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Exact time dependent probability density for a non-linear non-markovian stochastic process

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Abstract. The influence of an external colored noise on the evolution of non-linear stochastic processes is studied by means of a model. We obtain analytically the time dependent probability density of the process and discuss the evolution as a function of the covariance parameter λ of the applied noise. It is shown that the parameter λ plays a major role (bifurcation parameter) and, depending on its value, drives the system into markedly different stationary states.

1. Introduction

The phase transitions induced by the presence of fluctuating surroundings have recently gained much interest in the study of simple dynamical models. It is known that systems subject to white noise, the amplitude of which is itself controlled by the macro-variables, exhibit probabilistic behaviours not predictable from a deterministic analysis [1, 2]. The white noise process, being very erratic in its nature, may not, in certain cases, provide a good modelization of the real world. It is then worthwhile to study the dynamics of systems in the presence of colored noise stochastic processes, characterized by a finite correlation time λ^{-1}. The parameter λ can play a determinant role in the evolution of the system. This point has recently been discussed by Horsthemke [3]. In this paper, the stationary state reached by the system is approximately calculated and is shown to be drastically dependent on λ. We address ourselves to this question and provide a non-linear model for which the exact time dependent probability density can be calculated. In our example, the stationary state exhibits qualitatively the same behaviour than the model used by Horsthemke. The coincidence is in fact not accidental since, in the vicinity of the origin, the two models are identical.

We finally consider the white noise limit ($\lambda \to \infty$) and remark that the stationary probability density is the same as the one found when the Stratonovich prescription is used to calculate the stochastic integrals. This last property can be seen as an explicit illustration of the Wong, Zakai and Clark's theorem [4].

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2. The model

The model we propose to study reads:

$$\frac{dx}{dt} = -\alpha \operatorname{tgh}(\gamma x) - \frac{\beta}{\cosh(\gamma x)} w_t, \qquad x \in \mathbf{R}; \quad w_t \in \mathbf{R} \tag{1}$$

where α, β, γ are positive constants and w_t is the colored noise stochastic process defined by:

$$\langle w_t \rangle = 0; \qquad \langle w_t, w_{t'} \rangle = \frac{\mu^2 \lambda}{2} \exp\{-\lambda \, |t-t'|\} \tag{2}$$

μ and λ are fixed positive constants.

When $\beta = 0$, we can write the general deterministic solution of equation (1) in the form:

$$x(t) = \gamma^{-1} \operatorname{arsinh}\{(\sinh \gamma x_0) e^{-\alpha \gamma t}\} \tag{3}$$

where x_0 is the initial value. We conclude immediately from (3) that $x = 0$ is a stable solution for any initial value x_0.

When $\beta \neq 0$, the stochastic process (1) is non-markovian. This property is due to the finite correlations of w_t.

It is very useful to note that w_t can itself be generated by a stochastic differential equation [4];

$$dw = -\lambda w \, dt + \mu \lambda \, d\sigma_{,t} \qquad w \in \mathbf{R} \tag{4}$$

The notation $d\sigma_{,t}$ stands for the formal differential of the Wiener process $\sigma_{,t}$ and models therefore a white noise stochastic process. In order to get (2), we impose in (4) the initial value $\tilde{w}_0$ to be normally distributed with zero mean and variance $\mu^2\lambda/2$. We adopt the notation:

$$\tilde{w}_0 = \mathcal{N}\left(0, \frac{\mu^2 \lambda}{2}\right) \tag{5}$$

According to equation (4), we now rewrite the stochastic process (1) in the form*:

$$\begin{cases} dx = \left(-\alpha \operatorname{tgh}(\gamma x) - \dfrac{\beta}{\cosh(\gamma x)} w_t\right) dt & \text{(6a)} \\ dw = -\lambda w \, dt + \mu \lambda \, d\sigma_{,t} & \text{(6b)} \end{cases}$$

The pair (x, w) appearing in the equations (6a, b) constitutes now a markovian process whose realizations take place in $\mathbf{R}^2$.

Owing to the markovicity, we can write the Fokker–Planck equation (F.P.E.) associated to (6a, b). This equation reads:

$$\frac{\partial \mathrm{P}(x, w, t)}{\partial t} = \mathscr{F} \mathrm{P}(x, w, t); \qquad \iint_{\mathbf{R}^2} \mathrm{P}(x, w, t) \, dx \, dw = 1 \tag{7}$$

* Apart from its degeneracy, the diffusion process (6) is closely related to San-Miguel's class of models [6].

and $\mathcal{F}$ stands for the F.–P. operator which in this case takes the form:

$$\mathcal{F} = -\frac{\partial}{\partial x}\left[-\alpha \operatorname{tgh}(\gamma x) - \frac{\beta w}{\cosh(\gamma x)}\right] - \frac{\partial}{\partial w}[-\lambda w] + \frac{\mu^2\lambda^2}{2}\frac{\partial^2}{\partial w^2} \tag{8}$$

In order to solve (6), we introduce, following Chandrasekar [5], a one to one mapping T from $\mathbf{R}^2$ into $\mathbf{R}^2$;

$$T:(x, w) \mapsto (I_1, I_2), \tag{9}$$

where,

$$I_1 = e^{\alpha\gamma t}\left[\sinh(\gamma x) + \frac{\beta\gamma}{\alpha\gamma - \lambda} w\right]$$

$$I_2 = e^{\lambda t} w$$

I_1 and I_2 are two integrals of the deterministic ($\mu = 0$) system (6a, b).
In terms of the new coordinates (I_1, I_2) we have:

$$\iint_{\mathbf{R}^2} \mathrm{P}(x, w, t)\, dx\, dw = \iint_{\mathbf{R}^2} \hat{\mathrm{P}}(I_1, I_2, t)\, dI_1\, dI_2$$

$$= \iint_{\mathbf{R}^2} \hat{\mathrm{P}}(I_1(x, w, t), I_2(w, t), t)\frac{\partial(I_1, I_2)}{\partial(x, w)}\, dx\, dw = 1 \tag{10}$$

where

$$\frac{\partial(I_1, I_2)}{\partial(x, w)} = \gamma \cosh(\gamma x) e^{(\alpha\gamma+\lambda)t}$$

stands for the Jacobian of the mapping T.
From (10), we deduce straightforwardly:

$$T: P(x, w, t) \mapsto \hat{P}(I_1, I_2, t)\frac{\partial(I_1, I_2)}{\partial(x, w)} \tag{11}$$

and the differential operators appearing in (7) transform like:

$$\begin{aligned}
&T:\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} + \alpha\gamma I_1 \frac{\partial}{\partial I_1} + \lambda I_2 \frac{\partial}{\partial I_2}\\
&T:\frac{\partial}{\partial x} \mapsto \gamma\sqrt{1 + \left(e^{-\alpha\gamma t} I_1 - \frac{\beta\gamma}{\alpha\gamma - \lambda} I_2 e^{-\lambda t}\right)^2}\, e^{\alpha\gamma t}\frac{\partial}{\partial I_1}\\
&T:\frac{\partial}{\partial w} \mapsto \frac{\beta\gamma}{\alpha\gamma - \lambda} e^{\alpha\gamma t}\frac{\partial}{\partial I_1} + e^{\lambda t}\frac{\partial}{\partial I_2}\\
&T:\frac{\partial^2}{\partial w^2} \mapsto \left(\frac{\beta\gamma}{\alpha\gamma - \lambda}\right)^2 e^{2\alpha\gamma t}\frac{\partial^2}{\partial I_1^2} + \frac{2\beta\gamma e^{(\alpha\gamma+\lambda)t}}{\alpha\gamma - \lambda}\frac{\partial^2}{\partial I_1\, \partial I_2} + e^{2\lambda t}\frac{\partial^2}{\partial I_2^2}
\end{aligned} \tag{12}$$

Introducing (11) and (12) in the F.–P.E. (7) we end up with:

$$\frac{\partial \hat{P}(I_1, I_2, t)}{\partial t} = \frac{\mu^2\lambda^2}{2}\left[\left(\frac{\beta\gamma}{\alpha\gamma-\lambda}\right)^2 e^{2\alpha\gamma t}\frac{\partial^2}{\partial I_1^2} + \frac{2\beta\gamma}{\alpha\gamma-\lambda}\, e^{(\alpha\gamma+\lambda)t}\frac{\partial^2}{\partial I_1\,\partial I_2} + e^{2\lambda t}\frac{\partial^2}{\partial I_2^2}\right]\hat{P}(I_1, I_2, t) \quad (13)$$

The readily normalized solution of (13) reads [5]:

$$\hat{P}(I_1, I_2, t) = (2\pi\sqrt{\Delta})^{-1}\exp\left\{-\frac{A}{2\Delta}(I_1-I_{10})^2 - \frac{H}{\Delta}(I_1-I_{10})(I_2-I_{20}) - \frac{B}{2\Delta}(I_2-I_{20})^2\right\} \quad (14)$$

where:

$$\begin{aligned} A &= \frac{\mu^2\lambda}{2}(e^{2\lambda t}+C_1)\\ B &= \frac{\mu^2\lambda^2\beta^2\gamma^2}{2\alpha\gamma(\alpha\gamma-\lambda)^2}(e^{2\alpha\gamma t}+C_3)\\ H &= \frac{-\mu^2\lambda^2\beta\gamma}{(\alpha\gamma)^2-\lambda^2}(e^{(\alpha\gamma+\lambda)t}+C_2)\\ \Delta &= AB-H^2 \end{aligned} \quad (15)$$

and I_{10}, I_{20}, C_1, C_2, C_3 are constants to be fixed by the particular initial conditions of the problem.

The general, normalized solution of (7) then reads:

$$\begin{aligned} P(x, w, t) &= (2\pi\sqrt{\Delta})^{-1}\gamma\cosh(\gamma x)e^{(\alpha\gamma+\lambda)t}\\ &\times\exp\left\{-\frac{A}{2\Delta}\left[e^{\alpha\gamma t}\left(\sinh(\gamma x)+\frac{\beta\gamma w}{\alpha\gamma-\lambda}\right)-\left(\sinh(\gamma x_0)+\frac{\beta\gamma w_0}{\alpha\gamma-\lambda}\right)\right]^2\right.\\ &-\frac{H}{\Delta}\left[e^{\alpha\gamma t}\left(\sinh(\gamma x)+\frac{\beta\gamma w}{\alpha\gamma-\lambda}\right)-\left(\sinh(\gamma x_0)+\frac{\beta\gamma w_0}{\alpha\gamma-\lambda}\right)\right][we^{\lambda t}-w_0]\\ &\left.-\frac{B}{2\Delta}[we^{\lambda t}-w_0]^2\right\} \end{aligned} \quad (16)$$

Beside (5), which is fixed, we shall choose the following initial conditions:

$$x = 0 \quad \text{at time } t = 0 \quad (17)$$

and

$$P(x, w, t=0) = \delta(x)\mathcal{N}\left(0, \frac{\mu^2\lambda}{2}\right) = P_{Mx}(x, t=0)P_{Mw}(w, t=0) \quad (18)$$

where we have introduced the marginal densities P_{Mx}, P_{Mw} defined by:

$$\begin{aligned} P_{Mx}(x, t) &= \int_{\mathbf{R}} P(x, w, t)\,dw\\ P_{Mw}(w, t) &\int_{\mathbf{R}} P(x, w, t)\,dx \end{aligned} \quad (19)$$

According to (9), the conditions (5) and (17) reduce to

$$I_{10}=0 \quad \text{and} \quad I_{20}=0 \tag{20}$$

With (20), the total density (16) simplifies somewhat to finally give:

$$P(x, w, t)=\gamma \cosh(\gamma x) e^{(\alpha\gamma+\lambda)t}(2\pi\sqrt{\Delta})^{-1} \exp\left\{-\sum_1 (\sinh(\gamma x))^2 - \sum_2 \sinh(\gamma x) w - \sum_3 (w)^2\right\} \tag{21}$$

where

$$\sum_1 = (2\Delta)^{-1} A e^{2\alpha\gamma t} \tag{22a}$$

$$\sum_2 = [\Delta(\alpha\gamma-\lambda)]^{-1} A\beta\gamma e^{2\alpha\gamma t} + (\Delta)^{-1} H e^{(\alpha\gamma+\lambda)t} \tag{22b}$$

$$\sum_3 = [2\Delta(\alpha\gamma-\lambda)^2]^{-1} A\beta^2\gamma^2 e^{2\alpha\gamma t} + [\Delta(\alpha\gamma-\lambda)]^{-1} H\beta\gamma e^{(\alpha\gamma+\lambda)t} + (2\Delta)^{-1} B e^{2\lambda t} \tag{22c}$$

With the help of (21) and (22), we calculate the marginal densities:

$$P_{Mx}(x, t)=[2\pi\phi(\vec{\xi}, t)]^{-1/2}\gamma \cosh(\gamma x) \exp\{-[\sinh(\gamma x)]^2[2\phi(\vec{\xi}, t)]^{-1}\} \tag{23}$$

$$P_{Mw}(w, t)=[2\pi\psi(\vec{\xi}, t)]^{-1/2} \exp\{-w^2(2\psi(\vec{\xi}, t))^{-1}\} \tag{24}$$

where the vector $\vec{\xi}$ stands for the set $(\alpha, \beta, \gamma, \mu, \lambda)$ and:

$$\phi(\vec{\xi}, t)=(\alpha\gamma-\lambda)^{-2}\beta^2\gamma^2 e^{-2\lambda t} A + (\alpha\gamma-\lambda)^{-1} 2\beta\gamma e^{-(\alpha\gamma+\lambda)t} H + e^{-2\alpha\gamma t} B \tag{25}$$

$$\psi(\vec{\xi}, t)=A e^{-2\lambda t} \tag{26}$$

According to (26), the condition $P_{Mw}(w, t=0)=\mathcal{N}\left(0, \frac{\mu^2\lambda}{2}\right)$ takes the simple form

$$C_1=0 \tag{27}$$

Using (27), (18) and (22b) at time $t=0$, we obtain

$$C_2=\frac{\alpha\gamma+\lambda}{2\lambda}-1 \tag{28}$$

Finally $P_{Mx}(x, t=0)=\delta(x)$ implies $\phi(\vec{\xi}, t=0)=0$ which in view of (25) at time $t=0$, gives:

$$C_3=-1+\frac{\alpha\gamma}{\lambda} \tag{29}$$

Using (27), (28) and (29), the time dependent probability density of the non-markovian process x_t defined in (1) reads:

$$P_{Mx}(x, t)=[2\pi\hat{\phi}(\vec{\xi}, t)]^{-1/2}\gamma \cosh(\gamma x) \exp\{-[\sinh(\gamma x)]^2[2\hat{\phi}(\vec{\xi}, t)]^{-1}\} \tag{30}$$

and:

$$\begin{aligned}\hat{\phi}(\vec{\xi}, t)=&[2\alpha\gamma(\alpha\gamma+\lambda)]^{-1}\mu^2\lambda\beta^2\gamma^2 \\ &+(\alpha\gamma-\lambda)^{-2}\mu^2\lambda^2\beta^2\gamma^2\{[(2\lambda)^{-1}-(2\alpha\gamma)^{-1}]e^{-2\alpha\gamma t} \\ &+[2(\alpha\gamma+\lambda)^{-1}-\lambda^{-1}]e^{-(\alpha\gamma+\lambda)t}\}\end{aligned} \tag{31}$$

3. Discussion

We can study the shape of (30) as a function of $\hat{\phi}(\vec{\xi}, t)$. Let us then calculate the positions of the extrema which are given by the equation.

$$\frac{\partial P_{Mx}(x, t)}{\partial x} = \gamma^2 \frac{\sinh(\gamma x)}{\sqrt{2\pi\hat{\phi}(\vec{\xi}, t)}} \left\{1 - \frac{(\cosh(\gamma x))^2}{\hat{\phi}(\vec{\xi}, t)}\right\} \exp\left\{-\frac{(\sinh(\gamma x))^2}{2\hat{\phi}(\vec{\xi}, t)}\right\} = 0 \tag{31}$$

The solutions of (31) are:

$$\begin{aligned} &x_1 = 0 \qquad \forall \hat{\phi}(\vec{\xi}, t) \\ &x_{2,3}(t) = \mp\gamma^{-1} \operatorname{arcosh}(\sqrt{\hat{\phi}(\vec{\xi}, t)}), \quad \text{if} \quad \hat{\phi}(\vec{\xi}, t) > 1 \end{aligned} \tag{32}$$

The second derivative indicates if we deal with minima or maxima. We have:

$$\begin{aligned} \frac{\partial^2 P_M(x, t)}{\partial x^2} = &\frac{\gamma^3}{\sqrt{2\pi\hat{\phi}(\vec{\xi}, t)}} \cdot \cosh(\gamma x) \left\{\left[1 - \frac{(\cosh(\gamma x))^2}{\hat{\phi}(\vec{\xi}, t)}\right]\left[1 + \frac{1-(\cosh(\gamma x))^2}{\hat{\phi}(\vec{\xi}, t)}\right]\right. \\ &\left. + \frac{2[1-(\cosh(\gamma x))^2]}{\hat{\phi}(\vec{\xi}, t)}\right\} \exp\left\{-\frac{(\sinh(\gamma_x))^2}{2\hat{\phi}(\vec{\xi}, t)}\right\} \end{aligned}$$

So we obtain immediately:

$$\begin{aligned} &\left.\frac{\partial^2 P_M(x, t)}{\partial x^2}\right|_{x_1} \sim \gamma^3(2\pi\hat{\phi}(\vec{\xi}, t))^{-1/2}\{1-[\hat{\phi}(\vec{\xi}, t)]^{-1}\} \begin{cases} <0, & \text{if} \quad \hat{\phi}(\vec{\xi}, t) < 1 \\ >0, & \text{if} \quad \hat{\phi}(\vec{\xi}, t) > 1 \end{cases} \\ &\left.\frac{\partial^2 P_M(x, t)}{\partial x^2}\right|_{x_{2,3}} \sim \gamma^3(2\pi)^{-1/2}\{2[\hat{\phi}(\vec{\xi}, t)]^{-1} - 2\} < 0, \quad \text{for} \quad \hat{\phi}(\vec{\xi}, t) > 1 \end{aligned} \tag{34}$$

The situation is completely sketched in Fig. 1.

For the stationary state, we have:

$$\lim_{t\to\infty} \hat{\phi}(\vec{\xi}, t) = (2\alpha\gamma[\alpha\gamma + \lambda])^{-1}\mu^2\lambda\beta^2\gamma^2 = \hat{\phi}_\infty \tag{35}$$

Therefore if $\hat{\phi}_\infty > 1$, we shall have two maxima, corresponding to a bimodal density and in the contrary only one maximum centered at $x = 0$. When α, β, γ, μ are fixed, λ becomes the control parameter of a bifurcation problem and similarly

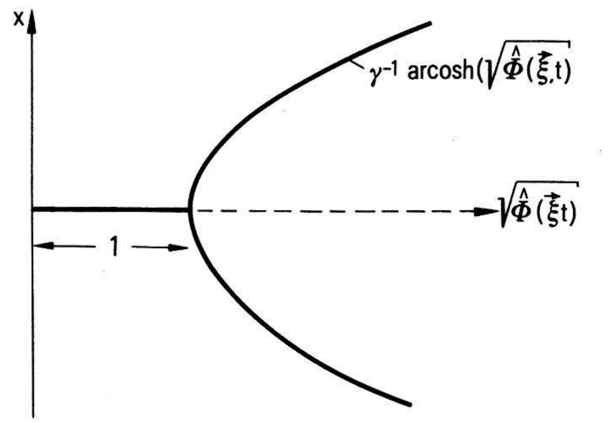

Figure 1

Position x of the extrema of P_{Mx} as a function of $\sqrt{\hat{\phi}(\vec{\xi}, t)}$ (—— maximum; --- minimum).

to Fig. 1, we can draw, for $\sqrt{\hat{\phi}_\infty}$, a global bifurcation diagram. The stationary state exhibits qualitatively the same behaviour as observed by Horsthemke [3], for the system:

$$\frac{dx}{dt} = -x + (x^2 - \tfrac{1}{4})w; \qquad x \in [-\tfrac{1}{2}, +\tfrac{1}{2}] \tag{36}$$

This similarity is not accidental. Indeed, the second order expansion of (1) in the vicinity of $x = 0$ coincides with (36) when $\alpha = \frac{1}{2}$, $\beta = \frac{1}{4}$ and $\gamma = 2$.

Let us finally calculate the stationary state of (30) in the white noise limit ($\lambda \to \infty$). We have immediately:

$$\lim_{\substack{\lambda \to \infty \\ t \to \infty}} P_{Mx}(x, t) = \left(\pi \frac{\mu^2 \gamma \beta^2}{\alpha}\right)^{-1/2} \gamma \cosh(\gamma x) \exp\{-\alpha(\mu^2 \beta^2 \gamma)^{-1} (\sinh(\gamma x))^2\} \tag{37}$$

Dividing both sides of (4) by λ and taking the limit gives:

$$\lim_{\lambda \to \infty} w_t = \mu \, d\sigma_{,t} \tag{38}$$

Using (38), we rewrite the process (1) in the form of a stochastic differential equation:

$$dx = [-\alpha \operatorname{tgh}(\gamma x)] \, dt - \frac{\beta \mu \, d\sigma_{,t}}{\cosh(\gamma x)} \tag{39}$$

The diffusion term appearing in (39) is not constant and we therefore have to specify the interpretation of the stochastic integral.

If (39) is interpreted in its Itô form, the associated stationary F.–P.E. reads:

$$0 = \frac{\partial}{\partial x}(\alpha \operatorname{tgh}(\gamma x) P_I(x)) + \frac{\beta^2 \mu^2}{2} \frac{\partial^2}{\partial x^2} \frac{P_I(x)}{(\cosh(\gamma x))^2} \tag{40}$$

which solution reads:

$$P_I(x) \sim (\cosh(\gamma x))^2 \exp\left\{\frac{-\alpha}{\beta^2 \mu^2 \gamma} (\sinh(\gamma x))^2\right\}$$

If (39) is interpreted in the Stratonovich form, we include the fluctuation induced drift, and the F.–P.E. reads:

$$0 = \frac{\partial}{\partial x}\left[\alpha \operatorname{tgh}(\gamma x) - \frac{\beta^2 \mu^2}{4} \frac{\partial}{\partial x}\left(\frac{1}{(\cosh(\gamma x))^2}\right) P_s(x)\right] + \frac{\beta^2 \mu^2}{2} \frac{\partial^2}{\partial x^2} \frac{P_s(x)}{(\cosh(\gamma x))^2} \tag{41}$$

which solution reads:

$$P_s(x) \sim \cosh(\gamma x) \exp\left\{-\frac{\alpha}{\beta^2 \mu^2 \gamma} (\sinh(\gamma x))^2\right\} \tag{42}$$

The solution (42) is identical with (37) which shows that the Stratonovich interpretation has to be used in this case. This situation provides an illustration of the Wong, Zakai and Clark's theorem [4].

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