Zeitschrift: Helvetica Physica Acta
Band: 51 (1978)
Heft: 5-6

Artikel: On a mathematical model for non-stationary physical systems
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DOI: https://doi.org/10.5169/seals-114968

Helvetica Physica Acta, Vol. 51 (1978), Birkhäuser Verlag, Basel

On a mathematical model for non-stationary physical systems[1])

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(28.VIII.1978)

1. Introduction

The perturbation theory of isolated eigenvalues is an old and common tool in physical investigations. Up to now, however, the perturbation theory of eigenvalues embedded into a continuous spectrum has received hardly any attention in the physics literature. There are two reasons for this: in the theory of unstable particles and theory of the emission of the light, where the role of embedded eigenvalues was realized, the problem was either buried in a difficult physical context or appeared too simple in exactly solvable models, while it was largely ignored in the rest of quantum physics, which is a main consumer of perturbation theory. And, secondly, this theory has been developed only in the last few years and even some of the principal questions have not yet been completed.

The aim of this paper is to develop the mathematical tools of this theory; a detailed discussion of the most important applications will be given in the following.

We shall now describe major physical situations where the problem of the perturbation of eigenvalues embedded into the continuous spectrum plays a central role. We begin with a class of important problems which have similar structure: decay of elementary particles, emission of photons by atoms, open systems. We will distinguish between two kinds of models corresponding to these problems: spectral models and physical models. The former are constructed on the basis of known results and the latter on the basis of physical principles.

In the spectral models of the physical problems listed above, the physical system is regarded as a system with many possible states some of which are associated with discrete spectrum, and others with continuous spectrum; these states interact with each other[4]). The desired physical result is the decay of discrete states into the continuum.

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[1]) Work supported in part by the U.S.-Israël Binational Science Foundation (BSF), Jerusalem, Israël.

[2]) This work was completed while the first author was Visiting Professor at the Département de Physique Théorique, Université de Genève, Switzerland.

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[4]) States are said to interact if the perturbation operator has non-vanishing matrix elements between them.

Mathematical analysis tells us that this can occur in either of the two following cases: The discrete eigenvalues are separated from the continuous spectrum and the interaction is rather strong, or the discrete eigenvalues are embedded into the continuum and the interaction is not necessarily strong. If one wants to consider interactions which are or can be very weak, then only the second possibility remains. We now proceed to describe the physical models. Fortunately, in the problems listed above, they can be reduced to the same structure as the corresponding spectral models.

Suppose that we have a system of m species, each one with its Fock Space F_i, Hamiltonian $\hat{H}_i$, and momentum $\hat{P}_i$, and that the momenta are constants of motion for these species:

$$[\hat{P}_i, \hat{H}_i] = 0. \tag{1.1}$$

The Hilbert space for the whole system is

$$F = \bigotimes_{i=1}^{m} F_i,$$

the Hamiltonian of the compound system is

$$\hat{H}_o = \sum_i I_1 \otimes \cdots \otimes \hat{H}_i \otimes \cdots \otimes I_m \tag{1.2}$$

and the total momentum is

$$\hat{P} = \sum_i I_1 \otimes \cdots \otimes \hat{P}_i \otimes \cdots \otimes I_m. \tag{1.3}$$

Suppose, furthermore, that some of the one-particle states are generalized eigenfunctions of $\hat{H}_o$. Because of (1.1), they become ordinary eigenfunctions after removal of the center of mass motion[5]). Note that the corresponding eigenvalues are, in general, imbedded into the continuous spectrum of $\hat{H}_o$ with center of mass motion removed (this continuous spectrum corresponds to multiparticle states).

Consider a perturbation of the compound system by a self-adjoint operator $\hat{V}$ on F which commutes with $\hat{P}$, such that $\hat{H} = \hat{H}_o + \hat{V}$ is also self-adjoint. $\hat{V}$ either annihilates one-particle eigenstates of $\hat{H}_o$, mixes them with some constant coefficients, or maps them into states which are not spanned by the vacuum and the one particle eigenstates.

In the first two cases, $\hat{H}$ has the same one-particle eigenspaces as $\hat{H}_o$. In the last case, already in this general setting, we have the problem of perturbation of eigenvalues imbedded into the continuous spectrum.

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[5]) The removal of the center of mass motion corresponds to the restriction into the invariant subspaces in the representation of F as a direct integral

$$F = \int \oplus F_p \, dp$$

with respect to the operator $\hat{P}$. Since all of the F_p are trivially isomorphic to each other, it suffices to consider the restriction to F_o, the subspace which is annihilated by $\hat{P}$.

1.1. Decay model

In order to obtain the decay model, we make this rather general situation more specific as follows: we will say that the system $(\hat{H}_o, \hat{H})$, as defined above, is a decay system if

(i) at least one one-particle eigenstate is mapped by $\hat{V}$ into a state which is not spanned by the vacuum and the one-particle eigenstates

(ii) the minimal subspace of F invariant with respect to $\hat{H}$, and containing a one-particle eigenstate of $\hat{H}_o$ of the type described in (i), does not contain any other states with the same kind of particles.

Naturally, when one studies the decay of unstable particles which are in our case the one-particle eigenstates described in (i), it is sufficient to consider the problem in this minimal invariant subspace. Often this subspace and the restriction of $\hat{H}$ onto it are much simpler than the original space F and the operator $\hat{H}$.

If we restrict all operators into this invariant subspace and then remove the center-of-mass motion, we complete the setting of the framework in which the decay problem may be stated.

1.2. Decay problem

Denote by $\mathscr{H}$ the invariant subspace described above, with center of mass motion removed, and by H the restriction of $\hat{H}$ on $\mathscr{H}$. The decay problem is then stated as an initial value problem for the Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi \tag{1.4}$$

with data $\psi(o)$ an exact discrete eigenstate (one of the one-particle states in the center of mass frame as described previously). The main quantity of interest is the probability, $P(t) = |(\psi(o), e^{-iHt}\,\psi(o))|^2$, for the unstable particle to be undecayed at the time t. By a system of particles, we will understand the pair, the space and the evolution operator; a one parameter semi-group of unitary operators, $U(t)$, will correspond to a system of stable particles and a positive function of contraction, $Z(t)$ (i.e. $\|Z(t)\| \leq 1$), will correspond to a system of unstable particles. We define the system of unstable particles in the model under consideration as the pair $(P\mathscr{H}, P\,e^{-iHt}P)$, where P is the projection operator on the one particle eigenspace[6]), and the system of decay products as the pair $(\bar{P}\mathscr{H}, e^{-i\bar{P}H\bar{P}t})$, where $\bar{P} = I - P$.

1.3. Emission of photons by an atom

Slightly modifying the previous definitions, we can obtain descriptions of completely different physical phenomena. Let one of the species in the previous

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[6]) Note that by Sz-Nagy theorem [17] any positive definite family $\{Z(t)\}$ of operators in a Hilbert space $\mathscr{K}$, with $Z(0) = I$ can be extended to a unitary group $U(t)$ in a bigger Hilbert space $\mathscr{H} \supset \mathscr{K}$ and this unitary extension can be chosen in such a way that $Z(t) = PU(t)\,|_{P\mathscr{H}}$, where P is a projection in $\mathscr{H}$ with the range $\mathscr{K}$.

paragraph be described by the usual quantum mechanical Hilbert space $\mathscr{H}_0$ and Hamiltonian h, which has bound states after removal of the center of mass motion.

The restriction (i) and (ii) of the decay model should be replaced by:

(i′) all bound states except the ground state are destroyed by $\hat{V}$

(ii′) a minimal invariant subspace of F does not contain, in the tensor product, states in $\mathscr{H}_0$ of higher energy than in the initial bound state, or those forbidden by selection rules (conservation laws).

This model may describe the emission of radiation by an atom (with Hamiltonian h). The various excited bound states of the atom in $\mathscr{H}_0$ multiplied by the vacuum state of the Fock space of the photon field play the role of one particle eigenstates of the decay model, and their decay corresponds to emission of photons by the atom. The invariant subspace generated by such states includes lower bound states in $\mathscr{H}_0$ in tensor product with non-vacuum states of the photon Fock space.

Remark. By a further small modification, the model for emission of radiation by an atom can be altered to describe emission of radiation and phonons by carriers in bound (localized) states in a crystal. In this case, (ii′) is modified to include the part of the continuous spectrum of h' (Hamiltonian of charged carrier in the crystal) which lies below the initial bound state in $\mathscr{H}_0'$ (Hilbert space for the charged carrier-crystal system). We have in this example, assumed in infinite crystal so that the allowed bands in the spectrum of h' are continuous, and the initial bound state corresponds to a trap level between the bands (due to some impurity). The invariant subspace may include other bound states in tensor product with non-vacuum states of the product of photon and phonon Fock spaces of the type allowed in (ii′).

1.4. Open systems

The formulation of simple models of open systems which led to the problem of perturbation of eigenvalues imbedded into a continuous spectrum can be found in [13].

1.5. Resonance scattering

One can use the decay model as it is described above as a framework for the statement of the resonance scattering problem. In discussing this application, we shall restrict ourselves here to heuristic remarks.

Since the operator of coupling between the unstable particles and decay products is self-adjoint the evolution of the decay system contains also the process, inverse to the decay-creation of unstable particles in collision of decay products. The created particles decay then into the decay products again. Thus, one has in this model the scattering, which can be described in terms of creation and annihilation of unstable particles. If the life-times of the unstable particles is large enough, the scattering has a resonance character. As was noted above the motion of the decay products is described by the Hamiltonian $\overline{P}H\overline{P}$ (on the space $\overline{P}\mathscr{H}$). Hence we come to the scattering problem for the pair $(\overline{P}H\overline{P}, H)$. The scattering matrix $S(\overline{P}H\overline{P}, H)$ for this pair can be written as

$$S_\lambda(\overline{P}H\overline{P}, H) = \langle\lambda|VP(H-(\lambda+i0)I)^{-1}PV|\lambda\rangle, \tag{1.5}$$

where $\langle\lambda|$ are the corresponding generalized eigenstates of $\overline{P}H\overline{P}$ and

$$V = PH\overline{P} + \overline{P}HP. \tag{1.6}$$

Recall that in our interpretation $P(H - zI)^{-1}P$ is a propagator for the motion of the unstable particles. Formula (1.5) justifies the heuristic interpretation of the scattering in the decay model as a creation and annihilation of unstable particles.

One can also consider the scattering problem for the pair $(H_o\overline{P}, H)$, where $H_o = \hat{H}_o \restriction \mathscr{H}$ and $H_o\overline{P}$ is a Hamiltonian of the free motion of the decay products. In this case the scattering in $(\overline{P}H\overline{P}, H)$ is the part, responsible for the resonance behaviour. This can be rigorously justified using the representation of the scattering matrix for $(H_o\overline{P}, H)$ through the scattering matrix for $(\overline{P}H\overline{P}, H)$ and the wave operators for $(H_o\overline{P}, \overline{P}H\overline{P})$.

The problem described above is a special case of a more general situation when one considers the scattering problems of a pair of operators (H_o, H) and there is another operator H_1 with a point spectrum embedded into the continuous one such that $H - H_1$ can be considered as a perturbation of H_1, and $H_1 - H_o$ of H_o. Then one can express the S-matrix of the pair (H_o, H) through the S-matrix for the pair $(H_1\overline{P}, H)$, where $\overline{P}$ is the projection into the subspace of $\mathscr{H}$ corresponding to the continuous spectrum of H_1.

Examples. (i) Nonleptonic uncharged K meson decay. The minimal subspace of principal interest is spanned by K^0, $\overline{K}^0$—one particle states multiplied by the vacuum of the π meson field—and the three π meson states and two π meson states each multiplied by the vacuum of the K meson field. This system, with center of mass removed, has a bound state (K-meson) of multiplicity two, embedded in a continuous spectrum (π meson final states), of multiplicity two, which corresponds to the minimal subspace of F invariant with respect to $\hat{H}$. The model describes the decay of K^0, $\overline{K}^0$ mesons into the 2π and 3π mesons states [2].

(ii) Resonant scattering. Pion proton scattering is characterized by a strong resonant behavior at about 1238 MeV in the center of mass system. The incident π meson-proton state, in this description, is converted by the interaction into the unstable particle, which then decays to the final π meson-proton state. The decay model can serve as a framework for this problem if one makes the obvious identifications (see also the previous example).

The problem of perturbation of eigenvalues embedded into a continuous spectrum occurs also in ordinary quantum mechanics.

(i) If an unperturbed Schrödinger operator admits a symmetry group then its spectrum is composed of the spectra of its parts on the invariant subspaces corresponding to the different types of irreducible representations of this group and isolated eigenvalues of one type of symmetry can occur on the continuous spectrum of another type [14], [15]. If a perturbation breaks the symmetry of the unperturbed system and one is interested in the fate of the eigenvalues mentioned above, he comes to the problem of perturbation of eigenvalues embedded into a continuum.

(ii) One meets a similar situation when one considers perturbation of a Hamiltonian which conserves particle number by an interaction which does not preserve the number of particles. In this case eigenvalues of the unperturbed Hamiltonian are always embedded into its continuous spectrum.

(iii) As was noted in [12] if one drops some interactions between electrons in an atom (for example, the interaction between electrons in helium) one can obtain a Hamiltonian with eigenvalues embedded into the continuous spectrum.

In all the cases considered above we have the mathematical problem of the perturbation of eigenvalues embedded into the continuous spectrum. What happens to these eigenvalues under a perturbation? As will be shown later, their multiplicities decrease (multiplicity zero means that there is no longer an eigenvalue). To follow the fate of these eigenvalues it is natural to consider an analytic continuation of the resolvent matrix elements into the second Riemann sheet in a neighborhood of the poles corresponding to the immersed eigenvalues. We shall show that the main results of the perturbation theory of the isolated poles of a resolvent are true for these also. However, some of them, or of their branches (split poles developed from degenerate eigenvalues of the unperturbed Hamiltonian) leave the real axis for the second sheet. They are no longer poles of the resolvent, but of some analytic function which is an analytic continuation (in which sense will be stated later) of the resolvent across the continuous spectrum, and therefore they are not eigenvalues of the operator. They manifest themselves in the perturbed operator through peaks in the concentration of the continuous spectrum near the disappeared eigenvalues.

These complex poles turn out to be responsible for the characteristic behaviour of the corresponding physical models (resonance effects in scattering, the behavior of unstable particles in the decay model, the approach to equilibrium in models of open systems, for example).

Much effort has been made by physicists to give a physical meaning for the complex poles of an analytic continuation of the resolvent into the second Riemann sheet. However, as the simple but characteristic mathematical model of a self-adjoint operator, the operator of multiplication by an independent variable in $L_2(-\infty, \infty)$, shows, the resolvent of a self-adjoint operator can in general have an analytic continuation through the continuous spectrum of the operator only in the sense of a sesquilinear form, and thus continuation depends on the choice of the form (i.e., the domain in $\mathscr{H} \times \mathscr{H}$ of the form $(R(z)f, \varphi)$).

If the sesquilinear forms for the resolvents of the perturbed H and the unperturbed H_o operators have the same domain and can be analytically continued into the second sheet, then the poles of the perturbed sesquilinear form which are not generated by the poles of the unperturbed one are obviously characteristic for the perturbation. However, in contrast to the real poles which are eigenvalues of H and are uniquely derived from H, they are connected with the pair (H_o, H) rather than with the single operator H.[7]) We shall show that these additional poles do not depend on the choice of the domain of the sesquilinear form.

The statement of the decay problem shows another way to find the desired complex poles. In the decay problem, as mentioned above, one is interested in the behavior of $P\,e^{-iHt}\,P$ for large t. This corresponds to a description of the singularities of the Fourier transform $P(H-z)^{-1}P$ of $\theta(t)P\,e^{-iHt}\,P$. If we have a finite number of unstable particles, then $\dim P < \infty$ and $P(H-z)^{-1}P$ has, therefore, a unique continuation in the second Riemann sheet (as an operator in a finite dimensional space $P\mathscr{H}$). The complex poles of this continuation coincide with the additional complex poles of an analytic continuation of the perturbed sesquilinear form discussed above.

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[7]) It was pointed out in [12] that the resonances should be assigned to the operator H with some additional structure specified.

This is proved by expressing the whole resolvent $(H - zI)^{-1}$ through its restriction $P(H - z)^{-1}P$ on $P\mathscr{H}$.

Note that, using the analytic continuation of $PR(z)P$, one can approximate $P\,e^{-iHt}P$ for intermediate times as follows (pole approximation)

$$P\,e^{-iHt}\,P = -\frac{1}{2\pi i}\oint_{\sigma(H)} e^{-izt}\,P(H - z)^{-1}P\,dz \approx \sum_j e^{-iz_jt}r_j, \tag{1.7}$$

where r_j are the residues of $-P(H - z)^{-1}P$, and z_j are the pole positions, in the lower half plane.

The natural question arises here, whether the analytic continuation of the resolvent across the continuous spectrum can be connected with the resolvent of some operator. The rather general situation when the answer is positive is described in [18]. We describe here one special case of it. Let there exist an operator B and a domain D in $\mathscr{H}$ such that

$$\begin{aligned} &D \subset D(H) \cap D(B), \qquad HD \subset D(B),\\ &\operatorname{Ker} B = \{f \in D(B),\, Bf = 0\} = \{0\}. \end{aligned} \tag{1.8}$$

Then

$$(H - zI)^{-1} = B^{-1}(BHB^{-1} - zI)^{-1}B \quad \text{on} \quad (H - zI)D \tag{1.9}$$

for $z \in \rho(H) \cap \rho(BHB^{-1})$, where the operator BHB^{-1} can be defined on the domain BD. Thus one can continue $(H - zI)^{-1}$ analytically as the sesquilinear form

$$((H - zI)^{-1}f, \varphi), \qquad f, \varphi \in D(B) \times D(B^{-1*}) \tag{1.10}$$

from, say $\mathbb{C}^+$ into $\rho(BHB^{-1})$. If

$$\rho(BHB^{-1}) \cap \sigma_c(H) \neq \phi \tag{1.11}$$

then this continuation is nontrivial and can be performed through $\rho(BHB^{-1}) \cap \sigma_c(H)$ into the corresponding singly connected pieces of the region $\rho(BHB^{-1}) \cap \mathbb{C}^-$ on the second Riemann sheet. The obtained continuation is equal to the sesquilinear form for the resolvent of the operator BHB^{-1};

$$((H - zI)^{-1}f, \varphi)^{\mathrm{II}} = ((BHB^{-1} - zI)^{-1}Bf, B^{*-1}\varphi) \tag{1.12}$$

We realize this general construction in the present paper utilizing the earlier Combes idea of the rotation of essential spectrum [18, 19].

As was mentioned above, the continuous spectrum of H near the disappeared eigenvalue of H_o (more precisely near the shifted value that one finds applying formally the perturbation theory and keeping only the first order) is highly 'concentrated'. In mathematical papers which deal with abstract operators this effect is described by the asymptotic properties of the resolution of identity $E(\lambda)$, for H as $H - H_o \to 0$ [20, 6, 9, 11]. Justifying the intuitive picture of the effect this way does not introduce, however, a constructive notion of the spectral concentration. In this paper we will not study the spectral concentration in the sense of [20]. In partial compensation, we give here a rough idea what happens to the continuous spectrum of H_o under perturbation when eigenvalues of H_o disappear.

We describe the behaviour of $\delta(H - \lambda)$ near $\operatorname{Re} z_j$, where z_j is a pole of an analytic continuation, $(PR(z)P)^{\mathrm{II}}$, of $PR(z)P$, $z \in \mathbb{C}^{\pm}$, into the second Riemann sheet. Using the equation

$$\delta(H-\lambda)=\frac{1}{2\pi i}[R(\lambda-io)-R(\lambda+io)] \tag{1.13}$$

and the representation of $R(z)$ through $PR(z)P$ (see equation (3.25)) we express $\delta(H-\lambda)-\overline{P}\,\delta(H-\lambda)\overline{P}$ in terms of $PR(\lambda\pm io)P$ and $\overline{P}R_o(\lambda\pm io)\overline{P}$. In particular this formula contains the factor $P\,\delta(H-\lambda)P$. Now we compute the main part of the contribution by this factor.

As it will be shown in Section 5, the main term in $(PR(z)P)^{\mathrm{II}}$, when z is close to the pole z_j is

$$(PR(z)P)^{\mathrm{II}}\sim\frac{Q_j}{z_j-z}, \tag{1.14}$$

where the operator Q_j satisfies

$$Q_j-Q_j^*=o(g^2Q_j) \tag{1.15}$$

and we have introduced the coupling constant $g:H=H_o+gV$. Recall that

$$(PR(\lambda-io)P)^{\mathrm{II}}=PR(\lambda+io)P \tag{1.16}$$

Hence the principal part of $P\,\delta(H-\lambda I)P$, when $\lambda\sim\mathrm{Re}\,z_j$, which contributes in $\delta(H-\lambda I)P_c$, is up to terms of the second order in g

$$P\,\delta(H-\lambda I)P\sim\frac{\mathrm{Im}\,z_jQ_j}{\pi|z_j-\lambda|^2} \tag{1.17}$$

Therefore, the difference between $\delta(H-\lambda)$ and $\delta(H_o-\lambda)\overline{P}$ for λ near $\mathrm{Re}\,z_j$ is, essentially, proportional to the factor $[(\mathrm{Re}\,z_j-\lambda)^2+(\mathrm{Im}\,z_j)^2]^{-1}$.

In the conclusion of the introduction we make some literary remarks. The theory of perturbation of eigenvalues embedded into a continuous spectrum has its roots in the classical Friedrichs paper [1], where the simple model of perturbation of an eigenvalue immersed into a continuum was introduced and the disappearance of this eigenvalue under perturbation was shown.

The original Friedrichs model was applied formally to various physical problems.

The most general approach stimulated by the decay and resonance scattering problem was given in [2]. The method of the study of the analytic continuation of the operator $(P(H-zI)^{-1}\,|_{P\mathscr{H}})^{-1}$ (where P is the projection operator on the eigenspace of the point spectrum of H_o immersed into the continuous one) into the second Riemann sheet, was proposed in [2] for the investigation of the behavior of embedded eigenvalues under perturbation.

The theory of the perturbation of eigenvalues embedded into continuous spectrum as a mathematical theory was initiated by J. S. Howland [3–8], where the important results of perturbation theory were found for finite rank or compact perturbation. The most general results were obtained by M. Baumgärtel and J. S. Howland [9–11] where a perturbation may be in principal unbounded. The condition imposed in these papers on operators, for the developed methods to be applicable, are implicit. Some questions of the perturbation theory of embedded eigenvalues were developed by B. Simon [12] for the case of multiparticle Schrödinger operators with dilatation analytic potentials. The present paper supplements the results of [9, 11].[8])

[8]) In fact, the manuscript of this paper was ready before the appearance of articles [9, 11].

We describe some classes of operators for which the theory developed in [9, 11] can be applied, i.e. the operator valued function $(P(H(g) - zI)^{-1} \mid_{P\mathscr{H}})^{-1}$, where $H(g)$ is a family of operators under investigation and P is the projection operator in $\mathscr{H}$ on the eigenspace corresponding to a finite system, σ_p, of eigenvalues of $H(0)$ immersed into $\sigma_c(H(0))$, has an analytic continuation into (a neighborhood of σ_p) $\times$ (a neighborhood of 0). We construct explicitly this continuation and show that the poles of this analytic continuation are semisimple (i.e. of the first order). We show also that the multiplicities of eigenvalues embedded into the continuous spectrum decrease, in general, under perturbation.

2. Mathematical preliminaries

2.1 Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$. As usual $D(A)$, $R(A)$, $\sigma(A)$ and $\rho(A)$ are its domain, range, spectrum and resolvent set, respectively. The spectrum, $\sigma(A)$, is divided into the three following parts: the absolute continuous, $\sigma_{\text{a.c.}}(A)$, the singular continuous, $\sigma_{\text{s.c.}}(A)$, and the point $\sigma_p(A)$, spectrum. The Hilbert space, $\mathscr{H}$, is divided into the three corresponding parts, $\mathscr{H}_{\text{a.c.}}$, $\mathscr{H}_{\text{s.c.}}$ and $\mathscr{H}_p$, with respect to this decomposition, $A_{\text{a.c.}}$ will denote the restriction of A to $\mathscr{H}_{\text{a.c.}}$. Let $R(z, A) = (A - zI)^{-1}$. We define the following positive function

$$\delta_\varepsilon(A - \lambda I) = \frac{1}{2\pi i}[R(\lambda + i\varepsilon, A) - R(\lambda - i\varepsilon, A)]$$

$$= \frac{\varepsilon}{\pi} R(\lambda + i\varepsilon, A) R(\lambda - i\varepsilon, A) \tag{2.1}$$

On the absolute continuous subspace $\mathscr{H}_{\text{a.c.}}$; this operator valued function has a weak limit as $\varepsilon \downarrow 0$

$$\delta(A_{\text{a.c.}} - \lambda I_{\text{a.c.}}) = w - \lim_{\varepsilon \downarrow 0} \delta_\varepsilon(A - \lambda I) \mid_{\mathscr{H}_{\text{a.c.}}}, \tag{2.2}$$

which is a weakly integrable function of $\lambda \in \mathbb{R}$, where $I_{\text{a.c.}}$ is identity operator in $\mathscr{H}_{\text{a.c.}}$.

We will use the theorem about the spectral representation of self-adjoint operators in the following form:

Theorem. *Let A be a self-adjoint operator on a Hilbert space $\mathscr{H}$. There are measures μ_i on $\mathbb{R}_i$, and Hilbert spaces X_i, where i runs through a countable index set I, and a unitary operator U from $\mathscr{H}$ into*

$$\hat{\mathscr{H}} = \bigoplus_{i \in I} L_2(\mathbb{R}_i, \mu_i, X_i),$$

where $L_2(\mathbb{R}, \mu, X,)$ is space of all μ measurable X-valued functions with finite norm $[\int \| f(x) \|_X^2 \, d\mu(x)]^{1/2}$, such that A is unitary equivalent to the operator of multiplication on the independent variable on $\hat{\mathscr{H}}$;

$$A = U^* \hat{A} U$$

$$\hat{A}\hat{f} = \{x f_i(x), i \in I\} \tag{2.3}$$

where

$$\hat{f} = \{f_i, i \in I\} \in D(\hat{A}) = \left\{\hat{f} \in \hat{\mathscr{H}}, \sum_{i \in I} \int |x|^2 \| f_i(x) \|_{X_i}^2 \, d\mu_i(x) < \infty \right\}. \tag{2.4}$$

Let $\hat{\mathscr{H}}_{\text{a.c.}}$ be the absolute continuous subspace of $\hat{\mathscr{H}}$ with respect to the operator $\hat{A}$ and $\hat{A}_{\text{a.c.}}$ is the restriction of $\hat{A}$ to $\hat{\mathscr{H}}_{\text{a.c.}}$. Note that $\hat{\mathscr{H}}_{\text{a.c.}} = U\mathscr{H}_{\text{a.c.}}$. $\hat{\mathscr{H}}_{\text{a.c.}}$ can be realized as

$$\bigoplus_{i \in I} L_2(\Delta_i, X_i)$$

where Δ_i's are Borel subsets of $\mathbb{R}$ and $L_2(\Delta, X) = L_2(\Delta, dx, X)$. dx is the Lebesgue measure on $\mathbb{R}$.

We restrict ourselves by the

Condition. The singular continuous spectrum of A is the empty set.

The resolvent, $R_{\text{a.c.}}(z, \hat{A})$, of $\hat{A}_{\text{a.c.}}$ has the form

$$R_{\text{a.c.}}(z, \hat{A})\hat{f} = \{(x - z)^{-1} f_i(x), i \in I\}, \tag{2.5}$$

where

$$\hat{f} = \{f_i, i \in I\} \in \hat{\mathscr{H}}_{\text{a.c.}}$$

It has an analytic continuation through the continuous spectrum into the second Riemann sheet only in the sense of a sesquilinear form. Now we construct this continuation. Let Ω be a domain in the lower complex semiplane $\mathbb{C}^-$ such that $\partial\Omega \cap \mathbb{R}$ contains an open set. We define $\hat{\mathscr{A}}(\Omega) = \{\hat{f} \in \hat{\mathscr{H}}_{\text{a.c.}}, f_i(x)|_{\partial\Omega \cap \mathbb{R}}$ is analytic and has an analytic continuation into Ω $\forall i \in I\}$.

Then for any $\hat{f}, \hat{\varphi}^* \in \hat{\mathscr{A}}(\Omega)$ one can define an analytic continuation of the form

$$(R_{\text{a.c.}}(z, \hat{A})\hat{f}, \hat{\varphi})$$

from $\mathbb{C}^+$ into Ω by

$$(R_{\text{a.c.}}^{\text{II}}(z, \hat{A})\hat{f}, \hat{\varphi}) \overset{\text{def.}}{\equiv} (R_{\text{a.c.}}(z, \hat{A})\hat{f}, \hat{\varphi})^{\text{II}} \overset{\text{def.}}{\equiv}$$

$$(R_{\text{a.c.}}(z, \hat{A})\hat{f}, \hat{\varphi}) + 2\pi i \sum_{i \in I} f_i(z)\varphi_i^*(z), \quad z \in \Omega, \tag{2.6}$$

where $f_i(z)$ and $\varphi_i(z)$, $i \in I$, are the analytic functions in Ω with values equal to $f_i(x)$ and $\varphi_i(x)$, respectively on $\partial\Omega \cap \mathbb{R}$. This expression can be written symbolically as

$$R_{\text{a.c.}}^{\text{II}}(z, \hat{A}) = R_{\text{a.c.}}(z, \hat{A}) + 2\pi i \, \delta(z, \hat{A}), \tag{2.7}$$

where $\delta(z, \hat{A})$ denotes the following sesquilinear form

$$(\delta(z, \hat{A})\hat{f}, \hat{\varphi}) = \sum_{i \in I} f_i(z)\varphi_i(z)^*, \quad \hat{f}, \hat{\varphi} \in \hat{\mathscr{A}}(\Omega) \tag{2.8}$$

Since the restriction of the resolvent of $\hat{A}$ onto the point subspace, $\hat{\mathscr{H}}_p$ with respect to $\hat{A}$, is an analytic operator function with the exception of poles at the eigenvalues of $\hat{A}$, we have for the full resolvent

$$R^{\text{II}}(z, \hat{A}) = R(z, \hat{A}) + 2\pi i \, \delta(z, \hat{A})\hat{P}_{\text{a.c.}}, \tag{2.9}$$

where $\hat{P}_{\text{a.c.}}$ is the projection operator on $\hat{\mathscr{H}}$ with the range $\hat{\mathscr{H}}_{\text{a.c.}}$.

In order to translate this description in terms of the operator A one should properly apply the unitary operator U. This gives the following definitions

$$\mathscr{A}(\Omega) = U^*(\hat{\mathscr{A}}(\Omega) \oplus \hat{\mathscr{H}}_p) \tag{2.10}$$

$$\delta(z, A) = U^* \, \delta(z, \hat{A})\hat{P}_{\text{a.c.}} U \tag{2.11}$$

$$R^{\text{II}}(z, A) = R(z, A) + 2\pi i \, \delta(z, A) \tag{2.12}$$

Obviously, if one replaces in the definition of $\hat{\mathscr{A}}(\Omega)$ the requirement of analyticity by the condition of meromorphicity one can get any desired poles for the analytic continuation $\hat{R}^{\text{II}}(z, A)$.

2.2 Now we describe another way of obtaining an analytic continuation of the resolvent through the continuous spectrum. To get this we need an additional restriction on the operator A:

Condition. The Borel subsets Δ_i in the spectral representation for A are semiaxes of the form

$$\Delta_i = [E_i, +\infty), \quad E_i \in \mathbb{R}, \; i \in I$$

We define the unitary operator of the dilatation of the continuous spectrum of A:

$$(\hat{U}(\theta)\hat{f})_i(x) = e^{-\theta/2}(\hat{P}_{\text{a.c.}}\hat{f})_i(E_i + (x - E_i)\, e^{-\theta}), \quad \theta \in \mathbb{R}, \hat{f} \in \hat{\mathscr{H}}, \tag{2.13}$$

and

$$U(\theta) = P_p + U^*\hat{U}(\theta)U \tag{2.14}$$

where $P_p = I - P_{\text{a.c.}}$ is the projection operator into the point subspace $\mathscr{H}_p$. One can then define the family of the unitary equivalent operators as

$$A(\theta) = U(\theta)AU(\theta)^{-1} \tag{2.15}$$

This definition implies

$$A(\theta) = U^*\hat{A}(\theta)U, \tag{2.16}$$

where

$$\hat{A}(\theta) = \hat{A}\hat{P}_p + [\hat{E} + (\hat{A} - \hat{E})\, e^{-\theta}]\hat{P}_{\text{a.c.}}$$

$$\hat{E}\{f_i, i \in I\} = \{E_i f_i, i \in I\} \tag{2.17}$$

Thus $A(\theta)$ has obviously an analytic continuation into the whole complex plane of θ and

$$\sigma_{\text{es.}}(A(\theta)) = \sigma_{\text{a.c.}}(A(\theta)) = \bigcup_{i \in I} \{E_i + e^{-\theta}\overline{\mathbb{R}^+}\},$$

$$\sigma_p(A(\theta)) = \sigma_p(A)$$

$$\sigma_d(A(\theta)) = \sigma_p(A) \backslash (\sigma_p(A) \cap \{E_i\}) \tag{2.18}$$

The space of analytic in a complex strip

$$\Sigma = \{z \in \mathbb{C}, |\text{Im}\, z| < a\}$$

vectors for the group $U(\theta)$, $\theta \in \mathbb{R}$, is defined as $D(\Sigma) = \{f \in \mathscr{H}, U(\theta)f, \theta \in \mathbb{R}$, has an analytic continuation from $\mathbb{R}$ into the complex strip $\Sigma\}$

The right side of the identity

$$(R(z, A)f, \varphi) = ((A(\theta) - zI)^{-1}U(\theta)f, U(\theta)\varphi), \quad \theta \in \mathbb{R}, z \in \mathbb{C}^+, \tag{2.19}$$

has an analytic continuation into a complex strip Σ for any pair $f, \varphi \in D(\Sigma)$:

$$(R(z, A)f, \varphi) = ((A(\theta) - zI)^{-1}U(\theta)f, U(\bar{\theta})\varphi), \quad \theta \in \Sigma, \quad z \in \mathbb{C}^+ \tag{2.20}$$

However the right side of this equality for any fixed nonreal $\theta \in \Sigma$ is analytic in $z \in \rho(A(\theta))$ (by (2.18), $\rho(A(\theta)) \supset \mathbb{C}^+$ for $\operatorname{Im} \theta < 0$) and therefore this equality can be used for an analytic continuation of $(R(z, A)f, \varphi)$, $z \in \mathbb{C}^+$, into the lower semiplane (of the second Riemann sheet), namely, into the domain

$$G_\theta = \rho(A(\theta)) \bigcap \mathbb{C}^- = \{z \in \mathbb{C}^-, z \notin \bigcup_i (E_i + \overline{\mathbb{R}^+ e^{-\theta}})\} \tag{2.21}$$

Note that

$$U(\theta)^{-1}(A(\theta) - zI)^{-1}U(\theta) \neq R(z, A) \quad \text{for} \quad \theta \in \Sigma, \operatorname{Im} \theta \neq 0, \quad z \in G_\theta, \tag{2.22}$$

where

$$G_\theta = \mathbb{C}^\pm \backslash \underset{i \in I}{U} \{E_i + \overline{e^{-\theta}\mathbb{R}^+}\}, \quad \theta \in \mathbb{C}^\mp.$$

2.3. **Proposition.** *Let V be an A-compact*[9]) *operator and $V(\theta) = U(\theta)VU(\theta)^{-1}$, $\theta \in \mathbb{R}$, have an A-compact analytic continuation into the complex strip Σ, then*
(*i*) *The operator*

$$B(\theta) = A(\theta) + V(\theta) = U(\theta)BU(\theta)^{-1}, \quad B = A + V \tag{2.23}$$

has an analytic extension from $\mathbb{R}$ into Σ
(*ii*) *The following relations hold:*

$$\begin{aligned} &\sigma_{\text{es}}(B(\theta)) = \sigma_{\text{a.c.}}(B(\theta)) = \sigma_{\text{a.c.}}(A(\theta)), \quad \theta \in \Sigma, \\ &\sigma_d(B(\theta)) \cap (\mathbb{C}^+ \cup \mathbb{R}) = \sigma_p(B) \backslash (\sigma_p(B) \cap \{E_i\}) \end{aligned} \tag{2.24}$$

The proof of this proposition is trivial. This proposition implies

2.4. **Proposition.** *Let the assumptions of Proposition 2.3 be satisfied. Then* (*a*) *the function*

$$F(z, f, \varphi) = (R(z, B)f, \varphi) \forall f, \varphi \in D(\Sigma) \tag{2.25}$$

has an analytic continuation, $F^{\text{II}}(z, f, \varphi)$, in z across $\sigma_{\text{a.c.}}(A)$ from the upper semiplane, $\mathbb{C}^+$ into the domain G_θ; (*b*)*$F^{\text{II}}(z, f, \varphi)$ has poles in G_θ at the values of $\sigma_d(B(\theta))$;* (*c*)*$F^{\text{II}}(z, f, \varphi)$ can be represented as*

$$F^{\text{II}}(z, f, \varphi) = ((B(\theta) - zI)^{-1}U(\theta)f, U(\bar{\theta})\varphi), \quad \theta \in \Sigma, z \in G_\theta \tag{2.26}$$

---

[9]) Compact from $D(A)$ to $\mathscr{H}$. (Scalar product in $D(A)$, $(f, \varphi)_D = (Af, A\varphi) + (f, \varphi)$).

Proof. (i). $U(\theta)$, $\theta \in \mathbb{R}$, are unitary operators in $\mathscr{H}$. Hence

$$((B - z)^{-1}f, \varphi) = ((B(\theta) - z)^{-1}U(\theta)f, U(\theta)\varphi), \quad \theta \in \mathbb{R}. \ \forall f, \varphi \in \mathscr{H}. \tag{2.27}$$

(ii) If $\operatorname{Im} z > 0$ and $f, \varphi \in D(\Sigma)$ the right side of this equality has an analytical extension in θ from $\mathbb{R}$ to the complex strip.

(iii) The resolvent $(B(\theta) - z)^{-1}$ is analytic for $z \in \mathbb{C}\backslash\sigma(B(\theta))$ and $G_\theta \subset \mathbb{C}\backslash\sigma_{es}(B(\theta))$. Hence the function

$$F(z, f, \varphi) = ((B(\theta) - z)^{-1}U(\theta)f, U(\overline{\theta})\varphi), \quad \theta \in \Sigma, \tag{2.28}$$

has an analytic extension in z from the upper semiplane to domain G_θ, which has poles in the points of the discrete spectrum of $B(\theta)$.

3. Decomposition with respect to a projection operator

3.1. We will say that an operator H on a Hilbert space $\mathscr{H}$ is decomposed with respect to a projection operator P in $\mathscr{H}$ if

$$H = H_o + V,$$

where

$$H_o = PHP + \overline{P}H\overline{P},$$
$$V = PH\overline{P} + \overline{P}HP$$

and

$$\overline{P} = I - P.$$

Note that

$$\{H_o, P] = [H_o, \overline{P}] = 0, \tag{3.1}$$
$$PV = V\overline{P}, \overline{P}V = VP \tag{3.2}$$

We consider now the eigenvalue problem for H:

$$(H - zI)f = 0$$

The components of this equation along $P\mathscr{H}$ and $\overline{P}\mathscr{H}$ are

$$(H_o - zI)Pf + PV\overline{P}f = 0$$
$$(H_o - zI)\overline{P}f + \overline{P}VPf = 0$$

The second equation shows that $\overline{P}VPf$ belongs to the domain of the inverse to $H_o - zI$ if $z \notin \sigma_p(H_o\overline{P})$. Hence this equation can be rewritten as follows

$$\overline{P}f = -(H_o - zI)^{-1}\overline{P}VPf \tag{3.3}$$

Inserting this expression for $\overline{P}f$ into the first equation one gets

$$(H_o - zI - V\overline{P}(H_o - zI)^{-1}\overline{P}V)Pf = 0$$

For the corresponding eigenfunction one gets from (3.3) the following equation

$$f = (I - (H_o - zI)^{-1}\overline{P}V)Pf$$

We introduce the operators

$$h(z) = (H_o - zI - V(H_o - zI)^{-1}V)P$$

and

$$n(z) = (I - (H_o - zI)^{-1}V)P$$

on the Hilbert space $P\mathscr{H}$ for $z \in \sigma_p(H_0)$, and rewrite in their terms the relations obtained above:

If λ is an eigenvalue of H and f the corresponding eigenvector, and $\lambda \notin \sigma_p(H_o)$, then Pf satisfies the equations

$$h(\lambda \pm i0)Pf = 0$$

and f is reconstructed from Pf by

$$f = n(\lambda \pm i0)Pf$$

3.2. Now we prove the converse statement:

If some element $\varphi \in P\mathscr{H}$ satisfies

$$h(z)\varphi = 0$$

for some z such that $n(z)\varphi$ is defined and

$$n(z)\varphi \in D(H) \tag{3.4}$$

then z is an eigenvalue of H and

$$\psi = n(z)\varphi \tag{3.5}$$

is a corresponding eigenfunction.

Indeed, using the simple operator identity

$$(A + B)(I - A^{-1}B) = A - BA^{-1}B$$

we get

$$(H - z)(I - (H_o - z)^{-1}V) = H_o - z - V(H_o - z)^{-1}V \tag{3.6}$$

where

$$[H_o - z - V(H_o - z)^{-1}V, P] = [H_o - z - V(H_o - z)^{-1}V, \bar{P}] = 0$$

Then (3.6) multiplied with P from right yields the identity

$$(H - z)n(z) = h(z) \tag{3.7}$$

which proves the desired statement.

So we get the recipe how to find eigenvalues and eigenfunctions of H.

(1) one finds the singular points of the operator $h(z)$ (in the space $P\mathscr{H}$) corresponding to the eigenvalue 0,

(2) then these singular points are eigenvalues of H and (3.5), corresponding eigenfunctions, if (3.4) holds.

Note that (3.4) implies that function $V\varphi$ in $H_o\bar{P}$—representation is zero at point z and sufficiently smooth near this point. More detail, let U_o be an unitary operator from

$\mathscr{H}$ to the space of spectral representation of H_o, thus (3.4) implies that

$$(U_o V\varphi)(z) = 0, \tag{3.8}$$

$$\exists\, \delta: \int_{|\lambda - z| < \delta} \frac{|U_0 V\varphi(\lambda)|^2}{|\lambda - z|^2}\, d\lambda < \infty.$$

Remarks. (1) If the projection operator P is finite-dimensional then $h(z)$ is an operator in a finite-dimensional space. In this case we can find singular points of $h(z)$ corresponding to the eigenvalue 0 as zeros of det $h(z)$;

(2) Because of (3.8) it is not important how we understand $h(v)$ and $n(v)$, at a zero of det $h(v)$; here

$$h(v + i0) = h(v - i0), \qquad n(v + i0) = n(v - i0).$$

3.3. Now we consider the operator $h(z)$. Denote

$$H^o = H_o\overline{P},\ R^o(z) = (H_o - zI)^{-1}\overline{P}, \lambda = \operatorname{Re} z$$

We have

$$h(z) = (H_o - \lambda I - V(H^o - \lambda)R^o(\bar{z})R^o(z)V)P$$
$$- i \operatorname{Im} z(I + VR^o(\bar{z})R^o(z)V)P$$

Note that

$$VR^o(\bar{z})R^o(z)V \geq 0,$$

i.e. $h(z)$ is dissipative if $\operatorname{Im} z < 0$ and accretive if $\operatorname{Im} z > 0$. Hence the equation

$$h(z)f = 0$$

has no nontrivial solution if $\operatorname{Im} z \neq 0$ (we can also see this directly from (3.7)) or, what is the same, the function det $h(z)$ has no zeros for $\operatorname{Im} z \neq 0$.

Note, that $n(z)$ is analytic in $\mathbb{C}\backslash\sigma(H^o)$ and

$$h(z)\mathscr{H} \subset D(H) \quad \text{if} \quad z \notin \sigma(H^o)$$

Further we want to study the boundary values of the operators $n(z)$ and $h(z)$ on $\sigma(H^o)$. For this we need one more condition on H (which is always satisfied in physical theories).

Condition. The essential spectrum of H^o contains only absolute continuous part. We have

$$n(\lambda \pm i0) = (I - R^o(\lambda)V)P \mp \pi i\, \delta(H^o - \lambda)VP, \tag{3.9}$$

where

$$R^o(\lambda) = \tfrac{1}{2}(R^o(\lambda + i0) + R^o(\lambda - i0)) \tag{3.10}$$

(3.9) shows that the condition $n(\lambda \pm i0)f \in D(H)(f \in P\mathscr{H})$ implies that

$$\delta(H^o - \lambda)Vf = 0 \tag{3.11}$$

Note further that (3.11) is the same as

$$V\,\delta(H^o - \lambda)Vf = 0 \tag{3.12}$$

Furthermore we have

$$h(\lambda \pm i0) = h, (\lambda) \mp ih_2(\lambda), \tag{3.13}$$

where

$$h, (\lambda) = (H_o - \lambda - VR^o(\lambda)V)P, \tag{3.14}$$

$$h_2(\lambda) = \pi V\,\delta(H^o - \lambda)VP \geq 0 \tag{3.15}$$

Because of (3.15) the equation

$$h(\lambda \pm i0)f = 0 \quad (f \in P\mathscr{H}) \tag{3.16}$$

implies

$$h_2(\lambda)f = 0 \tag{3.17}$$

and therefore also

$$h_1(\lambda)f = 0 \tag{3.18}$$

So we have

Proposition. *λ, $\lambda \notin \sigma_p(H_o)$, is an eigenvalue of the operator H with an eigenfunction ψ if and only if*

$$n(\lambda + i0)P\psi \in D(H) \tag{3.19}$$

and

$$h(\lambda + i0)P\psi = 0 \tag{3.20}$$

(One can replace $n(\lambda + i0)$ and $h(\lambda + i0)$ by $n(\lambda - i0)$ and $h(\lambda - i0)$ in (3.19), (3.20)).

Corollary. *If vectors from the subspace $P\mathscr{H}$ satisfy in H^o—representation some sufficient estimates of smoothness like*

$$\int \frac{|\varphi(\nu) - \varphi(\lambda)|^2}{|\nu - \lambda|^2}\,d\nu < C\;\forall \lambda \in \sigma_c(H^o), \tag{3.21}$$

then $\lambda \in \sigma_p(H)\backslash\sigma_p(H_o) \cap \sigma_p(H)$ if and only if equation (3.16) has a nontrivial solution, i.e. λ is a zero of det $h(z)$. *The multiplicity of λ is equal to the order of the corresponding zero* det $h(z)$.

3.4. Structure of the resolvent of H. Now we deduce the important formula for the resolvent $R(z) = (H - zI)^{-1}$ which plays an important role in this work. First of all we express $R(z)$ through its part $PR(z)P$ in $P\mathscr{H}$. Using equation

$$R = R_o - R_o VR$$

we get

$$\overline{P}RP = -\overline{P}R_o VPRP \tag{3.22}$$

and after conjugation

$$PR\overline{P} = -PRPV\overline{P}R_o\overline{P} \tag{3.23}$$

(we use here the known property of resolvent: $R^*(z) = R(\bar{z})$). Then

$$\bar{P}R\bar{P} = \bar{P}R_o\bar{P} - \bar{P}R_o VPR\bar{P} = \bar{P}R_o\bar{P} + \bar{P}R_o VPRPV\bar{P}R_o\bar{P} \tag{3.24}$$

Adding (3.22)–(3.24) and PRP we find

$$R(z) = \cdot R_o(z)\bar{P} + (I - R_o(z)\bar{P}V)PR(z)P(I - V\bar{P}R_o(z)) \tag{3.25}$$

In order to find $PR(z)P$ we apply the operator P from the right and left to the identity

$$R(z) = (I - R_o(z)V)(H_o - z - VR_o(z)V)^{-1} \tag{3.26}$$

and use (3.1), (3.2);

$$PR(z)P = h^{-1}(z) \tag{3.27}$$

(the inverse here is understood on $P\mathscr{H}$). Thus we have

$$R(z) = R^o(z) + (I - R^o(z)V)h^{-1}(z)(I - VR^o(z)) \tag{3.28}$$

4. Perturbation theory for eigenvalues embedded into the continuous spectrum. I. General remarks

4.1. We consider a family of self adjoint operators $H(g)$ in a Hilbert space $\mathscr{H}$, where the parameter g varies in a real neighborhood, Δ, of 0. We suppose that $H(0)$ has a continuous spectrum and a number of eigenvalues of finite multiplicity immersed into the continuous spectrum. The problem we study is how these eigenvalues behave when g changes.

Let P be the projection operator in $\mathscr{H}$ on the eigenspace of $H(0)$ corresponding to some finite system, σ_p, of the eigenvalues of $H(0)$ embedded into the continuous spectrum of $H(0)$. We decompose the operator $H(g)$ for any fixed $g \in \Delta$ with respect to the projection P as it was done in the previous section. All results of Section 3 can be applied to the pair $(H(g), P)$ with any fixed $g \in \Delta$. We restrict the choice of the family by the following

Condition. There are real neighborhoods Δ of 0 and V of σ_p such that the operator $H^o(g) = H_o(g)\bar{P} = \bar{P}H(g)\bar{P}$ has no eigenvalues in the neighborhood V for all $g \in \Delta$.

4.2. We repeat here the main definitions of the previous Section applying them to the pair $(H(g), P)$ as defined above. We have

$$H_o(g) = PH(g)P + \bar{P}H(g)\bar{P}, \qquad V(g) = PH(g)\bar{P} + \bar{P}H(g)P,$$

$$H^o(g) = H_o(g)\bar{P} = \bar{P}H(g)\bar{P}, \qquad R^o(z, g) = (H_o(g) - zI)^{-1}\bar{P}$$

$$h(z, g) = (H_o(g) - zI - V(g)R^o(z, g)V(g))P$$

$$n(z, g) = (I - R^o(z, g)V(g))P$$

4.3. As was shown in the previous section all eigenvalues of $H(g)$ for any fixed $g \in \Delta$ in the neighborhood V are singular points of the operator $h(z, g)$ (for properly chosen σ_p) in the finite dimensional space, i.e. zeros of the function

$$\Delta(z, g) = \det h(z, g) \tag{4.1}$$

The function $\Delta(z, g)$ has zeros at σ_p when $g = 0$: $h(z, 0) = (H(0) - zI)P$. In order to follow the behavior of these zeros when g changes it is necessary to know the analytic properties of $\Delta(z, g)$ in a complex neighborhood of the set $\sigma_p \times \{0\}$.

The operator-valued function $h(z, g)$, and therefore the function $\Delta(z, g)$, has the same analytic properties in z as the resolvent $R^o(z, g)$: it is analytic in $\rho(H^o(g))$ and has a cut along $\sigma_c(H^o(g))$ for any fixed g.

4.4. If the family $H(g)$ is analytic of type A in the sense of [21] in the domain Δ, $\Delta \subset \mathbb{R}$, then there exists a complex neighborhood, Q, of Δ, such that $h(z, g)$ is an analytic operator-function in $D \times Q$ for any compact $D \subset \mathbb{C}^{\pm}$.

This assertion is a consequence of the analyticity of the resolvent $R^o(z, g)$, as an operator function of two variables, unless $z \in \sigma(H^o(g))$, and the theorem about continuation of analytic function of mixed variables [22], applied to the function $h(z, g)$ (see Appendix, paragraph (a) of the proof).

Analogously, if $h(z, g)$ has an analytic continuation from $\mathbb{C}^+ \times \Delta$ (resp. $\mathbb{C}^- \times \Delta$) into $\Omega \times \Delta$, where Ω is a complex domain, containing V, then it can be further continued into $\Omega' \times Q$, where Ω' is any compact subset of Ω (or Ω itself if it is compact) and Q as above.

4.5. Now we assume that $h(z, g)$ has an analytic continuation into $\Omega \times Q$ and deduce some consequences of this conjecture.

We define the multiplicity of a singular point, z_o, of a family $A(z)$ of finite dimensional operators as dim Ker $A(z_o)$.

Proposition [11]. *If $h(z, g)$ has an analytic continuation, $h^{\mathrm{II}}(z, g)$ from $\mathbb{C}^+ \times \Delta$ into the neighborhood $\Omega \times Q$ through $\sigma_c(H^o(g))$, then there are in a neighborhood of $g = 0$ exactly* dim P *(counting multiplicities) singular points $z_{ij}(g)$($j = 1, \ldots$, dim Ker $(H(0) - \lambda_i I)P$, $\lambda_i \in \sigma(H(0)P)$) of the operator $h^{\mathrm{II}}(z, g)$, corresponding to the eigenvalue* 0. *They may be labelled in such way that each $z_{ij}(g)$ has a Puiseux series expansion in g.*

If

$$z_{ij}(g) = \lambda_i + \alpha_{i1}\omega^j g^{1/p} + \alpha_{i2}\omega^{2j} g^{2/p} + \cdots (j = 1, \ldots p)$$

is a given cycle, where $\omega = \exp(2\pi i/p)$, then either $p = 1$ and all the α_{in}'s are real, or the series has the form

$$z_{ij}(g) = \lambda_i + \alpha_{i,p} g + \cdots + \alpha_{i,2np} g^{2n} + \alpha_{i,2np+1}\omega^j g^{(2np+1)/p} + \cdots \tag{4.2}$$

where $\alpha_{i,p}, \ldots, \alpha_{i,(2np-1)}$ are real, and Im $\alpha_{i,2np} < 0$.

The same statement is true for a continuation from the lower semiplane, $\mathbb{C}^-$.

The proof of this proposition is based on the remark, that singular points of $h^{\mathrm{II}}(z, g)$, corresponding to the eigenvalues 0, are zeros of the analytic in $\Omega \times Q$ function

$$\Delta^+(z, g) = \det h^{\mathrm{II}}(z, g)$$

with the property

$$\Delta^+(z, 0) = \det (H(0) - zI)P$$

and the Weierstrass preparation theorem. The latter says [22, p 188]: Let $f(z, w)$ be analytic in a neighborhood of the origin, with

$$f(z, 0) = az^m \quad (m \text{ integer, positive})$$

Then there exist a neighborhood U of the origin and a function $F(z, w)$ which is analytic and nonvanishing in U such that throughout U, f can be expressed in the form

$$f(z, w) = (z^m + p_{m-1}(w)z^{m-1} + \cdots + p_o(w))F(z, w)$$

where the p's are analytic in a neighborhood of $w = 0$ and

$$P_j(0) = 0, \quad j = 0, 1, \ldots, m-1.$$

The functions $F, p_o, \ldots p_{m-1}$, are uniquely determined. Thus this theorem shows that the zeros of analytic functions are locally roots of polynomials. The special form (4.2) of the series is the consequence of the fact that Im $z_{ij}(g) \leq 0$ (since $(h^{\text{II}}(z))^{-1}$ is analytic in $\mathbb{C}^+$).

4.6. **Corollary.** *If $z_{ij}(g)$ is real for all small g, i.e. $z_{ij}(g)$ is an eigenvalue of $H(g)$ embedded into the continuous spectrum, then $z_{ij}(g)$ is analytic in g in a neighborhood of zero.*

4.7. Thus the real poles of $h^{\text{II}}(z, g)^{-1}$ are the eigenvalues of $H(g)$ belonging to σ_p when $g = 0$. Now we show that multiplicities of the eigenvalues of $H(0)$, embedded into the continuous spectrum, decrease, generally, under the perturbation $H(g) - H(0)$.

Proposition. *Let the condition of Proposition 4.5 be satisfied, $v \in \sigma_p$ and $\lambda_j(g) \in \sigma_p(H(g))$, $\lambda_j(0) = v, j = 1, \ldots J$. Let, in addition*

$$\Gamma^* \, \delta(H(0)\overline{P} - v)\Gamma \neq 0, \quad \textit{where} \quad \Gamma = \overline{P}H'(0)P. \tag{4.3}$$

Then

$$\sum_{j=1}^{J} \dim \operatorname{Ker} (H(g) - \lambda_j(g)I) < \dim \operatorname{Ker} (H(0) - vI).$$

Proof. By Proposition 3.3 for every j there exist $m_j = \dim \operatorname{Ker} (H(g) - \lambda_j(g)I)$ linear independent vectors $f_m^{(j)}(g)$, $g \neq 0$, satisfying

$$h(\lambda_j(g), g)f_m^{(j)}(g) = 0.$$

and therefore

$$\Gamma^*(g) \, \delta(H_o(g) - \lambda_j(g)I)\Gamma(g)f_m^{(j)}(g) = 0,$$

where we denoted

$$\Gamma(g) = g^{-1}\overline{P}V(g)P = g^{-1}\overline{P}(V(g) - V(0))P.$$

By Corollary 4.6 $\lambda_j(g)$ is an analytic function at $g = 0$. Then $f_m^{(j)}(g)$, $m = 1 \ldots m_j$, are analytic vector-functions at $g = 0$ as eigenvectors of the analytic family of the selfadjoint (for real g) operators

$$h(\lambda_j(g), g) - \lambda_j(g)P$$

(with eigenvalue $\lambda_j(g)$). Hence

$$\Gamma^*(0) \, \delta(H^o(0) - vI)\Gamma(0)f_m^{(j)}(0) = 0$$

This equation, the remark that $f_m^{(j)}(0)(m = 1 \ldots m_j, j = 1 \ldots J)$ are linearly independent, and Condition (4.3), imply that the number, $\Sigma_{j=1}^J m_j = \Sigma_{j=1}^J \dim \operatorname{Ker} (H(g)$

$-\lambda_j(g)I)$, of the vectors $f_m^{(j)}(0)$ must be less than the dimension of the space Ker $(H(o) - \nu I)$ where they lie.

Corollary. *If the conditions of Proposition 4.4 and 4.7 are satisfied then $h^{\mathrm{II}}(z, g)^{-1}$ in a neighborhood of $g = 0$, $g \neq 0$, has at least $\Sigma_{\nu \in \sigma_p} 1$ nonreal poles.*

Indeed, let $\nu \in \sigma_p$ and m_ν be multiplicity of ν. Then by Proposition 4.5 the total multiplicity of the poles of $h^{\mathrm{II}}(z, g)^{-1}$ split from ν is equal to m_ν. By proposition 4.7, the total multiplicity of those which are real is less than m_ν.

4.8. The values of the parameter g at which some different resonance eigenvalues of $H(g)$ coincide and equal ν will be called $(\nu -)$ exceptional points.

Corollary. *Let the conditions of Proposition 4.7 be satisfied. If $g = 0$ is not a ν-exceptional point of $H(g)$ then the eigenvalue ν of $H(0)$ disappears under perturbation, that is, there exists a neighborhood, U, of $g = 0$ such that $H(g)$ has no eigenvalue near ν for $g \in U \backslash \{0\}$.*

It follows from the definition that the set of exceptional points is not an invariant characteristic of the family $H(g)$. It depends on the chosen analytic continuation of $(H(g) - zI)^{-1}$ through the spectrum. This definition implies immediately the following

Proposition. *A compact subset of the domain of analyticity of $H(g)$ may contain at most a finite number of the exceptional points.*

Remark. It is not difficult to formulate a condition for an eigenvalue $\nu \in \sigma_p(\mathrm{H(o)})$ to disappear completely under small changes of g even if $g = 0$ is an exceptional point.

Remark. All considerations of this section can be carried out into the case when $H(g)$ is continuous in the generalized sense of [21]. The first part of Proposition 4.5 (the existence and the total multiplicity of the singular points of $h^{\mathrm{II}}(z, g)^{-1}$; here $h^{\mathrm{II}}(z, g)$ is an analytic continuation of $h(., g)$ in z into Ω for any $g \in \Delta$) remains valid also in this case. Proposition 4.7 will be valid in this case if the additional condition of the simplicity of the eigenvalues in question is imposed.

5. Perturbation of eigenvalues embedded into the continuous spectrum. II. Continuation of the resolvent into the second Riemann sheet.

5.1. General remarks

As was shown in Section 2, the resolvent $R(z, g) = (H(g) - zI)^{-1}$ may have an analytic continuation through the continuous spectrum (into the second Riemann sheet) only in sense of a sesquilinear form, $(R(z, g)f, \varphi)$, and the continuation depends on the chosen domain (in $\mathscr{H} \times \mathscr{H}$) of this form. On the contrary, $h(z, g)$ has, if so, a unique analytic continuation in the operator sense, since $h(z, g)$ is an operator on a finite dimensional Hilbert space.

Let $D_1(g) \times D_2(g)$ be a domain on which the form $(R^o(z, g)f, \varphi)$ has an analytic continuation across the continuous spectrum. Then if $V(g)P\mathscr{H} \subset D_1(g)$ the form

$(n(z, g)f, \varphi)$ on the domain $\mathcal{H} \times D_2(g)$ has also an analytic continuation across the continuum and the operator $n^*(z, g)$ has an analytic continuation in the operator sense. The last continuation is an unbounded operator with domain $D_2(g)$. Using equation (3.28) we can construct an analytic continuation of $R(z, g)$ for any fixed $g \in \Delta$ across the continuum as a sesquilinear form on the same domain as $R^o(z, g)$:

$$R^{\mathrm{II}}(z, g) = R^{o\mathrm{II}}(z, g) - n^{\mathrm{II}}(z, g)h^{\mathrm{II}}(z, g)^{-1}(n^*(\bar{z}, g))^{\mathrm{II}} \tag{5.1}$$

or more detail

$$R^{\mathrm{II}} = R^{o\mathrm{II}} + (I - R^{o\mathrm{II}}V)h^{\mathrm{II}-1}(I - VR^{o\mathrm{II}}) \tag{5.1'}$$

This equation shows that the poles of $R^{\mathrm{II}}(z, g)$, in addition to those of $R^{o\mathrm{II}}(z, g)$, are the poles of $h^{\mathrm{II}}(z, g)^{-1}$, i.e. the singular points of $h^{\mathrm{II}}(z, g)$ corresponding to the eigenvalue 0, which are also zeros of the analytic function

$$\Delta(z, g) = \det h^{\mathrm{II}}(z, g)$$

Thus the problem of finding the poles of an analytic continuation of the resolvent of the operator $H(g)$ which are not generated by poles of an analytic continuation of the resolvent of $H_o(g)$ is reduced to studying of the singular points of the operator valued function $h^{\mathrm{II}}(z, g)$ corresponding to the eigenvalue 0 in the finite dimension Hilbert space $P\mathcal{H}$, or, what is the same, zeros of the function $\Delta(z, g)$. Recall, that the following equation

$$h^{\mathrm{II}}(z, 0) = h(z, 0) = (H(0) - zI)P \tag{5.2}$$

is used as an initial condition.

The following statement, which is a special case of the lemma proved in Appendix, reduces the problem of an analytic expansion of $h(z, g)$ as a function of two complex variables to the problem of an analytic expansion of $h(z, g)$ as a function of z for any fixed $g \in \Delta$ (recall that $\Delta \subset \mathbb{R}$):

Lemma. *If for any $g \in \Delta$, $h(., g)$ has an analytic continuation, $h^{\mathrm{II}}(., g)$, from $\mathbb{C}^+$ through $\sigma_c(H^o(g))$ into an open domain Ω, $\Omega \cap \mathbb{C}^- \neq \phi$, (i.e. into the second Riemann sheet), then $h^{\mathrm{II}}(z, g)$ is an analytic operator-function of two complex variables in $\Omega^\varepsilon \times Q^\varepsilon$, where Q^ε is a complex neighborhood of Δ, $\Omega^\varepsilon = \{z \in \Omega, \operatorname{dist}(z, \partial\Omega) > \varepsilon\}$.*

5.2. Order of the poles on the second sheet. Now we give another proof Proposition 4.5 which allows as to get some additional information about poles of $h^{\mathrm{II}}(z)^{-1}$ (and therefore of $R^{\mathrm{II}}(z)$). We consider

$$W(z, g) = h^{\mathrm{II}}(z, g) - zI - H_o(g)P$$

as a perturbation of the operator $H_o(g)P$ (in the space $P\mathcal{H}$) and study an eigenvalue problem for the operator

$$A(z, g) = H_o(g)P + W(z, g) = h^{\mathrm{II}}(z, g) - zI$$

in the finite-dimension Hilbert space $P\mathcal{H}$. By definition $A(z, g)$ is an analytic operator valued function of two variables in the domain $\Omega \times Q$.

Let $\sigma(H(0)P) = \{\lambda_i\}_1^s$. In conformity with analytic perturbation theory [21] each eigenvalue λ_i of $H(0)P$ splits into r_i eigenvalues $\lambda_{ij}(z, g)(j = 1 \ldots r_i)$ of $A(z, g)$,

$$\lambda_{ij}(z, 0) = \lambda_i, \tag{5.3}$$

and $\lambda_{ij}(z, g)$'s as functions of each variable and therefore as functions of both variables are analytic in the principal values of fractional powers, say $z^{1/p}$ and $g^{1/q}$, of z and g. The total multiplicity for these eigenvalues is equal to the multiplicity of λ_i. $A(z, g)$ has no other eigenvalues aside from these.

Consider the dim P equations

$$\lambda_{ij}(z, g) = z \qquad j = 1,\ldots, r_i \quad i = 1,\ldots, s.$$

By the theorem about implicit functions [22, p 39] (or one can apply also the Weierstrass preparation theorem, see 4.5), each equation has a unique solution

$$z_{ij} = z_{ij}(g),$$

which is analytic in a neighborhood of 0 in a fractional power, say $g^{1/k}$, of g, where k/p is an integer. In what follows, we suppress reference to g

$$A(z)f_{ijk}(z) = \lambda_{ij}(z)f_{ijk}(z)$$

implies that

$$A(z_{ij})f_{ijk}(z_{ij}) = z_{ij}f_{ijk}(z_{ij})$$

i.e.,

$$h^{\mathrm{II}}(z_{ij})f_{ijk}(z_{ij}) = 0$$

Moreover, the eigenvectors $f_{ijk}(z)$ corresponding to the eigenvalues $\lambda_{ij}(z)$ form a basis in $P\mathscr{H}$. Hence all eigenvalues $\lambda_{ij}(z)$ are semisimple (the associated eigennilpotents are zeros, i.e. the poles of $(A(z) - w)^{-1}$ at $\lambda_{ij}(z)$ are of the first order). Consider the resolvent of $A(z)$ near the point $\lambda_{ij}(z)$. Its principal part near $\lambda_{ij}(z)$ is

$$(A(z) - wI)^{-1} \sim \frac{P_{ij}(z)}{\lambda_{ij}(z) - w},$$

where $P_{ij}(z)$ is the eigenprojection corresponding to $\lambda_{ij}(z)$. Hence

$$h^{\mathrm{II}}(z)^{-1} \sim \frac{P_{ij}(z)}{\lambda_{ij}(z) - z}$$

Since $h^{\mathrm{II}}(z)^{-1}$ may have poles of only integer order, the powers less than 1 do not occur in the series for $\lambda_{ij}(z)$. Since $\partial\lambda_{ij}/\partial z(z) \to 0(g \to 0)$ we find g, sufficiently small, such that

$$\left|\frac{\partial\lambda_{ij}}{\partial z}(z)\right| < \frac{1}{2}$$

Then

$$h^{\mathrm{II}}(z)^{-1} = \frac{P_{ij}(z_{ij})}{z_{ij} - z}\left(1 - \frac{\partial\lambda_{ij}}{\partial z}(z_{ij})\right)^{-1} + \text{term regular near } z_{ij}$$

Thus we get

Lemma. *The operator $h^{\mathrm{II}}(z)^{-1}$ has the following form $h^{\mathrm{II}}(z)^{-1} = \Sigma\, Q_{ij}/z_{ij} - z$ + term regular in any bounded subset of Ω, where*

$$Q_{ij} = \left(1 - \frac{\partial\lambda_{ij}}{\partial z}(z_{ij})\right)^{-1} P_{ij}(z_{ij})$$

5.3. The generalized eigenfunctions corresponding to complex poles of $R^{\mathrm{II}}(z)$. It is clear that equation (3.7) is valid still on the second sheet:

$$(H - z)n^{\mathrm{II}}(z) = h^{\mathrm{II}}(z) \tag{5.4}$$

Since $n^{\mathrm{II}}(z)$ is a form extension, (5.4) means, in fact, that

$$(n^{\mathrm{II}}(z)f, (H - \bar{z})\varphi) = (h^{\mathrm{II}}(z)f, \varphi), \quad f \in P\mathscr{H}, \varphi \in R(i)D_2.$$

If there exist z_o and f_o such that

$$h^{\mathrm{II}}(z_o)f_o = 0, \tag{5.5}$$

then the functional $n^{\mathrm{II}}(z_o)f_o$ (over D_2) is a generalized eigenvector of H with eigenvalue z_o:

$$(n^{\mathrm{II}}(z_o)f_o, H\varphi) = z_o(n^{\mathrm{II}}(z_o)f_o, \varphi)\forall\varphi \in \mathscr{H} \tag{5.6}$$

Thus we can construct the Gelfand-like triple[10])

$$D_2 \subset \mathscr{H} \subset D_2'$$

and the nonself-adjoint extension, H' of the operator H:

$$(H'f)(\varphi) = f(H\varphi), \quad \varphi \in D_2, f \in D_2'$$

But unfortunately, as it will be shown later, z_o is not an isolated eigenvalue of H'.

5.4. Summary of some general results. Proposition 4.5, Corollary 4.8 and Lemma 5.1 imply

Theorem. *Let for any $g \in \Delta$, $h(., g)$ have an analytic continuation, $h^{\mathrm{II}}(., g)$, from $\mathbb{C}^+$ into Ω through $\sigma_c(H_o(g))$. Then for g in a neighborhood of zero there are in a neighborhood $W \subset \Omega$ of σ_p exactly* dim *P (counting multiplicities) poles of $R(z, g)^{\mathrm{II}}$ which are not generated by poles of $R^o(z, g)^{\mathrm{II}}$. They are semisimple (have the first order). The real ones are analytic in g and for the complex ones the expansions of type (4.2) are valid. The generalized eigenfunctions of $H(g)$ correspond to the complex poles. These generalized eigenfunctions of $H(g)$ become usual eigenfunctions of $H(0)$ as $g \to 0$.*

If, in addition, Condition (4.3) is satisfied then at least $\Sigma_{\nu \in \sigma_p} 1$ of the poles described above are not real, i.e. are situated in the second Riemann (unphysical) sheet.

5.5. The first realization of an analytic continuation. We want to construct a continuation of $((H - z)^{-1}f, \varphi)$ on the domain $\mathscr{A}(H^o, \Omega) \in \mathscr{H}$, in which $((H^o - z)^{-1}\bar{P}f, \bar{P}\varphi)$ has an analytic continuation into the domain Ω on the second Riemann sheet. This requires an addition condition on the remaining part, $H - H_o$, of H:

Condition:

$$\bar{P}HP\mathscr{H} \subset \mathscr{A}(\bar{P}H\bar{P}, \Omega)$$

If this condition is satisfied then the operator $VR^o(z)VP$ in $P\mathscr{H}$ has an analytic continuation across $\sigma_c(H^o)(\Omega \cap \mathbb{R})$ from $\mathbb{C}^+$ into Ω which can be written symbolically

[10]) This triple differs from the Gelfand one by the properties of the inclusion $D_2 \subset \mathscr{H}$.

as

$$(VR^o(z)VP)^{\rm II} = VR^o(z)VP + 2\pi i V\,\delta(H^o, z)VP$$
$$= VR^o(z)^{\rm II}VP.$$

Hence the operator $h(z)$ has also an analytic continuation through spectrum of H^o from $\mathbb{C}^+$ into the domain Ω of the second Riemann sheet:

$$h^{\rm II}(z) = h(z) - 2\pi i V\,\delta(H^o, z)VP \tag{5.7}$$

Note that $h^{\rm II}(z)$ may have in $\mathbb{C}^- \cup \mathbb{R}$, and only there, singular points corresponding to eigenvalue 0.

The operator $n(z)$ has an analytic continuation from $\mathbb{C}^+$ into the second Riemann sheet in sense of sesquilinear form on the domain $P\mathscr{H} \times \mathscr{A}(H^o, \Omega)$. This continuation can be written as

$$n^{\rm II}(z) = n(z) - 2\pi i\,\delta(H^o, z)VP \tag{5.8}$$

The continuation of the adjoint operator $n^*(z)$ is an unbounded operator with the domain $\mathscr{A}(H^o, \Omega)$.

5.6. The second realization of an analytic continuation. Now we adopt the second method of the construction of an analytic continuation of $R(z)$, developed in Section 2. This method gives us the essential additional information about $R^{\rm II}(z)$, assigning to $(R^{\rm II}(z)f, \varphi)$ a quadratic form of the resolvent of some nonself-adjoint operator, but requires a different condition on H.

Condition. The representation space (for definitions see Section 2) for the absolute continuous part of H_o (or $H^o = H_o\overline{P}$) has the following form

$$\hat{\mathscr{H}} = \bigoplus_i L_2([E_i, +\infty), X_i)$$

where $E_i \in \mathbb{R}\,\forall i$ and X_i are some Hilbert spaces. The sum may be infinite.

We will repeat the main definitions of the corresponding part of Section 2 adapting them to the operator H_o. First we define two families of unitary operators on $\hat{\mathscr{H}}$ and $\mathscr{H}$, respectively:

$$(\hat{U}(\theta)\hat{f})_i(\lambda) = e^{-\theta/2}(\hat{f})_i(E_i + (\lambda - E_i)\,e^{-\theta}), \quad \theta \in \mathbb{R}, \hat{f} \in \hat{\mathscr{H}},$$

and

$$U(\theta) = Q + U_o^*\,\hat{U}(\theta)U_o\overline{Q}, \quad \theta \in \mathbb{R},$$

where $\overline{Q}$ is a projector in $\mathscr{H}$ on the absolute continuous subspace of H_o(or $H_o\overline{P}$), $Q = I - \overline{Q}$, U_o is an unitary operator from $\overline{Q}\mathscr{H}$ into $\hat{\mathscr{H}}$ related with spectral representation of $H_{o,\,\rm ac}$. Note that $P \subset Q$. Furthermore define the family of the unitary equivalent self-adjoint operators

$$H(\theta) = H_o(\theta) + V(\theta) = U(\theta)HU(\theta)^{-1}, \quad \theta \in \mathbb{R}$$

$H_o(\theta)$ has an analytic continuation in θ from $\mathbb{R}$ into the whole complex plane $\mathbb{C}$. To continue analytically $V(\theta)$ we need an additional assumption instead of Condition 5.5. Let O be a complex strip (in $\mathbb{C}$) such that $\mathbb{R} \subset$ O and

$$D(\mathrm{O}) = \{f \in \mathscr{H}\,;\, U(\theta)f \text{ has an analytic continuation into O}\}$$

Condition.

$VP\mathscr{H} \subset D(\mathrm{O})$

Then $V(\theta) = U(\theta)VU(\theta)^{-1}$ has an analytic continuation from $\mathbb{R}$ into O. Hence $H(\theta)$ also has an analytic extension from $\mathbb{R}$ into O.

The properties of the operators $H_o(\theta)$ and $H(\theta), \theta \in \mathrm{O}$, are described in Propositions 2.4 and 2.5 of the Section 2. The main result is:

Theorem. (*i*) *The form* $(R(z)f, \varphi)$ on $D(\mathrm{O}) \times D(\mathrm{O})$ *has an analytic continuation*

$$(R(z)f, \varphi)^{\mathrm{II}} \equiv (R^{\mathrm{II}}(z)f, \varphi)$$

in z from the upper (lower) semiplane $\mathbb{C}^+$ *to the domain* $G_\theta = \{z \in \mathbb{C}, \operatorname{Im} z \cdot \operatorname{Im} \theta < 0, z \neq E_i + \mathbb{R}^+ e^{\theta} \forall i\}$, $\operatorname{Im} \theta > 0 (\operatorname{Im} \theta < 0)$.
(*ii*) $(R^{\mathrm{II}}(z)f, \varphi)$ *has poles in* G_θ *for the values of* $\sigma_d(H(\theta)), \sigma_d(H(\theta) \cap \mathbb{R} = \sigma_p(H)\backslash (\sigma_p(H) \cap \{E_i\}^\theta)$.
(*iii*) *This continuation is given by the formula*

$$(R^{\mathrm{II}}(z)f, \varphi) = ((H(\theta) - z)^{-1} U(\theta)f, U(\overline{\theta})\varphi), \quad \theta \in \mathrm{O}, z \in G_\theta, f, \varphi \in D(\mathrm{O})$$

Now we construct the analytic continuation of the operators $h(z)$, $n(z)$ into the second Riemann sheet, using the method of rotation of spectrum. We have for $f, \varphi \in P\mathscr{H}$,

$$\begin{aligned}(h(z)f, \varphi) &= ((H_o - z)f, \varphi) - (R^o(z)Vf, V\varphi) \\ &= ((H_o - z)f, \varphi) - (R^o(z, \theta)U(\theta)Vf, U(\theta)V\varphi), \quad \theta \in \mathbb{R}.\end{aligned} \tag{5.9}$$

Let us continue the second term in the right side of (5.9) firstly in θ to O and then in z to G_θ. So we obtain continuation of $h(z)$ onto the second Riemann sheet. We can write it in the form

$$h^{\mathrm{II}}(z) = (H_o - z - VU(\theta)^{-1}(H^o(\theta) - z)^{-1}U(\theta)V)P \quad \theta \in \mathrm{O}, z \in G_\theta \tag{5.10}$$

$U(\theta), \theta \in \mathrm{O}$, in this expression is an analytic continuation of $U(\theta), \theta \in \mathbb{R}$. Note, that

$$(H^o(\theta) - z)^{-1} \neq U(\theta)(H^o - z)^{-1}U(\theta)^{-1} \quad (\text{on } \overline{P\mathscr{H}})$$

By Lemma 5.1, $h^{\mathrm{II}}(z, g)$ in this case is for $\theta \in \mathrm{O}$, $z \in G_\theta$ an analytic operator function in the domain $G_\theta \times Q$. In the same way we obtain

$$\begin{aligned}&(n^{\mathrm{II}}(z)f, \varphi) = (f, \varphi) - (R^o(z, \theta)U(\theta)Vf, U(\overline{\theta})\varphi), \\ &\theta \in \mathrm{O}, z \in G_\theta, \forall f \in P\mathscr{H}, \varphi \in D(\mathrm{O})\end{aligned} \tag{5.11}$$

(5.11) shows that $n^{\mathrm{II}}(z)f \forall f \in P\mathscr{H}$ is a functional on the basic space $D(\mathrm{O})$.

5.4. On the poles of the continuation of the resolvent into the second Riemann sheet.

Theorem. *Let* (*a*) E_i *be independent of g at* $g = 0$, (*b*) $U_o(g)$ *be continuous at* $g = 0$ *in the uniform operator topology,* (*c*) $H(g)$ *be continuous at* $g = 0$ *in the generalized sense (see* [21], *p.* 202). *Then* (*i*) *the points, at which* $R^{\mathrm{II}}(z, g, f, \varphi), f, \varphi \in D(\mathrm{O}), z \in G_\theta, \theta \in \mathrm{O}$, *has poles, are continuous at* $g = 0$. *For* $g \to 0$ *these points converge to eigenvalues of* $H(0)$ *embedded into the continuous spectrum;*
(*ii*) *The total multiplicity of the poles split from one eigenvalue of* $H(0)$ *is equal to the multiplicity of this eigenvalue;*

(*iii*) *Every eigenvalue of* $H(0)$, *embedded into the continuum, which does not belong to the set* $\{E_i\}$ *generates in such a way the poles of* $(R^{II}(z, g)f, \varphi)$;
(*iv*) *Real poles of* $(R^{II}(z, g)f, \varphi)$ *are eigenvalues of* $H(g)$.

Proof. (a) According to Theorem 5.6 {poles of $(R(z, g)f, \varphi)^{II}$, $\varphi, f \in D(O)$ which do not belong to $\{E_i\}\} = \sigma_d(H(g, \theta)) \subset G_\theta \cup \mathbb{R}$ and

$$\sigma_p(H(g))\backslash(\sigma_p(H(g)) \cap \{E_i\}) = \sigma_d(H(g, \theta)) \cap \mathbb{R}$$

(*b*) *Conditions* (a)–(c) of the Theorem imply that $H(g, \theta)$ for any $\theta \in O$ is continuous (in the generalized sense of [21] at $g = 0$). To obtain the statements of the theorem one may apply the theory of perturbation of isolated eigenvalues (but of nonseladjoint operators) to $\sigma_d(H(g, \theta))$.

Remark. If one supposes an analyticity of $U_0(g)$ and $H(g)$ at $g = 0$ instead of continuity, a stronger result about the dependence of the poles on g is valid:
The isolated eigenvalues of $H(g, \theta)$ and therefore the values of poles of $(R^{II}(z, g)f, \varphi)$, $z \in G_\theta$, $\theta \in O, f, \varphi \in D(O)$, are branches of one or several analytic functions at $g = 0$, which have only algebraic singularities and which are everywhere continuous at $g = 0$.

5.7. Resonance eigenvectors of $H(g)$. We saw above that if

$$h^{II}(z_o)f_o = 0,$$

then $n^{II}(z_o)f_o$ is a generalized eigenfunction of H corresponding to the value z_o. For $n^{II}(z_o)$ we have the two different expressions (5.8) and (5.11), which correspond to the different ways of the analytic continuation of $n(z)$. These expressions lead to the two different test spaces for $n^{II}(z)f, f \in P\mathscr{H}$.
In the first case we have for all $f \in P\mathscr{H}$ and $\varphi \in \mathscr{H}$ such that $U_o\bar{P}\varphi$, is analytic in a real domain and has an analytic continuation into a neighborhood of z,

$$|(n^{II}(z)f, \varphi)| \leq |(n(z)f, \varphi)| + 2\pi\left|\sum_i ((U_oVf)_i(z)(U_o\bar{P}\varphi)_i(z))_{X_i}\right|$$
$$\leq C\left(\|\varphi\|_{\mathscr{H}} + \sum_i \|(U_o\bar{P}\varphi)_i(z)\|_{X_i}\right), \ z \in \Omega$$

Let us denote by $\mathscr{H}_z$, $z \in \mathbb{C}$, the space which is obtained by completing of the following subset of $\mathscr{H}$, $\{\varphi \mid \varphi \in \mathscr{H}, (U_o\bar{P}\varphi)_i(\lambda)$ is analytic in a real domain and has an analytic continuation to a neighborhood of z and $\Sigma_i \|(U_o\bar{P}\varphi)_i(z)\|_{X_i} < \infty\}$, by the norm

$$\|\varphi\|_z = \|\varphi\|_{\mathscr{H}} + \sum_i \|(U_o\bar{P}\varphi)_i(z)\|_{X_i}$$

Then for any fixed point $z \in \mathbb{C}$ and any function $f \in P\mathscr{H}$, $n^{II}(z)f$ is the linear functional over the Banach space $\mathscr{H}_z$.
In the second case we get $\forall f \in P\mathscr{H}$, $\varphi \in D(O)$, $z \in G_\theta$

$$|(n^{II}(z)f, \varphi)| \leq |(f, \varphi)| + |(R^o(z, \theta)U(\theta)Vf, U(\bar{\theta})\bar{P}\varphi)|$$
$$\leq C(\|P\varphi\|_{\mathscr{H}} + \|U(\theta)\bar{P}\varphi\|_{\mathscr{H}})$$

and the test space $\mathscr{K}_\theta$ for any fixed $\theta \in 0$ is the completion of $D(O)$ in the norm

$$\|\varphi\|_\theta = \|P\varphi\|_{\mathscr{H}} + \|U(\theta)\bar{P}\varphi\|_{\mathscr{H}}$$

The both spaces are dense in $\mathscr{H}$. The space $\mathscr{K}_\theta$ is dense because in correspondence with Cartier-Dixmier, Nelson theorem the space of analytic functions with respect to a unitary representation of a semisimple Lie group is dense in $\mathscr{H}$. Obviously $\mathscr{K}_\theta \subset \mathscr{K}_z$ for $\theta > \arg z$ and therefore $\mathscr{K}_z$ is dense too.

This gives us a new way to provide a mathematical status for resonances. Indeed, we can define Gelfand-like triples[11])

$$\mathscr{K}_z \subset \mathscr{H} \subset \mathscr{K}'_z$$

or

$$\mathscr{K}_\theta \subset \mathscr{H} \subset \mathscr{K}'_\theta$$

and extend the operator H from $\mathscr{H}$ to $\mathscr{K}'_z(\mathscr{K}'_\theta)$:

$$(H'\Phi)(f) = \Phi(Hf),\ \Phi \in \mathscr{K}'_z(\mathscr{K}'_\theta), f \in \mathscr{K}_z \cap D(H)(\mathscr{K}_\theta \cap D(H))$$

The operator H' is already nonselfadjoint and $z_o, n^{\mathrm{II}}(z_o)f_o$ are a standard eigenvalue and corresponding eigenfunction of H'. Unfortunately, z_o is not an isolated eigenvalue. There exists a neighborhood of z_o, every point of which is an eigenvalue of H'. Indeed, define $\delta_{i,\lambda} \in \mathscr{K}'_z$ by the equality

$$\delta_{i,\lambda}(\varphi) = \{\delta_{i,j}(U_o\varphi)_i(\lambda)\}$$

The functional $\delta_{i,j}$ can be continuated analytically into a neighborhood of z_o:

$$\delta_{i,z}(\varphi) = \{\delta_{i,j}(U_o\varphi)_i(z)\},$$

where the analytic continuation of the function $(U_o\varphi)_i$ is denoted by the same symbol. Then

$$(H'_o\,\delta_{i,z})(\varphi) \equiv \delta_{i,z}(H_o\varphi) = (U_oH\varphi)_i(z) = z(U_o\varphi)_i(z)$$

i.e. $\delta_{i,z}$ is a generalized eigenfunction of H'_o. The same is true for $\mathscr{K}_\theta$. Hence, a neighborhood of z_o belongs to $\sigma_{\mathrm{es}}(H_o)$. Now, $V' = H' - H'_o$ is a finite dimensional operator on $\mathscr{K}'_z(\mathscr{K}'_\theta)$. Hence, [21, p. 244, Th. 5.35]

$$\sigma_{\mathrm{es}}(H'_o) = \sigma_{\mathrm{es}}(H'_o + V')$$

and therefore this neighborhood also belongs to $\sigma_{\mathrm{es}}(H')$.

6. The analytic extension of the complex poles

We show now the validity of the Howland's conjecture that for some nonreal g the values $z_{ij}(g)$ can be standard eigenvalues of the operator $H(g)$ ([8], see also Simon's elegant explanation, [12], of this phenomena in the example of n-particle dilatation analytic Hamiltonians). As it was noted by Howland, [8], this provides an invariant content for the resonance eigenvalues which now appear as analytic continuations of the usual eigenvalues (which have, of course, invariant sense).

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[11]) See footnote to the discussion, following equation (5.6)

As we have shown above $z_{ij}(g)$ are analytic functions in a neighborhood of $g = 0$ with the possible exception of the point 0. Using the equation

$$\lambda_{ij}(z, g) = (A(z, g)f_{ijk}(z, g), f_{ijk}(\bar{z}, \bar{g})),$$

where $f_{ijk}(z, g)$ is normalized in such a way that

$$f_{is} = f_{ijk}(\lambda_i, 0), s = s(j, k),$$

exists, we find

$$\begin{aligned}\operatorname{Im} \lambda_{ij}(z, g) &= \operatorname{Im} (A(z, g)f_{ijk}(z, g), f_{ijk}(z, g))\\ &= \operatorname{Im} (W(z, g)f_{ijk}(z, g), f_{ijk}(z, g))\end{aligned}$$

Using that $z_{ij}(g) \in \mathbb{C}^-$, $z_{ij}(g) \to \lambda_i - i0 (g \to 0)$, $\tilde{V}(g) \to V'(0) = d/dg\, V(g)\,|_{g=0}$, we have in the lowest order

$$\begin{aligned}\operatorname{Im} z_{ij}(g) &= \operatorname{Im} \lambda_{ij}(z_{ij}(g), g) \sim\\ &\quad -\operatorname{Im} \{g^2(R^o(\lambda_i - i0, 0, \theta)U((\theta, 0)V'(0)f_{is}, U(\overline{\theta}, 0)V'(0)f_{is})]\\ &= -\operatorname{Im} \{g^2(R^o(\lambda_i - i0, 0)V'(0)f_{is}, V'(0)f_{is})] \qquad (6.1)\end{aligned}$$

Here we used in the last equality that $\lambda_i \in \mathbb{R}$. (6.1) gives us finally,

$$\begin{aligned}\operatorname{Im} z_{ij}(g) \sim &-\operatorname{Re} g^2(\delta(H^o(0) - \lambda_i)V'(0)f_{is}, V'(0)f_{is})\\ &-\operatorname{Im} g^2 \operatorname{Re} (R^o(\lambda_i - i0, 0)V'(0)f_{is}, V'(0)f_{is})\end{aligned}$$

It is easy to see from this expression that for some sufficiently small $g^2 \in \mathbb{C}$, for example, negative ones, $Im z_{ij}(g) > 0$ and thus $z_{ij}(g)$ is a singular point of $h(z, g)$, corresponding to a zero eigenvalue, and is an eigenvalue of $H(g)$ if

$$n(z_{ij}(g), g)P\mathscr{H} \in D(H_o(g))$$

The latter is certainly true if the intersection of a neighborhood of λ_i (which contains λ_i) with $\mathbb{C}^+$ belongs to $\rho(H_o(g))$. (The last statement can be considered as an additional condition on $H_o(g)$.)

We stress that one can surely say that $z_{ij}(g)$ is an eigenvalue of $H(g)$ only for g such that $\operatorname{Im} z_{ij}(g) > 0$.

Acknowledgement

The second of the authors (I.M.S.) is indebted to Prof. R. Jost for the discussions of the theorem on analytic continuation, given in the Appendix. In fact, Profs. R. Jost and A. Huber have provided the proof of this theorem.

Appendix

Theorem. *Let D and Ω be convex domains in $\mathbb{C}$ such that $D \subset \Omega$, let Δ be a compact interval of the real axis and let a function f of two complex variables satisfy the following conditions*

(i) f is analytic in $D \times \Delta$;

(ii) $\forall \xi \in \Delta, f(., \xi)$ is analytic in Ω.

Then for any $\varepsilon > 0$ there exists a complex neighborhood P_ε of Δ such that f is analytic in $\Omega^{(\varepsilon)} \times P_\varepsilon$, where $\Omega^{(\varepsilon)} = \{z \in \Omega: \text{dist}(z, \partial\Omega) > \varepsilon\}$.

Proof. Let ζ and $a > b > 0$ be such that $\overline{K}_{\zeta,b} \subset D$ and $\overline{K}_{\zeta,a} \subset \Omega$, where we use the notation

$$K_{\zeta,a} = \{z \in \mathbb{C}, |z - \zeta| < a\}$$

for an open disc of the radius a with the center at ζ. In order to spare us two parameters we assume $\zeta = 0$ and $b = 1$. Then $a > 1$. Set $K_a = K_{o,a}$.

(a) There exists a complex neighborhood, P, of Δ such that f can be continuated analytically into $\overline{K}_1 \times P$. Indeed, $\forall(z, x) \in \overline{K}_1 \times \overline{\Delta}$, f has analytic continuation into an open, complex neighborhood, $Q_{z,x} \times P_{z,x}$ of (z, x) here $Q_{z,x}$ and $P_{z,x}$ are neighborhoods of z and x, respectively). $\{Q_{z,x} \times P_{z,x}\}$ is a covering of $\overline{K}_1 \times \overline{\Delta}$. Since $\overline{K}_1 \times \overline{\Delta}$ is compact, there exists a finite subsystem $\{Q_{z_i,x_i} \times P_{z_i,x_i}\}$ which covers $\overline{K}_1 \times \overline{\Delta}_1$. For any $\zeta \in \overline{K}_1$, define $\pi_\zeta = \{i: \zeta \in Q_{z_i,x_i}\}$ and $V_\zeta = \cap_{i\in\pi_\zeta} Q_{z_i,x_i}$. $\{V_\zeta, \zeta \in \overline{K}_1\}$ covers $\overline{K}_1$. Since $\overline{K}_1$ is compact, there is a finite subsystem $\{V_{\zeta_i}\}$ which covers K_1. Now define $P = \cap_{\zeta_k}(U_{i\in\pi_{\zeta_k}} P_{z_i,x_i})$. Obviously, for any $\zeta \in \overline{K}_1$, $\cup_{i\in\pi_\zeta} P_{z_i,x_i}$ is open and contains Δ. Therefore, so does P. Besides, as can easily be shown, $\overline{K}_1 \times P \subset U(Q_{z_i,x_i} \times P_{z_i,x_i})$. Since f is analytic in $\cup(Q_{z_i,x_i} \times P_{z_i,x_i})$, it is analytic in $\overline{K}_1 \times P$.

(b) Let

$$f(z,\zeta) = \sum a_n(\zeta) z^n$$

Because of (a), $a_n(\zeta)$ are analytic in P and

$$|a_n(\zeta)| \le M \text{ in any closed } P_1 \subset P.$$

Because of (ii),

$$|a_n(\zeta)| \le M a^{-n} \quad \text{for} \quad \zeta \in \Delta$$

(c) **Lemma.** *Let a set $\{a_n, n \in \mathbb{N}\}$ of function analytic in $\overline{K}_b$ satisfy*
(i) in $\overline{K}_b: |a_n| \le M$
(ii) in $K_b \cap \mathbb{R}: |a_n| \le M a^{-1}$, $a > 1$.
Then $\forall 0 < s < 1 \;\exists\, d = d(s, a), d(s, a) \to 0 (s \to 0): |a_n| \leqslant M\, d^{-n}$ in $\overline{K}_{sb}$.

Proof. Let $K_b^+ = K_b \cap \mathbb{C}^+$ and h be chosen to satisfy
(1) h is a harmonic function in K_b^+ bounded in $\overline{K_b^+}$
(2) $h|_{\partial K_b^+ \setminus [-b,b]} = 0$, $h|_{[-b,b]} = 1$
Let $\eta = \eta(s) = \inf_{\zeta\in\partial\overline{K}_{sb}^+} h(\zeta)$, $s > 0$. Then $0 < \eta < 1$, $\eta(s) > c(s)$, where $c = c(s)$ does not depend on b.

$\log|a_n|$ is a subharmonic function on K_b^+, bounded in $\overline{K_b^+}$,

$$\log|a_n|\,|_{\partial K_b^+ \setminus [-b,b]} \le \log M,$$

$$\log|a_n|\,|_{[-b,b]} \le \log M - n \log a$$

Therefore

$$\log|a_n|\,|_{\partial K_b^+} \le (\log M - hn \log a)|_{\partial K_b^+}$$

Hence, since $hn \log a$ is a harmonic function in $\overline{K}_b^+$,

$$\log|a_n| \le \log M - hn \log a$$

Therefore

$$\log |a_n| \,|_{\partial K_{sb}^+} \leq \log M - h|_{\partial K_{sb}^+} n \log a \leq \log M - cn \log a$$

Hence

$$\log |a_n| \,|_{\bar{K}_{sb}^+} \leq \log M - cn \log a = \log M a^{-cn}$$

and

$$|a_n| \leq M a^{-cn} \quad \text{in} \quad K_{sb}^+$$

Defining $d = a^{-c}$ we obtain

$$|a_n| \leq M\, d^{-n} \quad \text{in} \quad K_{sb}^+$$

The same result is true for K_b^-.

Construction of h. Set

$$h = 1 - \frac{2}{\pi} \arg \frac{b+\zeta}{b-\zeta} = 1 - \frac{2}{\pi} \operatorname{arctg} \frac{2rb \sin \varphi}{b^2 - r^2}, \quad \text{where} \quad \zeta = re^{i\varphi}$$

We find

$$\min_{\varphi} h(re^{i\varphi}) = 1 - \frac{2}{\pi} \max_{\varphi} \operatorname{arctg} \frac{2rb \sin \varphi}{b^2 - r^2} = 1 - \frac{2}{\pi} \operatorname{arctg} \frac{2rb}{b^2 - r^2}$$

$$= 1 - \frac{2}{\pi} \arcsin \frac{2rb}{b^2 + r^2},$$

$$\eta(s) = \min_{\varphi} h(sbe^{i\varphi}) = 1 - \frac{2}{\pi} \operatorname{arctg} \frac{2s}{1 - s^2} = 1 - \frac{2}{\pi} \arcsin \frac{2s}{1 + s^2}$$

Hence $\eta(s) + 0$ as $s \to 0$ and does not depend on b. Q.E.D.

(d) according to (b) and (c)

$$\forall\, x \in \Delta,\, \varepsilon > 0\, \exists r = r(\varepsilon, x),\, d = d(\varepsilon): |d - a| < \varepsilon,$$

$$|a_n(\zeta)| \leq M\, d^{-n} \quad \text{in} \quad \bar{K}_{x,r} \subset P$$

(e) $\{K_{x,r}, x \in \Delta\}$ is a covering of Δ. Since Δ is compact we can chose a finite subsystem $\{K_{xi, ri}\}$ which covers Δ. Let $P_\varepsilon = U.\ K_{xi, ri} \subset P$. Then P_ε is a complex neighborhood of Δ. By (d)

$$|a_n(\zeta)| \leq M\, d^{-n} \quad \text{in} \quad P_\varepsilon$$

Since $a_n(\zeta)$ is analytic in P this implies that f is analytic in $K_d \times P_\varepsilon$. In virtue of the condition on the domains D and Ω this implies the statement of theorem Q.E.D.

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