U_0)$ with $S = \Gamma/\Gamma_0 \cong G/H_0$ and $\mathcal{H}(U_0)$ the carrier spaces of the representations U_0 found in 2.

Each solution is found in this way and the observables found in Step 2 are the usual quantal ones as they correspond to projectors in separable Hilbert spaces whereas those found in step 3 play the role of superselection parameters or, in other words, are of the classical type, as they commute with each other and have a purely discrete point spectrum.

Let us now come back to the two examples discussed in Section 1, in order to show how our formalism applies and indeed provides the good framework to deal with.

4. Applications

(a) *The non relativistic spinless particles*

Using the above mentioned method we find that there are exactly two classes of spinless elementary $\mathbb{N}$-particles.

(i) *The classical particle*

Let first S be directly given by Γ. Then the step 2 of the procedure is trivial Γ_0 being identified with a single point, e.g. the origin, and H_0 being the rotation subgroup of $\mathbb{N}$.

$$S = \Gamma \cong \mathbb{N}/SO(3) \cong \mathbb{R}^6 \tag{4.1}$$

and

$$K = \bigvee_{s=\Gamma} \mathcal{H}_s \tag{4.2}$$

with $\mathcal{H}_s$ the one-dimensional carrier space of the trivial representation of $SO(3)$. The corresponding representation U of $\mathbb{N}$ in K is thus obtained from the defining representation σ given in (1.1)

$$(U(g)\psi)_s = (\psi)_{\sigma(g)^{-1}s}, \qquad s \in \Gamma \tag{4.3}$$

The observables $\vec{p}$, $\vec{q}$ and t are correspondingly given by the following operators

$$\begin{aligned} &\underline{\vec{p}} : \mathcal{B}(\mathbb{R}^3) \to \Gamma, \qquad \underline{\vec{p}}(\Delta_{\vec{p}}) = (\Delta_{\vec{p}}, \vec{q}, t) \\ &\underline{\vec{q}} : \mathcal{B}(\mathbb{R}^3) \to \Gamma, \qquad \underline{\vec{q}}(\Delta_{\vec{q}}) = (\vec{p}, \Delta_{\vec{q}}, t) \\ &\underline{t} : \mathcal{B}(\mathbb{R}) \to \Gamma, \qquad \underline{t}(\Delta_t) = (\vec{p}, \vec{q}, \Delta_t) \end{aligned} \tag{4.4}$$

or, equivalently these observables are defined as the inverses of the following mappings

$$\begin{aligned}(\vec{p}, \vec{q}, t) &\to \vec{p} \\ (\vec{p}, \vec{q}, t) &\to \vec{q} \\ (\vec{p}, \vec{q}, t) &\to t\end{aligned} \tag{4.5}$$

(ii) *The quantal particle*

This solution is obtained by generalized induction from the solution we have found in (2.14) (solution first restricted to the subgroup H_t given in (2.7), i.e. with $a^0 = 0$). That is, time plays the role of a superselection parameter and the space S is isomorphic to the time axis

$$S \cong \mathbb{R}_t \cong \mathbb{N}/H_t \tag{4.6}$$

Furthermore, with $\mathscr{H}_s$ given by the carrier space of the representation (2.14), i.e. $\mathscr{H}_s = \mathscr{L}^2(\mathbb{R}^3)$, the state space becomes

$$K = \bigvee_S \mathscr{H}_s \cong \bigvee_{t\in\mathbb{R}_t} (\mathscr{L}^2(\mathbb{R}^3))_t \tag{4.7}$$

The complete representation U of $\mathbb{N}$ in K is thus given, from (3.14), by

$$(U(\vec{w}, 0, \vec{a}, \alpha)\psi(\vec{x}))_t = (e^{-i\eta\lambda\vec{x}\cdot\vec{w}}\psi(\alpha^{-1}(\vec{x}-\vec{a})))_t \tag{4.8}$$

and

$$(U(\vec{0}, a^0, \vec{0}, 1)\psi(\vec{x}))_t = (\psi(\vec{x}))_{t-a^0} \tag{4.9}$$

Note at this point that

$$U(\vec{w}, 0, \vec{0}, 1)U(\vec{0}, 0, \vec{a}, 1) = e^{-i\eta\lambda\vec{w}\vec{a}}U(\vec{0}, 0, \vec{a}, 1)U(\vec{w}, 0, \vec{0}, 1) \tag{4.10}$$

so that, with $\eta\lambda = -\hbar^{-1}$, we recover the usual Weyl commutation relations.

The observables $\vec{p}$ and $\vec{q}$ are given from (2.15), by the following operators

$$\begin{aligned}(\underline{\vec{q}}\psi(\vec{x}))_t &= (\vec{x}\psi(\vec{x}))_t \\ (\underline{\vec{p}}\psi(\vec{x}))_t &= \left(\frac{i}{\eta\lambda}\vec{\partial}\psi(\vec{x})\right)_t = (-i\hbar\vec{\partial}\psi(\vec{x}))_t\end{aligned} \tag{4.11}$$

whereas the time observable is obtained, following (3.17) via the characteristic functions in the space S, i.e. via the supersystem of imprimitivity defined by

$$(P_{\Delta_t}\psi(\vec{x}))_t = (\chi_{\Delta_t}(t)\psi(\vec{x}))_t \tag{4.12}$$

with $\Delta_t \in \mathscr{B}(\mathbb{R}_t)$ and

$$\chi_{\Delta_t}(t) = \begin{cases} 1 & \text{if} \quad t \in \Delta_t \\ 0 & \text{else} \end{cases}$$

(b) *The relativistic spinless particles*

For the Einstein group $\mathbb{E}$, the situation is very similar to the previous case: we indeed again have exactly two solutions for the elementary spinless $\mathbb{E}$-particles.

(i) *The classical particle*

Here again Γ_0 reduces to a point, i.e. S is directly given by Γ

$$S=\Gamma \cong \mathbb{E}/H_0 \cong \mathbb{R}^8 \tag{4.13}$$

with $H_0 = SO(3, 1)$ the stability group of the origin. With $\mathcal{H}_s$ the one dimensional Hilbert space that carries the trivial representation of SO(3, 1)

$$K \cong \bigvee_{S=\Gamma} \mathcal{H}_s \cong \bigvee_{\gamma\in\mathbb{R}^8} \mathcal{H}_\gamma \tag{4.14}$$

the corresponding representation is obtained, again from the defining representation σ, as given by

$$\begin{aligned}(U(g)\psi)_{(p,q)} &= (\psi)_{\sigma(g)^{-1}(p\cdot q)} \\ &= (\psi)_{(\Lambda^{-1}(p-w),\Lambda^{-1}(q-a))}\end{aligned} \tag{4.15}$$

The observables p and q are obtained analogously as in (4.4) by

$$\begin{aligned}\underline{p}&: \mathcal{B}(\mathbb{R}^4) \to \Gamma, \qquad \underline{p}(\Delta_p) = (\Delta_p, q) \\ \underline{q}&: \mathcal{B}(\mathbb{R}^4) \to \Gamma, \qquad \underline{q}(\Delta_q) = (p, \Delta_q)\end{aligned} \tag{4.16}$$

or, equivalently, by the inverses of the following mappings

$$\begin{aligned}(p, q) &\to p \\ (p, q) &\to q\end{aligned} \tag{4.17}$$

(ii) *The quantal particle*

As follows from Proposition 2.5 and from the results of Appendix B (esp. (B.15)) there is a unique class of solutions, with no superselection parameter, i.e. within the usual frame discussed in Section 2. The space S is thus reduced to a single point, and the state space K is given by

$$K \cong \mathcal{H} \cong \mathcal{L}^2(\mathbb{R}^4) \tag{4.18}$$

the carrier space of the following irreducible unitary representation of $\mathbb{E}$

$$U(w, a, \Lambda)\psi(x) = e^{-i\eta\lambda g_{\mu\nu}w^\mu x^\nu}\psi(\Lambda^{-1}(x-a)) \tag{4.19}$$

where the constant $h^{-1} \equiv -\eta\lambda$ here also appears as a representation label, and $g_{\mu\nu}$ is taken with signature $(-,+,+,+)$. Note again here that we have, similarly as in (4.10) the following generalized Weyl commutation relations

$$U(w, 0, 1)U(0, a, 1) = e^{i\hbar^{-1}g_{\mu\nu}w^\mu a^\nu}U(0, a, 1)U(w, 0, 1) \tag{4.20}$$

The observables q and p are now again obtained via the characteristic functions in the x-space and in the Fourier transformed space respectively, that is

$$\begin{aligned}(\underline{q}(\Delta_q)\psi(x)) &= \chi_{\Delta_q}(x)\psi(x) \\ (\underline{p}(\Delta_p)\hat{\psi}(k)) &= \chi_{\Delta_q}(k)\hat{\psi}(k)\end{aligned} \tag{4.21}$$

where

$$\hat{\psi}(k) = (2\pi\hbar)^{-2} \int e^{-i\hbar^{-1} g_{\mu\nu} k^\mu x^\nu} \psi(x)\, d^4x \tag{4.22}$$

The corresponding selfadjoint operators then simply read

$$\begin{aligned} \underline{q}^\mu \psi(x) &= x^\mu \psi(x) \\ \underline{p}^\mu \psi(x) &= -i\hbar\, \partial^\mu \psi(x) \end{aligned} \tag{4.23}$$

so that now, in contradistinction to the usual Poincaré case, the space-time position operators exist and their (continuous) spectra are just the space-time variables.

Having thus shown the relevance of our approach, let us conclude with the following important remarks. First we have considered only spinless particles. It is however obvious that the solutions (4.2) and (4.7) easily extend to the case with spin, as SO(3) is the isotropy group of the origin in the classical case and as follows from the solution (B.7) in the quantal case (this incidentally shows that it is possible, within our frame, to treat problems where only spin has a quantal character). In the relativistic case however this is not possible so directly as the involved representations of the stability groups H_s are then necessarily infinite dimensional, H_s being then isomorphic to $SO(3, 1)$. We thus there have to consider additional observables or superselection rules. We just mention here that such a solution has been proposed in [14] (see also [15, 16]) and we refer to these papers on this point.

Next we want to emphasize that all the framework of this paper was purely kinematical, and that our construction, as well as the interpretation of the observables is independent of the dynamics. It will be a second step to construct in the just found state spaces the dynamics as realization of a two parameter family of operators, family that involves an additional dynamical parameter, a dynamical time, that is distinct as explained in the introduction from the time observable considered here. We refer to [5], [14–16] for a discussion of the dynamics within a frame corresponding to the point of views of the present analysis.

Appendix A

Central extensions of $\mathbb{N}$ *and* $\mathbb{E}$

As we are interested in the projective unitary representations of the kinematical symmetry groups G, we determine here briefly the corresponding central extensions. These can be obtained via the central extensions of the simply connected covering group, i.e. it is sufficient to determine all inequivalent exact sequences

$$0 \to \mathbb{R} \xrightarrow{\iota} g^\xi \to g \to 1\ ,\ \xi \tag{A.1}$$

with $\iota(\mathbb{R})$ in the center of g^ξ and g the Lie algebra of G. Such an extension is characterized by an infinitesimal exponent ξ, i.e. by a bilinear antisymmetric function

$$\xi : g \times g \to \mathbb{R} \tag{A.2}$$

satisfying the cocycle equations

$$\begin{aligned} d\xi(A_i, A_j, A_k) &\equiv \xi([A_i, A_j], A_k) + \xi([A_j, A_k], A_i) + \xi([A_k, A_i], A_j) \\ &= 0 \end{aligned} \tag{A.3}$$

with A_i, A_j, A_k the generators of g. Indeed, as is well known [17] there is a 1-to-1 correspondence between the equivalence classes of the exponents of a simply connected Lie group (i.e. the equivalence classes of projective representations) and the equivalence classes of the extensions (A.1).

These extensions can be obtained as follows (see [2]); let $A_1, \ldots, A_n$ be some basis of g and c_{ij}^k, $i, j, k = 1, \ldots, n$, the corresponding structure constants. The Lie brackets of an extension g^ξ are then given by

$$\begin{aligned} &[A_i, A_j] = c_{ij}^k A_k + \alpha_{ij} I \\ &[I, A_j] = 0 \end{aligned} \tag{A.4}$$

with I the generator of $\mathbb{R}$. The cocycle condition (A.3) is then equivalent with the condition that (A.4) is a Lie algebra, i.e. satisfies the Jacobi identities. Furthermore two extensions are equivalent if and only if the corresponding constants α_{ij} and α'_{ij} are related by

$$\alpha'_{ij} = \alpha_{ij} - c_{ij}^k \lambda_k \tag{A.5}$$

for n constants λ_k. This is equivalent to say that the two extensions are obtained from each other by a change of choice of section $g \to g^\xi$ in (A.1) which corresponds in (A.4) to translations of the generators along the center, i.e. to substitutions of the form

$$A_i \to A_i + \lambda_i I \tag{A.6}$$

Let us now apply this method to the Lie algebras of the Newton group $\mathbb{N}$ and of the Einstein group $\mathbb{E}$.

For $\mathbb{N}$, with Lie algebra (1.8), we first note that by translations (A.6) for the generators of the rotations, $\mathfrak{su}(2)$ can be seen to have no non-trivial extensions, as is well known. Similarly by translations of the generators P_i and V_i respectively the two subalgebras generated by $\{P_i, J_j\}$ and $\{V_i, J_j\}$, both isomorphic to $\mathfrak{e}(3)$, can also be brought in the original form, i.e. do not have non trivial extensions. We thus are left with the following possible commutation relations

$$\begin{aligned} &[V_i, P_0] = \alpha_i I \\ &[V_i, P_j] = \beta_{ij} I \\ &[P_0, P_i] = \gamma_i I \\ &[P_0, J_i] = \delta_i I \qquad i, j = 1, 2, 3 \end{aligned} \tag{A.7}$$

all others being as in (1.8). From (A.3), i.e. from the Jacobian identities

$$[A_i, A_j, A_k] \equiv [[A_i, A_j], A_k] + [[A_j, A_k], A_i] + [[A_k, A_i], A_j] = 0 \tag{A.8}$$

one can now straightforwardly obtain

$$\begin{aligned}
&[P_0, P_i, J_j] = 0 \Leftrightarrow \gamma_k = 0\\
&[J_i, V_j, P_0] = 0 \Leftrightarrow \alpha_k = 0\\
&[P_0, J_i, J_j] = 0 \Leftrightarrow \delta_k = 0\\
&[J_i, V_j, P_k] = 0 \Leftrightarrow \beta_{ij} = \lambda\delta_{ij}
\end{aligned} \tag{A.9}$$

whereas all other Jacobi identities are then satisfied. The central extensions (A.1) of the Lie algebra of $\mathbb{N}$ are thus characterized by the commutators of $\mathscr{n}$ (as in (1.8)) except for the following ones

$$[V_i, P_j] = \lambda\delta_{ij} \cdot I \tag{A.10}$$

Using (A.5) one directly finds that two such extensions are equivalent if and only if $\lambda = \lambda'$, hence the inequivalent classes form a one-dimensional family.

Similarly for $\mathbb{E}$, with Lie algebra (1.14), one may by translations along the center, show that $\mathfrak{sl}(2, \mathbb{C})$ as well as the two subalgebras generated by $\{P_\mu, M_{\rho\sigma}\}$ and by $\{V_\mu, M_{\rho\sigma}\}$ respectively do not have non-trivial central extensions, as can also be seen by noticing that these two subalgebras are isomorphic with the Poincaré algebra. Hence we are just left with the commutators (1.14) except for

$$[V_\mu, P_\nu] = \lambda_{\mu\nu} \cdot I, \qquad \mu, \nu = 0, 1, 2, 3 \tag{A.11}$$

The Jacobian identities $[V_\rho, P_\sigma, M_{\mu\nu}] = 0$ now give the equations

$$g_{\mu\nu}\lambda_{\rho\nu} - g_{\nu\sigma}\lambda_{\rho\mu} + g_{\mu\rho}\lambda_{\nu\sigma} - g_{\nu\rho}\lambda_{\mu\sigma} = 0 \tag{A.12}$$

whose only solutions can be found to be

$$\lambda_{\mu\nu} = \lambda \cdot g_{\mu\nu} \tag{A.13}$$

Again using (A.5) one concludes that two extensions characterized by two constants λ and λ' are equivalent if and only if $\lambda = \lambda'$, hence the inequivalent classes form a one dimensional family, with commutator structure given by (1.14) except for the following commutators

$$[V_\mu, P_\nu] = \lambda \cdot g_{\mu\nu} \cdot I \tag{A.14}$$

Appendix B

Projective unitary representations of $\mathbb{N}$ *and* $\mathbb{E}$

It follows from the results of Appendix A and from the so called lifting procedure [18] that the projective representations of $\mathbb{N}$ can be obtained (except for the usual $SU(2)$ spin factor systems that arises from the double covering of $SO(3)$), from the ordinary representations of the one dimensional family of central extensions found in (A.10) and that we shall denote by $\mathbb{N}^\lambda$. These groups have elements (θ, g), $\theta \in \mathbb{R}$, $g \in \mathbb{N}$ and the corresponding products read

$$(\theta_1, g_1)(\theta_2, g_2) = (\theta_1 + \theta_2 + \xi_\lambda(g_1, g_2), g_1 g_2) \tag{B.1}$$

with the factor systems ξ_λ obtained from (A.10) as given, for $g_i = (\vec{w}_i, a_i^0, \vec{a}_i, \alpha_i)$,

$i = 1, 2$, by

$$\xi_\lambda(g_1, g_2) = \lambda \vec{w}_1 \cdot \alpha_1 \vec{a}_2 \tag{B.2}$$

It is convenient at this point to consider the following exact sequences (split extensions)

$$0 \to \mathbb{R}^5 \to \mathbb{N}^\lambda \to E(3) \to 1, \qquad \varphi, m \sim 0 \tag{B.3}$$

with $\mathbb{R}^5$ generated by I (the center), P_0 and V_i, $i = 1, 2, 3$ whereas $E(3)$, the Euclidean group in 3 dimensions is associated with the subgroup of $\mathbb{N}^\lambda$ generated by $\{P_i, J_j\}$. Indeed we can calculate the representations of $\mathbb{N}^\lambda$ by the inducing technique [18][6]), from the corresponding normal abelian subgroup. An element of the dual of this subgroup will be denoted by $\hat{a}^{\eta,\vec{\kappa},p_0}$ with

$$\hat{a}^{\eta,\vec{\kappa},p_0}(\theta, \vec{w}, a^0) = e^{i\eta\theta} e^{i\vec{\kappa}\vec{w}} e^{ip_0 a^0} \tag{B.4}$$

and the actions $\hat{\phi}$ of the factor group of the extensions (B.3) are obtained using the group product law as given by

$$\begin{aligned}(\hat{\varphi}(\vec{a}, \alpha) \cdot \hat{a}^{\eta,\vec{\kappa},p_0})(\theta, \vec{w}, a^0) &\equiv \hat{a}^{\eta,\vec{\kappa},p_0}(\varphi^{-1}(\vec{a}, \alpha)(\theta, \vec{w}, a^0)) \\ &= \hat{a}^{\eta,\alpha\vec{\kappa}-\eta\lambda\vec{a},p_0}(\theta, \vec{w}, a^0)\end{aligned} \tag{B.5}$$

so that the orbits are characterized by constant η and p_0 and by the sets $\{(\alpha\vec{\kappa} - \eta\lambda\vec{a}), \forall \vec{a}, \alpha\}$. It is interesting to note at this point that in contradistinction to the Galilei group case, we here no longer have a dynamical type relationship as we then had for the orbits (see e.g. [2]) the characterization

$$E - \frac{1}{2m}\vec{p}^2 = \text{const.} \tag{B.6}$$

already implying in fact that the corresponding theory only would describe free particles.

With (B.5) the representations (B.4) can now be induced to the whole of $\mathbb{N}^\lambda$; there are obviously two cases:

Case 1. $\eta\lambda = 0$, implying either that the extension is trivial, or that the factor system ξ_λ is in the kernel of the representation. We thus then find, up to an uninteresting phase factor $\exp(i\eta\theta)$ the ordinary representations of $\mathbb{N}$ but, because of the Proposition 2.3 we may omit their analysis altogether.

Case 2. $\eta\lambda \neq 0$. We may then choose as orbit representatives the elements $\hat{a}^{\eta,\vec{0},p_0}$, whose little groups are isomorphic to $SO(3)$ (resp. $SU(2)$). The corresponding direct integral Hilbert spaces are $\mathcal{L}^2(\mathbb{R}^3) \otimes C_{2s+1}$, with $2s \in \mathbb{Z}_+$ and the induced representations are straightforwardly obtained as

$$\begin{aligned}&(\hat{a}^{\eta,\vec{0},p_0} \uparrow \mathbb{N}^\lambda)^{D_s}(\theta, \vec{w}, a^0, \vec{a}, \alpha) f_\sigma(\vec{x}) \\ &\quad = (\hat{a}^{\eta,\vec{0},p_0} \otimes D_s)(\theta - \lambda\vec{w}\vec{x}, \vec{w}, a^0, -\vec{x} + \vec{a} + \alpha\vec{x}', \alpha) f_\sigma(\vec{x}) \\ &\quad = e^{i\eta\theta} e^{-i\eta\lambda\vec{w}\vec{x}} e^{ip_0 a^0} (D_s)_{\sigma\sigma'}(\alpha) f_{\sigma'}(\alpha^{-1}(\vec{x} - \vec{a}))\end{aligned} \tag{B.7}$$

where we have used the little group condition for the coset representative $\vec{x}'$, and

[6]) For a brief survey of the method and more explicit notations, see [19], §3.

where $(D_s)_{\sigma\sigma'}$ are the usual spin s matrices of $SU(2)$. The projective representations of $\mathbb{N}$ are then obtained from (B.7) just by setting in this formula everywhere $\theta = 0$.

The situation is very analogous in the relativistic case. The central extensions found in (A.14) will be denoted $\mathbb{E}^\lambda$, and these groups have elements (θ, g), $\theta \in \mathbb{R}$, $g \in \mathbb{E}$ with corresponding products

$$(\theta_1, g_1)(\theta_2, g_2) = (\theta_1 + \theta_2 + \xi_\lambda(g_1, g_2), g_1 g_2) \tag{B.8}$$

where the factor systems ξ^λ can be obtained from (A.14), for $g_i = (w_i, a_i, \Lambda_i)$ $i = 1, 2$ as given by

$$\xi_\lambda(g_1, g_2) = \lambda g_{\mu\nu} w_1^\mu (\Lambda_1 a_2)^\nu \tag{B.9}$$

We again can decompose $\mathbb{E}^\lambda$ as follows

$$0 \to \mathbb{R}^5 \to \mathbb{E}^\lambda \to \mathscr{P} \to 1, \qquad \varphi, m \sim 0 \tag{B.10}$$

which is a split extension $(m \sim 0)$ and where $\mathbb{R}^5$ is generated by the center I and the momentum translations, whereas the factor group $\mathscr{P}$ is isomorphic with the Poincaré group. Denoting by $\hat{a}^{\eta,\kappa}$, $\eta \in \mathbb{R}$, κ a four covariant vector, an element of the dual of the corresponding normal subgroup, i.e.

$$\hat{a}^{\eta,\kappa}(\theta, w) = e^{i(\eta\theta + \kappa_\mu w^\mu)} \tag{B.11}$$

the action of $\mathscr{P}$ on this dual is then found to be given by

$$\begin{aligned}(\hat{\varphi}(a, \Lambda)\hat{a}^{\eta,\kappa})(\theta, w) &= \hat{a}^{\eta,\kappa}(\varphi^{-1}(a, \Lambda)(\theta, w)) \\ &= \exp i[\eta\theta + ((\Lambda\kappa)_\nu - \eta\lambda a^\mu g_{\mu\nu}) w^\nu]\end{aligned} \tag{B.12}$$

hence, with $(\tilde{g} \cdot a)_\nu \equiv g_{\mu\nu} a^\mu$

$$\hat{\varphi}(a, \Lambda)\hat{a}^{\eta,\kappa} = \hat{a}^{\eta, \Lambda\kappa - \eta\lambda(\tilde{g} \cdot a)} \tag{B.13}$$

Exactly as for the Newton group, and because the ordinary representations of $\mathbb{E}$ do not allow the desired systems of imprimitivity (see Proposition 2.5) we may restrict ourselves to the case where $\eta\lambda \neq 0$. The orbits defined by (B.13) are then characterized by a constant η, and by the set $\{\Lambda k - \eta\lambda(\tilde{g} \cdot a) \mid \forall a, \Lambda\}$ which is easily seen to generate the whole 4-dimensional space of covariant 4-vectors.

Note again at this point that similarly as in (B.6) we do not have a dynamical relationship characterizing the orbits as with the Poincaré group for which we had

$$p^2 = m^2 \tag{B.14}$$

which is bound to the fact that the representations of the Poincaré group only describe free particles.

Let us now induce the representations (B.13) to $\mathbb{E}^\lambda$: as $\eta\lambda \neq 0$ we may take as orbit representatives the elements $\hat{a}^{\eta,0}$ and the little groups are then $SO(3, 1)$ respectively $SL(2, \mathbb{C})$ and we find in this way a unique class of (true) projective representations given by

$$\begin{aligned}&(\hat{a}^{\eta,0} \uparrow \mathbb{E}^\lambda)^\Delta(\theta, w, a, \Lambda) f(x) \\ &\quad = \exp i\eta(\theta - \lambda g_{\mu\nu} w^\mu x^\nu)\, \Delta(\Lambda) f(\Lambda^{-1}(x - a))\end{aligned} \tag{B.15}$$

with $f \in \mathscr{L}^2(\mathbb{R}^4, \mathscr{H}(\Delta), d^4x)$, $\mathscr{H}(\Delta)$ the carrier space of an irreducible representation

Δ of $SL(2, \mathbb{C})$. In the spinless case, this gives the quantal solution (4.19) whereas for the case with spin, we refer to the remarks at the end of that Section 4.

REFERENCES

[1] E. P. Wigner, Ann. Math. *40*, 149 (1939).
[2] J. M. Lévy-Leblond, *Galilei Group and Galilean Invariance* in *Group Theory and Its Applications*, Vol. 2, ed. E. M. Loebl, Ac. Press N.Y. and London, 1971, p. 221 and references therein.
[3] G. Birkhoff and J. von Neumann, Ann. Math. *37*, 823, (1936).
[4] G. W. Mackey, *Mathematical Foundations of Quantum Mechanics*, Benjamin N.Y., (1963).
[5] C. Piron, *Foundations of Quantum Physics*, Benjamin Reading, Mass, (1976).
[6] N. Giovannini and C. Piron, in *Proc. of the VIIth Colloquium on Group Theoretical Methods in Physics*, at Austin (1978), Lecture Notes of Physics *94*, (1979), ed. Springer, p. 254.
[7] A. G. Kurosh, *Theory of Groups*, Vol. II, Chelsea New York 1960, Chap. 12.
[8] S. MacLane, *Homology*, Springer Berlin, 1963, chap. 4.
[9] G. W. Mackey, Proc. Natl. Acad. Sci. U.S.A. *35*, 537 (1949).
[10] G. W. Mackey, *The Theory of Unitary Group Representations*, University Chicago Press, 1976.
[11] C. Piron, in *Fundamenti di Meccanica Quantistica* Rend. Sc. Int. di Fisica E. Fermi, Vol. IL, ed. B. d'Espagnat, Acad. Press N.Y., 1971, p. 274.
[12] V. S. Varadarajan, *Geometry of Quantum Theory*, Vol. II, Van Nostrand Reinhold N.Y., 1970, chap. VIII and IX.
[13] R. N. Sen, Physica *94A*, 39 (1978).
[14] C. P. Horwitz, C. Piron and F. Reuse, Helv. Phys. Acta, *48*, 546 (1975).
[15] C. Piron and F. Reuse, Helv. Phys. Acta *51*, 146 (1978).
[16] F. Reuse, Helv. Phys. Acta *51*, 157 (1978).
[17] V. Bargmann, Ann. of Math. *59*, 1 (1954).
[18] G. W. Mackey, Trans. of Am. Soc. *85*, 134 (1957) and Acta Math. *99*, 265 (1958).
[19] N. Giovannini, Helv. Phys. Acta *50*, 349 (1977).