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Particle interpretation for external field problems in QED

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Abstract. Various Fock representations of the algebra of Fermi field operators in the presence of time-dependent external electromagnetic fields are investigated. The static representation obtained by freezing in the external field at any time t is of particular interest. It is proved that the system is always in this representation, describing the time evolution in the Schrödinger picture, which implies unitary implementation of the S-matrix. However, the attempt to use the static representation as a particle picture fails for reasons of relativistic covariance.

1. Introduction

The recent effort to test QED in strong electromagnetic fields has renewed the interest in the external field problem. The starting point of these investigations was the old observation that the vacuum structure changes from a neutral to charged vacuum if a *static* external field exceeds some critical strength. This statement has an unambiguous meaning in the static situation because there exists an essentially unique particle interpretation, usually called the Furry picture.

Unfortunately static supercritical fields do not exist. This has led to the proposal of looking for similar effects in time-dependent strong fields as they can be achieved in heavy ion collisions. Then, however, the meaning of a 'charged vacuum' is no longer clear. As a consequence, the meaning of 'critical fields' is also not clear. Therefore, the discussion of the problem must start from the analysis of the particle interpretation in time-dependent external fields.

In the next section, various Fock representations of the algebra of Fermi field operators are introduced. Besides the well-known in- and out-representations, there are at any time t interpolating in- and out-representations, obtained by the time evolution automorphism, and a static representation obtained by freezing in the external field at time t. It is proved in Section 3 that the interpolating and the static representations are unitary equivalent. This result has important physical consequences. First, it implies the unitary implementation of the out-representation in the in-Fock space, i.e. a unitary S-matrix. Second, it gives a justification of the Furry picture: if an external potential is switched on in a continuous manner and then remains constant $= A(\mathbf{x})$, the system is driven into the static representation corresponding to $A(\mathbf{x})$, which is just the Furry picture. For this reason, one might take the static representation as a candidate for a particle interpretation also in time-dependent fields. In fact, this particle picture has been used in almost all discussions of the external field problem in strong fields. However, the construction of the static representation is not a covariant procedure. As a consequence, the vacuum state for one reference frame will look as a many-particle state in a different frame of reference, which is unphysical. We therefore must retain the conservative point of view that the notion of particles is only related to the asymptotically in- and out-going states. But then, critical fields must be defined by some change in the structure of the S-matrix, as discussed below, and not by 'diving' of eigenvalues of a static Hamiltonian into the continuum.

2. Fock representations of the field-operator algebra

Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{A}$ the *-algebra of Fermi field operators $\psi(f)$ satisfying canonical anti-commutation relations

$$\{\psi(f), \psi^+(g)\} = (f, g), \qquad f, g \in \mathscr{H}. \tag{2.1}$$

Later on, $\mathscr{H}$ will be identified with the one-particle subspace, then we have $\mathscr{H} = (L^2(\mathbb{R}^3))^4$ for the Dirac field.

In $\mathscr{H}$ a time evolution is defined by the Dirac equation

$$i\frac{df(t)}{dt} = (H_0 + B(t))f(t), \tag{2.2}$$

where H_0 is the free Dirac Hamiltonian and

$$B(t, \mathbf{x}) = V(t, \mathbf{x}) - \boldsymbol{\alpha} \cdot \mathbf{A}(t, \mathbf{x})$$

represents a (possibly) time-dependent external electromagnetic field. We have to impose some mild restrictions on the potentials: Let $B(t)$ be H_0-bounded with norm <1 for all t, and differentiable in t such that the derivative $B'(t)$ is also H_0-bounded (with arbitrary bound). Then according to a general theorem [1] the Dirac equation (2.2) is solved by a strongly differentiable unitary propagator $U(t, t_0)$

$$f(t) = U(t, t_0)f_0, \qquad f_0 \in \mathscr{H} \tag{2.3}$$

satisfying

$$\begin{aligned} &U(t, t_1)U(t_1, t_0) = U(t, t_0) \\ &U(t, t) = \mathbb{1} \\ &U(t, t_0)^+ = U(t, t_0)^{-1} = U(t_0, t). \end{aligned} \tag{2.4}$$

By means of the propagator U, the time evolution can be carried over to the algebra $\mathscr{A}$

$$\psi_t(f_0) = \psi(U(t_0, t)f_0), \qquad f_0 \in \mathscr{H}. \tag{2.5}$$

This is a *-automorphism on $\mathscr{A}$ which solves the operator Dirac equation according to

$$\begin{aligned} i\frac{d}{dt}\psi_t(f_0) &= \psi\left(-i\frac{d}{dt}U(t, t_0)^+f_0\right) \\ &= \psi((H(t)U(t, t_0))^+f_0) = \psi(U(t_0, t)H(t)f_0) \\ &= \psi_t(H(t)f_0), \qquad f_0 \in D(H(t)). \end{aligned} \tag{2.6}$$

The abstract algebra $\mathscr{A}$ gets its physical meaning by representing it as an algebra of operators on a Hilbert space $\mathscr{F}$ of state vectors. If this representation is a Fock representation on

$$\begin{aligned} &\mathscr{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_n \\ &\mathscr{H}_0 = \{C\Omega, C \in \mathbb{C}\} \\ &\mathscr{H}_n = \mathscr{H}^{@n}, \end{aligned} \tag{2.7}$$

with vacuum Ω, we have a particle interpretation. For the definition of a Fock representation, it is sufficient to specify Ω. This can be done as follows: We choose an orthogonal direct decomposition of $\mathscr{H}$

$$\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_- \tag{2.8}$$

with corresponding projection operators $P_\pm$

$$\mathscr{H} \ni f = f_+ + f_-, \qquad f_\pm = P_\pm f \in \mathscr{H}_\pm, \tag{2.9}$$

and define annihilation operators by

$$b(f_+) = \psi(f_+), \qquad d(f_-) = \psi(f_-)^+,$$

such that

$$\psi(f) = b(f_+) + d(f_-)^+. \tag{2.10}$$

Then the vacuum Ω is defined by

$$b(f_+)\Omega = 0, \qquad d(f_-)\Omega = 0, \qquad \forall f_\pm \in \mathscr{H}.$$

The Fock space is constructed from Ω, applying b^+ and d^+ in the well-known way. We identify $\mathscr{H}_\pm$ with the electron and positron subspaces respectively.

The relationship between Ω and the projection operators $P_\pm$ is clearly seen in the structure of the N-point functions ($N = n + m$)

$$(\Omega, \psi(f_n)^+ \cdots \psi(f_1)^+ \psi(g_1) \cdots \psi(g_m)\Omega) = \delta_{nm} \det (f_i, P_- g_j). \tag{2.11}$$

Under the time evolution automorphism (2.5), the N-point functions transform as follows ($U = U(t) = U(t, t_0)$)

$$\begin{aligned} (\Omega, \psi(U^{-1}f_n)^+ \cdots \psi(U^{-1}g_m)\Omega) &= \delta_{nm} \det (U^{-1}f_i, P_- U^{-1}g_j) \\ &= \delta_{nm} \det (f_i, UP_- U^{-1}g_j). \end{aligned}$$

The right-hand side can be taken as the definition of a new Fock representation with a vacuum Ω_t, specified by the projector

$$P_-(t) = U(t)P_- U(t)^{-1}. \tag{2.12}$$

The corresponding Fock space $\mathscr{F}_t$ is obtained from $\mathscr{F}$ by the extension of the mapping

$$\Omega \rightsquigarrow \Omega_t, \qquad \prod_{jk} \psi^+(f_j)\psi(g_k)\Omega \rightsquigarrow \prod_{jk} \psi^+(U(t)f_j)\psi(U(t)g_k)\Omega_t \tag{2.13}$$

to the whole of $\mathscr{F}$. This interpretation of the time evolution as a change of the Fock representation with the field operators unchanged is the Schrödinger picture.

Every decomposition (2.8) or (2.10) characterizes a particular particle interpretation or Fock representation. The following representations will be important for what follows:

(i) The in-representation: We take the spectral projections $P_{\pm}^{0}$ on the positive and negative energy subspaces of the free Dirac Hamiltonian H_0, such that the decomposition (2.10) reads

$$\psi(f) = b_{\text{in}}(f_{+}^{0}) + d_{\text{in}}(f_{-}^{0})^{+} \tag{2.14}$$

with corresponding Ω_{in} and $\mathscr{F}_{\text{in}}$. Transforming $\psi(f)$ according to the free time evolution

$$U_0(t, t_0) = \exp - iH_0(t - t_0),$$

we write

$$\psi_t^{\text{in}}(f) = \psi(U_0(t_0, t)f). \tag{2.15}$$

If the (abstract) field is transformed with a non-trivial propagator U

$$\psi_t(f) = \psi_{t_0}(U(t_0, t)f), \tag{2.16}$$

we call it an interpolating field (using the LSZ terminology). To complete the definition of the interpolating field, an initial condition has to be imposed, which for the in-representation reads

$$\lim_{t \to -\infty} [\psi_t(f) - \psi_t^{\text{in}}(f)] = 0, \qquad \forall f \in \mathscr{H}. \tag{2.17}$$

Then, taking $t_0 \to -\infty$ in (2.16), we get the following expression for the interpolating field in the in-representation

$$\begin{aligned} \psi_t(f) &= \lim_{t_0 \to -\infty} \psi_{t_0}^{\text{in}}(U(t_0, t)f) \\ &= \lim_{t_0 \to -\infty} \psi_t^{\text{in}}(U_0(t, t_0)U(t_0, t)f) \\ &= \psi_t^{\text{in}}(W_-(t)^+ f), \end{aligned} \tag{2.18}$$

where

$$W_{\pm}(t) = s\text{-}\lim_{t_0 \to \pm\infty} U(t, t_0)U_0(t_0, t) \tag{2.19}$$

are the wave operators. At this point, some additional restrictions on the potentials, guaranteeing the existence of the wave operators, must be imposed. Such conditions are well-known in the static case [2] as well as for time-dependent potentials [3]. Furthermore, to simplify the discussion, we assume that $W_{\pm}$ are unitary operators on $\mathscr{H}$, which excludes bound states in the static case, but is no serious restriction for time-dependent fields going to 0 for $t \to \pm\infty$.

(ii) The interpolating in-representation: In this case the projection operators are

$$P_{\pm}(t) = s\text{-}\lim_{t_0 \to -\infty} U(t, t_0)P_{\pm}^{0}U(t_0, t) = W_-(t)P_{\pm}^{0}W_-(t)^+. \tag{2.20}$$

For $t=-\infty$ this coincides with the in-representation. For finite t, we get the decomposition

$$\begin{aligned}\psi_t(f) &= \lim_{t_0\to-\infty} \psi_{t_0}^{\text{in}}(U(t_0, t)f)\\ &= \lim_{t_0\to-\infty} \{\psi_{t_0}^{\text{in}}(P_+^0 U(t_0, t)f) + \psi_{t_0}^{\text{in}}(P_-^0 U(t_0, t)f)\}\\ &= \lim_{t_0\to-\infty} \{\psi_{t_0}(P_+^0 U(t_0, t)f) + \psi_{t_0}(P_-^0 U(t_0, t)f)\}\\ &= \psi_t(U(t, -\infty)P_+^0 U(-\infty, t)f) + \psi_t(U(t, -\infty)P_-^0 U(-\infty, t)f),\end{aligned}$$

where $-\infty$ is used as a short notation for the strong limit (2.20). This shows that the interpolating in-representation is obtained from the in-representation, taking the time-evolution into account in the Schrödinger picture (2.12).

(iii) The static representation: Here the projection operators $P_\pm$ are given by the positive and negative spectral parts of the static Hamiltonian

$$H = H_0 + B(t, \mathbf{x})_{t=\text{const}},$$

where the external field $B(t, \mathbf{x})$ is frozen in at some time t. The corresponding particle interpretation is known as the Furry picture in the case of a static field.

(iv) The out-representation: This representation is completely analogous to the in-representation with the only difference that the interpolating field is now defined by the asymptotic condition

$$\lim_{t\to+\infty} [\psi_t(f) - \psi_t^{\text{out}}(f)] = 0, \qquad \forall f \in \mathscr{H} \tag{2.21}$$

instead of (2.17). As a consequence, we now have

$$\psi_t(f) = \psi_t^{\text{out}}(W_+(t)^+ f). \tag{2.22}$$

Similar to (ii) an interpolating out-representation can be defined.

According to the construction, each Fock representation I–IV is realized on its own Fock space. Now, the natural question arises which representations are unitary equivalent. Let us consider two representations $\psi_1(f)$, $\psi_2(f)$ with Fock spaces $\mathscr{F}_1$, $\mathscr{F}_2$, defined by projection operators $P_\pm^1$, $P_\pm^2$. They are unitary equivalent if a unitary operator U from $\mathscr{F}_1$ onto $\mathscr{F}_2$ exists, such that

$$\psi_1(f) = U\psi_2(f)U^{-1}, \qquad \forall f \in \mathscr{H}. \tag{2.23}$$

Specializing the relation

$$b_1(P_+^1 f) + d_1(P_-^1 f)^+ = Ub_2(P_+^2 f)U^{-1} + Ud_2(P_-^2 f)^+ U^{-1} \tag{2.24}$$

for $f = P_\pm^2 f$, we obtain a Bogoliubov transformation on $\mathscr{F}_1$

$$\begin{aligned}b_2'(P_+^2 f) &= b_1(P_+^1 P_+^2 f) + d_1(P_-^1 P_+^2 f)^+ = Ub_2(P_+^2 f)U^{-1}\\ d_2'(P_-^2 f)^+ &= b_1(P_+^1 P_-^2 f) + d_1(P_-^1 P_-^2 f)^+ = Ud_2(P_-^2 f)^+ U^{-1},\end{aligned} \tag{2.25}$$

which is unitarily implemented. This implies that $\Omega_2' = U\Omega_2$ is the vacuum for the Fermi operators b_2', d_2' in $\mathscr{F}_1$. Therefore, the two representations are unitary equivalent if and only if there exists a unique vacuum Ω_2' for b_2', d_2' in $\mathscr{F}_1$. This problem has been studied in Ref. [4]. Writing

$$\begin{aligned} W_{++} &= P_+^1 P_+^2 \qquad & W_{-+} &= P_-^1 P_+^2 \\ W_{+-} &= P_+^1 P_-^2 \qquad & W_{--} &= P_-^1 P_-^2, \end{aligned} \tag{2.26}$$

the solution can be expressed in the following

Theorem 1. *Let*

$$\begin{aligned} b_2(f) &= b_1(W_{++}f) + d_1(W_{-+}f)^+ \\ d_2(f)^+ &= b_1(W_{+-}f) + d_1(W_{--}f)^+ \end{aligned} \tag{2.27}$$

be a Bogliubov transformation of Fermi operators generated by a unitary operator W on $\mathcal{H}$

$$W = \begin{pmatrix} W_{++} & W_{-+} \\ W_{+-} & W_{--} \end{pmatrix}, \qquad WW^+ = \mathbb{1}. \tag{2.28}$$

Then the two sets of operators have vacua Ω_1 *and* Ω_2 *in the same Fock space* $\mathcal{F}$ *if and only if* W_{-+} *and* W_{+-} *are Hilbert–Schmidt (H.S.) operators on* $\mathcal{H}$.

There exist several proof of this theorem in the literature [5]. The constructive proof given in [4] has the advantage of providing explicit expressions for the vacuum Ω_2 and the corresponding dressing transformation $\mathbb{V}$:

Theorem 2. *The conditions of Theorem 1 are necessary and sufficient for the existence of a unitary operator* $\mathbb{V}$ *on* $\mathcal{F}$, *satisfying*

$$\begin{aligned} &\Omega_2 = \mathbb{V}\Omega_1 \\ &\mathbb{V}\psi(f)\mathbb{V}^{-1} = \psi(Wf). \end{aligned} \tag{2.29}$$

$\mathbb{V}$ *is given in terms of W as follows: Let.*

$$\begin{aligned} \mathfrak{N}_+ &= \mathrm{Ker}\,(1 - W_{+-}^+ W_{+-}) = \mathrm{Ker}\, W_{++} \\ \mathfrak{N}_- &= \mathrm{Ker}\,(1 - W_{-+}^+ W_{-+}) = \mathrm{Ker}\, W_{--}, \end{aligned} \tag{2.30}$$

be the (finite dimensional) null-spaces and

$$\begin{aligned} A &= -W_{++}^{-1} W_{-+} \qquad & B &= W_{++}^{-1} \\ C &= W_{--}^{-1} \qquad & D &= (W_{+-} W_{++}^{-1})^T \end{aligned} \tag{2.31}$$

where the inverse operators are defined on the orthogonal complements $\mathfrak{N}'_\pm$. *Choose a basis*

$$\begin{aligned} &\varphi_{1'}^+ \ldots \varphi_n^+;\ \varphi_p^+, \qquad & p &= n+1, \ldots, \infty \\ &\varphi_{1'}^- \ldots \varphi_{n'}^-;\ \varphi_q^-, \qquad & q &= n'+1, \ldots, \infty \end{aligned} \tag{2.32}$$

in $\mathfrak{N}_+$, $\mathfrak{N}'_+$, $\mathfrak{N}_-$, $\mathfrak{N}'_-$ *respectively, and define*

$$\begin{aligned} b_p &= b_2(\varphi_p^+), \qquad & d_q &= d_2(\varphi_q^-), \qquad & p, q &= 1, \ldots, \infty \\ A_{pq} &= (\varphi_p^+, A\varphi_q^-), \qquad & B_{pq} &= (\varphi_p^+, B\varphi_q^+) \\ C_{pq} &= (\varphi_p^-, C\varphi_q^-), \qquad & D_{pq} &= (\varphi_p^+, D\varphi_q^-), \qquad & p &= n+1, \ldots, \infty \\ & & & & q &= n'+1, \ldots, \infty, \text{ etc.} \end{aligned} \tag{2.33}$$

Then

$$\mathbb{V} = C_0\mathbb{V}_0 \exp \sum_{pq} A_{pq} b_p^+ d_q^+ : \exp \sum_{pq} (B_{pq} - \delta_{pq}) b_p^+ b_q :$$
$$: \exp \sum_{pq} (C_{pq} - \delta_{pq}) d_p^+ d_q : \exp \sum_{pq} D_{pq} b_p d_q \quad (2.34)$$

with

$$C_0 = [\det(1 + A^+ A)]^{-1/2}$$
$$\mathbb{V}_0 = :\left(b_1^+ \mp \sum_q W_{+-}^{q1} d_q\right) \cdots \left(b_n^+ \mp \sum_q W_{+-}^{qn} d_q\right)$$
$$\left(d_1^+ \mp \sum_p W_{-+}^{p1} b_p\right) \cdots \left(d_{n'}^+ \mp \sum_p W_{-+}^{pn'} b_p\right): \quad (2.35)$$

− if $n + n'$ even, + if $n + n'$ odd. The double dots mean normal ordering.

Let us now return to the problem of unitary equivalence. We shall prove in the next section that for a large class of time dependent external fields the static representation (iii) and the interpolating (in-) representation (ii) are equivalent. This result has important physical consequences. First, it gives a justification of the Furry picture: The particle interpretation obtained by following the time-evolution in the Schrödinger picture, switching on the external field in a continuous manner, is equivalent to the Furry picture (up to a unitary dressing transformation). Second, since the static and the interpolating out-representation are equivalent as well, it follows that the interpolating in- and out-representations are also equivalent. The unitary transformation between them can be fixed by realizing both representations on the same Fock space, identifying the interpolating fields. Then equations (2.18) and (2.22) imply

$$\psi_t^{\text{in}}(W_-(t)^+ f) = \psi_t(f) = \psi_t^{\text{out}}(W_+(t)^+ f) \quad (2.36)$$

or

$$\psi_t^{\text{out}}(f) = \psi_t^{\text{in}}(W_-(t)^+ W_+(t) f) = \psi_t^{\text{in}}(S(t)^+ f), \quad (2.37)$$

where $S(t)$ is the 1-particle S-matrix. The unitary dressing transformation $\mathbb{S}(t)$ which exists according to Theorem 2 (2.29)

$$\psi_t^{\text{out}}(f) = \psi_t^{\text{in}}(S(t)^+ f) = \mathbb{S}(t)^+ \psi_t^{\text{in}}(f) \mathbb{S}(t) \quad (2.38)$$

is of course the S-matrix in Fock space.

The explicit structure of the S-matrix is given by (2.34). For weak external fields the factor $\mathbb{V}_0$ is $=1$. The main problem in connection with strong time-dependent fields is, whether the change in the structure of $\mathbb{S}$ with the appearance of a non-trivial $\mathbb{V}_0$ (2.35) or non-trivial null-space (2.30) actually occurs. This would correspond precisely to the charged vacuum in the static case [6], and will be discussed elsewhere.

Theorem 2 can also be understood as a criterion on the unitary implementation (2.29) of the *-automorphism

$$(\alpha \circ \psi)(f) = \psi(Wf) \quad (2.39)$$

within the *same* Fock space $\mathscr{F}$. It is important to notice that unitary correspondence can always be achieved between *different* Fock spaces $\mathscr{F}$ and $\mathscr{F}'$. In fact, let Ω and $\mathscr{F}$ be vacuum and Fock space corresponding to the decomposition

$$\psi(f)=b(f_+)+d(f_-)^+, \qquad f_\pm = P_\pm f \tag{2.40}$$

defined by $P_\pm$, and, in the same way, Ω', $\mathscr{F}'$ for

$$\psi'(f')=b'(f'_+)+d'(f'_-)^+, \qquad f'_\pm = P'_\pm f' \tag{2.41}$$

with

$$P'_\pm = WP_\pm W^{-1}, \tag{2.42}$$

such that

$$f'_\pm = Wf_\pm. \tag{2.43}$$

Extending the correspondence

$$\Omega \rightsquigarrow \Omega', \qquad \prod_{jk} b(f_{j+})^+ \, d(f_{k-})^+\Omega \rightsquigarrow \prod_{jk} b'(f'_{j+})^+ \, d'(f'_{k-})^+\Omega' \tag{2.44}$$

to the whole of $\mathscr{F}$, one gets a unitary mapping $\mathbb{V}$ of $\mathscr{F}$ onto $\mathscr{F}'$. Then

$$b'(f'_+)=\mathbb{V}b(f_+)\mathbb{V}^{-1}, \qquad d'(f'_-)=\mathbb{V}\, d(f_-)\mathbb{V}^{-1} \tag{2.45}$$

and

$$\psi'(Wf)=\mathbb{V}b(f_+)\mathbb{V}^{-1}+\mathbb{V}\, d(f_-)^+\mathbb{V}^{-1}=\mathbb{V}\psi(f)\mathbb{V}^{-1}, \tag{2.46}$$

which shows the unitary correspondence.

In scattering theory, one wants to calculate matrix elements between incoming and outgoing states. To carry this out, one needs unitary implementation of the S-matrix as discussed above and not only a unitary correspondence (2.46). On the other hand, in the discussion of symmetry transformations the trivial unitary correspondence may be sufficient. We shall return to this point in the last section. Moreover, as was pointed out in Ref. [7], the time evolution (2.5) is not implementable in general. Using the unitary correspondence (2.46) in this case, we are again in the Schrödinger picture (2.13).

3. Unitary equivalence of the static and interpolating representations

For simplicity we consider external potentials

$$B(t,\mathbf{x})=V(t,\mathbf{x})-\boldsymbol{\alpha}\cdot\mathbf{A}(t,\mathbf{x}) \tag{3.1}$$

which vanish identically for $t\le t_0$. Without loss of generality, we may assume $t_0=0$. The interpolating in-representation at some fixed time $T>0$ is then defined by $U(T,0)P^0_\pm U(0,T)$. According to Theorem 1 we have to prove

$$\begin{aligned} &P_+(T)U(T,0)P^0_- \in \text{H.S.}\\ &P_-(T)U(T,0)P^0_+ \in \text{H.S.}, \end{aligned} \tag{3.2}$$

where $P_\pm(T)$ are the projection operators of the static Hamiltonian $H(T)$, defining the static representation. We show that the operators (3.2) are integral operators with Hilbert–Schmidt kernels. In doing so, we have to deal with the more general Carleman integral operators [8]. A Carleman operator K on L^2 is defined by a measurable kernel $K(p, q)$ with

$$K(p, \cdot) \in L^2 \qquad \text{for almost all } p. \tag{3.3}$$

We shall only consider bounded Carleman operators, such that

$$(Kf)(p) = \int K(p, q) f(q)\, dq \in L^2 \quad \text{for all} \quad f \in L^2.$$

In addition we have to use the eigenfunction expansion for $H(T)$. If $B(T, \mathbf{x})$ is sufficiently restricted [9, 10], there exist normalizable eigenfunctions $\varphi(E_i, \mathbf{x}) \in L^2(\mathbb{R}^3)^4$ and generalized eigenfunctions $\varphi(p, \mathbf{x}) \in L^\infty(\mathbb{R}^3)^4$, $p = (\mathbf{p}, s, \varepsilon)$, $s, \varepsilon = \pm 1$, which define the generalized unitary Fourier transformation F

$$F : L^2(\mathbb{R}^3)^4 \to L^2(\mathbb{R}^3)^4 \oplus l^2$$

$$(Ff)(p) = \text{l.i.m.} \int d^3x \varphi(p, \mathbf{x})^+ f(\mathbf{x}) \in L^2(\mathbb{R}^3)^4 \tag{3.4}$$

$$(Ff)(i) = \int d^3x \varphi(E_i, \mathbf{x})^+ f(\mathbf{x}) \in l^2.$$

F diagonalizes $H(T)$:

$$\begin{aligned} &F(H(T)f)(p) = \varepsilon E(\mathbf{p})(Ff)(p) \\ &F(H(T)f)(i) = E_i(Ff)(i), \qquad E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}. \end{aligned} \tag{3.5}$$

To simplify the notation, we use the symbol p for both $(\mathbf{p}, s, \varepsilon)$ and the label i of the discrete spectrum, and $E(p)$ for $E(\mathbf{p})$ and E_i, etc. In contrast to F the ordinary Fourier transformation on $L^2(\mathbb{R}^3)^4$ is denoted by F_0. It transforms H_0 according to

$$\begin{aligned} &F_0(H_0 f)(\mathbf{k}) = E(\mathbf{k})(P^0_+(\mathbf{k}) - P^0_-(\mathbf{k}))(F_0 f)(\mathbf{k}) \\ &\quad P^0_\pm(\mathbf{k}) = \frac{1}{2}\left(1 \pm \frac{\boldsymbol{\alpha} \cdot \mathbf{k} + \beta m}{E(\mathbf{k})}\right). \end{aligned} \tag{3.6}$$

We sometimes write

$$\hat{f}(\mathbf{k}) = (F_0 f)(\mathbf{k}) \in L^2(\mathbb{R}^3)^4. \tag{3.7}$$

The norm of a 4×4 matrix B will be simply denoted by $|B|$ and the space of matrix functions $B(\mathbf{p})$ with

$$\|B\|_v = \left(\int |B(\mathbf{p})|^v \, d^3p\right)^{1/v} < \infty, \qquad v = 1, 2 \tag{3.8}$$

by $L^v(\mathbb{R}^3)^{16}$. The symbol F_0 is also used for the Fourier transform of $B(\mathbf{x}) \in L^2(\mathbb{R}^3)^{16}$

$$\hat{B}(\mathbf{k}) = (F_0 B)(\mathbf{k}) \in L^2(\mathbb{R}^3)^{16}. \tag{3.9}$$

We are going to prove

Theorem 3. *Suppose that the external potential* (3.1) *satisfies the following conditions:*

(1) $B(t, \mathbf{x}) = 0$ *for* $t \leq 0$,

(2) *$B(t,\cdot)$ is continuous and two times piecewise continuously differentiable with respect to t,*
(3) $\hat{B}^{(\alpha)}(t,\mathbf{k})\in L^2(\mathbb{R}^3)^{16}\cap L^1(\mathbb{R}^3)^{16}$, $\alpha=0,1,2$ *for all* $0\le t\le T$.
(4) $E=0$ *is not in the spectrum of* $H(T)$,
(5) $H(T)$ *has an eigenfunction expansion* (3.4), (3.5).

Then the static and the interpolating in-representations at any time $t\in[0,T]$ are unitary equivalent.

Remarks

(i) Condition (4) serves for the only purpose to have a unambiguous meaning of $P_\pm(T)$ in (3.2). If 0 is an isolated eigenvalue of finite multiplicity, the corresponding eigenspace may be included in $P_+(T)$, say, and the theorem is also valid.

(ii) Sufficient conditions for (5) have been given in Ref [9, 10]. These conditions are usually formulated in x-space in contrast to condition (3), and they are not automatically satisfied by (3). The additional condition $B(T,\mathbf{x})\in L^1(\mathbb{R}^3)^{16}$ guarantees the existence of the eigenfunction expansion, for example [9].

(iii) The idea of the proof is well-known from the similar problem for the S-matrix [11]. We express the operator (3.2) by an appropriate Dyson series and show convergence in Hilbert-Schmidt norm by partial integration in time. This works with the weak smoothness assumptions (2). The potential must vary discontinuously in time to be bad enough to destroy the Hilbert–Schmidt property.

Proof. We start from the following integral equation for the propagator $U(T)=U(T,0)$

$$U(T)=U_s(T)-i\int_0^T dt_1\,U_s(T-t_1)(B(t_1)-B(s))U(t_1) \tag{3.10}$$

where

$$U_s(t)=e^{-iH(s)t},$$

with s arbitrarily fixed. Combining the equations for $s=0$ and $s=T$ and interating, we get the following norm-converging Dyson series

$$U(T)=U_T(T)\sum_{n=0}^{\infty}(-i)^nU^{(n)}(T) \tag{3.11}$$

where $U^{(0)}(T)=1$

$$U^{(n)}(T)=\int_0^T dt_1\int_0^{t_1}dt_2\cdots\int_0^{t_{n-1}}dt_n\,U_T(-t_1)(B(t_1)-B(T))$$
$$\cdot\,U_0(t_1-t_2)B(t_2)\cdots B(t_{n-1})U_0(t_{n-1}-t_n)B(t_n)U_0(t_n). \tag{3.12}$$

To represent the relevant operators as Carleman integral operators in p-space, we often use mixed Fourier variables. Let for any bounded operator A

$$\tilde{A}=FAF_0^{-1}. \tag{3.13}$$

For $A = B^{(\alpha)}(s)$ we have

$$(\widetilde{B^{(\alpha)}}(s)f)(p) = \int \varphi(p, \mathbf{x})^* B^{(\alpha)}(s, \mathbf{x})(F_0^{-1}f)(\mathbf{x})\, d^3x$$
$$= \int F_0^{-1}(\varphi(p, \cdot)^* B^{(\alpha)}(s, \cdot))(\mathbf{k}) f(\mathbf{k})\, d^3k$$
$$\stackrel{\text{def}}{=} \int \tilde{B}^{(\alpha)}(p, \mathbf{k}) f(\mathbf{k})\, d^3k, \qquad f \in L^2(\mathbb{R}^3)^4, \tag{3.14}$$

using Parseval's equality for F_0. It follows from $\varphi \in L^\infty$, respectively L^2, and (3) that $\tilde{B}^{(\alpha)}(p, \mathbf{k})$ are Carleman kernels for $\alpha = 0, 1, 2$. Then

$$\widetilde{B^{(\alpha)}}(s)_{+-} \stackrel{\text{def}}{=} FP_+(T)B^{(\alpha)}(s)P_-^0 F_0^{-1} \tag{3.15}$$

are Carleman operators as well. The equation

$$(H(\widetilde{T)} - H_0)_{+-}(p, \mathbf{k}) = (E(p) + E(\mathbf{k}))P_+(\widetilde{T})P_-^0$$
$$= \widetilde{B(T)}_{+-}(p, \mathbf{k}) \tag{3.16}$$

shows that $P_+(\widetilde{T})P_-^0$ is also a Carleman operator with the kernel

$$(P_+(\widetilde{T})P_-^0)(p, \mathbf{k}) = \frac{\widetilde{B(T)}_{+-}(p, \mathbf{k})}{E(p) + E(\mathbf{k})}, \qquad E(p) > 0. \tag{3.17}$$

This implies that all operators we have to deal with in the following are Carleman operators.

We substitute the Dyson series (3.11) into (3.2) and study the Born term at first

$$(1 - i\widetilde{U^{(1)}}(T))_{+-}(p, \mathbf{k}) = (P_+(\widetilde{T})P_-^0)(p, \mathbf{k})$$
$$-i\int_0^T dt_1 e^{it_1(E(p)+E(\mathbf{k}))}(\widetilde{B(t_1)}_{+-}(p, \mathbf{k}) - \widetilde{B(\tau)}_{+-}(p, \mathbf{k})). \tag{3.18}$$

Partial integration in t_1 leads to

$$= (P_+(\widetilde{T})P_-^0)(p, \mathbf{k}) - \frac{\widetilde{B(T)}_{+-}(p, \mathbf{k})}{E(p) + E(\mathbf{k})}$$
$$+ \int_0^T dt_1 e^{it_1(E(p)+E(\mathbf{k}))} \cdot \frac{\widetilde{B^{(1)}}(t_1)_{+-}(p, \mathbf{k})}{E(p) + E(\mathbf{k})}, \tag{3.19}$$

where the first two terms compensate because of (3.16). The remaining term will be integrated a second time by parts. Assuming for a moment $B^{(1)}(t)$ to be continuous in t, we obtain

$$= -ie^{it_1(E(p)+E(\mathbf{k}))} \frac{\widetilde{B^{(1)}}(t_1)_{+-}(p, \mathbf{k})}{(E(p) + E(\mathbf{k}))^2} \Bigg|_{t_1=0}^{t_1=T}$$
$$+ i\int_0^T dt_1 e^{it_1(E(p)+E(\mathbf{k}))} \frac{\widetilde{B^{(2)}}(t_1)_{+-}(p, \mathbf{k})}{(E(p) + E(\mathbf{k}))^2}. \tag{3.20}$$

If $B^{(1)}(t)$ is only piecewise continuous, we get the analogous result for every interval of continuity. To see that the kernels (3.20) are Hilbert–Schmidt, we estimate

$$\left|\frac{B^{(\alpha)}(s)_{+-}(p,\mathbf{k})}{(E(p)+E(\mathbf{k}))^2}\right| \leq \left|\frac{\widetilde{B^{(\alpha)}(s)}_{+-}(p,\mathbf{k})}{E(\mathbf{k})^2}\right|$$
$$= \left|(\widetilde{B^{(\alpha)}(s)}H_0^{-2}P_-^0)_{+-}(p,\mathbf{k})\right|. \tag{3.21}$$

Since

$$(F_0B^{(\alpha)}(s)H_0^{-2}P_-^0F_0^{-1})(\mathbf{q},\mathbf{k}) = \frac{\widehat{B^{(\alpha)}(s)}(\mathbf{q}-\mathbf{k})}{E(\mathbf{k})^2}P_-^0(\mathbf{k}) \tag{3.22}$$

is a Hilbert–Schmidt operator

$$\|B^{(\alpha)}(s)H_0^{-2}P_-^0\|_{\text{H.S.}} \leq \|B^{(\alpha)}(s)\|_2\,\|E(\cdot)^{-2}\|_2 < \infty, \tag{3.23}$$

the same is true for (3.21) because of the unitarity of the eigenfunction expansion. The integral in (3.20) is also a Hilbert–Schmidt operator because the integrand is uniformly bounded and continuous in H.S.-norm.

Now we turn to the nth order term in (3.11). The corresponding kernel reads

$$\widetilde{U^{(n)}}(T)_{+-}(p,\mathbf{k}) = \int_\Delta dt_1\cdots dt_n e^{i(t_1E(p)+t_nE(\mathbf{k}))}I_n(t_1,\ldots,t_n;p,\mathbf{k}) \tag{3.24}$$

with

$$I_n(t_1,\ldots,t_n;p,\mathbf{k}) = \delta_{1\varepsilon}\int d^3k_1\cdots d^3k_{n-1}[\widetilde{B(t_1)}(p,\mathbf{k}_1)$$
$$-\widetilde{B}(T)(p,\mathbf{k}_1)]e^{i(t_2-t_1)H_0(\mathbf{k}_1)}\hat{B}(t_2)(\mathbf{k}_1-\mathbf{k}_2)\cdot\cdots \tag{3.25}$$
$$\cdot e^{i(t_n-t_{n-1})H_0(\mathbf{k}_{n-1})}\hat{B}(t_n)(\mathbf{k}_n-\mathbf{k})P_-^0(\mathbf{k}),$$

$$H_0(\mathbf{k}) = \boldsymbol{\alpha}\cdot\mathbf{k}+\beta m,$$
$$\Delta = \{0<t_n<t_{n-1}<\cdots<t_1<T\}\subset\mathbb{R}^n. \tag{3.26}$$

Since $\widetilde{B(s)}(p,\cdot)\in L^2$ and $\hat{B}(t,\cdot)\in L^1$, the convolutions (3.25) give $I_n(t_1,\ldots,t_n;p,\cdot)\in L^2$, such that $\tilde{U}^{(n)}$ (3.24) is a Carleman kernel. For partial integration of (3.24) we introduce new time variables

$$s_1 = \frac{1}{n}\sum_{i=1}^n t_i$$
$$s_j = t_j - t_{j-1}, \qquad j=2,3,\ldots,n \tag{3.27}$$

with

$$\left|\det\frac{\partial s_i}{\partial t_j}\right| = 1$$

and the inverse transformation

$$t_j = s_1+\frac{1}{n}\sum_{i=2}^{j}(i-1)s_i-\frac{1}{n}\sum_{i=j+1}^{n}(n-i+1)s_i. \tag{3.28}$$

Because of

$$t_1 = s_1 - \frac{1}{n}\sum_{i=2}^{n}(n-i+1)s_i$$
$$t_n = s_1 + \frac{1}{n}\sum_{i=2}^{n}(i-1)s_i \tag{3.29}$$

the integration of the exponential in (3.24) with respect to s_1 yields a denominator $E(p)+E(\mathbf{k})$ as in (3.19) above. Differentiation of I_n with fixed $s_2, \ldots, s_n$ acts on the potentials B only. In the first partial integration of (3.24) in s_1, the boundary terms vanish because they correspond to $t_1 = T$ and $t_n = 0$ in (3.29) and in both cases I_n (3.25) vanishes. For the second partial integration, let us assume for a moment $B(t)$ to be two times continuously differentiable. Then the final result consists of n^2 integrals over Δ (3.26) and $2n$ boundary terms with integrals over $\Delta \cap \{t_1 = T\}$ and $\Delta \cap \{t_1 = 0\}$ respectively. The integrands are of the following form

$$I_n^{(\alpha)}(t_1, \ldots, t_n; p, \mathbf{k}) = \frac{1}{(E(p)+E(\mathbf{k}))^2}\int d^3k_1 \cdots d^3k_{n-1}$$
$$\times [\widetilde{B^{(\alpha_1)}}(t_1)(p, \mathbf{k}_1) - \delta_{\alpha_1 0}\widetilde{B(T)}(p, \mathbf{k}_1)]e^{i(t_2-t_1)H_0(\mathbf{k}_1)} \cdots$$
$$e^{i(t_n-t_{n-1})H_0(k_{n-1})}\hat{B}^{(\alpha_n)}(t_n)(\mathbf{k}_n - \mathbf{k})P_-^0(\mathbf{k}),$$
$$\alpha = (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i = 0, 1, 2, \qquad \sum_i \alpha_i \le 2. \tag{3.30}$$

To show that $I_n^{(\alpha)}$ (3.30) is a Hilbert–Schmidt kernel, it is sufficient to consider the kernel

$$J^{(\alpha)}(\mathbf{p}, \mathbf{k}) = \frac{1}{E(\mathbf{k})^2}\int d^3k_2 \cdots d^3k_{n-1}\widehat{B^{(\alpha_1)}}(\mathbf{p}-\mathbf{k}_1)e^{i(t_2-t_1)H_0(\mathbf{k}_1)}$$
$$\cdot \widehat{B^{(\alpha_2)}}(\mathbf{k}_1 - \mathbf{k}_2) \cdots \widehat{B^{(\alpha_n)}}(\mathbf{k}_n - \mathbf{k})P_-^0(\mathbf{k})$$
$$\stackrel{\text{def}}{=} \frac{1}{E(\mathbf{k})^2}J_1(\mathbf{p}-\mathbf{k}) \tag{3.31}$$

instead, using again the unitarity of the eigenfunction expansion. We have

$$\|J^{(\alpha)}\|_{\text{H.S.}} = \left\|\frac{1}{E(\mathbf{k})^2}\right\|_2 \|J_1(\cdot)\|_2. \tag{3.32}$$

Since

$$\|J_1(\cdot)\|_2 \le \| \, |\hat{B}^{(\alpha_1)}| * |\hat{B}^{(\alpha_2)}| * \cdots * |\hat{B}^{(\alpha_n)}| \, \|_2, \tag{3.33}$$

Young's inequality gives

$$\|J_1(\cdot)\|_2 \le \prod_{j=1}^{n-1} \|\hat{B}^{(\alpha_j)}\|_1 \, \|\hat{B}^{(\alpha_n)}\|_2 \le C_1^{n-1}C_2, \tag{3.34}$$

$$C_i = \max_\alpha \|\hat{B}^{(\alpha)}\|_i, \qquad i = 1, 2, \tag{3.35}$$

which shows that $J^{(\alpha)}$ is H.S. It follows in particular that $I_n^{(\alpha)}$ (3.30) is continuous on Δ in the H.S.-norm.

Finally, the volumes of the domains of integration in $t_1, \ldots, t_n$ are

$$\mu_n(\Delta) = \frac{T^n}{n!}$$
$$\mu_{n-1}(\Delta \cap \{t_1 = T\}) = \frac{T^{n-1}}{(n-1)!} = \mu_{n-1}(\Delta \cap \{t_n = 0\}). \tag{3.36}$$

This implies for (3.24) the following bound

$$\|U^{(n)}(T)_{+-}\|_{\text{H.S.}} \leq 2 \cdot C_1^{n-1} C_2 \left\| \frac{1}{E(\mathbf{k})^2} \right\|_2 \cdot \left(\frac{n^2 T^n}{n!} + 2 \frac{T^{n-1}}{(n-1)!} \right). \tag{3.37}$$

and leads to convergence of the Dyson series (3.11) for $U(T)_{+-}$ in the H.S.-norm. This completes the proof in the case of a two times continuously differentiable potential. If $B(t)$ is only picewise differentiable, the only modification in the above estimates is the appearance of additional boundary terms. Let $B^{(1)}(t)$ be discontinuous at N points in $[0, T]$, then we get instead of (3.37) the bound

$$\|U^{(n)}(T)_{+-}\|_{\text{H.S.}} \leq 2 \cdot C_1^{n-1} C_2 \left\| \frac{1}{E(\mathbf{k})^2} \right\|_2 \cdot \left(\frac{n^2 T^n}{n!} + (2+2N) \frac{T^{n-1}}{(n-1)!} \right).$$

which leads also to a H.S.-converging Dyson series. The proof of the second condition in (3.2) $U(T)_{-+} \in$ H.S. runs along exactly the same lines.

4. Covariant particle interpretations

In this section we sometimes use relativistic notation, writing

$$f^0(x) = U_0(t, t_0) f_0(\mathbf{x}), \qquad x = (t, \mathbf{x}) \tag{4.1}$$

$$f(x) = U(t, t_0) f_0(\mathbf{x}). \tag{4.2}$$

As it is well-known, the restricted Lorentz group $L_+^\uparrow$ operates on the free test functions $f^0(x)$ as follows

$$(V_0(\Lambda) f^0)(x') = S(\Lambda) f^0(\Lambda^{-1} x') \tag{4.3}$$

where

$$x' = \Lambda x, \qquad \Lambda \in L_+^\uparrow \tag{4.4}$$

and $S(\Lambda)$ satisfies

$$\gamma^\mu \Lambda_\mu^\nu = S(\Lambda)^{-1} \gamma^\nu S(\Lambda). \tag{4.5}$$

This defines a unitary representation of $L_+^\uparrow$ which is most explicitly described in p-space: Let

$$f^0(x) = \int d\Omega(p) \sum_{s=\pm 1} \{ f_{s+}^0(p) u_s(p) e^{-ipx} + f_{s-}^0(p) v_s(p) e^{ipx} \} \tag{4.6}$$

be the time-dependent eigenfunction expansion in the free case, then

$$(V_0(\Lambda)f^0)(x') = \int d\Omega(p') \sum_{ss'} D^{1/2}_{s's}(W(\Lambda, p'))$$
$$\times \{f^0_{s+}(\Lambda^{-1}p')u_{s'}(p')e^{-ip'x'} + f^0_{s-}(\Lambda^{-1}p')^* v_{s'}(p')e^{ip'x'}\}, \tag{4.7}$$

where $p' = \Lambda p$ and $W(\Lambda, p') \in SU(2)$ is the Wigner rotation.

Let us now consider a time-dependent external potential $A^\mu(x)$, which transforms under Lorentz transformations as a 4-vector

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x'). \tag{4.8}$$

To simplify the discussion we assume $A^\mu(x)$ to be different from 0 only in the forward light-cone. Then there exist solutions $u_s(p, x)$, $v_s(p, x)$ of the Dirac equation with $A^\mu(x)$, defined uniquely by the initial conditions

$$u_s(p, x) = u_s(p)e^{-ipx}, \qquad v_s(p, x) = v_s(p)e^{ipx} \tag{4.9}$$

for $t \le 0$ and similarly for $A'^\mu(x')$:

$$u'_{s'}(p', x') = u_{s'}(p')e^{-ip'x'}, \qquad v'_{s'}(p', x') = v_{s'}(p')e^{ip'x'} \qquad \text{for} \quad t' \le 0. \tag{4.10}$$

The covariance of the Dirac equation with potential together with (4.9) imply the same transformation property as in the free case

$$S(\Lambda)u_s(p, \Lambda^{-1}x') = \sum_{s'} D^{1/2}_{s's}(W(\Lambda, p'))u'_{s'}(p', x'), \qquad \text{etc.,} \tag{4.11}$$

which leads to the following covariance formula for the test functions $f(x)$ (4.2)

$$(V(\Lambda)f)(x') = S(\Lambda)f(\Lambda^{-1}x')$$
$$= \int d\Omega(p') \sum_{s's} D^{1/2}_{s's}(W(\Lambda, p'))\{f_{s+}(\Lambda^{-1}p')u'_{s'}(p', x')$$
$$+ f_{s-}(\Lambda^{-1}p')v'_{s'}(p', x')\}. \tag{4.12}$$

If we take $t_0 = 0$ in equations (4.1), (4.2), then

$$f_{s+}(p) = \int u_s(p; 0, \mathbf{x})^+ f_0(\mathbf{x})\, d^3x = f^0_{s+}(p),$$

which we will assume from now on. $V(\Lambda)$ can be expressed in terms of the unitary propagators U, U' of the original and transformed Dirac equation as follows

$$V(\Lambda)f = U'(t', 0) \int d\Omega(p') \sum_{ss'} D^{1/2}_{s's}$$
$$\times \{f_{s+}(\Lambda^{-1}p')u'_{s'}(p'; 0, \mathbf{x}') + f_{s-}(\Lambda^{-1}p')v'_{s'}(p'; 0, \mathbf{x}')\}$$
$$= U'(t', 0)(V_0(\Lambda)f^0)(0, \mathbf{x}')$$
$$= U'(t', 0)(V_{00}(\Lambda)f_0)(\mathbf{x}') \tag{4.13}$$

where

$$(V_{00}(\Lambda)f_0)(\mathbf{x}') = V_0(\Lambda)(U_0(\cdot, 0)f_0)(0, \mathbf{x}'). \tag{4.14}$$

V_{00} maps the test function $f_0(\mathbf{x})$ on the plane $t = 0$ into the plane $t' = 0$ according to the free time evolution and Lorentz covariance. This is a unitary operator with respect to the invariant scalar product

$$(f, g)_\sigma = \int_\sigma d\sigma^\mu(x)\overline{f(x)}\gamma_\mu g(x), \tag{4.15}$$

where σ is any space-like hypersurface.

The restriction of $V(\Lambda)$ to the planes $t = t_0$ and $t' = t_0'$ defines a unitary operator $W(\Lambda)$

$$(W(\Lambda)f_0)(\mathbf{x}) = (V(\Lambda)f)(t_0', \mathbf{x}) \tag{4.16}$$

with

$$f(t, \mathbf{x}) = (U(t, t_0)f_0)(\mathbf{x}).$$

Because of (4.13), $W(\Lambda)$ can be written as the following product of unitary operators

$$W(\Lambda) = U'(t_0', 0)V_{00}(\Lambda)U(0, t_0). \tag{4.17}$$

The action of $W(\Lambda)$ on the test functions for $t = t_0$ gives rise to a $*$-automorphism α of the abstract field algebra $\mathscr{A}$:

$$\begin{aligned}\alpha(\Lambda)\circ(\psi_{t_0}(f_0)) &\overset{\text{def}}{=} \psi((WU(t_0, 0))^+ f_0)\\ &= \psi(U(0, t_0)W^+ f_0) = \psi_{t_0}(W^+ f_0).\end{aligned} \tag{4.18}$$

This is again a Bogoliubov automorphism and we can apply Theorem 2 (2.29) to decide whether it can be unitarily implemented in a Fock representation defined by projectors $P_\pm$. This is in general impossible. To see this explicitly, let us consider a purely electric field, smoothly switched on, which belongs for every time t to the class of regular potentials, studied in Ref. [7]. Then because of Theorem 3, the time evolution can be described for all t in the in-representation $P_\pm = P_\pm^0$. For the Lorentz automorphism (4.18) we have to examine

$$\begin{aligned}P_+^0 W P_-^0 = {} & P_+^0 U'(t', 0)P_+^0 V_{00}(\Lambda)U(0, t)P_-^0\\ & - P_+^0 U'(t', 0)P_-^0 V_{00}(\Lambda)U(0, t)P_+^0\\ & + P_+^0 U'(t', 0)P_-^0 V_{00}(\Lambda)U(0, t).\end{aligned} \tag{4.19}$$

According to Theorem 3, the propagators in (4.18) can be substituted by the corresponding static propagators. Then the first two operators in (4.19) are H.S., since.

$$P_+^0 V_{00}(\Lambda)U(0, t)P_-^0 = V_{00}(\Lambda)P_+^0 U(0, t)P_-^0 \in \text{H.S.}\,.$$

But the propagator U' corresponds to a non-vanishing magnetic field which is not in the regular class according to Theorem 1 in Ref. [7]. Consequently, the last term in (4.19) is not H.S., which means that $\alpha(\Lambda)$ (4.17) is not implementable.

We therefore are forced to adapt the more general point of view described at the end of Section 2, changing the representation under Lorentz transformation. If we start with the interpolating in-representation defined by

$$P_\pm(t) = U(t, 0)P_\pm^0 U(0, t), \tag{4.20}$$

a Lorentz transformation Λ causes a change to

$$P'_{\pm}(t') = W(\Lambda)P_{\pm}(t)W(\Lambda)^{-1} = U'(t', 0)P^0_{\pm}U'(0, t'), \tag{4.21}$$

which is just the interpolating in-representation for the transformed reference system. This covariance property one must have in a physically acceptable particle picture. On the other hand, if we start with a static representation $P_{\pm}$, the transformed representation

$$P'_{\pm} = W(\Lambda)P_{\pm}W(\Lambda)^{-1} \tag{4.22}$$

is not the static representation $(\tilde{P}_{\pm})$ in the transformed system! It is true that two are equivalent because of Theorem 3, but the dressing transformation is non-trivial $(\tilde{P}_+P'_- \neq 0,\ P'_+\tilde{P}_- \neq 0)$. That means, the transformed vacuum Ω' defined by $P'_{\pm}$ (4.22) is a many-particle state in the static representation $(\tilde{P}_{\pm})$. This is completely unphysical. For this reason, we must reject the static representation as a particle picture in the case of time-dependent external fields. The static representation does have a physical significance for time-independent (static) fields (Furry picture), since then a distinct reference frame exists in which the external field is static. But in the general time-dependent situation, there is no way to distinguish one static representation from the other.

We arrive at the conclusion that the only satisfactory particle interpretations are the in- and out-representations (or interpolating in- and out-representations in the Schrödinger picture) in general. This is in accordance with the conservative point of view that the notion of particles has only an asymptotic meaning. Concerning the problem of strong time-dependent fields, this implies that critical fields must be defined by the change in the structure of the S-matrix, as discussed in Section 2, and not by a 'diving' of an eigenvalue of the static Dirac Hamiltonian into the continuum.

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