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Equilibrium equations for classical systems with long range forces and application to the one dimensional Coulomb gas

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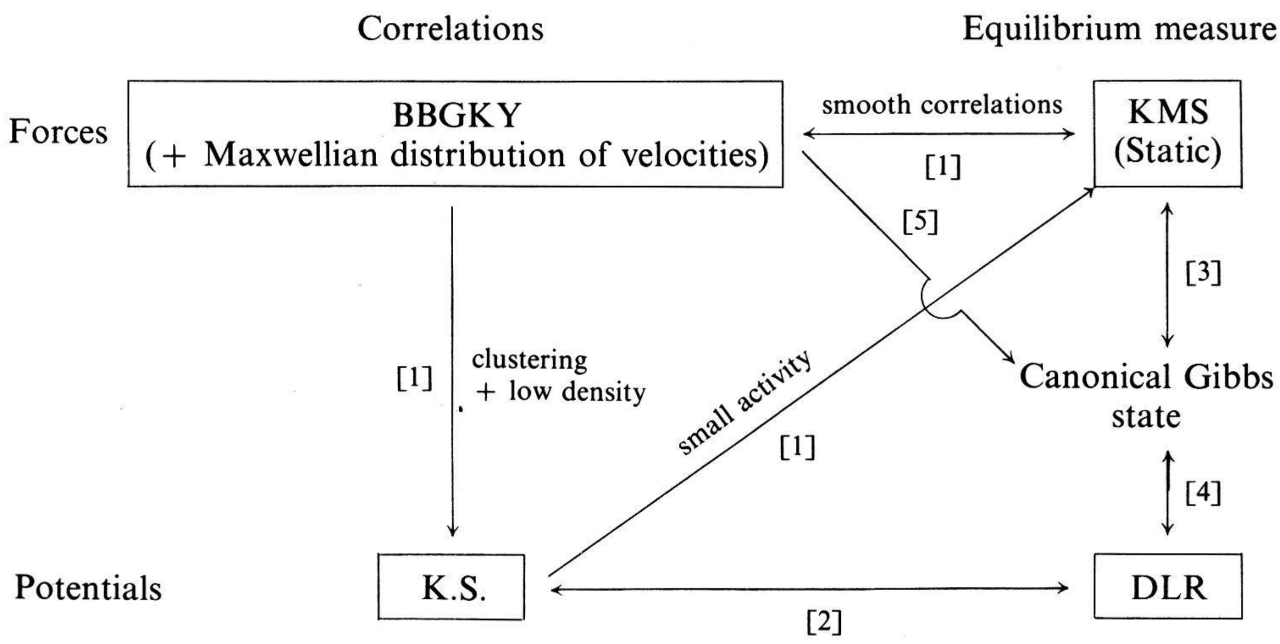
Abstract. We consider infinite classical systems with long range forces, i.e. forces which are not necessarily integrable at infinity, and propose a generalization of the classical KMS-condition to define equilibrium states. We then show the equivalence between this condition and the BBGKY-hierarchy for correlations, and discuss some consequences which are specific to long range forces. As an illustration, we use the method of functional integration to exhibit explicitly equilibrium states of the one dimensional Coulomb gas and give a new treatment of the one dimensional Jellium in the grand canonical formalism.

I. Introduction

This paper deals with the equations defining equilibrium states of infinite continuous classical systems with long range forces. By long range forces, we mean forces which decrease very slowly (or even do not decrease) at infinity. For instance the Coulomb force in one dimension is proportional to sign $(x\text{-}y)$, a quantity which is constant up to infinity. Precisely we shall call a force long range if it is *not integrable at infinity*. A short range force is not necessarily of finite range, but integrable at infinity.

The study of equilibrium states of infinite systems can usually be done in two ways. One can either define them by means of thermodynamic limit of finite volume Gibbs states with some boundary conditions, or one defines them as solutions of equilibrium equations which are assumed to be valid for infinite systems on reasonable physical grounds. In this work, we shall mainly be concerned with this second approach. There are four types of equations which have been proposed to define equilibrium states of classical systems. The two first ones, the static classical KMS condition [1] and the DLR [2] equations, are used to determine the equilibrium probability measure on infinite classical phase space. The two others, the BBGKY hierarchy and the Kirkwood Salzbourg (KS) equation, are equations for the correlation functions of the state. In the case of short range forces, the equivalence between these equations (under some smoothness assumptions on the potentials and correlation functions) is shown in the following diagram.



All these equations involve in one way or another the fact that the force (or the potential) exerted on a particle by the rest of the infinitely extended system is finite. It is clear that in presence of long range forces, this property is far from being obvious. Indeed one has to sum up the contributions to the force (or to the potential) of infinitely many other particles, a sum which is precisely not convergent. For instance in the one dimensional Coulomb gas the force on a particle of charge $+1$ at the origin due to an infinite configuration of particles

$$\{\sigma_i, y_i\}_{i=1}^{\infty}, \quad \sigma_i = \pm 1, y_i \in \mathbb{R}$$

is formally given by:

$$-\sum_{i=1}^{\infty} \sigma_i \operatorname{sign}(y_i)$$

i.e. an infinite sum of $+1$ and -1. Therefore the point is to find a natural generalization of the equilibrium equations by giving a prescription to make sense of such non convergent sums.

We notice that the KMS equation and the BBGKY hierarchy involve in their formulation the interparticle forces whereas DLR and KS are expressed in terms of the potentials (in this respect, DLR and KS can be viewed as integrated forms of KMS and BBGKY respectively). Moreover, the force is the basic physical quantity and it has certainly a better behaviour at infinity (faster decrease) than the potential. For this reason KMS or BBGKY lend themselves easier to the desired generalization, and we shall adopt these equations as the starting point of our investigation[1]). The generalization consists in assuming that KMS is asymptotically satisfied when the effects of the forces are first restricted to configurations enclosed in a sequence of finite

[1]) In DLR one would face the problem of giving a meaning to the interaction energy between the particles in a finite region with (almost all) external configurations. In KS one would have to deal with the non integrable kernel $(\exp(-\beta V(x)) - 1)$. We do not touch here the possibility of generalizing these two equations.

volumes V_λ which converges to the whole space as $\lambda \to \infty$ (for a precise formulation, see section 2.1). It is clear that this prescription can be dropped when the force is, say, of finite range. In this case, the particles in any finite region interact only with the particles located in some larger but still finite region. Thus, for λ large enough, our condition reduces to the usual KMS form. In the case of long range forces, a state verifying the generalized KMS condition may depend genuinely on the sequence V_λ: different sequences can distinguish different physical states. This is apparent in the one dimensional jellium which is treated in the second part of the paper. Therefore one has to consider such sequences of volume as participating to the determination of the various phases which can exist at a given temperature. We emphasize again that the prescription involving the sequence of volumes V_λ applies to the state of the infinite system, and must not be confused with the construction of such a state by taking the thermodynamic limit of finite Gibbs states. The relation between the two procedures is not studied here and deserves further investigation.

One has the following important feature which is specific to systems with long range forces.

According to our definition an equilibrium state is defined with respect to a *temperature T*, a *family* of *k-body forces* $\mathbf{F}^{(k)}$, $k = 1, \ldots, n < \infty$, and a *sequence of finite volumes* $V_\lambda \to \mathbb{R}^\nu$. It happens (see section 2.5) that the same state can also be an equilibrium state with respect to the same temperature T but for another family of forces $\mathbf{F}'^{(k)}$ together with a different sequence of volumes V'_λ (with however $\mathbf{F}'^{(n)} = \mathbf{F}^{(n)}$).

In the case of two body long range forces, this means that one body force has no definite meaning without the specification of the sequence V_λ. This feature reflects the fact that for a given state there are various ways of dealing with the particles at infinity, giving rise to different effective one body forces (because far away particles have still a direct influence everywhere locally). Similarly, three body long range forces will modify the two body part, and so on. Thus one must be aware that the external fields and the forces which define the equilibrium state of the infinite system state are not necessarily the fields and forces which define the Hamiltonian of the finite system. A concrete example of this situation will be found in [6].

We shall motivate our generalized KMS condition in two ways: Firstly, we show explicitly in the second part of the paper that it is satisfied by certain states of the one dimensional Coulomb gas (the two component plasma and the jellium). Therefore, our condition covers the simplest known examples of states with long range forces. Then, in another paper [7] we derive some remarkable consequences of the generalized KMS condition for ν dimensional Coulomb systems: the neutrality and the non normality of charge fluctuations.

The present paper is organized as follows. In the first part we state the generalized KMS condition and prove its equivalence with the BBGKY hierarchy (with the equivalent prescription for summing the long range forces). This is done in two steps. First we show that, irrespective of the nature of the forces, the KMS equation implies a Maxwellian velocity distribution. Therefore the configuration part and the velocity dependent part of the probability measure decouple completely. The KMS condition for the configuration part takes a simple form which, in turn, is seen to be equivalent with BBGKY. In establishing this equivalence, no smoothness assumptions are made neither on the forces (which can be n-body) nor on the correlation functions, and BBGKY is understood to hold in the sense of distributions. This result can be considered as an extension of part of reference [1]. Then we show that if the forces are of finite range and smooth, the correlation measures are in fact C^1-functions and

BBGKY holds in the usual sense. Finally, we discuss some aspects of the equilibrium equation which are intrinsically linked to long range forces.

In the second part of the paper we compute explicitly the density distribution functions of the one dimensional Coulomb gas. These density distributions are obtained as thermodynamic limits of grand canonical states, calculated by the method of the Wiener integral which has been used in [8]. Although the method of functional integration for Coulomb systems has been powerfully extended beyond one dimension [9, 10], we find instructive that the state of the one-dimensional gas can be explicitly exhibited and shown to verify the equilibrium equations. In particular, we give a treatment of the one dimensional jellium which is mathematically analogous to that of the two component system. This treatment differs from Kunz's study of the same system in several respects [11]: the finite system is not assumed to be neutral, the formalism is grand canonical and functional integration is essentially used. This state exhibits the same features as the state obtained in the canonical formalism: screening, and spatial periodicity.

II. Equilibrium equations for long range forces

2.1. Notation and definition of KMS-states

We consider an infinite classical system of N different types of particles without hard core, located in the physical space $\mathbb{R}^\nu$ and interacting by means of n-body potentials $\phi^{(n)}$, $n = 1, 2\ldots$

Let $\Gamma = \mathbb{R}^\nu \times \mathbb{R}^\nu \times \{1,\ldots, N\}$ be the one particle phase-space which is the space of variables $u = (p, q) = (p, x, \sigma)$.

The states of such systems are defined as (regular) probability measures on the phase space $\{\Omega, \mathfrak{B}\}$ where $\Omega = \{U = \{u_i\} \subset \Gamma;\quad u_i = (p_i, x_i, \sigma_i),\quad |x_i - x_j| > 0$ for all $i \neq j$, and for all bounded $\Lambda \subset \mathbb{R}^\nu$, $|U_\Lambda| < \infty\}$ with $U_\Lambda = \{u_i = (p_i, x_i, \sigma_i) \in U$, $x_i \in \Lambda\}$, $|U_\Lambda|$ = cardinality of the set U_Λ.

$\mathfrak{B}$ is the σ-algebra of subsets of Ω generated by means of the symmetric Borel subsets of

$$[\mathbb{R}^\nu \times \Lambda \times \{1,\ldots, N\}]^n_{\neq} = \{U^{(n)}\} = (u_1,\ldots, u_n) \in [\mathbb{R}^\nu \times \Lambda \times \{1,\ldots, N\}]^n; |x_i - x_j| > 0\}$$

where Λ will always denote a bounded Borel subset of $\mathbb{R}^\nu$ [3].

The *interaction* ϕ is defined by means of symmetric functions

$$\phi^{(n)} = \phi^{(n)}(q_1,\ldots, q_n) \text{ on } [\mathbb{R}^\nu \times \{1,\ldots, N\}]^n, \geq 1,$$

which are assumed to be of class C^1 on

$$[\mathbb{R}^\nu \times \{1,\ldots, N\}]^n_{\neq} = \{Q^{(n)} = (q_1,\ldots, q_n); |x_i - x_j| > 0\}$$

and which are zero except for finitely many n. The forces might be singular at points of coincident particles in physical space.

To define KMS-states we introduce the algebra $\tilde{\mathfrak{A}}^{(r)}$ of real functions on Ω which are given by:

$$f(U) = \sum_{k \geq 0} \sum_{\overline{U}^{(k)} \subset U} f_T^{(k)}(\overline{U}^{(k)}) \quad \text{or simply} \quad f = \mathscr{S} f_T$$

where $f_T^{(k)}$ are C^r-functions[2]) on Γ^k which are symmetric, bounded, have compact support in the Q variables and are identically zero except for finitely many k.

We introduce also the algebra $\mathfrak{A}^{(r)}$ generated by elements of the form: $\exp(i\mathscr{S}\vartheta)f$ with $f \in \tilde{\mathfrak{A}}^{(r)}$ and ϑ a C^∞-function on Γ with compact support in q. For any $\Lambda \subset \mathbb{R}$ we denote by χ_Λ the function

$$\chi_\Lambda(U) = \begin{cases} 1 & \text{if } U = U_\Lambda \\ 0 & \text{otherwise} \end{cases}$$

In the following definition we shall introduce the condition that $\mathfrak{A}^{(0)} \subset \mathscr{L}^1[\mu]$ for any KMS-state μ; it thus follows that any KMS-state μ defines unique, regular, symmetric, Borel measures, $\rho^{(k)}[dU^{(k)}] = \rho^{(k)}_{\sigma_1,\ldots,\sigma_k}[dP^k\, dX^k]$, $\sigma_i \in \{1,\ldots, N\}$, on each $[\mathbb{R}^\nu \times \mathbb{R}^\nu]^k$, called *Correlation Measures* and such that:

(1) for all

$$f = \mathscr{S} f_T \in \mathfrak{A}^{(0)}$$
$$\int_\Omega d\mu\, f = \int \rho[dU] f_T(U) \tag{1}$$

where

$$\int \rho[dU] f_T(U) = \sum_{k \geq 0} \frac{1}{k!} \int_{\Gamma^k} \rho^{(k)}[du_1,\ldots, du_k] f_T^{(k)}(u_1,\ldots, u_k)$$

(2) for all

$$f = \mathscr{S} f_T \in \mathfrak{A}^{(0)} \tag{2}$$
$$\|f_T\|_{\mathscr{L}^1[\rho]} \geq \|f\|_{\mathscr{L}^1[\mu]}.$$

Definition. A state μ is a *static KMS-state* with respect to the *interaction* ϕ, the *inverse temperature* β and the *sequence of volumes* $\{V_\lambda\}$, $V_\lambda \to \mathbb{R}^\nu$ if:

(C1) $\mathfrak{A}^{(0)} \subset \mathscr{L}^1[\mu]$.

(C2) The correlation measures satisfy:

(a) $$\int \rho^{(k)}[dU^{(k)}] |p_{i,\alpha}| \chi_{V\lambda}(U) < \infty$$

for all $i = 1,\ldots, k$, $\alpha = 1,\ldots, \nu$

(b) $$\int \rho^{(n+l)}[dU^{(n)}\, dV^{(l)}] \left| \frac{\partial \phi^{(n)}(q_1,\ldots, q_n)}{\partial x_{i,\alpha}} \right| \chi_{V_\lambda}(U^{(n)} V^{(l)}) < \infty$$

for all $i = 1,\ldots, n$, $\alpha = 1,\ldots, \nu$

(C3) for all h and g in $\mathfrak{A}^{(1)}$

$$\mu[\{h, g\}] = \beta \lim_{\lambda \to \infty} \mu[g\{h, H\}_{V_\lambda}]$$

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[2]) To simplify we use the same notation $U^{(k)}$ to denote k points subsets of Γ and to denote one point in Γ^k.

where $\{h, g\}$ denotes the Poisson Bracket given formally by

$$\{h, g\}(U) = \sum_{u_i \in U} \sum_{\alpha=1}^{\nu} \left[\frac{\partial h}{\partial x_{i,\alpha}} \frac{\partial g}{\partial p_{i,\alpha}} - \frac{\partial h}{\partial p_{i,\alpha}} \frac{\partial g}{\partial x_{i,\alpha}}\right](U)$$

and

$$\{h, H\}_V(U) = \{h, H\}(U_V) = \sum_{u_i \in UV} \sum_{\alpha=1}^{\nu} \left[\frac{\partial h(U_V)}{\partial x_{i,\alpha}} \frac{p_{i,\alpha}}{m} - \frac{\partial h(U_V)}{\partial p_{i,\alpha}} \sum_{\substack{\bar{U} \subset UV \\ \bar{U} \ni u_i}} \frac{\partial \phi(\bar{U})}{\partial x_{i,\alpha}}\right].$$

Remarks

(1) From conditions C1 and C2 follows that h and g in $\mathfrak{A}^{(1)}$ implies that $\{h, g\}$ and $g\{h, H\}_{V_\lambda}$ are in $\mathscr{L}^1[\mu]$ (see Appendix A). These conditions imply that for any finite volume, the expected number of particles, the absolute value of momentum, and the absolute value of the (cut-off) force are finite.

(2) If the potential has finite range, the condition C3 reduces to

$$\mu[\{h, g\}] = \beta\mu[g\{h, H\}]$$

which is the usual KMS-Condition.

(3) Introducing the *n-body force* $\mathbf{F}_{u_i}^{(n)}(U)$ on the particle u_i in the configuration $U \ni u_i$ as

$$F_{u_i,\alpha}^{(n)}(U) = -\sum_{\substack{\bar{U}^{(n)} \subset U \\ \bar{U}^{(n)} \ni u_i}} \frac{\partial \phi^{(n)}(\bar{U}^{(n)})}{\partial x_{i,\alpha}} \tag{3}$$

we have as soon as V_λ is sufficiently large

$$\{h, H\}_{V_\lambda}(U) = \sum_{u_i \in U} \sum_{\alpha=1}^{\nu} \left[\frac{\partial h(U)}{\partial x_{i,\alpha}} \frac{p_{i,\alpha}}{m} + \frac{\partial h(U)}{\partial p_{i,\alpha}} \sum_{n \geq 1} \sum_{\substack{\bar{U}^{(n)} \subset U \\ \bar{U}^{(n)} \ni u_i}} F_{u_i,\alpha}^{(n)}(\bar{U}^{(n)})\chi_{V_\lambda}(\bar{U}^{(n)})\right]. \tag{4}$$

Therefore, formally, the cut-off we have introduced to define KMS-states consists in replacing the force $\mathbf{F}_{u_i}^{(n)}(U^{(n)})$ on $u_i \in U^{(n)}$ by the force $\mathbf{F}_{u_i}^{(n)}(U^{(n)})\chi_{V_\lambda}(U^{(n)})$. We could have considered other types of cut-off functions $\chi_\lambda(\mathrm{U}^{(n)}) \to 1$; the characteristic function we have chosen appears to be the simplest one which yields the usual KMS-condition for finite range potentials.

(4) We shall need the algebra $\mathfrak{A}^{(1)}$ rather than $\tilde{\mathfrak{A}}^{(1)}$ only to derive the Maxwellian distribution of velocities.

We first show that the KMS condition remains true if only one of the functions h or g is symmetric. This property will be useful in the following.

Property 1. *Let $h = \mathscr{S}h_T$ where the $h_T^{(k)}$ are C^1-functions on Γ^k which are not necessarily symmetric, then for any KMS-state*

$$\mu[\{h, g\}] = \beta \lim_{\lambda \to \infty} \mu[g\{h, H\}_{V_\lambda}] \quad \text{for all } g \in \mathfrak{A}^{(1)}$$

Proof.

(i) By definition for a cylindrical function f with basis Λ which is not symmetric

$$\mu[f] = \sum_{k \geq 0} \frac{1}{k!} \int_{\Gamma^k} \mu_\Lambda^{(k)}[du_1, \ldots, du_k] f(u_1, \ldots, u_k)$$

where $\mu_\Lambda^{(k)}[du_1, \ldots, du_k]$ are the *density distributions* associated with Λ i.e.

$$\mu[f] = \mu[f_{sym}].$$

(ii) For any function g which is symmetric

$(fg)_{\text{sym}} = f_{\text{sym}} g$ and $\{f, g\}_{\text{sym}} = \{f_{\text{sym}}, g\}$.

(iii) For $h = \mathscr{S} h_T$ then $h_{\text{sym}} = \mathscr{S} h_{T,\text{sym}}$.

Therefore

$$\begin{aligned}
\mu[\{h, g\}] &= \mu[\{h, g\}_{\text{sym}}] = \mu[\{\text{h}_{\text{sym}}, g\}] = \mu[\{\mathscr{S} h_{T,\text{sym}}, g\}] \\
&= \beta \lim_{\lambda \to \infty} \mu[g\{\mathscr{S} h_{T,\text{sym}}, H\}_{V_\lambda}] \\
&= \beta \lim_{\lambda \to \infty} \mu[g\{h_{\text{sym}}, H\}_{V_\lambda}] \\
&= \beta \lim_{\lambda \to \infty} \mu[g\{h, H\}_{V_\lambda}].
\end{aligned}$$

2.2. *Maxwellian distribution of velocities*

In this section we show that any KMS-state has Maxwellian distributions of velocities.

Theorem 1. *For any KMS-state and any bounded open set Λ in $\mathbb{R}^\nu$ the density distributions $\mu_\Lambda^{(n)}$ are Maxwellian in the velocities, i.e.*

$$\mu_\Lambda^{(n)}[dU^{(n)}] = \prod_{i=1}^{n} \frac{\exp(-\beta p_i^2/2m)}{(2m\pi/\beta)^{\nu/2}} dp_i \tilde{\mu}_\Lambda^{(n)}[dQ^{(n)}].$$

Corollary. *For any KMS-state the correlation measures $\rho^{(k)}$ are Maxwellian in the velocities:*

$$\rho^{(k)}[dU^{(k)}] = \prod_{i=1}^{k} \frac{\exp(-\beta p_i^2/2m)}{(2m\pi/\beta)^{\nu/2}} dp_i \tilde{\rho}^{(k)}[dQ^{(k)}].$$

Proof. Let us first prove the corollary using Theorem 1.

For any $f = \mathscr{S} f_T^{(k)} \in \mathfrak{A}^{(0)}$ with $f_T^{(k)}(U^{(k)}) = \tilde{f}_T(Q^{(k)}) g_T(P^{(k)})$ where $\tilde{f}_T^{(k)}$ has support in $[\Lambda \times \{1, \ldots, N\}]^k$

$$\int_\Omega d\mu f = \sum_{n \geq 0} \frac{1}{n!} \int_{\Omega_\Lambda} \mu_\Lambda^{(n)} [dU^{(n)}] f(U^{(n)})$$

$$= \sum_{n \geq 0} \frac{1}{n!} \int_{\Omega_\Lambda} \tilde{\mu}_\Lambda^{(n)} [dQ^{(n)}] \prod_{i=1}^{n} \frac{\exp(-\beta p_i^2/2m)}{(2m\pi/\beta)^{\nu/2}} dp_i$$

$$\left[\sum_{\bar{U}(k) \subset U(n)} \tilde{f}_T(Q^{(k)}) g_T(P^{(k)}) \right]$$

$$= \left[\int_{\mathbb{R}^{\nu k}} g_T(P^{(k)}) \prod_{i=1}^{k} \frac{\exp(-\beta p_i^2/2m)}{(2m\pi/\beta)^{\nu/2}} dp_i \right]$$

$$\left[\sum_{n \geq 0} \frac{1}{n!} \int_{\Omega_\Lambda} \tilde{\mu}_\Lambda^{(n)} [dQ^{(n)}] \mathscr{S} \tilde{f}_T(Q^{(n)}) \right].$$

Therefore we have

$$\int_\Omega d\mu f = \left[\int_{\mathbb{R}^{\nu k}} g_T(P^{(k)}) \prod_{i=1}^{k} \frac{\exp(-\beta p_i^2/2m}{(2m\pi/\beta)^{\nu/2}} dp_i \right]$$

$$\left[\frac{1}{k!} \int_{[\mathbb{R}^\nu \times \{1, \ldots, N\}]^k} \tilde{\rho}^{(k)} [dQ^{(k)}] \tilde{f}_T(Q^{(k)}) \right].$$

But by definition of the correlation measures

$$\int_\Omega d\mu f = \frac{1}{k!} \int_{[\mathbb{R}^\nu \times \mathbb{R}^\nu \times \{1, \ldots, N\}]^k} \rho^{(k)} [dP^{(k)} \, dQ^{(k)}] g_T(P^{(k)}) \tilde{f}_T(Q^{(k)})$$

which concludes the proof of the corollary.

To prove Theorem 1 we then proceed in a way similar to [3]:

Let

$$h = \mathscr{S} h_T^{(k)}, \qquad g = \mathscr{S} g_T^{(k)}$$

be two functions in $\mathfrak{A}^{(1)}$ such that $h_T^{(k)}$ and $g_T^{(k)}$ have compact support in Λ (in the x variable) and $h_T^{(k)}$ is independent of p.

Let $\vartheta^r = \vartheta^r(x)$

be a sequence of C^∞-functions with compact support in Λ converging pointwise to $\vartheta \chi_\Lambda$, $\vartheta \in \mathbb{R}$.

Since

$$\{h, g \exp(i\mathscr{S}\vartheta^r)\} = \exp(i\mathscr{S}\vartheta^r)\{h, g\} + g\{h, \exp(i\mathscr{S}\vartheta^r)\}$$
$$= \exp(i\mathscr{S}\vartheta^r)\{h, g\}$$

it follows that for any KMS-state

$$\mu[\exp(i\mathscr{S}\vartheta^r)(\{h, g\} - \beta g\{h, K\})] = 0$$

where K denotes the kinetic energy function.

Furthermore, since $(\{h, g\} - \beta g\{h, K\}) \in \mathscr{L}^1[\mu]$ (Appendix), dominated convergence theorem implies, in the limit $r \to \infty$

$$\sum_{n \geq 0} \exp(in\vartheta) \int_{\Omega_{\Lambda,n}} d\mu_\Lambda^{(n)}[U^{(n)}](\{h, g\} - \beta g\{h, K\})(U^{(n)}) = 0$$

and thus

$$\mu_\Lambda^{(k)}[\{h, g\} - \beta g\{h, K\}] = 0.$$

The positive Borel measure $\bar{\mu}_\Lambda^{(k)} = \exp(\beta K(P^{(k)}))\mu_\Lambda^{(k)}$ satisfies therefore

$$\bar{\mu}_\Lambda^{(k)}[\{h, g \exp(-\beta K)\}] = 0$$

and by density

$$\bar{\mu}_\Lambda^{(k)}[\{h, g\}] = 0$$

for all C^1-functions $h = h(Q^{(k)})$ with support in Λ and $g = g(U^{(k)})$ with compact support in $\{\mathbb{R}^\nu \times \Lambda \times \{1, \ldots, N\}\}^k$.

Repeating the argument of [3, prop 2] we then obtain

$$\frac{\partial}{\partial p_{i,\alpha}} \bar{\mu}_\Lambda^{(k)} = 0 \quad \text{as distribution}$$

and therefore

$$d\bar{\mu}_\Lambda^{(k)} = dp_1, \ldots, dp_k \tilde{\mu}_\Lambda^{(k)}[dQ^{(k)}]$$

which yields:

$$\mu_\Lambda^{(k)}[dU] = \prod_{j=1}^{k} G(p_j)\, dp_j \tilde{\mu}_\Lambda^{(k)}[dQ^{(k)}]$$

with

$$G(p) = \frac{\exp(-\beta p^2/2m)}{(2m\pi/\beta)^{\nu/2}}$$

Remark. Since the velocity and the configuration part of the measure decouple completely with the above choice of the functions h and g, the result is independent of any assumption on the potential and remains valid for any sequence of cut-off $\chi_\lambda(U)$.

Moreover, it is clear that the Maxwellian distribution of velocities implies that the state is flow less (i.e. $\int \rho^{(k)}[dU^{(k)}]p_{i,\alpha} = 0$). This follows from our formulation of the KMS condition and is not an additional requirement as in [1]. (Notice, however, that we use the slightly larger algebra $\mathfrak{A}^{(1)}$.)

2.3. *Equivalence of KMS-condition and weak BBGKY-equation*

In section 2.2, we have seen that any KMS-state has Maxwellian distribution of the velocities, i.e.

$$\mu_\Lambda^{(k)}[dU] = G(P)\, dP \tilde{\mu}_\Lambda^{(k)}[dQ]$$

$$\rho^{(k)}[dU] = G(P)\, dP \tilde{\rho}^{(k)}[dQ] \tag{5}$$

where

$$G(P)\,dP = \prod_{j=1}^{k} \frac{\exp(-\beta p_j^2/2m)}{(2m\pi/\beta)^{\nu/2}}\,dp_j. \tag{6}$$

We shall now discuss the properties of $\tilde{\rho}^{(k)}[dQ]$ and $\tilde{\mu}_\Lambda^{(k)}[dQ]$; in particular we shall show that the correlation measures $\tilde{\rho}^{(k)}$ are solutions of the weak BBGKY-equation.

Theorem 2. *Let μ be a state which is Maxwellian in the velocities and satisfies the conditions C1 and C2, then μ is a KMS-state if and only if its correlation measures satisfy the following BBGKY-equation as distributions of order 1:*

$$\frac{\partial}{\partial x_{j,\alpha}}\tilde{\rho}^{(k)}[dQ] = \beta F_{q_{j},\alpha}(Q)\tilde{\rho}^{(k)}[dQ]$$
$$+ \beta \lim_{\lambda\to\infty}\sum_{r\geq 1}\frac{1}{r!}\int \tilde{\rho}^{(k+r)}[dQ^{(k)}\,d\overline{Q}^{(r)}]\chi_{V_\lambda}(\overline{Q})$$
$$\times \sum_{\substack{Q^{(s)}\subset Q^{(k)}\\ Q^{(s)}\ni q_j}} F_{q_j,\alpha}^{(r+s)}(\tilde{Q}^{(s)}\overline{Q}^{(r)})$$

where

$$F_{q_j,\alpha}(Q) = \sum_{n=1}^{k} F_{q_j,\alpha}^{(n)}(Q) = -\sum_{\substack{\tilde{Q}\subset Q\\ \tilde{Q}\ni q_j}} \frac{\partial\phi(\tilde{Q})}{\partial x_{j,\alpha}} \tag{7}$$

Proof. (i) Using property 1 of section 1, let us choose $g = 1$ and $h = \mathscr{S} h_T^{(k)}$ where

$$h_T^{(k)}(U) = \tilde{h}_T^{(k)}(Q)\overline{h}(p_{i,\alpha}),\ u_i = (p_i, q_i) \in U,$$

$\tilde{h}_T^{(k)}(Q)$ has support in $[\Lambda \times \{1,\ldots,N\}]^k$ and let us consider only the λ such that $V_\lambda \supset \Lambda$.

For any KMS-state we then have:

$$\lim_{\lambda\to\infty}\mu[\{h, H\}_{V_\lambda}] = 0.$$

Using lemma A4 of Appendix A together with $g_T^{(k)} = \delta_{k,0}$ and

$$C = \int_{\mathbb{R}} G(p)\,dp\,\frac{p}{m}\,\overline{h}(p)$$

we have

$$\mu[\{h, H\}_{V_\lambda}] = \frac{C}{k!}\left\{\int \tilde{\rho}^{(k)}[dQ^{(k)}]\,\frac{\partial \tilde{h}_T(Q^{(k)})}{\partial x_{i,\alpha}}\right.$$
$$\left. + \beta \iint \tilde{\rho}[dQ^{(k)}\,d\overline{Q}]\chi_{V_\lambda}(\overline{Q})\Bigg[\sum_{\substack{\tilde{Q}\subset Q^{(k)}\\ \tilde{Q}\ni q_i}}\left(-\frac{\partial\phi(\tilde{Q}\overline{Q})}{\partial x_{i,\alpha}}\right)\Bigg]\tilde{h}_T(Q^{(k)})\right\}.$$

Therefore the KMS-condition implies that:

$$0 = \lim_{\lambda\to\infty} \int\left\{ -\frac{\partial}{\partial x_{i,\alpha}} \tilde{\rho}^{(k)}[dQ^{(k)}] + \beta \int \tilde{\rho}[dQ^{(k)}\, d\,\overline{Q}]\chi_{V_\lambda}(\overline{Q})\left[\sum_{\tilde{Q}\subset Q^{(k)}}\left(-\frac{\partial\phi(\tilde{Q}\overline{Q})}{\partial x_{i,\alpha}}\right)\right]\right\} h_T(\tilde{Q}^{(k)}).$$

Using the condition on the potential together with the above equation which is valid for any function $\tilde{h}_T^{(k)}(Q)$ of class C^1 with compact support, we conclude that:

$$\frac{\partial}{\partial x_{i,\alpha}} \tilde{\rho}^{(k)}[dQ^{(k)}] = \beta \lim_{\lambda\to\infty} \int \tilde{\rho}[dQ^{(k)}\, d\overline{Q}]\chi_{V_\lambda}(\overline{Q}) \sum_{\substack{\tilde{Q}\subset Q^k \\ \tilde{Q}\ni q_i}} \left[-\frac{\partial\phi(\tilde{Q}\overline{Q})}{\partial x_{i,\alpha}}\right] \tag{8}$$

in the sense of distribution, which is identical to equation 7.

(ii) Conversely for any state which is Maxwellian in the velocities[3])

$$\begin{aligned}
\mu[\{h, g\}] &= \iiint \rho[dU\, dU'\, dU'']\{h_T(UU''), g_T(UU')\} \\
&= \iiint \rho[dU\, dU'\, dU''] \sum_{u_j\in UU''}\left[\frac{\partial}{\partial x_{j,\alpha}} h_T(UU'')\frac{\partial}{\partial p_{j,\alpha}} g_T(UU') \right. \\
&\qquad \left. -\frac{\partial}{\partial p_{j,\alpha}} h_T(UU'')\frac{\partial}{\partial x_{j,\alpha}} g_T(UU')\right] \\
&= \iiint \sum_{u_j\in UU''} \frac{\partial}{\partial x_{j,\alpha}} \rho[dU\, dU'\, dU'']\frac{\partial h_T(UU'')}{\partial p_{j,\alpha}} g_T(UU') \\
&\quad - \iiint \sum_{u_j\in UU''} \frac{\partial}{\partial p_{j,\alpha}} \rho[dU\, dU'\, dU'']\frac{\partial h_T(UU'')}{\partial x_{j,\alpha}} g_T(UU').
\end{aligned}$$

Therefore

$$\begin{aligned}
\mu[\{h, g\}] &= \iiint \rho[dU\, dU'\, dU''] \sum_{u_j\in UU''} \frac{\beta p_{j,\alpha}}{m}\frac{\partial}{\partial x_{j,\alpha}} h_T(UU'')g_T(UU') \\
&\quad + \iiint \sum_{u_j\in UU''} \frac{\partial}{\partial x_{j,\alpha}} \rho[dU\, dU'\, dU'']\frac{\partial h_T(UU'')}{\partial p_{j,\alpha}} g_T(UU').
\end{aligned}$$

Using lemma A4 a state μ which is Maxwellian in the velocities is a KMS-state if and only if

$$\begin{aligned}
&\iiint \sum_{u_j\in UU''}\left[\frac{\partial}{\partial x_{j,\alpha}} \rho[dU\, dU'\, dU'']\right]\frac{\partial h_T(UU'')}{\partial p_{j,\alpha}} g_T(UU') \\
&= \beta \lim_{\lambda\to\infty} \iiint \sum_{u_j\in UU''}\left[\int \rho[dU\, dU'\, dU''\, d\overline{U}]\chi_{V_\lambda}(\overline{U}) \right. \\
&\qquad \left. \times \sum_{\substack{\tilde{U}\subset UU'U'' \\ \tilde{U}\ni u_j}} \left(-\frac{\partial\phi(\tilde{U}\overline{U})}{\partial x_{j,\alpha}}\right)\right]\frac{\partial h_T(UU'')}{\partial p_{j,\alpha}} g_T(UU')
\end{aligned}$$

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[3]) In the following the notation $\sum_{u_j}$ means $\sum_{u_j}\sum_{\alpha=1}^{\nu}$.

Thus $\tilde{\rho}^{(k)}$ solution of the BBGKY-equation in the sense of distribution implies that μ is a KMS-state.

From the proof of theorem 2 we have also the following property:

Corollary 1. *Let μ be a state which is Maxwellian in the velocities and satisfies the conditions C1 and C2, then μ is a KMS-state if and only if*

$$\lim_{\lambda\to\infty} \mu[\{h, H\}_{V\lambda}] = 0 \quad \textit{for all } h \in \mathfrak{A}^{(1)}$$

Corollary 2. (*i*) *Let μ be a state which is Maxwellian in the velocities and satisfies conditions C1 and C2; if the density distribution $\tilde{\mu}_\Lambda^{(k)}$ satisfy the equation*

$$\frac{\partial}{\partial x_{i,\alpha}} \tilde{\mu}_\Lambda^{(k)}[dQ_\Lambda] = \beta \lim_{\lambda\to\infty} \int_{V\lambda/\Lambda} \tilde{\mu}_{V\lambda}[dQ_\Lambda\, d\overline{Q}] F_{qi,\alpha}(Q_\Lambda \overline{Q}) \tag{9}$$

then μ is a KMS-state.

(*ii*) *If the forces have finite range μ is a KMS-state if and only if equation* 9 *holds.*

Proof.

(i) For any h in $\mathfrak{A}^{(1)}$ with basis in Λ we have (equation 4)

$$\mu[\{h, H\}_{V\lambda}] = \sum_{k\geq 0} \frac{1}{k!} \int_{V\lambda} \tilde{\mu}_{V\lambda}^{(k)}[dQ] G(P)\, dP$$
$$\sum_{i=1}^{k} \sum_{\alpha=1}^{\nu} \left(\frac{\partial h(Q_\Lambda P)}{\partial x_{i,\alpha}} \frac{p_{i,\alpha}}{m} + \frac{\partial h(Q_\Lambda P)}{\partial p_{i,\alpha}} F_{qi,\alpha}(Q) \right).$$

Therefore μ is a KMS-state if and only if [Corollary 1]

$$\sum_{k\geq 0} \frac{1}{k!} \int_\Lambda \sum_{i=1}^{k} \sum_{\alpha=1}^{\nu} \frac{\partial}{\partial x_{i,\alpha}} \tilde{\mu}_\Lambda^{(k)}[dQ] \int_{\mathbb{R}^{\nu k}} G(P)\, dP h(Q_\Lambda P) \frac{p_{i,\alpha}}{m}$$
$$= \beta \lim_{\lambda\to\infty} \sum_{k\geq 0} \frac{1}{k!} \int_\Lambda \sum_{i=1}^{k} \sum_{\alpha=1}^{\nu} \int_{V\lambda/\Lambda} \tilde{\mu}_{V\lambda}[dQ_\Lambda^{(k)}\, d\overline{Q}_{V\lambda/\Lambda}] F_{qi,\alpha}(Q_\Lambda^{(k)} \overline{Q}_{V\lambda/\Lambda})$$
$$\times \int_{\mathbb{R}^{\nu k}} G(P)\, dP h(Q_\Lambda P) \frac{p_{i,\alpha}}{m}. \tag{10}$$

In conclusion if equation 9 is satisfied equation 10 is verified and μ is a KMS-state.

(ii) Conversely let $h = \mathscr{S} h_T^{(k)}$ with $h_T^{(k)} = \tilde{h}_T^{(k)}(Q)\overline{h}(p_{i,\alpha})$ and $g = \exp(i\mathscr{S}\vartheta^r)$ where $\vartheta = \vartheta^r(x)$ is a sequence of C^∞-functions converging pointwise to $\vartheta\chi_\Lambda(x)$. Then if the forces have finite range the limit $\lambda\to\infty$ disappears and we obtain equation 9 using dominated convergence theorem as in Section 2.

Remarks.

(1) The converse statement of Corollary 2 will remain valid for infinite range forces whenever it is possible to permute the limit $\lambda\to\infty$ and $r\to\infty$.

(2) Equation 9 can be written in a more suggestive form:

$$\frac{\partial}{\partial x_{i,\alpha}} \tilde{\mu}_\Lambda^{(k)}[dQ_\Lambda^{(k)}] = \beta F_{q_{i,\alpha}}(Q_\Lambda^{(k)}) \tilde{\mu}_\Lambda^{(k)}[dQ_\Lambda^{(k)}]$$

$$+ \beta \lim_{\lambda \to \infty} \sum_{n \geq 1} \frac{1}{n!} \int_{V_\lambda 1 \Lambda} \tilde{\mu}_{V_\lambda}^{(k+n)}[dQ^{(k)}\, d\overline{Q}^{(n)}] F_{q_{i,\alpha}}^{\Lambda, \Lambda c}(Q_\Lambda^{(k)}; \overline{Q}^{(n)}) \tag{11}$$

where

$$F_{q_{i,\alpha}}^{\Lambda, \Lambda^c}(Q_\Lambda; \overline{Q}) = - \sum_{\substack{\tilde{Q} \subset Q_\Lambda \\ \tilde{Q} \ni q_i}} \sum_{\substack{\bar{\bar{Q}} \subset \overline{Q} \\ \bar{\bar{Q}} \neq \phi}} \frac{\partial \phi(\tilde{Q}\bar{\bar{Q}})}{\partial x_{i,\alpha}}. \tag{12}$$

The first term on the RHS is the contribution from the particles inside Λ while the second term represents the effect of the outside of Λ.

2.4. *Equivalence of KMS and BBGKY equation*

The next result one would like to establish is that the $\tilde{\rho}^{(k)}$ are absolutely continuous with respect to the Lebesque measure, i.e. that there exist *correlation functions* $\tilde{\rho}^{(k)}(X^k \sigma^k)$ satisfying the usual BBGKY equation. However, to derive such a result we shall need extra assumptions on the potentials (finite range potentials) and it is not known how such conditions could be relaxed.

To show that the $\tilde{\rho}^{(k)}$ are absolutely continuous with respect to the Lebesque measure we shall show that all derivatives of rank 1 considered as distributions over $\mathbb{R}_{\neq}^{\nu k}$ are measures [12].

We shall simplify the notation in the following by restricting ourselves to the case of 1 and 2-body potentials; however as we have seen in section 3, we could discuss exactly in the same way the general case of n-body potentials; for 1 and 2-body potentials theorem 2 becomes:

$$\frac{\partial}{\partial x_{j,\alpha}} \tilde{\rho}^{(k)}[dX^{(k)}; \sigma^{(k)}] = -\beta \Bigg[\frac{\partial}{\partial x_{j,\alpha}} \phi^{(1)}(x_j \sigma_j)$$

$$+ \sum_{i \neq j} \frac{\partial}{\partial x_{j,\alpha}} \phi^{(2)}(x_j \sigma_j; x_i \sigma_i) \Bigg] \tilde{\rho}^{(k)}\, [dX^{(k)}; \sigma^{(k)}]$$

$$- \beta \lim_{\lambda \to \infty} \sum_{\bar{\sigma}=1}^{N} \int_{V_\lambda} \tilde{\rho}^{(k+1)}[dX^{(k)}\, d\bar{x}; \sigma^{(k)} \bar{\sigma}]$$

$$\times \frac{\partial}{\partial x_{j,\alpha}} \phi^{(2)}(x_i \sigma_j; \bar{x}\bar{\sigma}). \tag{13}$$

For a given k let

$$\underline{m} = (m_1, \ldots, m_k) \qquad m_j = (m_{j1}, \ldots, m_{j\nu})\ m_{j,\alpha} \in \{0, 1\}$$

$$|\underline{m}| = \sum_{j,\alpha} m_{j,\alpha}$$

$$D_{\underline{m}} = \prod_{j=1}^{k} \prod_{\alpha=1}^{\nu} \left[\frac{\partial}{\partial x_{j,\alpha}}\right]^{m_{j,\alpha}}$$

We then have to prove that $D_{\underline{m}}\tilde{\rho}^{(k)}$ is a measure.

By definition of KMS-state (condition b) we know that for each bounded V_λ

$$T_\lambda^{(k)}[dQ] = \int \tilde{\rho}^{(k+1)}[dQ\, d\bar{q}]\chi_{V_\lambda}(\bar{q}) \frac{\partial}{\partial x_{j,\alpha}} \phi^{(2)}(q_j, \bar{q})$$

is a measure over $(\mathbb{R}^\nu \times \{1, 2, \ldots, N\})^k$ and that $T_\lambda^{(k)}[dQ] \xrightarrow[\lambda\to\infty]{} T^{(k)}[dQ]$ in the sense of *distribution* of order 1. To be able to conclude that $T^{(k)}[dQ]$ is a measure we shall introduce the condition that $\phi^{(2)}$ is a finite range potential.

Theorem 3. *If the potentials $\phi^{(n)}$, $n \geq 1$, have all their derivatives of rank 1 in C^0 and have finite range for $n \geq 2$, then any KMS-state has correlation measures which are absolutely continuous with respect to the Lebesque measure; furthermore the correlation functions are of class C^1 and satisfy the usual BBGKY-equation.*

$$\begin{aligned} \frac{\partial}{\partial x_{j,\alpha}} \tilde{\rho}^{(k)}(X, \sigma) = {} & -\beta \frac{\partial}{\partial x_{j,\alpha}}\left[\phi^{(1)}(x_j\sigma_j) + \sum_{i \neq j} \phi^{(2)}(x_j\sigma_j; \mathrm{x}_i\sigma_i)\right]\tilde{\rho}^{(k)}(\mathrm{X}, \sigma) \\ & - \beta \sum_{\bar{\sigma}=1}^{N} \int d\bar{x}\tilde{\rho}^{(k+1)}(X\bar{x}, \sigma\bar{\sigma}) \frac{\partial}{\partial x_{j,\alpha}} \phi^{(2)}(x_j\sigma_j; \bar{x}\bar{\sigma}). \end{aligned} \tag{14}$$

Proof (By induction)

(1) For $|\underline{m}| = 1$ Theorem 2 implies that

$$\frac{\partial}{\partial x_{j,\alpha}} \tilde{\rho}^{(k)}(dQ) = F_{j,\alpha}\tilde{\rho}^{(k)}(dQ) - \beta \sum_{\bar{\sigma}=1}^{N} \int \tilde{\rho}^{(k+1)}(dQ; d\bar{x}\bar{\sigma}) \frac{\partial}{\partial x_{j,\alpha}} \phi^{(2)}(q_j, \bar{x}\bar{\sigma})$$

where $F_{j,\alpha}$ is a continuous function. It thus follows that $(\partial/\partial x_{j,\alpha})\tilde{\rho}^{(k)}$ are regular measures since with our conditions on the potentials

$$F_{j,\alpha}\tilde{\rho}^{(k)}(dQ) \quad \text{and} \quad \int \tilde{\rho}^{(k+1)}(dQ; d\bar{x}\bar{\sigma}) \frac{\partial}{\partial x_{j,\alpha}} \phi^{(2)}(q_j, \bar{x}\bar{\sigma})$$

are regular measures.

(2) Let us then assume that $D_{\underline{m}}\tilde{\rho}$ is a regular measure for any $\underline{m}$ with $|\underline{m}| = M$. For $m_{s,\alpha} = 0$ we then consider $D_{\underline{m}}(\partial/\partial x_{s,\alpha})\tilde{\rho}^{(k)}$;

$$D_{\underline{m}} \frac{\partial}{\partial x_{s,\alpha}} \tilde{\rho}^{(k)}(dQ) = D_{\underline{m}}\left[F_{s,\alpha}(Q)\tilde{\rho}^{(k)}(dQ) - \beta \sum_{\bar{\sigma}}^{N} \int \tilde{\rho}^{(k+1)}(dQ; d\bar{x}\bar{\sigma}) \frac{\partial}{\partial x_{s,\alpha}} \phi^{(2)}(q_s, \bar{x}\bar{\sigma})\right]$$

Therefore by the induction hypothesis and the condition on the potentials the RHS is a regular measure and $D_{\underline{m}}\tilde{\rho}$ is thus a regular measure for any $\underline{m}$ with $|\underline{m}| = M + 1$.

In conclusion all derivatives of rank 1 are measures from which follows that $\tilde{\rho}[dX, \sigma]$ are locally bounded functions [12, p. 189]; furthermore from theorem 2 and the finite range condition on the potential all derivatives of rank 1 are functions from which we conclude that $\tilde{\rho}^{(k)}(X, \sigma)$ are functions which are absolutely continuous [12, p. 189]; at last all derivatives of order 1 being continuous functions it follows that $\tilde{\rho}^{(k)}(X, \sigma)$ are of class C^1 [12, p. 61].

Remark. Theorem 3 can be extended to include potentials with singularities at coinciding points; indeed it follows from the above proof that if the potentials $\phi^{(n)}$ have all their derivatives of *rank 1* which are locally integrable with respect to $\rho^{(k)}[dQ]$, $k \geq n$, then $\rho^{(k)}(X, \sigma)$ is an absolutely continuous function on $\mathbb{R}^{\nu k}$ which is of class C^1 on $\mathbb{R}^{\nu k}_{\neq}$. (We remark that in the definition of KMS-state we have only imposed this condition on the derivatives of order 1.)

2.5. *Remarks on KMS-states*

In this section we discuss some general features of KMS-state under the additional assumption of clustering. To simplify the discussion here, we suppose that the correlations are C^1 functions and that we have only one and two body forces.

$$F_\alpha^{(1)}(q) = -\frac{\partial}{\partial x_\alpha} \phi^{(1)}(q), \qquad F_\alpha^{(2)}(q, \bar{q}) = -\frac{\partial}{\partial x_\alpha} \phi^{(2)}(q, \bar{q}).$$

The two body force is assumed to be bounded at infinity and locally integrable

$$|F_\alpha^{(2)}(q, \bar{q})| < C \qquad |x - \bar{x}| > R$$

$$\int_{|x-\bar{x}|<R} d\bar{q} |F^{(2)}(q, \bar{q})| < \infty \tag{15}$$

According to the Theorem 2, the states verify the KMS condition with respect to the forces $F_\alpha^{(1)}(q)$, $F_\alpha^{(2)}(q\bar{q})$ and the sequence of volumes V_λ if and only if

$$\frac{\partial}{\partial x_{j,\alpha}} \tilde{\rho}^{(k)}(Q) = \beta \left[F_\alpha^{(1)}(q_j) + \sum_{i \neq j} F_\alpha^{(2)}(q_j, q_i) \right] \tilde{\rho}^{(k)}(Q) + \beta \lim_{\lambda \to \infty} \int d\bar{q} F_\alpha^{(2)}(q_j, \bar{q}) \tilde{\rho}^{(k+1)}(Q, \bar{q}) \chi_{V_\lambda}(\bar{q}) \tag{16}$$

χ_{V_λ} is the characteristic function of V_λ.

We say that the state is $\mathscr{L}^1$-clustering if the truncated correlation functions satisfy

$$\int dq_2, \ldots, dq_k |\tilde{\rho}_T^{(k)}(q_1, q_2, \ldots, q_k)| < \infty. \tag{17}$$

The main observation is that for a $\mathscr{L}^1$-clustering state the limit in the right hand side of (16) exists for the whole hierarchy of correlation functions if and only if this limit exists for the one point function, i.e. if

$$\lim_{\lambda \to \infty} \int_{V_\lambda} d\bar{q} F_\alpha^{(2)}(q, \bar{q}) \tilde{\rho}^{(1)}(\bar{q}) \text{ exists.} \tag{18}$$

Indeed $\tilde{\rho}^{(k+1)}(Q, \bar{q}) - \tilde{\rho}^{(1)}(\bar{q})\tilde{\rho}^{(k)}(Q)$ is a sum of products of truncated correlation functions where $\bar{q}$ occurs always in conjunction with some other argument $q_i \in Q$. Thus, by (17), this function is integrable in $\bar{q}$; the condition (15) on the force with the dominated convergence theorem allow to conclude that the limit in (16) exists as soon as (18) holds.

Consider now a $\mathscr{L}^1$-clustering KMS-state with respect to the sequence of volumes V_λ, and let us investigate in what sense the same state can be KMS with respect to another sequence of cut-offs. Let $\chi_\lambda(\bar{q})$ be a sequence of functions with compact support on $\mathbb{R}^\nu \times \{1, \ldots, N\}$, converging to 1 as $\lambda \to \infty$, and such that

$$\lim_{\lambda \to \infty} \int d\bar{q}\, \tilde{\rho}^{(1)}(\bar{q}) F_\alpha^{(2)}(q, \bar{q}) \chi_\lambda(\bar{q}) \text{ exists.} \tag{19}$$

As above, the clustering implies that the $\lim_{\lambda \to \infty} \int d\bar{q}\tilde{\rho}^{(k+1)}(Q, \bar{q}) F_\alpha^{(2)}(q_j, \bar{q}) \chi_\lambda(\bar{q})$ exists for all correlation functions and we have

$$\begin{aligned}
&\lim_{\lambda \to \infty} \int d\bar{q}\tilde{\rho}^{(k+1)}(Q, \bar{q}) F_\alpha^{(2)}(q_j, \bar{q}) \chi_{V_\lambda}(\bar{q}) \\
&= \lim_{\lambda \to \infty} \int d\bar{q}\tilde{\rho}^{(k+1)}(Q, \bar{q}) F_\alpha^{(2)}(q_j, \bar{q}) \chi_\lambda(\bar{q}) \\
&\quad + \lim_{\lambda \to \infty} \int d\bar{q}\tilde{\rho}^{(k+1)}(Q, \bar{q}) F_\alpha^{(2)}(q_j, \bar{q}) (\chi_{V\lambda} - \chi_\lambda)(\bar{q}).
\end{aligned} \tag{20}$$

Using the clustering and dominated convergence again the last term of (20) is of the form $\tilde{F}_\alpha^{(1)}(q_j)\tilde{\rho}^{(k)}(Q)$ with

$$\tilde{F}_\alpha^{(1)}(q_j) = \lim_{\lambda \to \infty} \int d\bar{q} F_\alpha^{(2)}(q_j, \bar{q}) \tilde{\rho}^{(1)}(\bar{q}) (\chi_{V_\lambda} - \chi_\lambda)(\bar{q}). \tag{21}$$

We see that BBGKY equation (16) holds for the new cut-off $\chi_\lambda(\bar{q})$ provided that the one body force is replaced by $F_\alpha^{(1)} + \tilde{F}_\alpha^{(1)}$.

The following conclusions are in order[4])

(i) If $\tilde{F}^{(1)}(q)$ is non zero, the one body force depends essentially on the limiting process used to define KMS-states. If the two body force has long range, the particles at infinity can still produce an effective field everywhere. This effective field may depend on the cut-offs used to sum up the contributions to the force of these faraway particles.

(ii) Assume that the state has some translation invariance property (say, under some subgroup of the translations), then the first equation of the BBGKY hierarchy yields

$$\int_{\text{Cell}} dq \tilde{\rho}^{(1)}(q) F_\alpha^{(1)}(q) + \lim_{\lambda \to \infty} \int_{\text{Cell}} dq \int_{V_\lambda} d\bar{q} \tilde{\rho}^{(2)}(q, \bar{q}) F_\alpha^{(2)}(q, \bar{q}) = 0.$$

This expresses that the total force (irrespective of a decomposition in one and two body effects) acting on the cell is zero, as it should be the case in an equilibrium state.

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[4]) These features will be discussed in more detail in [7].

(iii) We see that a given state $\tilde{\rho}$ can be considered as an equilibrium state with respect to the parameters $F^{(1)}$, $F^{(2)}$, $\{\chi_{V_\lambda}\}$ as well as an equilibrium state with respect to the parameters $F^{(1)} + \tilde{F}^{(1)}$, $F^{(2)}$, $\{\chi_\lambda\}$. This shows that there are different equivalent parametrisations of the same equilibrium state (section Gibbs states with Hamiltonian H_Λ, the one body force with respect to which $\tilde{\rho}$ satisfies the KMS condition may differ from the one body force entering in H_Λ. An example of such a situation can be found in [6].

III. Examples—one and two component plasma in $\mathbf{R}^1$.

3.1. Definition of the systems

In this part we give an explicit representation of the density distributions $\mu_\Lambda^{(n)}(Q)$[5]) for the one and two component plasma. We then check explicitly that the corresponding states satisfy the KMS-condition. To compute these density distributions we use the method of functional integration developed by Lénard [8], a method which has been widely extended for Coulomb systems [9]. However the application of this method to the calculation of the state of the Jellium (=one component system) is new. Let us recall that:

$$\mu_\Lambda^{(n)}(q_1, \ldots, q_n) = \sum_{p \geq 0} \frac{(-1)^p}{p!} \int_{\Lambda^p} d\bar{q}_1 \ldots d\bar{q}_p \tilde{\rho}^{(n+p)}(q_1, \ldots, q_n, \bar{q}_1, \ldots, \bar{q}_p) \tag{1}$$

where

$$\int_\Lambda d\bar{q} = \int_\Lambda d\bar{x} \sum_{\bar{\sigma}=1}^{N} .$$

In this first section we define the systems and summarize briefly the part of Reference [8] which is relevant for our purpose.

The energy of a *two component plasma* with charges $\sigma = \pm 1$ in the interval $[L_1, L_2] \subset \mathbb{R}^1$ is:

$$H(Q^{(n)}) = -\sum_{1 \leq k < l \leq n} |x_k - x_l| \sigma_k \sigma_l. \tag{2}$$

The energy of a *one component plasma*, or jellium, of positive charges in the interval $[L_1, L_2] \subset \mathbb{R}^1$ in the field of a uniform background of negative charges with

[5]) We always assume Maxwellian distribution of velocities, and write simply $\mu_\Lambda^{(n)}(Q^n)$, $\rho^{(n)}(Q^n)$ for $\tilde{\mu}^{(n)}(Q^n)$, $\tilde{\rho}^{(n)}(Q^n)$.

fixed density ρ_B is:

$$H(Q^{(n)}) = -\sum_{1\le k<l\le n} |x_k - x_l| + \rho_B \sum_{k=1}^{n} \int_{L_1}^{L_2} |x_k - y|\, dy - \frac{1}{2}\rho_B^2 \int_{L_1}^{L_2} dx \int_{L_1}^{L_2} dy |x-y|$$

$$= -\sum_{1\le k<l\le n} |x_k - x_l| + \rho_B \sum_{k=1}^{n} \left(x_k - \frac{1}{2}(L_1 + L_2)\right)^2 + \frac{n\rho_B}{4}(L_2 - L_1)^2 - \frac{\rho_B^2}{6}(L_2 - L_1)^3. \tag{3}$$

Since we will treat both systems with the same formalism it is useful to write the Hamiltonian for both cases in the form

$$H(Q^{(n)}) = -\frac{1}{2}\int_{L_1}^{L_2} dx \int_{L_1}^{L_2} dy\; c(x; Q^{(n)})|x-y|c(y; Q^{(n)}) \tag{4}$$

where $c(x; Q^{(n)})$ denotes the charge density and $c_n = \int_{L_1}^{L_2} c(x, Q^{(n)})\, dx$ the total charge:

$$c(x; Q^{(n)}) = \sum_{k=1}^{n} \sigma_k\, \delta(x - x_k) \qquad c_n = \sum_{k=1}^{n} \sigma_k \qquad \text{(two component)}$$

$$c(x; Q^{(n)}) = \sum_{k=1}^{n} \delta(x - x_k) - \rho_B \quad c_n = n - \rho_B(L_2 - L_1) \quad \text{(one component).}$$

The main observation is that with

$$-\frac{|x-y|}{2} = \min(x, y) - \frac{x+y}{2}$$

the energy can be written in the form

$$H(Q^{(n)}) = V(Q^{(n)}) - c_n \int_{L_1}^{L_2} (x - L_1)c(x; Q^{(n)})\, dx$$

and

$$V(Q^{(n)}) = \int_{L_1}^{L_2} dx \int_{L_1}^{L_2} dy\; c(x; Q^{(n)}) \min(x, y) c(y; Q^{(n)}) - c_n^2 L_1 \tag{5}$$

is the covariance of the Wiener integral when $L_1 = 0$.

It is important for the following to notice that for $n = 0$, $V^{(0)} = 0$ for the two component plasma whereas

$$V^{(0)} = \rho_B^2 \tfrac{1}{3}(L_2 - L_1)^3$$

for the jellium.

As in ref. [8], we shall consider the modified system described by $V(Q^{(n)})$ instead of $H(Q^{(n)})$. In the strictly neutral case ($c_n = 0$) both energy functions coincide. In the non neutral case, $V(Q^{(n)})$ differs from $H(Q^{(n)})$ by the effect of a suitable external field[6]). States arising from other classes of boundary conditions are studied in [6].

For practical calculations we shall first choose $L_2 = L, L_1 = 0$, we shall then come back to arbitrary intervals by a shift of the origin.

One introduces the space of trajectories $\varphi(x)$ (Brownian paths), and the joint probability distributions for trajectories starting in $\varphi = 0$ at $x = 0$ to be found in $d\varphi_1 \ldots d\varphi_n$ around $\varphi_1 \ldots \varphi_n$ at $x_1 \ldots x_n$.

$$P(\varphi_1 x_1, \ldots, \varphi_n x_n) = P(\varphi_n - \varphi_{n-1}, \ x_n - x_{n-1}) \ldots P(\varphi_2 - \varphi_1, \ x_2 - x_1) P(\varphi_1, \ x_1)$$

$$P(\varphi, x) = \frac{1}{\sqrt{4\pi\beta x}} \exp\left(-\frac{\varphi^2}{4\beta x}\right) \qquad x > 0. \tag{6}$$

We denote simply by $\langle - \rangle$ averages calculated with the probability measure defined by (6). The Boltzmann factor is then represented as

$$\exp(-\beta V(Q^{(n)})) = \langle \exp i \int_0^L c(x; Q^{(n)}) \varphi(x) \, dx \rangle$$

$$= \begin{cases} \langle \exp(i\sigma_1 \varphi(x_1)) \ldots \exp(i\sigma_n \varphi(x_n)) \rangle & \text{two component} \quad (7a) \\ \langle \exp(i\varphi(x_1)) \ldots \exp(i\varphi(x_n)) \exp\left(-i\rho_B \int_0^L \varphi(x) \, dx\right) \rangle & \text{jellium.} \quad (7b) \end{cases}$$

With the help of (7) we can express all statistical mechanical quantities as averages of the form $\langle \exp(\int_0^x F(\varphi(x'), x') \, dx') \rangle$ which can be evaluated by the Wiener–Kac formula

$$\langle \exp \int_0^x F(\varphi(x'), x') \, dx' \rangle = \int_{-\infty}^{\infty} d\varphi U_x(\varphi, 0). \tag{8}$$

$U_x(\varphi, \varphi')$ is the kernel (in the configuration representation $\mathscr{L}^2(\mathbb{R}, d\varphi)$) of the 'propagator' U_x solution of

$$\frac{dU_x}{dx} = \Gamma(x) U_x, \qquad U_{x=0} = I \tag{9}$$

where $\Gamma(x)$ is the differential operator

$$\Gamma(x) = \beta \frac{d^2}{d\varphi^2} + F(\varphi, x) \tag{10}$$

acting on $\mathscr{L}^2(\mathbb{R}, d\varphi)$.

$F(\varphi, x)$ acts as a (possibly x-dependent) multiplicative potential on $\mathscr{L}^2(\mathbb{R}, d\varphi)$ and we set $U_x^0 = \exp(\Gamma_0 x)$ for the semigroup generated by the free part $\Gamma_0 = \beta(d^2/d\varphi^2)$.

We denote $p = -i(d/d\varphi)$ the operator canonically conjugated to φ on $\mathscr{L}^2(\mathbb{R}, d\varphi)$.

In connexion with (8) we shall make use of the following property. We say that an operator A on $\mathscr{L}^2(\mathbb{R}, d\varphi)$ is periodic if A commutes with the discrete translations

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[6]) This external field can be produced by external charges at the boundaries.

$\exp(2i\pi np)$, n integer. If A is represented by a kernel $A(\varphi, \varphi')$ this is equivalent with

$$A(\varphi + 2\pi n, \varphi') = A(\varphi, \varphi' - 2\pi n). \tag{11}$$

To each periodic A on $\mathscr{L}^2(\mathbb{R}, d\varphi)$ we associate the operator $\bar{A}$ acting on $\mathscr{L}^2([-\pi, \pi], d\varphi)$ defined by

$$\bar{A} = \sum_{n=-\infty}^{\infty} \mathscr{P} \exp(2i\pi np) A \,|_{\mathscr{L}^2([-\pi, \pi], d\varphi)}$$

where $\mathscr{P}$ is the natural projection of $\mathscr{L}^2(\mathbb{R}, d\varphi)$ on $\mathscr{L}^2([-\pi, \pi], d\varphi)$.

If A has a kernel, one has

$$\bar{A}(\varphi, \varphi') = \sum_{n=-\infty}^{\infty} A(\varphi + 2\pi n, \varphi'). \tag{12}$$

If A and B are periodic, we have $\overline{AB} = \bar{A}\bar{B}$ and $\overline{A^*} = (\bar{A})^*$. In particular $\exp(ik\phi)$, k integer, and p are periodic.

It is easy to check that $\bar{p}$ is simply the operator $-i(d/d\varphi)$ with periodic boundary conditions on $\mathscr{L}^2([-\pi, \pi], d\varphi)$.[7]

If the propagator U_x is periodic, then (8) becomes

$$\left\langle \exp\left(\int_0^x F(\varphi(x'), x')\, dx'\right)\right\rangle = \int_{-\pi}^{\pi} d\varphi\, \bar{U}_x(\varphi, 0). \tag{13}$$

We define also the Fourier representation of an operator $\bar{A}$ on $\mathscr{L}^2([-\pi, \pi], d\varphi)$ by

$$(k|\bar{A}|l) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} d\varphi\, d\varphi' \exp(-i(k\varphi - l\varphi'))\bar{A}(\varphi, \varphi'), \qquad k, l \text{ integers}.$$

3.2. *Density distributions of the two-component plasma*

We treat first the case of the two component plasma which is simpler. We consider the grand-canonical ensemble with partition function

$$Z(L) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\sigma_1, \ldots, \sigma_n} \int_0^L dx_1 \ldots \int_0^L dx_n \exp(-\beta V(Q^{(n)}))$$
$$= \left\langle \exp\left(\int_0^L 2z\cos\varphi(x')\, dx'\right)\right\rangle. \tag{14}$$

The corresponding semigroup $U_x = \exp(\Gamma x)$ is generated by $\Gamma = \Gamma_0 + 2z\cos\phi$ and is periodic for all x. Hence from (13)

$$Z(L) = \int_{-\pi}^{\pi} d\varphi\, \bar{U}_L(\varphi, 0). \tag{15}$$

[7] We do not discuss here under what general conditions $\bar{A}$ is well defined. In all applications A will be either $\exp(i\phi)$ or a bounded function of p (see also [13, XIII 16]).

The finite volume correlation functions are

$$\rho_L(Q^{(n)}) = \frac{z^n}{Z(L)}\left\langle \exp(i\sigma_1\varphi(x_1))\dots\exp(i\sigma_n\varphi(x_n))\right.$$
$$\left.\exp\left(\int_0^L 2z\cos\varphi(x')\,dx'\right)\right\rangle$$

From (1) we find the finite volume density distribution functions

$$\mu^{(n)}_{\Lambda,L}(Q^{(n)}) = \frac{z^n}{Z(L)}\left\langle \exp(i\sigma_1\varphi(x_1))\dots\exp(i\sigma_n\varphi(x_n))\right.$$
$$\left.\exp\left(\int_0^L F(\varphi(x'),x')\,dx'\right)\right\rangle$$

with

$$F(\varphi, x) = \begin{cases} 2z\cos\varphi & x\notin\Lambda \\ 0 & x\in\Lambda \end{cases} \tag{16}$$

for any $\Lambda\subset[0, L]$.

To calculate $\mu^{(n)}_{\Lambda,L}$ by the Wiener–Kac formula we see on (16) that we have to consider an x dependent but piece-wise constant interaction, i.e. we have to use the semigroup U_x for $x\notin\Lambda$, *but the free semigroup* U^0_x for $x\in\Lambda$, The full 'propagator' entering in (16) is then for $\Lambda = [a, b]$, $x_1,\dots,x_n\in[a, b]$

$$U_{L-b}W(Q^{(n)}\,|\,b, a)U_a$$

with[8])

$$W(Q^{(n)}\,|\,b, a) = T(U^0_{b-x_n}\exp(i\sigma_n\phi)\dots U^0_{x_2-x_1}\exp(i\sigma_1\phi)U^0_{x_1-a}). \tag{17}$$

Clearly $W(Q^{(n)}\,|\,b, a)$ is periodic, and thus from (13), we find

$$\mu^{(n)}_{\Lambda,L}(Q^{(n)} = z^n\frac{\int_{-\pi}^{\pi}d\varphi(\bar U_{L-b}\bar W(Q^{(n)}\,|\,b, a)\bar U_a)(\varphi, 0)}{\int_{-\pi}^{\pi}d\varphi\bar U_L(\varphi, 0)}.$$

In order to take the thermodynamic limit

(i) We shift the origin from 0 to $L/2$, replacing

$$L-b \quad\text{by}\quad \frac{L}{2}-b,\qquad a\quad\text{by}\quad\frac{L}{2}+a,\qquad x_i\quad\text{by}\quad x_i+\frac{L}{2}.$$

(ii) We know [8] that $\bar\Gamma = -\beta\bar p^2 + 2z\,\overline{\cos\varphi}$ has a maximal non degenerate eigenvalue γ_0 with eigenvector $|\Omega)$. (Actually the eigenvalue equation for $\bar\Gamma$ is the well known Mathieu equation.) Hence as $L\to\infty$, $\bar U_L$ behaves as

$$\bar U_L = \exp(\gamma_0 L)|\Omega)(\Omega| + o(\exp(\gamma_0 L)) \tag{18}$$

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[8]) T is the ordering operator, i.e.

$$W(Q^{(n)}\,|\,b, a) = U^0_{b-x_{\pi(n)}}\exp(i\sigma_{\pi(n)}\varphi)\dots U^0_{x_{\pi(2)}-x_{\pi(1)}}\exp(i\sigma_{\pi(1)}\varphi)\,U^0_{x_{\pi(1)}-a}$$

where $a\le X_{\pi(1)}\le x_{\pi(2)}\le\dots\le x_{\pi(n)}\le b$.

From (i) and (ii) one gets

$$\mu_\Lambda^{(n)}(Q^{(n)}) = \lim_{L\to\infty} \mu_{\Lambda,L}^{(n)}(Q^{(n)}) = z^n(\Omega|\bar{U}_{-b}\bar{W}(Q^{(n)}\,|\,b,a)\bar{U}_a|\Omega)$$
$$= z^n \exp(-\gamma_0(b-a))(\Omega|\bar{W}(Q^{(n)}\,|\,b,a)|\Omega). \tag{19}$$

(19) expresses the system of density distributions $\mu_\Lambda^{(n)}$ in terms of known quantities, the fundamental vector $|\Omega)$ and the free propagator U_x^0. We notice that $(z^n/n!)W(Q^{(n)}\,|\,b,a)$ is simply the nth order term of the perturbation expansion of the semigroup U_{b-a}

$$U_{b-a} = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\sigma_1,\ldots,\sigma_n} \int_a^b dx_1 \ldots \int_a^b dx_n W(Q^{(n)}\,|\,b,a). \tag{20}$$

From (20) it is easy to check the normalization and the compatibility relations of the $\mu_\Lambda^{(n)}(Q^{(n)})$.

We can write the $\mu_\Lambda^{(n)}(Q^{(n)})$ in a very suggesting way by looking at the Fourier representation of $\bar{W}(Q^{(n)}\,|\,b,a)$. After a straightforward calculation one finds:

$$(l|\bar{W}(Q^{(n)}\,|\,b,a)|k) = \exp[-\beta H(Q^{(n)}, ka, -lb)]\,\delta_{c_n+k-l,0} \tag{21}$$

where $H(Q^{(n)}, ka, -lb)$ is the Coulomb energy of a neutral system of $(n+2)$ particles located at $a, x_1, \ldots, x_n, b$ with charges $k, \sigma_1, \ldots, \sigma_n, -l$.

$$H(Q^{(n)}, ka, -lb) = -\sum_{1\leq i<j\leq n} \sigma_i\sigma_j|x_i - x_j| - k\sum_{j=1}^{n} \sigma_j|x_j - a|$$
$$+ l\sum_{j=1}^{n} \sigma_j|x_j - b| + kl|b-a|. \tag{22}$$

Then

$$\mu_\Lambda^{(n)}(Q^{(n)}) = z^n \exp(-\gamma_0(b-a)) \sum_{k,l\in\mathbb{Z}} \Omega_k\Omega_l \exp(-\beta H(Q^{(n)}, ka, -lb))\,\delta_{c_n+k-l}.$$

$\Omega_k = (k\,|\,\Omega)$ are the Fourier coefficients of the fundamental vector Ω. They are real and positive:

$$\Omega_k = \overline{\Omega_k} > 0.$$

Thus $\mu_\Lambda^{(n)}$ is a convex combination of neutral Boltzmann factors with boundary charges k and $-l$ at a and b. The boundary charges k and $-l$ represent the net effect of the external configuration of particles on the finite region $[a, b]$.

Finally we check explicitly that the state is invariant under the charge conjugation operation $\sigma \to -\sigma$ as it should be since it is already invariant for a finite volume. From equation 16 we notice that the charge conjugation is implemented in $\mathscr{L}^2(\mathbb{R}, d\varphi)$ by the operator C

$$(C\psi)(\varphi) = \psi(-\varphi) \qquad C\bar{U}_x C^* = \bar{U}_x. \tag{24}$$

Since the eigenvector Ω is even, we have $C\Omega = \Omega$ and the relation

$$\Omega_k = (C\Omega)_k = \Omega_{-k}. \tag{25}$$

3.3. Density distributions of the one component plasma

The grand-canonical partition function is now with (7b)

$$Z(L) = \left\langle \exp\left(\left(\int_0^L (z \exp(i\varphi(x')) - i\rho_B\varphi(x'))\, dx'\right)\right)\right\rangle. \tag{26}$$

We have to define here the semigroup $U_x = \exp(\Gamma x)$ generated by $\Gamma = \Gamma_0 + z\exp(i\phi) - i\rho_B\phi$. We consider first the semigroup U'_x generated by $\Gamma'_1 = \Gamma_0 - i\rho_B\phi$. U'_x can be written in alternative forms

$$\begin{aligned} U'_x &= \exp\left(\frac{-\beta\rho_B^2x^3}{12}\right)\exp\left(\frac{-i\rho_B x\phi}{2}\right)U_x^0\exp\left(\frac{-i\rho_B x\phi}{2}\right) \\ &= \exp\left(\frac{-\beta\rho_B^2x^3}{12}\right)\exp\left(-\beta\left(p+\frac{\rho_B x}{2}\right)^2 x\right)\exp(-i\rho_B x\phi). \end{aligned} \tag{27}$$

Taking (27) as a definition of U'_x, we check directly on (27) that $U'_x, x \geq 0$ is a strongly continuous semigroup of contractions. U'_x takes into account the presence of the background density ρ_B only. We note the following useful relations[9])

$$U'_{-x}\exp(i\phi)U'_x = \exp(-\beta\rho_B x^2)U^0_{-x}\exp(i\phi)U^0_x \tag{28}$$

$$U'_{-x}pU'_x = p - \rho_B x. \tag{29}$$

Then U_x is defined as a perturbation of U'_x by the operator norm convergent series

$$U_x = \sum_{n\geq 0}\frac{z^n}{n!}\int_0^x dx_1\ldots\int_0^x dx_n T(U'_{x-x_n}\exp(i\phi)\ldots U'_{x_2-x_1}\exp(i\phi)U'_{x_1}). \tag{30}$$

Here occurs the main difference with the two component case. U_x (as well as U'_x) is not periodic for all values of x, since

$$\begin{aligned} &\exp(-i\alpha p)U'_x\exp(i\alpha p) = \exp(i\alpha\rho_B x)U'_x \\ &\exp(-i\alpha p)U_x(z)\exp(i\alpha p) = \exp(i\alpha\rho_B x)U_x(z\exp(-i\alpha)) \end{aligned} \tag{31}$$

implies that for $\alpha = 2\pi n$, U_x is periodic only if $x = k\rho_B^{-1}$, k integer.

This fact, which is due to the presence of the continuous background, is at the origin of the periodic spatial structure of the jellium.

To take the thermodynamic limit, we consider a discrete sequence of boxes $L = l\rho_B^{-1}$, l integer. Then $U_{l\rho_B^{-1}}$ is periodic and we get in view of (13)

$$Z(l\rho_B^{-1}) = \int_{-\pi}^{\pi} d\varphi\,\bar{U}_{l\rho_B^{-1}}(\varphi, 0) = \int_{-\pi}^{\pi} d\varphi\,\bar{U}^l(\varphi, 0) \tag{32}$$

where we write simply $\bar{U} = \bar{U}_{\rho_B^{-1}}$. The existence of the thermodynamic limit is insured by the next lemma proved in Appendix B.

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9) Occasionally we use the unbounded semigroups U'_{-x}, U^0_{-x}, $x > 0$. Whenever they occur in the following, there will be no problems of domain.

Lemma B1. *$\overline{U}$ is compact and has a unique eigenvector Ω with positive Fourier coefficients Ω_k. The corresponding eigenvalue* $\exp(\gamma_0)$ *is positive and has the largest modulus of the characteristic numbers of $\overline{U}$. $\overline{U}^*$ has a unique eigenvector Ω^c, $\Omega_k^c > 0$, with the same eigenvalue.*

As $l \to \infty$, l integer,

$$\overline{U}_{l\rho_B^{-1}} = \overline{U}^l = \exp(\gamma_0 l)P + o(\exp(\gamma_0 l)) \tag{34}$$

with eigen (but non self adjoint) projection

$$P = \frac{|\Omega)(\Omega^c|}{(\Omega^c \mid \Omega)} \tag{35}$$

In fact $\overline{U}$ and $\overline{U}^*$ are related by the charge conjugation. The charge conjugation $\sigma \to -\sigma$, $\rho_B \to -\rho_B$ is again implemented by the operator C (24). One has now

$$C\overline{U}C^* = \overline{U}^*, \qquad C\Omega = \Omega^c \tag{36}$$

and

$$\Omega_k^c = (C\Omega)_k = \Omega_{-k}. \tag{37}$$

With the lemma we find the Gibbs free energy g in the thermodynamic limit

$$g = \lim_{l \to \infty} -\frac{1}{\beta l \rho_B^{-1}} \log Z(l\rho_B^{-1}) = -\frac{\rho_B}{\beta}\gamma_0. \tag{38}$$

We study now the correlation functions. Their expression for finite volume is

$$\rho_L(Q^{(n)}) = \frac{z^n}{Z(L)} \left\langle \exp(i\varphi(x_1)) \ldots \exp(i\varphi(x_n)) \right.$$
$$\left. \exp\left(\int_0^L (z \exp(i\varphi(x')) - i\rho_B \varphi(x'))\, dx'\right)\right\rangle. \tag{39}$$

In order to exhibit the spatial periodicity of the jellium, we calculate first the one point function. The application of the Wiener–Kac formula gives

$$\rho_L^{(1)}(x) = z\,\frac{\int_{-\infty}^{\infty} d\varphi (U_{L-x} \exp(i\varphi) U_x)(\varphi, 0)}{\int_{-\infty}^{\infty} d\varphi U_L(\varphi, 0)}. \tag{40}$$

To take the thermodynamic limit

(i) We shift $L - x \to L/2 - x$, $x \to x + L/2$ and set $L/2 = l\rho_B^{-1}$, l integer.
(ii) The operator entering in (40) is $U_{l\rho_B^{-1}} U_{-x} \exp(i\varphi) U_x U_{l\rho_B^{-1}}$; (31) shows that $U_{l\rho_B^{-1}}$ and $U_{-x} \exp(i\varphi) U_x$ are periodic[10])
(iii) We use the lemma with (34) and (35) and find

$$\rho^{(1)}(x) = \frac{z}{(\Omega^c \mid \Omega)} (\Omega^c | \overline{U_{-x} \exp(i\varphi) U_x} | \Omega). \tag{41}$$

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[10]) It is important to note that the product U_x is periodic for all x although U_x alone is not.

The fact that $\rho^{(1)}(x)$ is periodic with period ρ_B^{-1} appears now clearly on the structure of (41)

$$\begin{aligned}\rho^{(1)}(x+\rho_B^{-1}) &= \frac{z}{(\Omega^c\,|\,\Omega)}(\Omega^c|\overline{U_{-x-\rho_B^{-1}}\exp(i\phi)U_{x+\rho_B^{-1}}}|\Omega)\\ &= \frac{z}{(\Omega^c\,|\,\Omega)}(\Omega^c|\overline{U}^{-1}\overline{U_{-x}\exp(i\phi)U_x}\overline{U}|\Omega)\\ &= \frac{z}{(\Omega^c\,|\,\Omega)}\exp(-\gamma_0)(\Omega^c|\overline{U_{-x}\exp(i\phi)U_x}|\Omega)\exp(\gamma_0)\\ &= \rho^{(1)}(x).\end{aligned}$$

It remains to show that $\rho^{(1)}(x)$ is non trivially periodic. This is done in the following lemma proved in Appendix B.

Lemma B2.
(*i*) *For every* x, $\rho_\beta^{(1)}(x)$ *is a holomorphic function of* β *in a neighbourhood of the positive real axis* $\beta > 0$.
(*ii*) *For any open set of values of* β, $\rho_\beta^{(1)}(x)$ *is not a constant.*

To find the density distributions $\mu_\Lambda^{(n)}$ one proceeds along the same lines as for the two component system. The density distributions are now from (1) and (39).

$$\mu_{\Lambda,L}^{(n)}(Q^{(n)}) = \frac{z^n}{Z(L)}\left\langle \exp(i\varphi(x_1))\ldots\exp(i\varphi(x_n))\exp\left(\int_0^L F(\varphi(x'),x')dx'\right)\right\rangle$$

with

$$F(\varphi,x) = \begin{cases} z\exp(i\varphi) - i\rho_B\varphi & x\notin\Lambda \\ -i\rho_B\varphi & x\in\Lambda\end{cases}. \tag{42}$$

The Wiener–Kac formula for (42) involves now the 'propagator' $U_{L-b}W'(Q^{(n)}\,|\,b,a)U_a$ with

$$W'(Q^{(n)}\,|\,b,a) = T(U'_{b-x_n}\exp(i\phi)\ldots U'_{x_2-x_1}\exp(i\phi)U'_{x_1-a}). \tag{43}$$

To go to the thermodynamic limit, we shift the origin as in (i) setting $L/2 = l\rho_B^{-1}$, l integer. We notice that the product $U_{-b}W'(Q^{(n)}\,|\,b,a)U_a$ is periodic (although U_{-b}, $W'(Q^{(n)}\,|\,b,a)$, U_a are not periodic separately). Then we use (13) and the lemma to find

$$\mu_\Lambda^{(n)}(Q^{(n)}) = \frac{z^n}{(\Omega^c\,|\,\Omega)}(\Omega^c|\overline{U_{-b}W'(Q^{(n)}\,|\,b,a)U_a}|\Omega). \tag{44}$$

The density distributions of the jellium have a structure similar to those of the two component system (19) except for two important differences. U_x^0 is replaced by U'_x which takes the background into account and they are invariant only under discrete translations $l\rho_B^{-1}$, l integer.

We can also express the $\mu_\Lambda^{(n)}$ as convex combinations of neutral Boltzman factors. From (28) we have

$$W'(Q^{(n)}\,|\,b,a) = U'_bU^0_{-b}W(Q^{(n)}\,|\,b,a)U^0_a\,U'_{-a}\exp\left(-\beta\rho_B\sum_{i=1}^n x_i^2\right). \tag{45}$$

Inserting (45) in (44) and using (21) one gets

$$\mu_\Lambda^{(n)}(x_1, \dots, x_n) = z^n \sum_{k,l \in \mathbb{Z}} c_k(a) c_l(b) \exp\left(-\beta \rho_B \sum_{i=1}^n x_i^2\right) \times \exp(-\beta H((x_1, \dots, x_n); ka, -lb))\, \delta_{n+k-l,0} \tag{46}$$

where $H((x_1, \dots, x_n); ka, -lb)$, is as in (22) (with a single type of charges). The coefficients in the superposition (46) are

$$\begin{aligned} c_k(a) &= \frac{(k \,|\, \overline{U_a^0}\, \overline{U'_{-a} U_a} \,|\, \Omega)}{(\Omega^c \,|\, \Omega)^{1/2}} \\ c_l(b) &= \frac{(\Omega^c | \overline{U_{-b} U'_{-b}}\, \overline{U_{-b}^0} | l)}{(\Omega^c \,|\, \Omega)^{1/2}}. \end{aligned} \tag{47}$$

Notice in (46) that as expected the background acts as a harmonic one body force.

It is clear that in addition to the state that we have just discussed, we have also all its translates in the period ρ_B^{-1}. These translated states are simply obtained by shifting the origin in the process of the thermodynamic limit (i).

3.4. Properties of the equilibrium states of the Jellium

We shall now show that *the state of the Jellium defined by equation 44 is neutral and independent of z for $z \neq 0$.*

To study the z dependence of the semi-group $U_x(z)$ equation 30, we remark that $U_x(z)$ transforms under the unitary transformation $\exp(i\alpha p)$ as in equation 31.

Since $\exp(i\alpha p)$ is periodic, the same relation holds in $\mathscr{L}^2([-\pi, \pi], d\varphi)$ showing that $\overline{U}(z)$ and $\exp(i\alpha)\overline{U}(z \exp(-i\alpha))$ equation 32, are unitarily equivalent. From this we deduce that the eigenvalue of Lemma B1 verifies the identity

$$\exp(\gamma_0(z)) = \exp(\gamma_0(z \exp(-i\alpha)) + i\alpha).$$

Moreover, by the same arguments as those used in Lemma B2 this eigenvalue is a holomorphic function of z in a neighbourhood of the positive axis $z > 0$. Then this identity implies that the Gibbs free energy is necessarily of the form

$$g = f(\rho_B, \beta) - \frac{\rho_B}{\beta} \log z$$

where $f(\rho_B, \beta)$ is independent of z.

If g_L denotes the finite volume Gibbs free energy, we have by definition

$$-\beta z \frac{d}{dz} g_L = \frac{1}{L} \int_0^L \rho_L^{(1)}(x)\, dx$$

and hence

$$\begin{aligned} \lim_{L \to \infty}\left(-\beta z \frac{d}{dz} g_L\right) &= -\beta z \frac{d}{dz} g = \rho_B \\ &= \lim_{L \to \infty} \frac{1}{L} \int_0^L \rho_L^{(1)}(x)\, dx = \rho_B \int_0^{\rho_B^{-1}} \rho^{(1)}(x)\, dx \end{aligned}$$

showing that the average density of particles equals that of the bath. The exchange of the limit and of the z-derivative is justified by the fact that g is a differentiable function of z which is the limit of concave functions of z. The second limit results of the spatial periodicity of $\rho^{(1)}(x)$, and of the estimate $|\rho_L^{(1)}(x) - \rho^{(1)}(x)| < C_1 \exp(-C_2(L-x))$, $x > 0$, which can be obtained on the basis of Lemma B1.

Finally, let us show that the correlation functions are indeed independent of z. We note that (41) can be written in the form of a trace which is invariant under the unitary transformation $\exp(i\alpha\bar{p})$. We find

$$\begin{aligned}\rho^{(1)}(x, z) &= z \operatorname{Tr} P(z)\overline{U_{-x}(z) \exp(i\phi) U_x(z)} \\ &= z \operatorname{Tr} \exp(-i\alpha\bar{p}) P(z) \overline{U_{-x}(z) \exp(i\phi) U_x(z)} \exp(i\alpha\bar{p}) \\ &= z \operatorname{Tr} P(z \exp(-i\alpha)) \\ &\quad \times \overline{\exp(-i\alpha p) U_{-x}(z) \exp(i\phi) U_x(z) \exp(i\alpha p)} \\ &= z \exp(-i\alpha) \operatorname{Tr} P(z \exp(-i\alpha)) \\ &\quad \times \overline{U_{-x}(z \exp(-i\alpha)) \exp(i\phi) U_x(z \exp(-i\alpha))} \\ &= \rho^{(1)}(x, z \exp(-i\alpha)).\end{aligned}$$

Here $P(z)$ is the projection (35) which transforms as $\exp(-i\alpha\bar{p})P(z)\exp(i\alpha\bar{p}) = P(z\exp(-i\alpha))$. Since for x fixed $\rho^{(1)}(x, z)$ is analytic in a neighbourhood of $z > 0$ (proof as in Lemma B2) this equality shows that $\rho^{(1)}(x, z)$ is in fact independent of z. The same is true for the full set of correlation functions. Thus in the thermodynamic limit the state is determined by ρ_B alone.

As a last remark we note that

$$-\frac{\beta}{\rho_B}\left(z\frac{d}{dz}\right)^2 g_L = \frac{\langle (N - \langle N \rangle)^2 \rangle_L}{L}$$

is the square of the fluctuation of the number of particles by unit volume. The fact that $z(d/dz)g_L$ becomes independent of z as $L \to \infty$ shows that $\langle (N - \langle N \rangle)^2 \rangle_L / L$ tends to zero as $L \to \infty$ indicating that the particles fluctuations are not normal in the jellium. Indeed by neutrality $\langle N \rangle \sim L\rho_B$ as $L \to \infty$ the particles fluctuations coincide with the charge fluctuations for the jellium.

To conclude this discussion we note that the state of the Jellium is solution of the BBGKY equation, as we shall prove in the next section. It is then a general result that every clustering solution of the BBGKY-equation for Coulomb systems is neutral, with charge fluctuations which are not normal (see [7]).

3.5. *Verification of equilibrium equation*

We shall now verify that the equilibrium states defined by equation 23 (two component plasma) and equation 46 (Jellium) satisfy the KMS-condition with respect to the *forces defined for the original model* by equation 2 and equation 3.

Using the result of section 2.3 it is sufficient to verify that the $\mu_\Lambda^{(n)}(Q^{(n)})$ are solutions of equation 2.11 for an appropriate sequence of volumes $V \to \mathbb{R}^1$. In our examples equation 2.11 reads:

$$\frac{\partial}{\partial x_i} \mu_\Lambda^{(n)}(Q) = \beta F_i(Q)\mu_\Lambda^{(n)}(Q) + \beta \lim_{V \to \mathbb{R}^1} \sum_{k \geq 1} \frac{1}{k!} \int_{V \setminus \Lambda} d\bar{q}_1 \dots d\bar{q}_k \mu_V^{(n+k)}(Q\bar{Q}) F_i^{(2)}(q_i, \bar{Q}) \tag{48}$$

where

$$\bar{Q} = (\bar{q}_1, \dots, \bar{q}_k) \qquad \bar{q}_i = (\bar{x}_i, \bar{\sigma}_i).$$

$$\int_{V \setminus \Lambda} d\bar{q} = \int_{V \setminus \Lambda} d\bar{x}_i \sum_{\sigma_i = 1}^{N} \qquad N = 1 \text{ (Jellium)}, N = 2 \text{ (2-component)}$$

$$F_i^{(2)}(q_i, \bar{Q}) = \sum_{j=1}^{k} \sigma_i \bar{\sigma}_j \operatorname{sign}(x_i - \bar{x}_j)$$

$$F_i(Q) = \sum_{j \neq i} \sigma_i \sigma_j \operatorname{sign}(x_i - x_j) \quad \text{2-component.}$$

For the Jellium the sequence of volumes $V = [L_1, L_2]$ must be such that $\frac{1}{2}(L_1 + L_2) = x_0$ is fixed; for a given x_0

$$F_i(Q) = \sum_{j \neq i} \operatorname{sign}(x_i - x_j) - 2\rho_B(x_i - x_0).$$

We shall first treat the two component plasma and then give the necessary modifications for the Jellium.

We calculate first $(\partial/\partial x_i)\mu_\Lambda(Q)$. For this we use the representation (23) of $\mu_\Lambda(Q)$ in terms of Boltzmann factors. Since $-(\partial/\partial x_i)H(Q^{(n)}, ka, -lb) = F_i(Q^{(n)}) + (k + l)\sigma_i$ one has

$$\frac{\partial}{\partial x_i} \mu_\Lambda^{(n)}(Q^{(n)}) = \beta F_i(Q^{(n)})\mu_\Lambda^{(n)}(Q^{(n)}) + \beta v_\Lambda^{(n)}(Q^{(n)}) \tag{49}$$

where

$$v_\Lambda^{(n)}(Q^{(n)}) = \sigma_i z^n \exp(-\gamma_0(b - a)) \sum_{k, l \in \mathbb{Z}} (k + l) \quad \Omega_k \Omega_l \exp(-\beta H(Q^{(n)}, ka, -lb))\, \delta_{c_n + k - l}.$$

The first term of (49) is the usual force for the particles in Λ. $v_\Lambda^{(n)}(Q^{(n)})$ represents the effect of the infinite system on Λ and has to be shown equal to the last term of (48). We notice that by (21) and $\exp(-\gamma_0 b) l \Omega_l = (\Omega | \overline{U}_{-b} \bar{p} | l)$, $v_\Lambda^{(n)}(Q^{(n)})$ can be written in the more compact form

$$v_\Lambda^{(n)}(Q^{(n)}) = \sigma_i z^n (\Omega | [\bar{p} \overline{W}(Q^{(n)} \mid b, a) + \overline{W}(Q^{(n)} \mid b, a) p] | \Omega) \exp(-\gamma_0(b - a)). \tag{50}$$

Let us then evaluate the remaining term of equation 48. For V we choose an interval $[L_1, L_2] \supset [a, b]$ and we denote by $R = (r_1, \dots, r_u)$, $r_i = (x_i', \sigma_i')$ and $S = (s_1, \dots, s_v)$,

$s_i = (x_i'', \sigma_i'')$ the configurations of particles in $[L_1, a]$ and $[b, L_2]$ respectively, $\bar{Q} = R \cup S$.

The force $F_i(\bar{Q})$ can be decomposed into

$$F_i(\bar{Q}) = F_i(R) + F_i(S) = \sigma_i \left(\sum_{r_j \in R} \sigma_j' - \sum_{s_j \in S} \sigma_j'' \right) \tag{51}$$

which is simply the difference of the total charge of the left and the right of $[a, b]$. $F_i(\bar{Q})$ is independent of the position of the i particle in Λ. The evaluation of the force due to the outside of Λ, i.e. the last term of (48) is given by the following lemma.

Lemma 1.

$$\int_{V \setminus \Lambda} \mu_V(QRS) F_i(S)\, dR\, dS$$
$$= -\sigma_i z^n (\Omega | \bar{U}_{-L_2} [\bar{p}, \bar{U}_{L_2 - b}] \bar{W}(Q \mid b, a) \bar{U}_a | \Omega)$$

Proof. The integral over the R variables is just a compatibility relation. In an explicit notation and introducing (19) we are left with

$$z^n \sum_{v \geq 1} \frac{z^v}{v!} \sum_{\sigma_1'', \ldots, \sigma_v''} \int_b^{L_2} dx_1'' \ldots \int_b^{L_2} dx_v''$$
$$\times (\Omega | \bar{U}_{-L_2} \bar{W}(S^{(v)} Q \mid L_2, a) \bar{U}_a | \Omega) \sum_{j=1}^{v} \sigma_j''.$$

Using $W(S^{(v)} Q \mid L_2, a) = W(S^{(v)} \mid L_2, b) W(Q \mid b, a)$, we see that we have to sum the series

$$\sum_{v \geq 1} \frac{z^v}{v!} \sum_{\sigma_1'', \ldots, \sigma_v''} \int_b^{L_2} dx_1'' \ldots \int_b^{L_2} dx_v'' \bar{W}(S^{(v)} \mid L_2, b) \sum_{j=1}^{v} \sigma_j''.$$

The result is obtained if we use (20) and the fact that

$$[\bar{p}, \bar{W}(S^{(v)} \mid L_2, b)] = \begin{cases} \sum_{j=1}^{v} \sigma_j'' \bar{W}(S^{(v)} \mid L_2, b) \\ 0 \quad \text{if } v = 0. \end{cases} \tag{52}$$

The next proposition shows that (48) holds for an arbitrary sequence of intervals $V \to \mathbb{R}^1$.

Proposition 1. *The state of the two component plasma defined by equation 23 is a KMS-state with respect to the forces defined by equation 2 and any sequence of boxes $V \to \mathbb{R}^1$.*

Proof. Choose an arbitrary sequence of intervals $V = [L_1, L_2]$ converging to $\mathbb{R}^1$. It remains to show that the limit in the last term (48) exists and is equal to $v_\Lambda(Q)$.

If we apply the Lemma 1 to the evaluation of

$$\int_{V \setminus \Lambda} \mu_V(Q\bar{Q}) F_i(\bar{Q})\, d\bar{Q} = \int_{V \setminus \Lambda} \mu_V(QRS)(F_i(R) + F_i(S))\, dR\, dS$$

we find exactly $v_\Lambda(Q)$ plus the two terms

$$-\sigma_i z^n(\Omega|\bar{U}_{-L_2}\bar{p}\bar{U}_{L_2}\bar{U}_{-b}\bar{W}(Q\,|\,b, a)\bar{U}_a|\Omega)$$

$$-\sigma_i z^n(\Omega|\bar{U}_{-b}\bar{W}(Q\,|\,b, a)\bar{U}_a\bar{U}_{-L_1}\bar{p}\bar{U}_{L_1}|\Omega).$$

They converge to zero as $L_2 \to \infty$ and $L_1 \to -\infty$ in view of the fact that from (18) and (25)

$$(\Omega|\bar{U}_{-L_2}\bar{p}\bar{U}_{L_2} = [(\Omega|\bar{p}|\Omega) + \exp(-\gamma_0 L_2)o(\exp(\gamma_0(L_2))](\Omega|$$

and

$$(\Omega|\bar{p}|\Omega) = \sum_k k\Omega_k^2 = 0.$$

To verify the equilibrium equations for the jellium we calculate $(\partial/\partial x_i)\mu_\Lambda^{(n)}(Q^{(n)})$ from (46), using successively equations 47, 21, 45, 29:

$$\frac{\partial}{\partial x_i}\mu_\Lambda^{(n)}(Q^{(n)}) = \beta(F_i(Q^{(n)}) - 2\rho_B x_i)\mu_\Lambda^{(n)}(Q^{(n)}) + \beta v_\Lambda^{(n)}(Q^{(n)}) \tag{53}$$

$$v_\Lambda^{(n)}(Q^{(n)}) = \frac{z^n}{(\Omega^c\,|\,\Omega)}\,\overline{(\Omega^c|U_{-b}pW'(Q^{(n)}|b, a) + W'(Q^{(n)}|b, a)pU_a\,|\Omega)}$$
$$+\ \rho_B(b + a)\mu_\Lambda^{(n)}(Q^{(n)}).$$

In addition to the two body force occurs the one body harmonic force $-2\rho_B x$ which can be seen on the Hamiltonian (3) (with $L_1 = -L_2$). The lemma 1 and proposition 1 are modified as follows:

Lemma 2.

$$\int_{V|\Lambda}\mu_V(Q^{(n)}RS)F_i(S) = -\frac{z^n}{(\Omega^c\,|\,\Omega)}(\Omega^c|U_{-L_2}[p, U_{L_2-b}]W'(Q^{(n)}\,|\,b, a)U_a|\Omega)$$
$$-\ \rho_B(L_2 - b)\mu_\Lambda^{(n)}(Q^{(n)}).$$

The proof is the same as that of Lemma 1 with $W'(Q)$ replacing $W(Q)$. This causes only a modification of (52). With (45), (52) and (29), we have

$$\sum_{j=1}^{v}\sigma_j'' W'(S^{(v)}\,|\,L_2, b) = vW'(S^{(v)}\,|\,L_2, b)$$
$$= [p, W'(S^{(v)}\,|\,L_2, B)] + \rho_B(L_2 - b)W'(S^{(v)}\,|\,L_2, b)$$

giving rise to the additional term $(L_2 - b)\mu_\Lambda(Q)$ in the Lemma 2.

Proposition 2. *Let $V = [L_1, L_2]$ be a sequence of boxes such that $L_1 + L_2 = 2x_0$ is fixed.*

(*i*) *The state of the jellium defined by equation 46 is a KMS-state with respect to the forces defined by equation 3 and any sequence of boxes V such that $2x_0 = k\rho_B^{-1}$, where k is integer.*

(*ii*) *This same state is not a KMS-state with respect to the forces equation 3 for a sequence of boxes such that $2x_0\rho_B$ is not integer. However, if $2x_0\rho_B$ is not* *integer, this state is a KMS-state with respect to a modified one body force which corresponds to add an external field.*

Proof.

(i) Let us first consider a sequence V with $L_1 = l_1\rho_B^{-1}$, $L_2 = l_2\rho_B^{-1}$ where l_1, l_2 are integers, $l_1 \to -\infty$, $l_2 \to \infty$, $l_1 + l_2 = k$ fixed.

The proof is then the same as that of proposition 1 because the operators are periodic and we use now the charge conjugation relation equation 36. Indeed it follows from Lemma 2 that

$$\nu_\Lambda^{(n)}(Q^{(n)}) = \lim_{V\to\mathbb{R}^1}\int d\bar{Q}\mu_V(Q\bar{Q})F_i(q_i\bar{Q}) + \rho_B(L_2+L_1)\mu_\Lambda^{(n)}(Q^{(n)})$$

and thus equation 53 is identical with equation 48.

To conclude that the same result holds for arbitrary sequences of $V = [L_1, L_2]$ centered around $x_0 = \frac{1}{2}k\rho_B^{-1}$ we use the fact the state is $\mathscr{L}^1$-clustering and equation 48 is equivalent to BBGKY:

$$\frac{\partial}{\partial x_i}\rho^{(k)}(X) = \beta F_i(X)\rho^{(k)}(X) + \beta\lim_{\substack{L_1\to-\infty\\L_2\to+\infty}}\int_{L_1}^{L_2} d\bar{x}\rho^{(k+1)}(X\bar{x})\,\mathrm{sign}\,(x_i-\bar{x}). \tag{54}$$

But

$$\lim_{\substack{L_1\to-\infty\\L_2\to+\infty}}\int_{L_1}^{L_2} d\bar{x}\rho^{(k+1)}(X\bar{x})\,\mathrm{sign}\,(x_i-\bar{x}| = \lim_{\substack{L_1\to-\infty\\L_2\to+\infty}}\left\{\int_{[L_1]}^{[L_2]} d\bar{x}\rho^{(k+1)}(X\bar{x})\,\mathrm{sign}\,(x_i-\bar{x}) - \int_{[L_2]}^{L_2} d\bar{x}\rho^{(k+1)}(X\bar{x}) + \int_{L_1}^{[L_1]} d\bar{x}\rho^{(k+1)}(X\bar{x})\right\}$$

where $[L_1] = l_1\rho_B^{-1}$, $[L_2] = l_2\rho_B^{-1}$ with l_1, l_2 integers such that $L_1 = [L_1] - \varepsilon$, $L_2 = [L_2] + \varepsilon$, $\varepsilon\in[0, \rho_B^{-1}]$ (see Figure 1) and thus

$$\lim_{\substack{L_2\to-\infty\\L_2\to\infty}}\int_{L_1}^{L_2} d\bar{x}\rho^{(k+1)}(X\bar{x})\,\mathrm{sign}\,(x_i-\bar{x}) = \lim_{\substack{L_1\to-\infty\\L_2\to+\infty}}\int_{[L_1]}^{[L_2]} d\bar{x}\rho^{(k+1)}(X\bar{x})\,\mathrm{sign}\,(x_i-\bar{x}) - - \rho^{(k)}(ix)\left[\left[\int_0^{\varepsilon} d\bar{x}\rho^{(1)}(\bar{x}) - \int_{-\varepsilon}^{0} d\bar{x}\rho^{(1)}(\bar{x})\right]\right.$$

(since the state is clustering, and periodic with period ρ_B^{-1}).

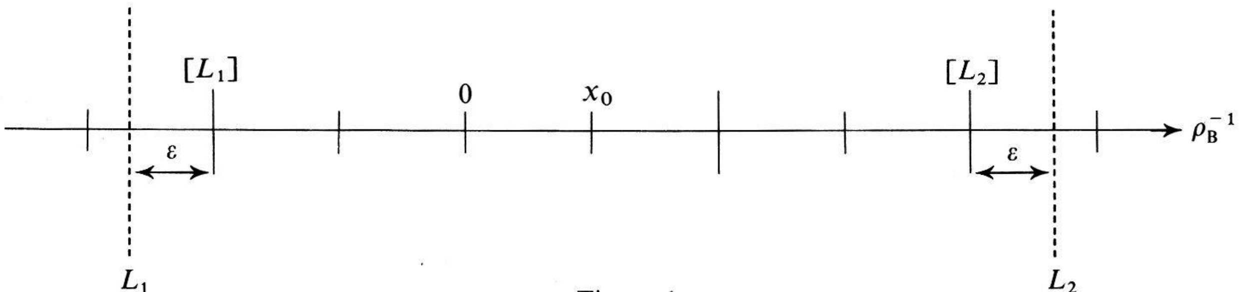

Figure 1

The proof is thus concluded since $\rho^{(1)}(x) = \rho^{(1)}(-x)$.

(ii) Let us then consider a sequence $V = [L_1, L_2]$ centered around x_0 such that $(L_1 + L_2)\rho_B = 2x_0\rho_B$ is not integer, but $L_2 - L_1 = N\rho_B^{-1}$. We consider again equation 53 which we now write as:

$$\frac{\partial}{\partial x_i}\rho^{(k)}(X) = \beta F_i(X)\rho^{(k)}(X) + \beta \lim_{\substack{L_1 \to -\infty \\ L_2 \to +\infty}} \left[\int_{L_1-\delta}^{L_2-\delta} + \int_{L_2-\delta}^{L_2} - \int_{L_1-\delta}^{L_1}\right] \times d\bar{x}\rho^{(k+1)}(X\bar{x}) \operatorname{sign}(x_i - \bar{x})$$

where $L_1 - \delta = n\rho_B^{-1}$

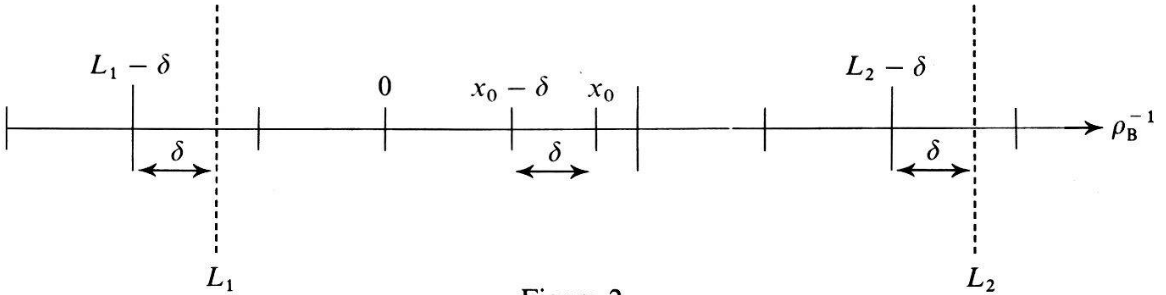

Figure 2

We thus obtain

$$\frac{\partial}{\partial x_i}\rho^{(k)}(X) = \beta F_i(X)\rho^{(k)}(X) + \beta \lim_{\substack{L_1 \to -\infty \\ L_2 \to +\infty}} \int_{L_1-\delta}^{L_2-\delta} d\bar{x}\rho^{(k+1)}(X\bar{x}) \quad \operatorname{sign}(x_i - \bar{x}) - 2\beta\rho^{(k)}(X)\int_0^\delta d\bar{x}\rho^{(1)}(\bar{x})$$

or

$$\frac{\partial}{\partial x_i}\rho^{(k)}(X) = \beta\left[\sum_{j\neq i}\operatorname{sign}(x_i - x_j) - 2\rho_B\left(x_i - \frac{L_1 + L_2 - 2\delta}{2}\right)\right]\rho^{(k)}(X) + \beta \lim_{\substack{L_1 \to -\infty \\ L_2 \to +\infty}} \int_{L_1-\delta}^{L_2-\delta} d\bar{x}\rho^{(k+1)}(X\bar{x}) \operatorname{sign}(x_i - \bar{x}) - 2\beta\rho^{(k)}(X)\int_0^\delta d\bar{x}(\rho^{(1)}(\bar{x}) - \rho_B). \tag{55}$$

In conclusion $\mu_\Lambda(Q)$ given by equation 46 is not a KMS-state with respect to the force equation 3 if $\int_0^\delta d\bar{x}(\rho^{(1)}(\bar{x}) - \rho_B) \neq 0$, since as we have seen in (i) its correlation functions satisfy equation 54 without the last term.

Remarks.

(1) We would like to stress the fact that the state we have constructed using the modified Hamiltonian $V(Q^{(n)})$ is, as we have shown in the two propositions, a *KMS-state with respect to the original Hamiltonian.*

(2) The example of the Jellium illustrates the remarks discussed in section 2.5 namely that a given KMS-state with respect to $\{F_1, F_2, \{V\}\}$ may also be a KMS-state with respect to $\{F_1', F_2, \{V'\}\}$:

let

$$V = [L_1, L_2] \quad \text{with} \quad L_1 + L_2 = 2x_0 = k\rho_B^{-1}$$

then

$$F_1 = -2\rho_B(x_1 - x_0)$$

let

$$V' = [L_1', L_2'] \quad \text{with} \quad L_1' + L_2' = 2x_0' = k\rho_B^{-1} + 2\varepsilon$$

then

$$F_1' = -2\rho_B(x_1 - x_0') + 2\int_0^{\varepsilon} d\bar{x}(\rho^{(1)}(\bar{x}) - \rho_B).$$

Appendix A

Lemma A.1. *If the correlation measures of a state μ such that $\mathfrak{A}^{(0)} \subset \mathscr{L}^1[\mu]$ satisfy the condition*

$$\int_{\Gamma^n} \rho^{(n)}[dU^{(n)}]|p_{i,\alpha}|\chi_{V_\lambda}(U^{(n)}) < \infty$$

then, for all h and g in $\mathfrak{A}^{(1)}$ with basis in V_λ, $g\{h, K\} \in \mathscr{L}^1[\mu]$ where K denotes the kinetic energy function, and

$$\mu[g\{h, K\}] = \mu[g\{h, K\}_{V_\lambda}] = \iiint \rho[dU\, dU'\, dU'']\, g_T(UU') \sum_{\substack{u_i \in UU'' \\ 1 \le \alpha \le v}} \frac{\partial h_T(UU'')}{\partial x_{i,\alpha}} \frac{p_{i,\alpha}}{m}.$$

Proof. Following the algebraic derivation of Gallavotti and Verboven [1] we have the following combinatorial identities:

(i) For any $h = \mathscr{S}h_T$ and $g = \mathscr{S}g_T$ then $h \cdot g = \mathscr{S}(h \cdot g)_T$ and $\{h, g\} = \mathscr{S}[\{h, g\}]_T$ where

$$[hg]_T(U) = \sum_{R \subset U} \sum_{V \subset U \backslash R} h_T(RV) g_T(U \backslash V) \tag{A.1}$$

$$[\{h, g\}]_T(U) = \sum_{R \subset U} \sum_{V \subset U \backslash R} \{h_T(RV), g_T(U \backslash V)\} \tag{A.2}$$

(ii)

$$\int \rho[dU]G(U) \sum_{U_1 \subset U} F_1(U_1)F_2(U \backslash U_1)$$

$$= \iint \rho[dU_1 dU_2]G(U_1 U_2)F_1(U_1)F_2(U_2) \tag{A.3}$$

To establish Lemma A.1, we just need to consider those g and h of the following form:

$$g = \mathscr{S} g_T^{(k)} \qquad h = \mathscr{S} h_T^{(l)}$$

we then have:

$$\int \rho[dU] |[g \cdot \{h, K\}]_T(U)| \leq \int \rho[dU] \sum_{R \subset U} \sum_{V \subset U \backslash R} |g_T(RV) \{h, K\}_T(U | V)|$$

$$\leq \iiint \rho[dR\, dV\, dW] |g_T(RV)| \, |\{h, K\}_T(RW)|$$

$$\leq \iiint \rho[dR\, dV\, dW] \sum_{\substack{u_i \in RW \\ 1 \leq \alpha \leq v}} |g_T^{(k)}(RV)| \left| \frac{\partial h^{(l)}(RW)}{\partial x_{i,\alpha}} \right| \frac{|p_{i,\alpha}|}{m}$$

$$\leq \| g^{(k)} \|_\infty \sup_{i,\alpha} \left\| \frac{\partial h^{(l)}}{\partial x_{i,\alpha}} \right\|_\infty \sum_{n=0}^{\min\{k,l\}} \frac{1}{n!} \frac{1}{(k-n)!}$$

$$\frac{l}{(l-n)!} \sum_{\alpha=1}^{v} \int \rho[dR^{(n)}\, dV^{(k-n)}\, dW^{(l-n)}] \frac{|p_{i,\alpha}|}{m} \chi_{V_\lambda}(RVW)$$

Therefore $[g \cdot \{h, K\}]_T \in \mathscr{L}^1[\rho]$ and thus $\mathscr{S}[g \cdot \{h, K\}]_T = g \cdot \{h, K\} \in \mathscr{L}^1[\mu]$ since $\|g_T\|_{\mathscr{L}^1[\rho]} \geq \|\mathscr{S} g_T\|_{\mathscr{L}^1[\mu]}$ for any $g_T \in \mathscr{L}^1[\rho]$.

Lemma A.2. *If the correlation measures of a state μ such that $\mathfrak{A}^{(0)} \subset \mathscr{L}^1[\mu]$ satisfy*

$$\int \rho^{(n+l)}[dU^{(n)}\, dV^{(l)}] \left| \frac{\partial \phi_T^{(n)}(U^{(n)})}{\partial x_{i,\alpha}} \right| \chi_{V_\lambda}(U^{(n)} V^{(l)}) < \infty$$

then for all h and g in $\mathfrak{A}^{(1)}$ with basis in V_λ, $g\{h, U\}_{V_\lambda} \in \mathscr{L}^1[\mu]$ where U denotes the potential energy function and

$$\mu[g\{h, U\}_{V_\lambda}] = - \iiint \rho[dR\, dV\, dW] g_T(RV) \sum_{S \subset RW} \sum_{\Delta \subset RW \backslash S}$$

$$\times \sum_{\substack{r_i \in RW \backslash \Delta \\ 1 \leq \alpha \leq v}} \frac{\partial \phi}{\partial x_{i,\alpha}} (S\Delta) \frac{\partial h_T}{\partial p_{i,\alpha}} (RW \backslash \Delta) \times \chi_{V_\lambda}(\Delta).$$

Proof. Form the identities equations A.1, A.2, and A.3 above

$$\left| \int \rho[dU][g \cdot \{h, U\}]_T(U) \right| = \left| \iiint \rho[dR\, dV\, dW] g_T(RV) \{h, U\}_{V_\lambda}(RW) \right|$$

$$\leq \iiint \rho[dR\, dV\, dW] |g_T(RV)| \sum_{S \subset RW} \sum_{\Delta \subset RW \backslash S} \sum_{\substack{r_i \in RW \backslash \Delta \\ 1 \leq \alpha \leq v}}$$

$$\times \left| \frac{\partial \phi(S\Delta)}{\partial x_{i,\alpha}} \right| \left| \frac{\partial h_T(RW \backslash \Delta)}{\partial p_{i,\alpha}} \right| \chi_{V_\lambda}(RVW)$$

and the proof is concluded as before.

Lemma A.3. *For any h and g in $\mathfrak{A}^{(1)}$ with basis in V_λ*

$$\mu[g\cdot\{h, U\}_{V_\lambda}] = \iiiint \rho[dU\, dU'\, dU''\, d\bar{U}] \sum_{\substack{u_i\in UU''\\ 1\le\alpha\le\nu}} \chi_{V_\lambda}(\bar{U}) \times \sum_{\tilde{U}\subset UU'U''}\left(-\frac{\partial\phi(\tilde{U}\bar{U})}{\partial x_{i,\alpha}}\right)\frac{\partial h_T(UU'')}{\partial p_{i,\alpha}} g_T(UU').$$

Proof. Let us consider the expression of $\mu[g\cdot\{h, U\}_{V_\lambda}]$ as given in Lemma A.2 and write:

$$\sum_{S\subset RW}\sum_{\Delta\subset RW\setminus S}\frac{\partial\phi(S\Delta)}{\partial x_{i,\alpha}}\frac{\partial h_T(RW\setminus\Delta)}{\partial p_{i,\alpha}} = \sum_{\bar{U}\subset W}\sum_{U\subset R}\sum_{S\subset UW\setminus\bar{U}}\frac{\partial\phi(SR\setminus U\bar{U})}{\partial x_{i,\alpha}}\frac{\partial h_T(UW\setminus\bar{U})}{\partial p_{i,\alpha}}.$$

Using equation A.3 successively for $\bar{U}\subset W$ and $U\subset R$ yields

$$\iiint \rho[dR\, dV\, dW]\left(\sum_{S\subset RW}\sum_{\Delta\subset RW\setminus S}\sum_{u_i\in RW\setminus\Delta} \times \frac{\partial\phi(S\Delta)}{\partial x_{i,\alpha}}\frac{\partial h_T(RW\setminus\Delta)}{\partial p_{i,\alpha}}\right)\chi_{V_\lambda}(W) g_T(RV)$$

$$= \iiint \rho[dR\, dV\, dW] \sum_{\bar{U}\subset W}\sum_{U\subset R}\sum_{S\subset U\cdot W\setminus\bar{U}}\sum_{u_i\in UW\setminus\bar{U}}\chi_{V_\lambda}(W) \times g_T(RV)\frac{\partial\phi(SR\setminus U\bar{U})}{\partial x_{i,\alpha}}\frac{\partial h_T(UW\setminus\bar{U})}{\partial p_{i,\alpha}}$$

$$= \iiiint \rho[dR\, dV\, dU''\, d\bar{U}] g_T(RV)\chi_{V_\lambda}(\bar{U})\sum_{U\subset R}\sum_{S\subset UU''}\sum_{u_i\in UU''} \times \frac{\partial\phi(SR\setminus U\bar{U})}{\partial x_{i,\alpha}}\frac{\partial h_T(UU'')}{\partial p_{i,\alpha}}$$

$$\iiiiint \rho[dU\, dR_1\, dV\, dU''\, d\bar{U}] g_T(UR_1V)\chi_{V_\lambda}(\bar{U})\sum_{S\subset UU''} \times \sum_{u_i\in UU''}\frac{\partial\phi(SR_1\bar{U})}{\partial x_{i,\alpha}}\frac{\partial h_T(UU'')}{\partial p_{i,\alpha}}.$$

(Setting $R_1V = U'$ and using equation A.3 again)

$$= \iiiint \rho[dU\, dU'\, dU''\, d\bar{U}] g_T(UU')\chi_{V_\lambda}(\bar{U})\sum_{S\subset UU''}\sum_{T\subset U'}\sum_{u_i\in UU''} \times \frac{\partial\phi(ST\bar{U})}{\partial x_{i,\alpha}}\frac{\partial h_T(UU'')}{\partial p_{i,\alpha}}$$

$$= \iiiint \rho[dU\, dU'\, dU''\, d\bar{U}] g_T(UU')\chi_{V_\lambda}(\bar{U})\sum_{\tilde{U}\subset UU'U''}\sum_{u_i\in UU''} \times \frac{\partial\phi(\tilde{U}\bar{U})}{\partial x_{i,\alpha}}\frac{\partial h_T(UU'')}{\partial p_{i,\alpha}}.$$

Combining Lemma A.1 and A.3, we thus have:

Lemma A.4. *For all h and g in $\mathfrak{A}^{(1)}$ with basis in V_λ, then*

$$\mu[g\{h, H\}_{V_\lambda}] = \iiint \rho[dU\, dU'\, dU''] \sum_{\substack{u_j \in UU'' \\ 1 \leq \alpha \leq v}} \frac{p_{j,\alpha}}{m} \frac{\partial h_T(UU'')}{\partial x_{j,\alpha}} g_T(UU')$$

$$+ \iiiint \rho[dU\, dU'\, dU''\, d\bar{U}] \sum_{\substack{u_j \in UU'' \\ 1 \leq \alpha \leq v}} \chi_{V_\lambda}(\bar{U}) \sum_{\tilde{U} \subset UU'U''}$$

$$\left[-\frac{\partial \phi(\tilde{U}\bar{U})}{\partial x_{j,\alpha}} \right] \frac{\partial h_T(UU'')}{\partial p_{j,\alpha}} g_T(UU').$$

Appendix B

Proof of Lemma B1. To have the result of the lemma, it is enough to show by theorem (β'') [14, p. 279] that $\bar{U}$ has the following properties:

(i) $\bar{U}$ is compact.
(ii) The matrix elements $\bar{U}_{lm}$ of $\bar{U}$ in Fourier representation are non negative.
(iii) For any pair of indices (l, m) there exists an integer $N(l, m)$ such that for $s \geqslant N$, $(\bar{U}^s)_{lm} > 0$.

To study $\bar{U}$, it is convenient to introduce first $K_x = \exp(i\phi\rho_B x)U_x$. K_x is periodic and in particular:

$$\bar{U} = \overline{\exp(-i\phi)}\, \overline{K_{\rho_B^{-1}}} \tag{B1}$$

Since U_x is defined by the perturbation series (3.30), we have the integral relation

$$U_x = U'_x + \int_0^x U'_{x-y} z \exp(i\phi) U_y\, dy \tag{B2}$$

we find from (B2) and (3.27) the corresponding relation for K_x

$$K_x = G(x, 0) + \int_0^x G(x, y) z \exp(i\phi) K_y\, dy \tag{B3}$$

with

$$G(x, y) = \exp(-\beta\rho_B^2(x-y)^3/12) \exp\left[-\beta\left(p - \rho_B \frac{(x+y)}{2} \right)^2 (x-y) \right]. \tag{B4}$$

We notice that $G(x, y)$ (acting as a multiplicative function of p) is periodic. $\bar{G}(x, y)$ acts in $\mathscr{L}^2([-\pi, \pi], d\varphi)$ by multiplication in Fourier space by the same function and $\bar{G}(x, y)$ is bounded with $\|\bar{G}(x, y)\| \leq 1$ for $x \geq y$.

Hence $\bar{K}_x$ is given by the norm convergent series

$$\bar{K}_x = \sum_{n \geq 0} z^n \int_0^x dx_n \int_0^{x_n} dx_{n-1} \ldots \int_0^{x_2} dx_1 \bar{G}(x, x_n)\, \overline{\exp(i\phi)}\, \bar{G}(x_n, x_{n-1})$$
$$\ldots \bar{G}(x_2, x_1)\, \overline{\exp(i\phi)} \bar{G}(x_1, 0) \tag{B5}$$

Since the function (B4) is positive and $(\exp(i\phi)\psi)_k = \psi_{k-1}$ is the shift operator in Fourier space, all terms of $\bar{K}_x$ are positivity preserving. Hence $\bar{U}$ is positively preserving and this proves (ii).

We see on (B1) and (B5) that the nth term $\bar{U}^{(n)}$ of the perturbation series of $\bar{U}$ involves $n-1$ shifts, therefore $\bar{U}^{(n)}$ acts necessarily as

$$(\bar{U}^{(n)}\psi)_k = (F_n \exp(i(n-1)\phi)\psi)_k = F_n(k)\psi_{k-n+1}$$

where $F_n(k)$ is a multiplicative function of k. One finds from the structure of (B5) that $F_n(k)$, $n > 0$, is of the form

$$F_n(k) = \exp(-\beta k^2 \rho_B^{-1}) z^n \int_0^{\rho_B^{-1}} dx_n \int_0^{x_n} dx_{n-1} \cdots$$
$$\times \int_0^{x_2} dx_1 \exp(f(x_1, \ldots, x_n)k + g(x_1, \ldots, x_n)) > 0$$

$f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ being polynomials. From this we get the estimate

$$F_n(k) \leq c_n \frac{(\rho_B^{-1} z)^n}{n!} \exp(-\beta k^2 \rho_B^{-1} + d_n|k|) \to 0, \quad \text{for } |k| \to \infty.$$

Thus the operator F_n has discrete spectrum accumulating on zero, and is compact. Consequently, $\bar{U}^{(n)}$ and the norm convergent series of the $\bar{U}^{(n)}$ are compact[11]). This proves (i).

We obtain from (B5) the matrix elements of $\bar{U}$

$$\bar{U}_{lm} = \begin{cases} F_{l-m+1}(l) > 0 & m \leq l+1 \\ 0 \end{cases}$$

from which one deduces that (iii) is satisfied.

Proof of Lemma B2. Define $K_x^c = \exp(-i\rho_B x\phi)U_x^*$ for $x > 0$ the charge conjugate of the operator K_x introduced in Lemma B1. Then it is easily checked that $\rho_\beta^{(1)}(x)$ can be written in the form

$$\rho_\beta^{(1)}(x) = \exp(-\gamma_0 l) z \frac{(\bar{K}^c_{l\rho_B^{-1}-x} P^*\chi, \exp(-i\phi(l-1))\bar{K}_x P\psi)}{(\chi, P\psi)} \tag{B6}$$

where P is the eigenprojection (3.35) of $\bar{U}$ and ψ, χ are two fixed vectors with $P\psi \neq 0$, $P^*\chi \neq 0$ and l is an integer chosen such that $l\rho_B^{-1} - x > 0$. Since $\bar{G}(x, y)$, $x \geq y$, is uniformly bounded and holomorphic for $\mathrm{Re}\,\beta > 0$, we conclude that $\bar{K}_x$ and $\bar{U}$ are bounded holomorphic in the half plane $\mathrm{Re}\,\beta > 0$. For any real positive β_0, the eigenvector Ω_{β_0} of $\bar{U}$ is non degenerate, therefore the Kato–Rellich theorem insures that the corresponding eigenprojection P and the eigenvalue γ_0 are holomorphic in a neighbourhood of β_0. The same statements hold true for $\bar{K}_x^c$ and P^*, and the holomorphy of $\rho_\beta^{(1)}(x)$ follows from the formula (B6).

To prove (ii) we calculate explicitly the first and the second derivatives of $\rho_\beta(x)$ in $x = 0$. Using $\Omega_k^c = \Omega_{-k} > 0$ we find

$$\left.\frac{d\rho_\beta^{(1)}(x)}{dx}\right|_{x=0} = 0$$

[11]) $\bar{U}^{(0)} = \overline{\exp(-i\phi)\bar{G}(\rho_B^{-1}, 0)}$ is compact.

and

$$\left.\frac{d^2\rho_\beta^{(1)}(x)}{dx^2}\right|_{x=0} = \frac{\beta z}{(\Omega_c \mid \Omega)}\left\{2z \sum_{k\in\mathbb{Z}} \Omega_{-k}\Omega_{k-2} + \sum_{k\in\mathbb{Z}} (\beta(2k-1)^2 - 2\rho_B)\Omega_{-k}\Omega_{k-1}\right\}$$

a quantity which is clearly positive for $\beta > 2\rho_B$. These derivatives are well defined in view of the fact that Ω is in the domain of p^s for all $s > 0$. We deduce indeed from the integral equation

$$\bar{p}^s\overline{K}_x = \bar{p}^s\overline{G}(x, 0) + \int_0^x \overline{G}(x, y)z \exp(i\phi)(\bar{p} + 1)^s\overline{K}_y\, dy$$

that the following estimate holds

$$\|\bar{p}^s\overline{K}_x\| \leq C\|\bar{p}^s\overline{G}(x, 0)\| < \infty, \quad x > 0.$$

This implies with (B1) that $\|\bar{p}^s\overline{U}\| < \infty$, and thus $\|\bar{p}^s\Omega\| < \infty$. The same argument shows that $\overline{K}_x$ and $\overline{K}_x^c$ are infinitely strongly differentiable for $x \neq 0$, and therefore it follows from its representation (B6) that $\rho_\beta^{(1)}(x)$ is a C^∞-function of x for every $\beta > 0$. We conclude that $\rho_{\beta_0}(x)$ is not constant for $\beta_0 > 2\rho_B$. Choose x such that $\rho_{\beta_0}(x) \neq \rho_{\beta_0}(0)$. Then by (i) the holomorphic function $\rho_\beta(x) - \rho_\beta(0)$ cannot vanish on an open set of the real positive axis $\beta > 0$, proving (ii).

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