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A remark about weakly coupled one-dimensional Schrödinger operators

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Abstract. We discuss the asymptotic behavior of the ground state of weakly coupled one-dimensional Schrödinger operators for various potential classes, in particular long range potentials.

Our recent work [1] about spectral properties of the infinite harmonic crystal led us to this reconsideration of the weak coupling limit of one-dimensional Schrödinger operators. We show how the method of [1] can be used as an alternative, in momentum spaces to the position space methods used in [2] and [3]. As regards the fall-off of the potential at infinity the method allows for a unified treatment of all important cases. In particular, for applications to Schrödinger operators with magnetic fields, potentials that behave as $|x|^{-1}$ for large $|x|$ are of interest [4]. Moreover, we will disprove a conjecture in [2] concerning potentials that fall off as $|x|^{-\alpha}$, $0<\alpha<1$. We hope that this paper completes the overall picture of this subject. We recall that we study the bound states of

$$\frac{-d^2}{dx^2}+\lambda V \tag{1}$$

as $\lambda \downarrow 0$. The peculiarity of one dimension compared to three dimensions is that the ground state in one dimension is asymtotically separated from the other bound states while in three dimensions all bound states behave in the same way. However, even in one dimension this is only true if the potential falls off fast enough. The borderline is given by the $|x|^{-1}$-tail. Notice that $E<0$ is a bound state of (1) if and only if 1 is an eigenvalue of $-\lambda Q_E$ where λQ_E is in momentum space given by

$$\lambda(2\pi)^{-1/2}(p^2-E)^{-1/2}\hat{V}(p-p')(p'^2-E)^{-1/2} \tag{2}$$

where

$$\hat{V}(p)=(2\pi)^{-1/2}\int e^{ipx}V(x)\,dx \tag{2}$$

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In [2] and [3] the operator $K_E = +\lambda\,|V|^{1/2}(p^2-E)^{-1}V^{1/2}$ where $V^{1/2} = |V|^{1/2}(\mathrm{sgn}\ V)$ was studied. We remark that K_E and Q_E are *iso-spectral*, since $Q_E = AB$ and $K_E = BA$ where $A = (p^2-E)^{-1/2}V^{1/2}$ and $B = \lambda(\mathrm{sgn}\ V)A^+$. Thus $\sigma(K_E)\setminus\{0\} = \sigma(Q_E)\setminus\{0\}$ [6]. We introduce the following potential classes:

$$
\begin{aligned}
&(A): \quad \int (1+|x|)\,|V(x)|\,dx < \infty \\
&(B): \quad \int |V(x)|\,dx < \infty \\
&(C): \quad V = -\frac{a}{|x|+b} + W, \qquad a>0, \qquad b>0 \\
&(D): \quad V = -\frac{a}{|x|^\alpha} + W, \qquad a>0, \qquad 0<\alpha<1\,.
\end{aligned}
\tag{3}
$$

Conditions on W will be given later. Cases (A) and (B) will only be considered briefly for they have been studied extensively in [2] and [3]. Case (C) and (D) show some features that one could perhaps not expect at first sight. We will see that one must not interchange the limits $\lambda \downarrow 0$ and $a \downarrow 0$ in the asymptotic series. Most of our work will be devoted to case (D).

The main idea in this paper is a perturbation argument which we briefly recall here. We decompose Q_E as

$$Q_E = P_E + R_E \tag{4}$$

Where, in cases (A)–(C), $P_E = -\psi_E(\psi_E, \cdot)$ with $\|\psi_E\| \to \infty$ as $E\uparrow 0$. $\|R_E\|$ will stay bounded or blow up at a smaller rate than $\|\psi_E\|^2$ as $E\uparrow 0$ (so that $\|R_E\|/\|\psi_E\|^2 \to 0$). $1 \in \sigma(-\lambda Q_E)$ implies

$$\lambda(\psi_E, (1+\lambda R_E)^{-1}\psi_E) = 1$$

The leading order is implicity given by $\lambda\,\|\psi_E\|^2 = 1$. Perturbation theory tells us that $\|\lambda R_E\| < 1$, whenever E lies in an interval around the E-value given by the leading order (the length of the interval also being of this order). To find the higher order terms, we expand $(1+\lambda R_E)^{-1}$. Compared to [2] we avoid the use of $\det(1+\lambda Q_E)$.

Case (D) differs from this scheme in that P_E will be a compact operator whose eigenvalues spread apart at a larger rate than $\|R_E\|$ blows up.

Case (A). Since $\hat{V} \in C^1$ this case is completely analogous to the problem in [1].

If $\int V\,dx \neq 0$ we define

$$P_E = \frac{1}{\hat{V}(0)\sqrt{2\pi}}\,\frac{\hat{V}(p)\hat{V}(-p')}{(p^2-E)^{1/2}(p'^2-E)^{1/2}}\; E<0 \tag{6}$$

$$R_E = Q_E - P_E \tag{7}$$

If $\int V = 0$,

$$P_E = \frac{1}{\sqrt{2\pi}}\,\frac{\hat{V}(p)+\hat{V}(-p')}{(p^2-E)^{1/2}(p'^2-E)^{1/2}} \tag{8}$$

$V \in (A)$ implies $|R_0(p, p')| \leq \text{const.}/\max(|p|, |p'|)$ so that R_0 is bounded (see Appendix). Then $\|R_E\| \leq \|R_0\|$ $(E \leq 0)$ and as $\lambda \downarrow 0$, $-\lambda Q_E$ can only have eigenvalue $+1$ if $\int V \leq 0$. For $\int V = 0$ this follows as in [1]. Setting $\|\lambda P_E\| = 1$ we find in leading order

$$(-E)^{1/2} = -\frac{\lambda}{2} \int V(x)\, dx \tag{9}$$

in accordance with [2, 3].

Case (B). The same decompositions work. We only need remark that (H.S. = Hilbert Schmidt)

$$\|R_E\|_{\text{H.S.}} = E^{-1/2} o(E)$$

by dominated convergence, while $\|P_E\| = 0(E^{-1/2})$. Hence P_E will dominate R_E as $E \uparrow 0$. We do not give the details of the asymptotics in this case.

Case (C). Here $\hat{V}(p)$ is logarithmically divergent at the origin. Hence the decomposition (7) is no longer meaningful but this can be remedied. We assume for simplicity that $W \in L_1$ and $\hat{W}(0) \neq 0$. The right decomposition is

$$P_E = +\frac{a \ln \sqrt{-E}}{\pi (p^2 - E)^{1/2} (p'^2 - E)^{1/2}}, \qquad R_E = Q_E - P_E \tag{10}$$

where a is the constant in (C). $\|P_E\| \sim -(\ln \sqrt{-E})/\sqrt{-E}$ while $\|R_E\| \sim 1/\sqrt{-E}$. The latter fact follows from a consideration of $R_E(p, p')$ whose explicit form can easily be worked out. The first order correction to the eigenvalue $a \ln \sqrt{-E}/\sqrt{-E}$ of P_E is $(\psi_E, R_E \psi_E)$ where $\psi_E = (-E)^{1/4}/(p^2 - E)^{1/2} \pi^{1/2}$. We get

$$\sqrt{-E}(\psi_E, R_E \psi_E) = \sqrt{-E}(\psi_E, Q_E \psi_E) - \sqrt{-E}(\psi_E, P_E \psi_E)$$

$$= -a \ln \sqrt{-E} + \frac{E}{\pi \sqrt{2\pi}} \int \frac{\hat{V}(p - p')}{(p^2 - E)(p'^2 - E)}\, dp\, dp'$$

$$= -a \ln \sqrt{-E} + \frac{1}{2} \int \left(\frac{-a}{b + |x|} + W(x) \right) e^{-2\sqrt{-E}|x|}\, dx$$

$$\to -a(C - \ln(2b)) + \frac{1}{2} \int W(x)\, dx \equiv c \quad \text{as} \quad E \uparrow 0. \tag{11}$$

Here

$$C = \int_0^1 \frac{e^{-u} - 1}{u}\, du + \int_1^\infty \frac{e^{-u}}{u}\, du$$

is Euler's constant. To find the ground state we have to solve

$$-\lambda a \ln x - \lambda c = x, \qquad x = \sqrt{-E} \tag{12}$$

Iteration gives

$$\sqrt{-E} = +\lambda a \ln(1/\lambda) - \lambda a \ln\ln\left(\frac{1}{\lambda}\right) - \lambda(a \ln a + c)$$
$$+\lambda a \frac{\ln\ln(1/\lambda)}{\ln(1/\lambda)} - \frac{\lambda}{\ln(1/\lambda)}(a \ln(1/a) - c) + 0\left(\frac{\ln\ln(1/\lambda)}{\ln^2(1/\lambda)}\right) \tag{13}$$

These few terms are consistent with the first order perturbation of $P_E(11)$. We observe that as $a \downarrow 0$ the first three terms reduce to $-\lambda \int W\,dx$ in accordance with (9) but the first five terms go into $-\lambda(1-1/\ln(1/\lambda)) \int W\,dx$!

Case (D). It is crucial to write V in the form (D). Then, if $W=0$,

$$Q_E = (-E)^{(\alpha/2)-1}\, Q_{-1}, \qquad E<0. \tag{14}$$

by performing a dilation.

$$Q_1(p, p') = \frac{aC_\alpha}{(p^2-E)^{1/2}\,|p-p'|^{1-\alpha}\,(p'^2-E)^{1/2}}, \qquad E<0, \qquad 0<\alpha<1 \tag{15}$$

where $C_\alpha = \cos(\pi(1-\alpha)/2)\Gamma(1-\alpha)/\pi$. $-Q_{-1}$ is compact and positive. Let $(-Q_1/a)\psi_s = \sigma_s\psi_s$ $(s=0, 1, \ldots)$, $\sigma_0 > \sigma_1 > \cdots \|\psi_s\| = 1$. Denote by E_s^0 $(s=0, 1, \ldots)$ the bound states of $p^2 - \lambda a/|x|^\alpha$ in increasing order. Then

$$(-E_s^0)^{1/2} = (\lambda a \sigma_s)^{1/(2-\alpha)} \tag{16}$$

Adding $W \in (A)$ means perturbing Q_E by an operator whose norm is bounded by const. $E^{-1/2}$ as $E \uparrow 0$. This cannot affect the leading order (16). We denote by E_s the bound states of $p^2 - \lambda a/|x|^\alpha + \lambda W$. Then, $E_s = E$ obeys

$$(-E)^{1-(\alpha/2)} = a\lambda\sigma_s(\psi_s, (1-\lambda B)^{-1}\psi_s) \tag{17}$$

$$B = (1 - \lambda E^{(\alpha/2)-1} a \sum_{i \neq s} \sigma_i P_i)^{-1} \tilde{Q}_E \tag{18}$$

$$\tilde{Q}_E = -\frac{1}{\sqrt{2\pi}\sqrt{-E}} \frac{\hat{W}(\sqrt{-E}(p-p'))}{(p^2+1)^{1/2}(p'^2+1)^{1/2}} \tag{19}$$

$$P_i(\cdot) = \psi_i(\psi_i,). \tag{20}$$

$\tilde{Q}_E$ is related to Q_E in (2) by a dilation. The inverse in (17) is well defined for we need only consider $\lambda \approx E^{1-\alpha/2}/(\sigma_s a)$ corresponding to the leading order (16). Now

$$(-E_s)^{1-(\alpha/2)} = \lambda\sigma_s a + \lambda^2\sigma_s a(\psi_s, B\psi_s) + \lambda^3\sigma_s a(\psi_s, B^2(1-\lambda B)^{-1}\psi_s) \tag{21}$$

s even: Assuming $W \in (A)$ and $\hat{W}(0) \neq 0$

$$(\psi_s, B\psi_s) = (\psi_s, \tilde{Q}_E\psi_s) = \frac{-1}{\sqrt{-E}} \int W(x)\chi_s^2(x\sqrt{-E})\,dx$$
$$= \frac{-\chi_s^2(0)}{\sqrt{-E}} \int W(x)\,dx + 0(1). \tag{22}$$

χ_s was introduced as the Fourier inverse of $(p^2+1)^{-1/2}\psi_s$. We also have $(p^2-(1/\sigma_s)|x|^{-\alpha})\chi_s=-\chi_s$. $\chi_s(0)\neq 0$ by a Wronski argument. The 0(1) term follows from dominated convergence. The second term on the r.h.s. of (21) is $0(\lambda^{(3-2\alpha)/(2-\alpha)})+0(\lambda^2)$ while the third is $0(\lambda^{(4-3\alpha)/(2-\alpha)})$. It follows

$$(-E_s)^{1/2}=(a\lambda\sigma_s)^{1/(2-\alpha)}-\frac{\lambda\chi_s^2(0)}{2-\alpha}\int W(x)\,dx+0(\lambda^{5-3\alpha/2-\alpha}) \tag{23}$$

Similar as in (C), the second order term does not reduce to (9) as $a\downarrow 0$ although χ_s does not depend on a. To see this note that

$$\sqrt{2\pi}\chi_s(0)=((p^2+1)^{-1/2},\psi_s)\leq\sqrt{\pi}$$

by the Schwarz inequality. Thus

$$\frac{\chi_s^2(0)}{2-\alpha}\leq\frac{1}{2(2-\alpha)}<\frac{1}{2}.$$

s odd: Assuming $\int|W(x)|\,(1+x^2)\,dx<\infty$ we find

$$(\psi_s,B\psi_s)=-\sqrt{-E}(\chi_s'(0))^2\int W(x)x^2\,dx+0(1) \tag{24}$$

where $x_s'(0)\neq 0$ for s odd. The third term in (21) is $0(\lambda^3)$ if $\hat{W}'(0)\neq 0$ and smaller otherwise. Thus

$$(-E_s)^{1/2}=(a\lambda\sigma_s)^{1/(2-\alpha)}-\lambda^{(4-\alpha)/(2-\alpha)}\frac{(\sigma_s a)^{2/(2-\alpha)}}{2-\alpha}(\chi_s'(0))^2 \times\int x^2W(x)\,dx+0(\lambda^{5-\alpha/2-\alpha}) \tag{25}$$

Remarks. s odd also covers the three dimensional case for s-waves since the odd functions satisfy a Dirichlet boundary condition at the origin. One can make the limit $\alpha\uparrow 1$ in (25) using the fact that on odd functions $C_\alpha\,|p-p'|^{\alpha-1}$ goes over into $\log(|p-p'|/|p+p'|)$ which is familiar as the s-wave part of $|p-p'|^{-2}$.

In the very special case $W(x)=-W(-x)$, s odd,

$$(\psi_s,B\psi_s)=0$$

identically in E. We have to look at

$$\lambda^3\sigma_s a(\psi_s,B^2\psi_s).$$

If $\int xW(x)\,dx\neq 0$ this term approaches

$$-c\lambda^3\left(\int xW(x)\,dx\right)^2(\chi_s'(0))^2\sum_{k=0}^{\infty}\rho_{2k}(\chi_{2k}(0))^2 \tag{26}$$

as $E_s\uparrow 0$, where $\rho_{2k}=(1-(\sigma_{2k}/\sigma_s))^{-1}$. (26) would give the second order if it were *non-zero*. However there are always positive and negative terms in the sum which might cancel. We don't know the second order in this case.

The response of the asymptotic series to a replacement of $-a/|x|^\alpha$ by $-a/(|x|^\alpha+b)$, $b>0$, is surprising. It amounts to choosing

$$W=\frac{ab}{|x|^\alpha\,(|x|^\alpha+b)}$$

in (D). If $1>\alpha>1/2$ and s is even, the second order is still given by (22) and (23). If $0<\alpha<1/2$, $(\psi_s, B\psi_s)$ is $0((-E)^{\alpha-1})$ and the second order is (23) becomes $0(\lambda^{(\alpha+1)/(2-\alpha)})$. This accords with the limit $\alpha\downarrow 0$ $(b>0)$ where one expects $-E_0\sim 0(\lambda)$. If $\alpha=1/2$ we get an $0(\lambda \ln \lambda)$ term.

Similar effects occur if s is odd.

We conclude with the remark that the momentum space method also allows us to discuss the threshold behavior of potentials like $\sin|x|/|x|$. This has been discovered recently and will be discussed elsewhere.

Appendix

This appendix considers a more general threshold behavior than the weak coupling limit. We show how the methods of this paper can be used to give a quick proof of a well-known fact about the $\sim 1/1+x^2$-potential. The appendix also explains some steps in [1]. Let

$$H=p^2-\lambda/(d^2+x^2), \qquad d>0, \qquad \lambda>0.$$

We show

(i) H has *one* bound state if $\lambda\le 1/4$.
(ii) H has an infinite number of bound states if $\lambda>1/4$.

Since H serves parity, we consider H on the odd or even function separately. Accordingly, we introduce $Q_E^{\pm}$ $(+(-)=\text{even (odd)})$ and consider these operators on $L^2(R^+)$ (after an obvious unitary transformation). Then

$$Q_E^-(P,p')=-\frac{1}{2d}\frac{e^{-d|p-p'|}-e^{-d|p+p'|}}{(p^2-E)^{1/2}(p'^2-E)^{1/2}},\ p,p'\ge 0\ E\le 0. \tag{A.1}$$

$$Q_E^+-Q_E^-=-\frac{1}{2d}\frac{e^{-dp}e^{-dp'}}{(p^2-E)^{1/2}(p'^2-E)^{1/2}},\ E<0. \tag{A.2}$$

$Q_E^+-Q_E^-$ is a rank one operator whose negative eigenvalue goes to $-\infty$ as $E\uparrow 0$. Anticipating that Q_0^- is bounded, we note that $Q_E^-\to Q_0^-$ strongly and $\|Q_E^-\|\uparrow\|Q_0^-\|$. Moreover, Q_0^- gets unitarily transformed when d varies (dilation) and as $d\downarrow 0$

$$Q_0^-\to -A_0(p,p')=-1/\max(p,p') \tag{A.3}$$

strongly and $\|Q_0^-\|\le\|A_0\|$. The limit $d\downarrow 0$ serves as an auxiliary tool to discover the spectrum of Q_0^- from that of $-A_0$. Substituting $p\to e^s$ $(s\in(-\infty\,\infty))$ we can unitarily transform A_0 into the integral operator with kernel $\exp(-|x-y|/2)$ on $L^2(\mathbb{R})$. Hence $\sigma(A_0)=[0,4]=\sigma(Q_0^-)$ since, on the one hand, $\sigma(Q_0^-$ has to "fill up" $[0,4]$ as $d\downarrow 0$ (strong convergence) [5, p. 290]) and on the other $\sigma(Q_0^-)$ is invariant as d changes. (ii) follows now from the fact that $\dim(\text{RamP}_{(1,\infty)}(-Q_0^-))=\infty$ if $\lambda>1/4$. (i) follows since $(1,\infty)\in\rho(-Q_0^-)$ for $\lambda\le 1/4$ and since Q_E^+ differs from Q_E^- by a rank one operator which creates one single bound state.

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