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General solution of multichannel partial-wave dispersion relations I. Coincident thresholds, pole approximation

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Abstract. We show how to obtain the general solution for the S-matrix of s-wave amplitudes, in the case of n coupled two-body channels whose thresholds coincide. The physical S-matrix is assumed to be the boundary value from above of a function analytic in the whole complex plane with the physical cut and the positions of a finite number of simple poles removed. The input consists of the positions of these poles and the residue matrix at each pole. Competing channels are neglected, so that the physical S-matrix is required to be unitary at each energy. Necessary and sufficient conditions are obtained for a solution to exist, and three types of soluble problem are distinguished. To obtain these results we have generalized some parts of the Schur–Pick–Nevanlinna interpolation theory to analytic matrices. Some special cases are treated in detail and various results of physical interest pointed out. Particular attention is paid to the characterization of the diagonal components of a solution of a multichannel problem, considered as solutions of inelastic one-channel problems.

1. Introduction

Our aim in this paper (and in others which will follow it) is to solve partial-wave dispersion relations for $n > 1$ coupled two-body channels by using function theoretical methods which yield the same sort of detailed information about the existence and non-uniqueness of solutions as was obtained in [1, 2] for the case $n = 1$. As a first stage we consider here the case of coincident thresholds (the sum of the masses of the two particles in each channel is the same for all channels). We confine ourselves in the text to s-waves. The generalization to higher partial waves is possible, but then complexities arise which obscure the simplicity of the method. Competing channels are neglected, so that the physical $n \times n$ S-matrix of s-wave amplitudes is taken to be unitary at each energy.

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The 'unphysical' singularities in this paper are taken to be a finite set of N simple poles on the real axis. This is what is done in all attempts at practical dynamical calculations. It turns out that in this case a natural matrix generalization of the method of [1] is the appropriate mathematical tool to use. The method of this paper can be extended to deal with cases where the 'unphysical' singularities include multiple poles or pairs of complex conjugate poles; such approximations are appropriate to the more complicated 'unphysical' singularities which occur for example in pion-nucleon scattering (see [3]). We also want to point out that we treat only 'unphysical' singularities ('driving forces' in more physical language) whose effect is so small that there are no bound states. In other words, we confine ourselves strictly to the mathematical problem which we shall define. Consideration of cases in which there are bound states would require a more careful analysis even of the one channel problem.

The problem of finding solutions of partial-wave dispersion relations has been extensively studied in the past by the so-called ND^{-1} method. We do not want to comment on this work because it was recognized very early (see for example page 403 of Amati and Fubini [4]) that it is difficult to establish the analyticity properties of the solution by this method, because of possible zeros in D (or in det D in multichannel cases). Moreover, there is a possible arbitrariness in D which it is very difficult to characterize precisely. Though valuable work has been done by Bjorken [5, 6], Warnock [7, 8], Atkinson [9–13] and others, these problems have not been solved. For this reason we looked for other methods which avoid these difficulties.

For $n = 1$, with the 'unphysical' singularities consisting of a finite set of N simple poles on the real axis, the problem of finding solutions was solved by us in a general and complete way [1, 2]. When the effect of competing channels is specified by the inelasticity parameter η $(0 \leq \eta \leq 1)$, Nenciu [1] showed that the Schur-Pick-Nevanlinna interpolation theory gives a comprehensive description of the non-uniqueness of solutions and also gives necessary and sufficient conditions for a solution to exist. To apply this theory, Nenciu used a conformal mapping which transforms the complex plane with the physical cut $[1, \infty)$ removed into the open unit disc (see (1.2) below). The problem then became one of finding functions analytic in the open disc, continuous on the closed unit disc, and having prescribed values at the N transformed pole positions. The modulus of the function on the unit circle is the specified function η, and a Froissart transformation may be made to a function whose modulus on the unit circle is always unity (see Section 4). The problem was solved by making a sequence of invertible transformations of the functions which remove the prescribed values one at a time. Finally one arrives at a function with the same properties as the original function, but with *no* prescribed values, and this function is easily characterized. Necessary and sufficient conditions on the input data (pole positions and residues, and the function η) for a solution to exist arise naturally in making this sequence of transformations, on applying the maximum modulus principle for analytic functions. The input data may be such that there is no solution, or a unique solution, or an infinity of solutions. In the last case, the number of zeros of a solution in the unit disc is at least N, and there is just one solution (called the isolated solution) with exactly N such zeros.

It is also possible in the one channel case to specify the effect of competing channels by the ratio R of the total cross-section to the total elastic cross-section for the partial wave in question $(R \geq 1)$. We showed in [2] how it was possible to obtain a very general class of solutions in this case. The conformal transformation was not used, and it was possible to characterize the solutions as the boundary values on the physical cut from above of analytic functions, which could be represented as the product of a Herglotz function and the quotient of two polynomials. The Herglotz function could be written in a more explicit form involving the prescribed function R. However, a set of necessary and sufficient conditions for solutions to exist could be given only in a very indirect form.

For the case to be considered in this paper, with $n > 1$ coupled two-body channels whose thresholds coincide, and the left-hand singularities again consisting of N poles with prescribed residue matrices, the method of obtaining solutions is a natural matrix generalization of the method used in [1]. Since the thresholds coincide, one can begin by making the same conformal mapping to the unit disc as in the one channel case. The problem then becomes one of finding *matrix* functions analytic in the open unit disc, continuous on the closed unit disc and having prescribed values at the N transformed pole positions. The effect of competing channels is neglected and the matrix functions are required to be unitary everywhere on the unit circle. There are some further conditions to be satisfied by the matrix functions which are given later (see (1.3) and (1.6)).

Following the method of [1], in Section 2 we make a sequence of invertible transformations of the analytic matrix function being sought, each transformation removing one prescribed value. These transformations are best described as 'matrix Blaschke transforms' and the result (if the sequence proceeds for the full N steps) is that the problem is reduced to finding analytic matrix functions with the same properties as the original function, but with *no* prescribed values. Application of the maximum principle for analytic matrix functions (given in the Appendix along with other mathematical results) at each step enables a set of necessary and sufficient conditions for a solution to exist to be given. The input data may be such that there is no solution or a unique solution; in either case the sequence of transformations stops short of the full N steps.

If the sequence proceeds for the full N steps, however, there is an infinity of solutions, which can be divided into sets characterized by the number of zeros of the *determinant* of the matrix function solution in the unit disc. The solutions are obtained from the matrix analytic functions with no prescribed values by reversing the steps in the sequence of transformations. We shall give in Section 3 a representation theorem which shows how to construct all the matrix functions having the properties desired, but no prescribed values in the open unit disc. The dimension of these matrix functions may be n, the number of coupled channels, in which case we say that the solution is completely non-unique (CNU). However, the dimension may be reduced from n to some lower value in the course of the sequence of transformations; in such a case we say that the solution is partially non-unique (PNU). In either case there is a single solution whose determinant has the smallest number of zeros in the open unit disc (this number is nN for a CNU case and is less than nN for a PNU case). This solution is called the isolated solution; it is given in terms of the input data only (pole positions and residue matrices), with no arbitrary parameters.

Section 4 will specialize the results of Sections 2 and 3 to the cases of one pole and of two poles, and will discuss some consequences of physical interest. A particular problem which we shall consider is the following. Suppose that, from a solution of the n-channel problem, the inelasticity parameters $\eta_j = |S_{jj}|$ $(j = 1, \ldots, n)$ are computed for the n elastic scattering processes $j \to j$. Each S_{jj} is then a solution of the one-channel problem whose input consists of η_j and the (jj) components of the residue matrices at the poles. We call this the equivalent inelastic one-channel problem j (or EIOCP j for short). But what kind of solution is it? In particular, are there cases where, for the isolated solution of the n-channel problem, one at least of the S_{jj}, considered as a solution of EIOCP j, has CDD zeros? There is a large body of work on these questions (see footnote 14 of Warnock [7] and Refs. [11, 14, 15–17]). In the cases of one pole and of two poles the isolated solution will be constructed explicitly and the second of the questions just given will be studied.

Consider the S-matrix of s-wave amplitudes and assume that it is the boundary value from above of a matrix function $S(s)$ which is analytic on $\mathbb{C}$, with the interval $[1, \infty)$ and the positions s_i ($i = 1, \ldots, N$) of the simple poles removed. The Mandelstam variable s has been scaled so that the common threshold is at 1, as in [2], and it is assumed that $S(s)$ is continuous onto $[1, \infty)$ from above in the precise sense discussed in [2]. In addition we assume that the following mathematical properties hold:

$$
\begin{aligned}
&\text{(a)}\ S(\sigma + i0)S^*(\sigma + i0) = \mathbb{1}_n, \quad \sigma \geq 1;\\
&\text{(b)}\ S(s) = S^T(s);\\
&\text{(c)}\ S(1) = \mathbb{1}_n;\\
&\text{(d)}\ S(s) = \overline{S(\bar{s})};\\
&\text{(e)}\ \lim_{s \to s_i} (s - s_i)S(s) = A_i', \quad A_i' \neq 0.
\end{aligned}
\tag{1.1}
$$

The star in the (unitarity) condition (a) denotes the adjoint. Time reversal invariance leads to the symmetry of $S(\sigma + i0)$ and (b) follows by analytic continuation. Condition (c) is the threshold condition appropriate to s-waves; for $l > 0$ further conditions must be imposed. The reflection property (d) is integral to the Mandelstam representation. We do not assume any particular ordering of the pole positions s_i; the s_i are real, distinct and each less than 1. By (b) and (d), the residue matrices A_i' are real and symmetric.

The next step is to make the standard conformal mapping (as in [1])

$$
z = \frac{1 - (1 - s)^{1/2}}{1 + (1 - s)^{1/2}}, \tag{1.2}
$$

which maps $\mathbb{C} - [1, \infty)$ bijectively to $\Delta(0; 1)$ (the open unit disc). The semi-infinite interval $(-\infty, 1)$ is mapped into $(-1, 1)$, the upper half-plane ($\operatorname{Im} s > 0$) into $\Delta(0; 1) \cap \{\operatorname{Im} z > 0\}$ and the lower half-plane into $\Delta(0; 1) \cap \{\operatorname{Im} z < 0\}$. The upper side of the cut $(1, \infty)$ is mapped into $C(0; 1) \cap \{\operatorname{Im} z > 0\}$ and the lower side into $C(0; 1) \cap \{\operatorname{Im} z < 0\}$, $C(0; 1)$ being the unit circle. Finally, 1 maps into 1 and the point at infinity into -1. The pole positions s_i map into x_i ($i = 1, \ldots, N$); the x_i are real and distinct, and $|x_i| < 1$. We did not specify the behaviour of $S(s)$ at infinity. However, we now require that the new function $S(z)$ (S is used again for convenience) be defined at $z = -1$, unitary *everywhere* on $C(0; 1)$, and continuous on $\overline{\Delta(0; 1)} - \{x_i\}$. Using (1.1) we now write the full set of conditions on $S(z)$.

(a) $S(z)$ is continuous on $\overline{\Delta(0; 1)} - \{x_i\}$;

(b) $S(z)$ is analytic in $\Delta(0; 1) - \{x_i\}$;

(c) $S(e^{i\theta})S^*(e^{i\theta}) = \mathbb{1}_n, \quad 0 \leq \theta \leq \pi;$

(d) $S(z) = S^T(z);$

(e) $S(1) = \mathbb{1}_n;$

(f) $S(z) = \overline{S(\bar{z})};$

(g) $\lim_{z \to x_i} (z - x_i)S(z) = A_i'', \quad A_i'' \neq 0, \quad A_i'' = \overline{A_i''} = A_i''^T.$ (1.3)

It is neater to work with the function $W(z)$ defined by

$$W(z) = \left(\prod_{i=1}^{N} b(x_i; z)\right) S(z), \tag{1.4}$$

where

$$b(x_i; z) = (z - x_i)/(1 - x_i z). \tag{1.5}$$

The transformation (1.5) maps $\Delta(0; 1)$ bijectively to $\Delta(0; 1)$ and $C(0; 1)$ bijectively to $C(0; 1)$. The conditions (1.3) on $S(z)$ become the following conditions on $W(z)$:

(a) $W(z)$ is continuous on $\overline{\Delta(0; 1)}$;

(b) $W(z)$ is analytic in $\Delta(0; 1)$;

(c) $W(e^{i\theta})W^*(e^{i\theta}) = \mathbb{1}_n, \quad 0 \leq \theta \leq \pi;$

(d) $W(z) = W^T(z);$

(e) $W(1) = \mathbb{1}_n;$

(f) $W(z) = \overline{W(\bar{z})};$

(g) $W(x_i) = A_i, \quad A_i \neq 0,$ (1.6)

with x_i real and distinct, $|x_i| < 1$, $A_i = \overline{A_i} = A_i^T$, $i = 1, \ldots, N$. Our problem is to find all the matrix functions which satisfy the conditions (1.6).

2. Reduction of the problem

We solve the problem posed at the end of Section 1 in two steps. In this section we show how condition (g) of (1.6) may be removed, and the problem reduced to finding functions $U(z)$ satisfying conditions (a)–(f) of (1.6). In Section 3 we give a general representation theorem for such functions $U(z)$. In the course of removing condition (g) we shall collect a sequence of necessary conditions for a solution to exist. These conditions together will be *sufficient* for a solution to exist. We shall also be able to distinguish three classes of problems (that is, of prescribed x_i, A_i) for which a solution exists.

We begin by applying the maximum principle given in the Appendix to prove

Lemma 2.1. *A necessary condition for the existence of a solution $W(z)$ is that either*

$\|A_i\| < 1, \quad i = 1, \ldots, N$ (case I) *or*

$\|A_i\| = 1, \quad i = 1,\ldots, N$ (case II).

Proof. From (c) and (g) of (1.6) and the maximum principle it follows that $\|A_i\| \leq 1, i = 1,\ldots, N$, is a necessary condition for a solution $W(z)$ to exist. Moreover, if one of the A_i has norm 1, then $\|W(z)\| = 1$ for $z \in \overline{\Delta(0; 1)}$ and thus all the other A_i must also have norm 1. $\square$

We next analyse case II in more detail and obtain a further necessary condition in this case.

Lemma 2.2. *If the condition of case II of Lemma 2.1 holds and if a solution* $W(z)$ *exists, then there is a real orthogonal matrix* Ω *and an integer* m *satisfying* $0 \leq m \leq n - 1$, *such that*

$$\Omega^T A_i \Omega = \begin{pmatrix} \mathbb{1}_{n-m} & 0 \\ 0 & B_i \end{pmatrix}, \quad \|B_i\| < 1, i = 1,\ldots, N. \tag{2.1}$$

Moreover

$$\Omega^T W(z)\Omega = \begin{pmatrix} \mathbb{1}_{n-m} & 0 \\ 0 & V(z) \end{pmatrix},$$

where $V(z)$ *is an* $m \times m$ *matrix function satisfying the same conditions as* $W(z)$, *but with* $n \to m$ *and with condition* (g) *of* (1.6) *replaced by*

$$V(x_i) = B_i, \quad i = 1,\ldots, N.$$

Proof. Let Ω be a real orthogonal matrix which diagonalizes A_N in such a way that

$$\Omega^T A_N \Omega = \begin{pmatrix} J & 0 \\ 0 & B_N \end{pmatrix}, \qquad \Omega^T\Omega = \mathbb{1}_n,$$

where J is a diagonal $(n - m) \times (n - m)$ matrix with diagonal elements $J_i = \pm 1$ and B_N is a diagonal $m \times m$ matrix with $\|B_N\| < 1$. Since $\|A_N\| = 1$, there is at least one eigenvalue with absolute value 1 and so $m \leq n - 1$.

Now the matrix function $\Omega^T W(z)\Omega$ satisfies the same conditions as $W(z)$, except for the obvious change in (1.6) (g). Application of the maximum principle to its first $(n - m)$ diagonal elements gives

$$(\Omega^T W(z)\Omega)_{jj} = J_{jj}, \quad j = 1,\ldots, n - m, \quad z \in \overline{\Delta(0; 1)}.$$

It follows from (1.6) (c) that, for all θ,

$$(\Omega^T W(e^{i\theta})\Omega)_{jk} = 0, \quad j \neq k, \quad 1 \leq j \leq n - m \quad \text{or} \quad 1 \leq k \leq n - m.$$

Thus, by the maximum principle again,

$$(\Omega^T W(z)\Omega)_{jk} = 0, \quad z \in \overline{\Delta(0; 1)}, \quad j \neq k,$$
$$1 \leq j \leq n - m \quad \text{or} \quad 1 \leq k \leq n - m,$$

and so

$$\Omega^T W(z)\Omega = \begin{pmatrix} J & 0 \\ 0 & V(z) \end{pmatrix}.$$

From (1.6) (e) we have $J = \mathbb{1}_{n-m}$, $V(1) = \mathbb{1}_m$. Further, from (1.6) (g),

$$\Omega^T A_i \Omega = \begin{pmatrix} \mathbb{1}_{n-m} & 0 \\ 0 & B_i \end{pmatrix}, \quad i = 1,\ldots, N-1,$$

$$V(x_i) = B_i, \quad i = 1,\ldots, N.$$

From Lemma 2.1, $\|B_i\| < 1$, $i = 1,\ldots, N-1$, since $\|B_N\| < 1$. □

Thus in case II a further necessary condition on the A_i is that *each* A_i should have the same number $(n-m)$ of eigenvalues equal to $+1$ and the same number m of eigenvalues with absolute value less than 1. Moreover each A_i must be reducible to the form on the right side of (2.1) by the *same* orthogonal matrix Ω. In the proof we used an Ω for which B_N was diagonal, but it is clear that this was not necessary.

Two subcases of case II may now be distinguished. First, it is possible that $m = 0$, in which case $A_i = \mathbb{1}_n$, $i = 1,\ldots, N$ and the problem has the unique solution

$$W(z) = \mathbb{1}_n, \quad z \in \overline{\Delta(0;1)}.$$

On the other hand, if $1 \leq m \leq n-1$, we have for $V(z)$ exactly the same problem as we had originally for $W(z)$, but with a lower dimensionality. Moreover, since $\|B_i\| < 1$, $i = 1,\ldots, N$, we are back to a case I situation for $V(z)$.

We now show how the prescribed values in condition (g) of (1.6) may be successively removed. We start with a set $\{A_i\}$ and write

$$A_i = A_i^{(0)}, \qquad W(z) = W^{(0)}(z), \qquad n = n^{(0)}.$$

If it is necessary to make the transformation of Lemma 2.2 we write further

$$B_i = B_i^{(0)}, \qquad V(z) = V^{(0)}(z), \qquad m = m^{(0)}.$$

To include case I as well and so give a general discussion we make the obvious convention that in this case

$$B_i = A_i, \qquad V(z) = W(z), \qquad m = n.$$

The next lemma shows how to remove one prescribed value.

Lemma 2.3. *Define the function* $W^{(1)}(z)$ *by*

$$W^{(1)}(z) = [b(x_N; z)]^{-1}(\mathbb{1}_m - B_N^2)^{-1/2}(V(z) - B_N) \times (\mathbb{1}_m - B_N V(z))^{-1}(\mathbb{1}_m - B_N^2)^{1/2}. \tag{2.2}$$

Then $W^{(1)}(z)$ *satisfies the same conditions as* $V(z)$, *except that its values are prescribed only at the points* $x_1,\ldots, x_{N-1}$.

Proof. The necessary background for the matrix operations in (2.2) is given in the Appendix. The only parts of the proof which are not trivial are given in Lemmas A.1 and A.2. It is clear that $W^{(1)}(x_N)$ is not prescribed. □

It follows from (2.2) that

$$W^{(1)}(x_i) = A_i^{(1)}, \quad i = 1, \ldots, N-1,$$

where

$$A_i^{(1)} = [b(x_N; x_i)]^{-1}(\mathbb{1}_m - B_N^2)^{-1/2}(B_i - B_N)(\mathbb{1}_m - B_N B_i)^{-1}(\mathbb{1}_m - B_N^2)^{1/2}. \tag{2.3}$$

The procedure described in Lemmas 2.1–2.3 may now be implemented again to define successively matrix functions $W^{(j)}(z)$, $j = 0, 1, \ldots, N$, and $V^{(j)}(z)$, $j = 0, 1, \ldots, N-1$, of dimension $n^{(j)}$ and $m^{(j)}$ respectively, where

$$n = n^{(0)} \geq m^{(0)} = n^{(1)} \geq m^{(1)} = \cdots \geq m^{(N-1)} = n^{(N)} \geq 1.$$

The function $W^{(j)}(z)$ has the prescribed values

$$W^{(j)}(x_i) = A_i^{(j)}, \quad j = 0, 1, \ldots, N-1 \quad \text{and} \quad i = 1, \ldots, N-j,$$

while the function $V^{(j)}(z)$ has the prescribed values

$$V^{(j)}(x_i) = B_i^{(j)}, \quad j = 0, 1, \ldots, N-1 \quad \text{and} \quad i = 1, \ldots, N-j.$$

The function $W^{(N)}(z)$ has no prescribed values.

Generalizing (2.2) and (2.3) and the results in Lemma 2.2, the required recursions are

$$\begin{aligned} W^{(j+1)}(z) &= [b(x_{N-j}; z)]^{-1}(\mathbb{1}_{m^{(j)}} - B_{N-j}^{(j)2})^{-1/2}(V^{(j)}(z) - B_{N-j}^{(j)}) \\ &\quad \times (\mathbb{1}_{m^{(j)}} - B_{N-j}^{(j)} V^{(j)}(z))^{-1}(\mathbb{1}_{m^{(j)}} - B_{N-j}^{(j)2})^{1/2}, \quad j = 0, 1, \ldots, N-1 \end{aligned} \tag{2.4}$$

$$V^{(j)}(z) = W^{(j)}(z) \quad \text{if} \quad m^{(j)} = n^{(j)}, \tag{2.5}$$

$$\begin{pmatrix} \mathbb{1}_{n^{(j)}-m^{(j)}} & 0 \\ 0 & V^{(j)}(z) \end{pmatrix} = \Omega^{(j)T} W^{(j)}(z) \Omega^{(j)} \quad \text{if} \quad m^{(j)} < n^{(j)}, \tag{2.6}$$

$$\begin{aligned} A_i^{(j+1)} &= [b(x_{N-j}; x_i)]^{-1}(\mathbb{1}_{m^{(j)}} - B_{N-j}^{(j)2})^{-1/2}(B_i^{(j)} - B_{N-j}^{(j)}) \\ &\quad \times (\mathbb{1}_{m^{(j)}} - B_{N-j}^{(j)} B_i^{(j)})^{-1}(\mathbb{1}_{m^{(j)}} - B_{N-j}^{(j)2})^{1/2}, \\ &\qquad j = 0, 1, \ldots, N-2 \quad \text{and} \quad i = 1, \ldots, N-j-1, \end{aligned} \tag{2.7}$$

$$B_i^{(j)} = A_i^{(j)} \quad \text{if} \quad m^{(j)} = n^{(j)}, \tag{2.8}$$

$$\begin{pmatrix} \mathbb{1}_{n^{(j)}-m^{(j)}} & 0 \\ 0 & B_i^{(j)} \end{pmatrix} = \Omega^{(j)T} A_i^{(j)} \Omega^{(j)} \quad \text{if} \quad m^{(j)} < n^{(j)}. \tag{2.9}$$

It has been assumed in the preceding two paragraphs that the recursion continues for the full N steps. It is possible that, for some J such that $0 \leq J \leq N-1$,

$$A_i^{(J)} = \mathbb{1}_{n^{(J)}}, \quad i = 1, \ldots, N-J. \tag{2.10}$$

The recursion will then terminate at this stage, with the *unique* function

$$W^{(J)}(z) = \mathbb{1}_{n^{(J)}}, \quad z \in \overline{\Delta(0; 1)},$$

giving a unique solution $W^{(0)}(z) = W(z)$ of the original problem on reversing the steps.

We now distinguish the three classes of problems for which a solution exists.

(1) Suppose that the prescribed x_i, A_i are such that the recursion (2.7), starting with $A_i^{(0)} = A_i$, yields matrices $A_i^{(j)}$ which satisfy

$$\|A_i^{(j)}\| < 1, \quad j = 0, 1, \ldots, N-1 \quad \text{and} \quad i = 1, \ldots, N-j.$$

Then $m^{(j)} = n^{(j)}$, $B_i^{(j)} = A_i^{(j)}$, $V^{(j)}(z) = W^{(j)}(z)$, $j = 0, 1, \ldots, N-1$ (as in (2.5) and (2.8)). The problem has been reduced to that of finding all $n \times n$ matrix functions $U(z)$ satisfying conditions (a)–(f) of (1.6). From this set of functions $\{U(z)\}$ the full set $\{W(z)\}$ of solutions of the original problem is obtained by setting $U(z) = W^{(N)}(z)$ and reversing the steps (2.4) and (2.5) of the procedure. We shall say that in this case the problem has a *completely non-unique* solution (or, for short, that it is a CNU problem).

(2) Suppose that the recursion continues for the full N steps but that $n^{(N)} < n^{(0)}$. This means that at one or more steps we have $\|A_i^{(j)}\| = 1$, $i = 1, \ldots, N-j$, so that the 'sideways' step of Lemma 2.2 (given in (2.6) and (2.9)) is required. In this case the problem has been reduced to that of finding all matrix functions $U(z)$ satisfying conditions (a)–(f) of (1.6), but with the dimension of $U(z)$ now less than n. Again the set $\{W(z)\}$ of solutions of the original problem is obtained from this set of functions $\{U(z)\}$ by setting $U(z) = W^{(N)}(z)$ and reversing the steps of the procedure. In this case we say that the problem has a *partially non-unique* solution (or that it is a PNU problem).

(3) The prescribed x_i, A_i may be such that the solution is *unique* (a U problem for short). The condition for a U problem is given in (2.10).

We have now given, in cases (1)–(3) above, necessary and sufficient conditions on the prescribed x_i, A_i for a matrix function $W(z)$ to exist which satisfies conditions (a)–(g) of (1.6). If none of the conditions for cases (1), (2) or (3) is satisfied, no such function exists. In the one-channel case, as was pointed out in [1], the conditions on x_i, A_i for a solution to exist can be put in a compact form. Although we shall not write it here explicitly, a similar compact form can be written down for the n-channel case, using the powerful theory of unitary dilatations of contractions [18].

One naturally asks what happens if a different procedure is used for successively removing the prescribed values of $W(z)$. For a U problem it is clear that any procedure will terminate short of N steps with a set of unit matrices. However, for a CNU or PNU problem the succession of transformations (2.4) and (2.7) could be carried out in $N!$ different ways, simply by taking the pole positions in any order. Further, if transformations of the type (2.6) are required, there is an arbitrariness in the choice of the $\Omega^{(j)}$. Denote by $\mathscr{P}$ any such procedure which removes the prescribed values of $W(z)$ at $x_1, \ldots, x_N$. Since it is quickly verified that each step of a procedure is invertible and thus one-to-one, it follows that, for a CNU or PNU problem, *all* solutions $W(z)$ of the problem with N prescribed values are obtained by taking *all* distinct solutions $U(z)$ of the problem without any prescribed values and reversing the steps of any *fixed* procedure $\mathscr{P}$. Moreover, if $W_1(z)$ and $W_2(z)$ are two solutions of the problem with N prescribed values and $U_1(z)$ and $U_2(z)$ are the corresponding solutions of the problem with no prescribed values (a fixed procedure being assumed), then

$$W_1(z) \neq W_2(z) \quad \text{if and only if} \quad U_1(z) \neq U_2(z). \tag{2.11}$$

Consider next what happens to the number of zeros in $\Delta(0; 1)$ of det $W^{(j)}(z)$ as j increases from 0 to N. From (2.6) it is clear that det $V^{(j)}(z)$ and det $W^{(j)}(z)$ have the same (finite by Lemma A.5) number of zeros in $\Delta(0; 1)$. From Lemma A.7 and (2.4) it follows that

$$p(W^{(j+1)}) = p(W^{(j)}) - n^{(j+1)},$$

where $p(W^{(j)})$ is the number of zeros of det $W^{(j)}(z)$ in $\Delta(0; 1)$. Hence

$$p(\mathscr{P}(W)) = p(W) - \sum_{i=1}^{N} n^{(i)}. \tag{2.12}$$

Thus $p(W) - p(\mathscr{P}(W))$ is independent of W; denote this quantity by $p(\mathscr{P})$. Then $p(\mathscr{P})$ is independent of $\mathscr{P}$. For, if $\mathscr{W}$ is the set of all solutions of the problem with N prescribed values, then

$$p(\mathscr{P}) = \min_{W \in \mathscr{W}} p(W) - \min_{W \in \mathscr{W}} p(\mathscr{P}(W)).$$

But in the set $\mathscr{U}$ of all solutions of the problem with no prescribed values is the function which is identically $\mathbb{1}_{n(N)}$ on $\overline{\Delta(0; 1)}$; its determinant has no zeros in $\Delta(0; 1)$. Thus

$$p(\mathscr{P}) = \min_{W \in \mathscr{W}} p(W),$$

which is clearly independent of $\mathscr{P}$; call it p.

Note further that the function which is identically $\mathbb{1}_{n(N)}$ on $\overline{\Delta(0; 1)}$ is the only function in $\mathscr{U}$ whose determinant has no zeros in $\Delta(0; 1)$. For if $U(z)$ is such a function, it follows that det $U(z)$ is identically 1 on $\overline{\Delta(0; 1)}$, since $U(1) = \mathbb{1}_{n(N)}$. Since $U(x_0)$ is real and symmetric for each x_0 in $(-1, 1)$, the result follows from Lemma A.4. It now follows from (2.11) that there is just *one* function $W_0(z)$ in $\mathscr{W}$ for which

$$p(W_0) = \mathrm{p}.$$

This function is called, in the standard terminology, the *isolated solution*. If

$$U_0(z) = \mathbb{1}_{n(N)}, \quad z \in \overline{\Delta(0; 1)},$$

then

$$W_0(z) = \mathscr{P}^{-1}(U_0(z)),$$

and the same $W_0(z)$ is obtained, whatever the procedure $\mathscr{P}$. Note also that from the preceding discussion it follows that $\mathscr{W}$ may be written as

$$\mathscr{W} = \bigcup_{k=0}^{\infty} \mathscr{W}^{(k)},$$

where $\mathscr{W}^{(k)}$ is the set of all solutions whose determinant has $(p + k)$ zeros in $\Delta(0; 1)$. The set $\mathscr{W}^{(0)}$ consists of the single function $W_0(z)$. Moreover,

$$\mathscr{W}^{(k)} = \mathscr{P}^{-1}(\mathscr{U}^{(k)}), \tag{2.13}$$

where $\mathscr{U}^{(k)}$ is the set of all solutions of the problem with no prescribed values whose determinant has k zeros in $\Delta(0; 1)$.

A further question may be raised. From (2.12) we see that

$$\sum_{i=1}^{N} n^{(i)} = p, \tag{2.14}$$

so that the sum on the left side is independent of the procedure $\mathscr{P}$. For a CNU problem, of course, the sum is nN. For a PNU problem, however, one may ask whether different procedures can give rise to different sequences $\{n^{(i)}\}$. We have not been able to show that this is not possible. It is clear that $n^{(1)}$ is the same for all procedures, and it may also be shown (though we do not give the proof here) that the same is true for $n^{(2)}$. Moreover, we shall show at the end of Section 3 that $n^{(N)}$ is independent of $\mathscr{P}$; from this it follows that the sets $\mathscr{U}^{(k)}$, $k = 0, 1, 2, \ldots$, are also independent of $\mathscr{P}$. Further than this, together with (2.14), we have not been able to go.

We conclude this section with a result needed in Section 4; its proof requires a preliminary lemma.

Lemma 2.4. *Suppose that there exists a function $T(z)$ which satisfies conditions* (a), (b), (d), (f) *and* (g) *of* (1.6) *and the further condition*

$$\sup_{|z|\leq 1} \|T(z)\| = \sup_{\theta} \|T(e^{i\theta})\| < 1.$$

Then the prescribed x_i, A_i define a CNU problem.

Proof. It follows immediately that $\|A_i\| < 1$, $i = 1, \ldots, N$. Define

$$T^{(1)}(z) = [b(x_N; z)]^{-1}(\mathbb{1}_n - A_N^2)^{-1/2}(T(z) - A_N) \times (\mathbb{1}_n - A_N T(z))^{-1}(\mathbb{1}_n - A_N^2)^{1/2}.$$

From the manipulations of Lemma A.1 we know that

$$\mathbb{1}_n - T^{(1)}(e^{i\theta})^* T^{(1)}(e^{i\theta}) = (\mathbb{1}_n - A_N^2)^{-1/2}(\mathbb{1}_n - T(e^{i\theta})^* A_N)^{-1} \times (\mathbb{1}_n - T(e^{i\theta})^* T(e^{i\theta}))(\mathbb{1}_n - A_N T(e^{i\theta}))^{-1}(\mathbb{1}_n - A_N^2)^{1/2},$$

since $|b(x_N; e^{i\theta})| = 1$. Now, using the equivalence of $\mathbb{1}_n - T^*T > 0$ and $\|T\| < 1$, and the fact that, if H is Hermitian and $\det M \neq 0$, then

$$M^*HM > 0 \quad \text{if and only if} \quad H > 0,$$

we see that

$$\sup_{\theta} \|T^{(1)}(e^{i\theta})\| < 1.$$

Thus, from the maximum principle,

$$\sup_{|z|\leq 1} \|T^{(1)}(z)\| < 1,$$

and, in particular, since $A_i^{(1)} = T^{(1)}(x_i)$,

$$\|A_i^{(1)}\| < 1, \quad i = 1, \ldots, N-1.$$

Continuing this process through a further $N - 2$ steps, we obtain the required result. □

The converse of Lemma 2.4 is also true, but we do not need it in what follows. We now use Lemma 2.4 to prove

Lemma 2.5. *If the problem with prescribed x_i, A_i, $i = 1, \ldots, N$, has a solution, then the problem with prescribed x_i, λA_i, $i = 1, \ldots, N$, has a completely non-unique solution for $0 \leq \lambda < 1$.*

Proof. Let $W(z)$ be a solution of the problem with prescribed x_i, A_i. If $0 < \lambda < 1$ and $T(\lambda; z) = \lambda W(z)$, then $T(\lambda; z)$ satisfies the conditions on $T(z)$ in Lemma 2.4, but with $A_i \to \lambda A_i$. It follows from that lemma that the problem with prescribed $x_i, \lambda A_i$ has a CNU solution. The case $\lambda = 0$ also has a CNU solution, since it follows from (2.4) and (2.5) that, if $U(z)$ is any function in the set $\mathscr{U}$, the corresponding function in $\mathscr{W}$ is

$$W(z) = \left(\prod_{i=1}^{N} b(x_i; z)\right) U(z). \quad \square$$

3. Representation theorem

We turn now to the problem of finding functions $U(z)$ of dimension m ($1 \leq m \leq n$) satisfying conditions (a)–(f) of (1.6). We use m instead of $n^{(N)}$; for a CNU problem $m = n$, while for a PNU problem $m < n$. We shall show how to construct each of the sets $\mathscr{U}^{(k)}$ of (2.13). To motivate the construction, we start with a function $U(z)$ from $\mathscr{U}^{(k)}$, so that det $U(z)$ has k zeros in $\Delta(0; 1)$. If $k = 0$ we know that $U(z)$ is identically $\mathbb{1}_m$, by Lemma A.4. If, however, $k > 0$, then, for each x_0 in $(-1, 1)$, $U(x_0)$ has at least one eigenvalue with absolute value less than 1; for otherwise det $U(x_0) = 1$, which by Lemma A.4 contradicts the assumption that $k > 0$. Take $x_0 = 0$ for convenience and let $U(0) = \mathrm{B}$. Then either $\|B\| < 1$ or $\|B\| = 1$. In the latter case an orthogonal matrix Ω exists such that

$$B = \Omega \begin{pmatrix} \mathbb{1}_{m-\tilde{m}} & 0 \\ 0 & \tilde{B} \end{pmatrix} \Omega^T, \qquad \Omega\Omega^T = \mathbb{1}_m,$$

where $0 < \tilde{m} < m$ and $\tilde{B}$ is a matrix of dimension $\tilde{m}$ for which $\|\tilde{B}\| < 1$. Moreover, in this case we know from Lemma 2.2 that $U(z)$ may be written in the form

$$U(z) = \Omega \begin{pmatrix} \mathbb{1}_{m-\tilde{m}} & 0 \\ 0 & \tilde{U}(z) \end{pmatrix} \Omega^T,$$

with $\tilde{U}(0) = \tilde{B}$. If $\|B\| < 1$ this step is not necessary; in this case we make the convention that $\tilde{U}(z) = U(z), \tilde{B} = B, \tilde{m} = m$.

Now define a new function

$$U^{(1)}(z) = z^{-1}(\mathbb{1}_{\tilde{m}} - \tilde{B}^2)^{-1/2}(\tilde{U}(z) - \tilde{B}) \\ \times (\mathbb{1}_{\tilde{m}} - \tilde{B}\tilde{U}(z))^{-1}(\mathbb{1}_{\tilde{m}} - \tilde{B}^2)^{1/2}. \tag{3.1}$$

From the arguments in Section 2, $U^{(1)}(z)$ satisfies conditions (a)–(f) of (1.6), with $n \to \tilde{m}$, and det $U^{(1)}(z)$ has $k - \tilde{m}$ zeros in $\Delta(0; 1)$. If $k - \tilde{m} > 0$ we may repeat the procedure until we arrive at a function whose determinant has no zeros in $\Delta(0; 1)$; this function is then identically $\mathbb{1}$. By reversing this procedure we see that each $U(z)$ can be constructed in a finite number of steps from a function which is identically $\mathbb{1}$, and possibly of lower dimensionality. To express this more precisely, note that the inverse relation to (3.1) is

$$\tilde{U}(z) = (\mathbb{1}_{\tilde{m}} - \tilde{B}^2)^{-1/2}(zU^{(1)}(z) + \tilde{B})(\mathbb{1}_{\tilde{m}} + z\tilde{B}U^{(1)}(z))^{-1}(\mathbb{1}_{\tilde{m}} - \tilde{B}^2)^{1/2}.$$

This enables us to formulate the

Representation theorem. The set $\mathscr{U}$ of matrix functions $U(z)$ of dimension m satisfying conditions (a)–(f) of (1.6) is the union of disjoint sets $\mathscr{U}^{(k)}$:

$$\mathscr{U} = \bigcup_{k=0}^{\infty} \mathscr{U}^{(k)},$$

where the determinant of each function in $\mathscr{U}^{(k)}$ has k zeros in $\Delta(0; 1)$. The following construction gives all the functions $U(z)$ in $\mathscr{U}^{(k)}$. Let $\{m_j\}$, $j = 0, 1, \ldots, q$, be a sequence of integers satisfying

$$1 \leq m_0 \leq m_1 \leq \cdots \leq m_q = m, \qquad \sum_{j=0}^{q-1} m_j = k.$$

Let B_j, $j = 0, 1, \ldots, q-1$, be real symmetric matrices of dimension m_j satisfying $\|B_j\| < 1$. Further, whenever $m_{j-1} < m_j$, $j = 1, \ldots, q$, let $\Omega^{(j)}$ be a real orthogonal matrix of dimension m_j. Then define $U(z)$ by the recursion

$$\begin{aligned} V^{(j+1)}(z) &= (\mathbb{1}_{m_j} - B_j^2)^{-1/2}(zU^{(j)}(z) + B_j) \\ &\quad \times (\mathbb{1}_{m_j} + zB_j U^{(j)}(z))^{-1}(\mathbb{1}_{m_j} - B_j^2)^{1/2}, \quad j = 0, \ldots, q-1, \\ U^{(j)}(z) &= V^{(j)}(z) \quad \text{if } m_j = m_{j-1}, \\ U^{(j)}(z) &= \Omega^{(j)} \begin{pmatrix} \mathbb{1}_{m_j - m_{j-1}} & 0 \\ 0 & V^{(j)}(z) \end{pmatrix} \Omega^{(j)T} \quad \text{if } m_j > m_{j-1}, j = 1, \ldots, q-1 \\ U^{(0)}(z) &= \mathbb{1}_{m_0}, \quad z \in \overline{\Delta(0; 1)}, \quad U^{(q)}(z) = U(z). \end{aligned}$$

While we know that all functions $U(z)$ in $\mathscr{U}^{(k)}$ can be constructed in this way it is not true that all the $U(z)$ so constructed will be distinct. We look now at the way in which the distinct functions in $\mathscr{U}^{(1)}$ may be indexed. For $k = 1$, we must have $q = 1$ and

$$m_0 = 1, \qquad m_1 = m \ (\geq 1).$$

The case $m = 1$ is trivial; we look at $m > 1$ and construct $U^{(0)}(z) = 1$, $V^{(1)}(z) = (z + b)(1 + bz)^{-1}$, b real, $|b| < 1$,

$$U(z) = U^{(1)}(z) = \Omega \begin{pmatrix} \mathbb{1}_{m-1} & 0 \\ 0 & V^{(1)}(z) \end{pmatrix} \Omega^T,$$

where Ω is a real orthogonal matrix of dimension m. Writing

$$\Omega = \begin{pmatrix} M_{m-1} & x \\ y & \eta \end{pmatrix},$$

$\Omega\Omega^T = \mathbb{1}_m$ is equivalent to

$$M_{m-1} M_{m-1}^T + xx^T = \mathbb{1}_{m-1}, \qquad M_{m-1} y^T + \eta x = 0, \qquad yy^T + \eta^2 = 1.$$

Thus

$$U(z) = \mathbb{1}_m - (1 - V^{(1)}(z)) \begin{pmatrix} xx^T & \eta x \\ \eta x^T & \eta^2 \end{pmatrix},$$

so that the functions $U(z)$ depend on the choice of b and of the unit vector $\mathbf{e}$ in $\mathbb{R}^m$ given by

$$\mathbf{e} = \begin{pmatrix} x \\ \eta \end{pmatrix}.$$

It is not difficult to see that $U(z) = U'(z)$ if and only if $b = b'$ and $\mathbf{e} = \pm\mathbf{e}'$. Thus the distinct functions of $\mathscr{U}^{(1)}$ may be indexed by the Cartesian product of the open interval $(-1, 1)$ and the set of pairs $\{\mathbf{e}, -\mathbf{e}\}$ of unit vectors in $\mathbb{R}^m$.

This result may be used to prove the statement in the paragraph preceding Theorem 2.4, that for a PNU problem $n^{(N)}$ is independent of the procedure $\mathscr{P}$. From (2.11) and (2.13), it must be possible, for a fixed k, to index the distinct sets $\mathscr{U}^{(k)}$ for the various possible procedures by equivalent index sets. However, the index sets for $\mathscr{U}^{(1)}$ for different $n^{(N)}$ are clearly inequivalent.

4. Consequences of physical interest

We begin by returning to Lemma 2.5. Consider an n-channel case with N poles for which the prescribed x_i, A_i define a CNU problem. Then the problem with x_i, λA_i prescribed is a CNU problem for $0 \leq \lambda \leq 1$. Denote by $W(\lambda; z)$, $0 \leq \lambda \leq 1$, $z \in \overline{\Delta(0; 1)}$, the isolated solution of this problem. Recall from (2.4)–(2.9) (but using the inverse of (2.4)) that $W(\lambda; z)$ is given by

$$W^{(j)}(\lambda; z) = (\mathbb{1}_n - A_{N-j}^{(j)}(\lambda)^2)^{-1/2}(b(x_{N-j}; z)W^{(j+1)}(\lambda; z) + A_{N-j}^{(j)}(\lambda))$$
$$\times (\mathbb{1}_n + b(x_{N-j}; z)A_{N-j}^{(j)}(\lambda)W^{(j+1)}(\lambda; z))^{-1}(\mathbb{1}_n - A_{N-j}^{(j)}(\lambda)^2)^{1/2},$$
$$j = N-1, \ldots, 0,$$

with

$$W^{(N)}(\lambda; z) = \mathbb{1}_n, \quad z \in \overline{\Delta(0; 1)}, \quad W^{(0)}(\lambda; z) = W(\lambda; z),$$

where

$$A_i^{(j+1)}(\lambda) = [b(x_{N-j}; x_i)]^{-1}(\mathbb{1}_n - A_{N-j}^{(j)}(\lambda)^2)^{-1/2}(A_i^{(j)}(\lambda) - A_{N-j}^{(j)}(\lambda))$$
$$\times (\mathbb{1}_n - A_{N-j}^{(j)}(\lambda)A_i^{(j)}(\lambda))^{-1}(\mathbb{1}_n - A_{N-j}^{(j)}(\lambda)^2)^{1/2},$$
$$j = 0, \ldots, N-2, \quad i = 1, \ldots, N-j-1,$$

with

$$A_i^{(0)}(\lambda) = \lambda A_i, \quad i = 1, \ldots, N.$$

In particular,

$$W(0; z) = \left(\prod_{i=1}^{N} b(x_i; z)\right)\mathbb{1}_n. \tag{4.1}$$

It is clear from the recursions of the previous paragraph that $W(\lambda; z)$ is continuous in λ on $[0, 1]$, uniformly with respect to z in $\overline{\Delta(0; 1)}$. Note that it is the hypothesis that x_i, A_i define a CNU problem which secures this continuity on the *closed* interval $[0, 1]$. Applying Lemma A.6 to $f_\lambda(z) = W_{jj}(\lambda; z)$, $j = 1, \ldots, n$, we see that, provided

$$\inf_{\substack{0 \leq \lambda \leq 1 \\ 0 \leq \theta \leq 2\pi}} |W_{jj}(\lambda; e^{i\theta})| > 0,$$

the function $W_{jj}(\lambda; z)$ has the same number of zeros in $\Delta(0; 1)$ for all λ in $[0, 1]$. From (4.1), $W_{jj}(0; z)$ has N zeros in $\Delta(0; 1)$; so then does $W_{jj}(1; z)$, the isolated solution of the CNU problem with x_i, A_i prescribed.

We now take up the question introduced in Section 1. This may be formulated precisely as follows:

(a) take a solution $W(z)$ of an n-channel problem with prescribed x_i, A_i;
(b) compute, for each j in $\{1,\ldots,n\}$, the inelasticity parameter $\eta_j(\theta)$ for the one-channel elastic scattering process $j \to j$, namely

$$\eta_j(\theta) = |W_{jj}(e^{i\theta})|;$$

(c) consider the one-channel problem with prescribed x_i, $(A_i)_{jj}$ and $\eta_j(\theta)$ (which we shall call the *equivalent inelastic one-channel problem* j, or EIOCP j for short) and seek to characterize the solution $W_{jj}(z)$ of this problem.

In particular, when $W(z)$ is the isolated solution of an n-channel CNU or PNU problem, we are interested in finding conditions for which $W_{jj}(z)$ is the isolated solution of the EIOCP j.

Let us digress for a moment to note that by a Froissart transformation [19] an inelastic one-channel problem can be transformed into an elastic one-channel problem. Define the function $L(z)$ on $\Delta(0; 1)$ by

$$L(z) = \exp\left[\frac{1}{2\pi}\int_0^{2\pi} dt\,\frac{e^{it}+z}{e^{it}-z}\ln\frac{1}{\eta(t)}\right], \quad z \in \Delta(0; 1),$$

and a new function $\tilde{W}(z)$ by

$$\tilde{W}(z) = L(z)W(z).$$

Then $|\tilde{W}(e^{i\theta})| = 1$ if $|W(e^{i\theta})| = \eta(\theta)$. Since $L(z)$ does not vanish in $\Delta(0; 1)$, $W(z)$ and $\tilde{W}(z)$ have the same number of zeros in $\Delta(0; 1)$. Thus, if we say that an inelastic one-channel problem is a CNU problem whenever the transformed elastic one-channel problem is a CNU problem, then the isolated solution of such a problem has N zeros in $\Delta(0; 1)$, and the other solutions have more than N zeros in $\Delta(0; 1)$. There are clearly no PNU problems in the one-channel case. However, there are problems with a unique solution, and in such cases the solution has less than N zeros in $\Delta(0; 1)$. Note too that for an n-channel case with $n > 1$, the determinant of the isolated solution of a CNU problem has nN zeros in $\Delta(0; 1)$, while the determinant of the isolated solution of a PNU problem or of the solution of a unique problem has less than nN zeros.

We may now state the result in the paragraph following (4.1) in the form of a lemma.

Lemma 4.1. *Let x_i, A_i, $i = 1,\ldots,N$, define a CNU problem with $n(>1)$ channels. Denote by $W(\lambda; z)$, $0 \le \lambda \le 1$, $z \in \overline{\Delta(0; 1)}$, the isolated solution of the CNU problem with prescribed x_i, λA_i. Let*

$$\eta_j(\lambda; \theta) = |W_{jj}(\lambda; e^{i\theta})|, \quad 0 \le \lambda \le 1, \quad 0 \le \theta \le 2\pi.$$

If

$$\inf_{\substack{0\le\lambda\le 1\\ 0\le\theta\le 2\pi}} \eta_j(\lambda; \theta) > 0, \tag{4.2}$$

then $W_{jj}(1; z)$ is the isolated solution of EIOCP j (with prescribed x_i, $(A_i)_{jj}$ and $\eta_j(1; \theta)$).

This is a convenient point at which to give the natural generalization of Levinson's theorem for the multichannel case we have considered. In view of the remarks immediately preceding Lemma 4.1 (which apply to det $W(z)$, $W(z)$ being the isolated solution of a CNU or PNU problem, or the solution of a unique problem), and the relation (1.4) between $W(z)$ and $S(z)$, it follows that, if D_S is the difference between the number of zeros and the number of poles of det $S(z)$ in $\Delta(0; 1)$, then $D_S = 0$ for the isolated solution of a CNU problem, while $D_S < 0$ for the isolated solution for a PNU problem or for the solution of a unique problem. In particular, if det $S(e^{i\theta}) = \exp(i\Delta(\theta))$, then

$$\Delta(2\pi) - \Delta(0) = 0$$

for the isolated solution for a CNU problem.

We now apply Lemma 4.1 to a special case.

Lemma 4.2. *Take a one-pole CNU problem with $n(>1)$ channels; let $W(x_1) = A$, with $\|A\| < 1$. Then if $W(z)$ is the isolated solution of the n-channel problem, $W_{jj}(z)$ is the isolated solution of EIOCP j.*

Proof. If $A = \Omega D \Omega^T$, with Ω a real orthogonal matrix, then

$$W(\lambda; z) = \Omega(b(x_1; z)\mathbb{1}_n + \lambda D)(\mathbb{1}_n + \lambda b(x_1; z)D)^{-1}\Omega^T,$$

$$W_{jj}(\lambda; z) = \sum_{k=1}^{n} \Omega_{jk}^2 (b(x_1; z) + \lambda d_k)(1 + \lambda b(x_1; z)\, d_k)^{-1}.$$

The matrix D is diagonal, and its diagonal elements d_k are real and satisfy $|d_k| < 1$, $k = 1, \ldots, n$.

It is easy to verify that

$$\operatorname{Im} W_{jj}(\lambda; e^{i\theta}) = \sum_{k=1}^{n} \frac{\Omega_{jk}^2 (1 - \lambda^2 d_k^2)}{|1 + \lambda b(x_1; e^{i\theta}) d_k|^2} \frac{(1 - x_1^2)\sin\theta}{|1 - x_1 e^{i\theta}|^2}.$$

Further,

$$W_{jj}(\lambda; 1) = 1, \quad W_{jj}(\lambda; -1) = -1.$$

Thus (4.2) is satisfied and the result follows from Lemma 4.1. □

We turn now to the next most simple case, that of two poles at x_i, x_2 with $n \times n$ matrices A_1, A_2 prescribed. We shall consider only CNU cases, so that

$$\|A_i\| < 1, \quad i = 1, 2.$$

It is more convenient to use the matrices P_i defined by

$$P_i = (\mathbb{1} + A_i)(\mathbb{1} - A_i)^{-1},$$

which satisfy,

$$P_i > 0, \quad i = 1, 2, \tag{4.3}$$

Using the recursion at the beginning of this section, the isolated solution is given by

$$W^{(2)}(z) = \mathbb{1}_n, \quad z \in \overline{\Delta(0; 1)},$$

$$W^{(1)}(z) = (b(x_1; z)\mathbb{1}_n + A_1^{(1)})(\mathbb{1}_n + b(x_1; z)A_1^{(1)})^{-1},$$

$$W(z) = W^{(0)}(z)$$
$$= (\mathbb{1}_n - A_2^2)^{-1/2}(b(x_2\,;z)W^{(1)}(z) + A_2)$$
$$\times (\mathbb{1}_n + b(x_2\,;z)A_2 W^{(1)}(z))^{-1}(\mathbb{1}_n - A_2^2)^{1/2},$$

with

$$A_1^{(1)} = [b(x_2\,;x_1)]^{-1}(\mathbb{1}_n - A_2^2)^{-1/2}(A_1 - A_2)(\mathbb{1}_n - A_2A_1)^{-1}(\mathbb{1}_n - A_2^2)^{1/2}$$
$$= [b(x_2\,;x_1)]^{-1}(P - \mathbb{1}_n)(P + \mathbb{1}_n)^{-1}, \tag{4.4}$$
$$P = P_2^{-1/2}P_1P_2^{-1/2}. \tag{4.5}$$

From now on we write b instead of $b(x_2\,;x_1)$. For a CNU problem we also require $\|A_1^{(1)}\| < 1$; by (4.4) this is equivalent to

$$(1 - |b|)/(1 + |b|) < p_i < (1 + |b|)/(1 - |b|), \tag{4.6}$$

where p_i, $i = 1,\ldots, n$, are the eigenvalues of the positive matrix P defined in (4.5). A tedious computation, which uses the identity

$$b(x_2\,;x_1)b(x_1\,;z)b(x_2\,;z) = b(x_2\,;x_1) + b(x_1\,;z) - b(x_2\,;z),$$

shows that the isolated solution may be put in the form

$$W(z) = (Q(z) - q(z)\mathbb{1}_n)(Q(z) + q(z)\mathbb{1}_n)^{-1},$$

where

$$Q(z) = (z - x_2)(1 - x_2z)(1 - x_1^2)P_1 - (z - x_1)(1 - x_1z)(1 - x_2^2)P_2,$$
$$q(z) = (x_1 - x_2)(1 - x_1x_2)(1 - z^2).$$

The conditions (4.3) and (4.6) ensure that $\det(Q(z) + q(z)\mathbb{1}_n)$ has all its $2n$ zeros outside $C(0;1)$ and that $\det(Q(z) - q(z)\mathbb{1}_n)$ has all its zeros inside $C(0;1)$. That each of these statements implies the other can be seen from the relations

$$Q(z^{-1}) = z^{-2}Q(z), \qquad q(z^{-1}) = -z^{-2}q(z), \qquad z \neq 0. \tag{4.7}$$

To obtain results of physical interest we confine ourselves from now on to the case of *two* channels. Then

$$W_{12}(z) = 2[\det(Q(z) + q(z)\mathbb{1}_2)]^{-1}q(z)Q_{12}(z),$$

where

$$Q_{12}(z) = (z^2 + 1)[-x_2(1 - x_1^2)(P_1)_{12} + x_1(1 - x_2^2)(P_2)_{12}]$$
$$+ z[(1 + x_2^2)(1 - x_1^2)(P_1)_{12} - (1 + x_1^2)(1 - x_2^2)(P_2)_{12}].$$

The product of the two zeros of $Q_{12}(z)$ is 1. There are thus cases where $Q_{12}(z)$ has complex conjugate zeros on $C(0;1)$. Physically, this means that there is an energy above threshold at which the two channels decouple. The simplest such case occurs when $(P_1)_{12} = (P_2)_{12}$, since

$$2|-x_2(1 - x_1^2) + x_1(1 - x_2^2)| > |(1 + x_2^2)(1 - x_1^2) - (1 + x_1^2)(1 - x_2^2)|.$$

Condition (4.6) can also be satisfied at the same time, since it holds for P_1 and P_2 in a neighbourhood of $P_1 = P_2$. We have shown that there are two pole, two channel cases for which the 'dynamical input' x_i, $A_i (i = 1, 2)$ gives an isolated solution for which the two channels decouple at an energy above threshold.

We consider now the characterization, for the isolated solution of a CNU two pole, two channel case, of the solutions $W_{jj}(z)$ of the two EIOCPs. Suppose that $W_{11}(z)$ has m_i zeros inside $C(0; 1)$, m_c zeros on $C(0; 1)$ and m_0 zeros outside $C(0; 1)$. Denote the corresponding quantities for $W_{22}(z)$ by n_i, n_c, n_0 respectively. Since

$$W_{jj}(z) = [\det(Q(z) + q(z)\mathbb{1}_2)]^{-1} \times [\det Q(z) - q^2(z) - (-)^j q(z)(Q_{11}(z) - Q_{22}(z))], \quad j = 1, 2,$$

it follows from (4.7) that

$$m_i = n_0, \quad m_c = n_c, \quad m_0 = n_i.$$

Also,

$$m_i + m_c + m_0 = 4.$$

Since $W_{jj}(1) = W_{jj}(-1) = 1, j = 1, 2$, it follows that only the following cases can occur:

$$m_c = 4, \quad m_i = 0, \quad m_0 = 0; \tag{4.8a}$$

$$m_c = 2, \quad m_i = 2, \quad m_0 = 0; \tag{4.8b}$$

$$m_c = 2, \quad m_i = 0, \quad m_0 = 2; \tag{4.8c}$$

$$m_c = 0, \quad m_i = 4, \quad m_0 = 0; \tag{4.8d}$$

$$m_c = 0, \quad m_i = 2, \quad m_0 = 2; \tag{4.8e}$$

$$m_c = 0, \quad m_i = 0, \quad m_0 = 4. \tag{4.8f}$$

In case (4.8a) both EIOCPs are unique. In case (4.8b), W_{11} is the isolated solution of a CNU EIOCP, while W_{22} is the solution of a unique EIOCP. These characterizations of W_{11} and W_{22} are reversed in case (4.8c). In case (4.8e) both W_{11} and W_{22} are the isolated solutions of CNU EIOCPs. In case (4.8d), W_{11} is a solution of 'CDD' type [20] (with two extra zeros in $\Delta(0; 1)$) of a CNU EIOCP, while W_{22} is the solution of a unique EIOCP. The characterizations of W_{11} and W_{22} are reversed in case (4.8f). We now give explicit examples of all six cases.

We shall first look for an example of case (4.8a). Note that if $W_{11}(e^{\pm i\theta}) = 0$, then, by unitarity, $W_{22}(e^{\pm i\theta}) = 0$ and so

$$\det Q(e^{i\theta}) - q^2(e^{i\theta}) = 0 \quad \textit{and} \quad Q_{11}(e^{i\theta}) = Q_{22}(e^{i\theta}).$$

If $m_c = 4$ there must be two pairs of zeros of W_{11} on $C(0; 1)$, $e^{\pm i\theta_1}$ and $e^{\pm i\theta_2}$. This means that $[Q_{11}(z) - Q_{22}(z)]$, which is a second degree polynomial, has four zeros and thus is identically zero. This can occur if and only if

$$(P_j)_{11} = (P_j)_{22}, \quad j = 1, 2.$$

But then both P_1 and P_2 can be diagonalized by the orthogonal matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

In such a case we need to look at the zeros of $\det Q(z) - q^2(z)$. The products of the four zeros is 1. Since W_{11} and W_{22} must have an even number of zeros on the real axis between -1 and 1, the only possible subcases are (4.8a) and (4.8e). We now give an example to show that both cases occur.

Take

$$(A_i)_{11} = (A_i)_{22} = \frac{1}{2}\left[\left(\frac{x_i - y}{1 - x_i y}\right)^2 + \left(\frac{x_i + y}{1 + x_i y}\right)^2\right], \quad i = 1, 2,$$

$$(A_i)_{12} = (A_i)_{21} = \frac{1}{2}\left[\left(\frac{x_i + y}{1 + x_i y}\right)^2 - \left(\frac{x_i - y}{1 - x_i y}\right)^2\right], \quad i = 1, 2,$$

where $0 < y < 1$. It is not difficult to check that the isolated solution is given by

$$W_{11}(z) = W_{22}(z) = \frac{1}{2}\left[\left(\frac{z - y}{1 - yz}\right)^2 + \left(\frac{z + y}{1 + yz}\right)^2\right],$$

$$W_{12}(z) = W_{21}(z) = \frac{1}{2}\left[\left(\frac{z + y}{1 + yz}\right)^2 - \left(\frac{z - y}{1 - yz}\right)^2\right].$$

It is a bit more difficult, but still routine, to check that (4.6) holds. It is easy to verify that the four zeros of $W_{jj}(z)$ are at

$$(2y)^{-1}[\pm i(1 - y^2) \pm \{4y^2 - (1 - y^2)^2\}^{1/2}].$$

Thus, if $0 < y < (\sqrt{2} - 1)$, the four zeros are all on the imaginary axis, with two inside $C(0; 1)$ and two outside. However, if $(\sqrt{2} - 1) \leq y < 1$, all four zeros are on $C(0; 1)$.

The next lemma gives another example of case (4.8e).

Lemma 4.3. *Consider a two channel, two pole CNU case with prescribed* x_i, A_i, $i = 1, 2$, *and assume that*

$$A_1 A_2 - A_2 A_1 = 0,$$

so that the A_i *can be diagonalized by a real orthogonal matrix*

$$\Omega = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$ *If* $\cos 2\omega \neq 0$, *the isolated solution of the two channel problem has diagonal elements* $W_{11}(z)$, $W_{22}(z)$ *which are the isolated solutions of the two EIOCPs.*

Proof. Let $\tilde{w}_i(z)$, $i = 1, 2$, be the isolated solutions of the two one-channel problems in the diagonal representation. Each of them has two zeros in $\Delta(0; 1)$ and

$$|\tilde{w}_i(e^{i\theta})| = 1 \quad \text{for all } \theta.$$

The isolated solution of the two channel, two pole problem is

$$W(z) = \Omega \begin{pmatrix} \tilde{w}_1(z) & 0 \\ 0 & \tilde{w}_2(z) \end{pmatrix} \Omega^T,$$

so that

$$W_{11}(z) = \cos^2\omega\,\tilde{w}_1(z) + \sin^2\omega\,\tilde{w}_2(z),$$

$$W_{22}(z) = \sin^2\omega\,\tilde{w}_1(z) + \cos^2\omega\,\tilde{w}_2(z).$$

If $\cos 2\omega \neq 0$, then $\cos^2\omega \neq \sin^2\omega$. If $\cos^2\omega > \sin^2\omega$, it follows that

$$|\cos^2\omega\,\tilde{w}_1(e^{i\theta})| > |\sin^2\omega\,\tilde{w}_2(e^{i\theta})|$$

and so, by Rouché's theorem, $W_{11}(z)$ has the same number of zeros in $\Delta(0;1)$ as $\tilde{w}_1(z)$ has, namely 2. The same conclusion follows if $\sin^2\omega > \cos^2\omega$, and the same argument shows that $W_{22}(z)$ also has 2 zeros in $\Delta(0;1)$. □

We now give examples of the other cases in (4.8). Take

$$(P_1)_{11} = (P_2)_{11} = p_1, \qquad (P_1)_{22} = (P_2)_{22} = p_2, \quad p_1 \neq p_2,$$
$$(P_1)_{12} = \varepsilon d, \qquad (P_2)_{12} = 0, \quad \varepsilon = \pm 1, d > 0.$$

To have $P_1 > 0$ we must take $p_1 > 0, p_2 > 0, p_1p_2 > d^2$. The matrix P of (4.5) is

$$P = \begin{pmatrix} 1 & \varepsilon d/(p_1p_2)^{1/2} \\ \varepsilon d/(p_1p_2)^{1/2} & 1 \end{pmatrix},$$

so that (4.6) is satisfied if and only if

$$d/(p_1p_2)^{1/2} < 2|b|/(1+|b|) \quad (<1 \text{ since } |b|<1).$$

This more restrictive condition replaces $p_1p_2 > d^2$. It is more convenient to use

$$d' = d(1-b^2)^{1/2}/2|b|,$$

in terms of which the condition becomes

$$d'^2/p_1p_2 < (1-|b|)/(1+|b|). \tag{4.9}$$

If we write further

$$d'^2 = 1 + \delta,$$

then

$$\begin{aligned} &Q_{11}(z)Q_{22}(z) - Q_{12}^2(z) - q^2(z) + q(z)(Q_{11}(z) - Q_{22}(z)) \\ &= [(z^2+1)(1+x_1x_2) - 2z(x_1+x_2)] \\ &\quad \times [(z^2+1)\{p_1p_2(1-x_2^2)(1+x_1x_2) - (1-3x_1x_2+3x_2^2-x_1x_2^3)\} \\ &\quad - 2z\{p_1p_2(1-x_2^2)(x_1+x_2) - (3x_2 - x_1 - 3x_1x_2^2 + x_2^3)\} \\ &\quad + (1-x_1x_2)(1-x_2^2)(p_1-p_2)(1-z^2)] \\ &\quad - 4\delta(1-x_1^2)[(z^2+1)x_2 - z(1+x_2^2)]^2. \end{aligned} \tag{4.10}$$

If $\delta = 0$, the quartic in (4.10) has two zeros on $C(0;1)$, namely $\zeta_0, \bar{\zeta}_0$, with

$$\zeta_0 = \frac{(x_1+x_2) + i(1-x_1^2)^{1/2}(1-x_2^2)^{1/2}}{1+x_1x_2}.$$

The other two zeros are those of the other quadratic in the term in (4.10) which does not involve δ. We denote this quadratic by

$$\varphi(z) = \alpha z^2 + 2\beta z + \gamma.$$

When $\delta = 0$, we must choose

$$p_1p_2 > (1+|b|)/(1-|b|), \tag{4.11}$$

by (4.9). Further,

$$\begin{aligned} \varphi(1)\varphi(-1) &= (\alpha+\gamma)^2 - 4\beta^2 \\ &= 4(1-x_2^2)^2(1-x_1x_2)^2[(p_1p_2-1)^2 - (p_1p_2+1)^2b^2] > 0, \end{aligned}$$

by (4.11), which, together with the reality of α, β, γ implies that the two zeros of $\varphi(z)$ are both inside, both outside, or both on $C(0; 1)$. The product of the two zeros is γ/α, which has the form

$$[a + c(p_1 - p_2)][a - c(p_1 - p_2)]^{-1},$$

where a is real and c is real and positive. Further, from (4.11),

$$\begin{aligned} a &= (p_1 p_2 + 1)(1 - x_2^2)(1 + x_1 x_2) - 2(1 - x_1 x_2)(1 + x_2^2) \\ &> \frac{2(1 - x_2^2)|x_1 - x_2|}{(1 - x_1 x_2)(1 - |b|^2)}[1 + x_1 x_2 + \operatorname{sgn}(x_1^2 - x_2^2)|x_1 + x_2|] > 0. \end{aligned}$$

Thus, if $p_1 < p_2$, $|\gamma/\alpha| < 1$ and $\varphi(z)$ has two zeros inside $C(0; 1)$, while if $p_1 > p_2$, $|\gamma/\alpha| > 1$ and $\varphi(z)$ has two zeros outside $C(0; 1)$ (one of these zeros may be at infinity for exceptional values of p_1, p_2 for which $\alpha = 0$). We have thus given examples of cases (4.8b) and (4.8c).

It is easy to give examples of cases (4.8d) and (4.8f) by making δ in (4.10) different from zero. Consider what happens to the zeros ζ_0, $\bar{\zeta}_0$ of the quartic (4.10) as δ moves slightly from zero. Suppose that ζ_0 moves to $\zeta_0(1 + \eta)$. Then

$$0 \approx \zeta_0 \eta(\zeta_0 - \bar{\zeta}_0)\varphi(\zeta_0) - 4\delta(1 - x_1^2)[(x_2 - \zeta_0)(1 - x_2\zeta_0)]^2.$$

But

$$(x_2 - \zeta_0)(1 - x_2\zeta_0) = (1 + x_1 x_2)^{-1}(1 + 3x_1 x_2 + 3x_2^2 + x_1 x_2^3)\zeta_0$$

and

$$(1 + 3x_1 x_2 + 3x_2^2 + x_1 x_2^3) \neq 0 \quad \text{since} \quad (x_2 - \zeta_0)(1 - x_2\zeta_0) \neq 0.$$

Thus

$$\begin{aligned} 2i\eta\zeta_0(\operatorname{Im}\zeta_0)\varphi(\zeta_0) \approx 4(1 - x_1^2)(1 + 3x_1 x_2 + 3x_2^2 + x_1 x_2^3)^2 \\ \times (1 + x_1 x_2)^{-2}\zeta_0^2\,\delta. \end{aligned}$$

Since $\bar{\zeta}_0 = \zeta_0^{-1}$ and $\zeta_0\varphi(\zeta_0^{-1}) = \alpha\zeta_0^{-1} + 2\beta + \gamma\zeta_0$, it follows that

$$\operatorname{Re}\eta \approx \frac{2(1 - x_1^2)(1 + 3x_1 x_2 + 3x_2^2 + x_1 x_2^3)^2(\gamma - \alpha)}{(1 + x_1 x_2)^2\,|\varphi(\zeta_0)|^2}\,\delta,$$

so that

$$\operatorname{sgn}(\operatorname{Re}\eta) = \operatorname{sgn}(p_1 - p_2)\operatorname{sgn}\delta.$$

Thus, if $p_1 < p_2$, $W_{11}(z)$ has two zeros which remain inside $C(0; 1)$, and as δ increases through zero the other two zeros move through $C(0; 1)$ from the outside to the inside. For δ slightly greater than zero, then, $W_{11}(z)$ has four zeros inside $C(0; 1)$ (case (4.8d)). On the other hand, if $p_1 > p_2$, $W_{11}(z)$ has four zeros outside $C(0; 1)$ when δ is slightly greater than zero (case (4.8f)).

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Appendix

We use freely in the paper some elementary results about $n \times n$ matrices, considered as linear operators on the finite dimensional Hilbert space $\mathbb{C}^n$, with the usual scalar product, namely

$$(x, y) = \sum_{i=1}^{n} \bar{x}_i y_i, \quad x, y \in \mathbb{C}^n.$$

These results include the norm continuity of the product, the existence of a unique non-negative Hermitian square root of a non-negative Hermitian matrix, and the fact that $\mathbb{1}_n - A^*A > 0$ is equivalent to $\|A\| < 1$.

Two lemmas follow, involving a kind of matrix Blaschke transform. Let A, X be $n \times n$ matrices satisfying

$$\|A\| < 1, \qquad A = \bar{A} = A^T, \qquad \|X\| \leq 1.$$

Define [21]

$$Y = (\mathbb{1}_n - A^2)^{-1/2}(X - A)(\mathbb{1}_n - AX)^{-1}(\mathbb{1}_n - A^2)^{1/2}. \tag{A.1}$$

Lemma A.1 [21]. *The transformation* (A.1) *is invertible and thus is one-to-one. Moreover,*

$$X^*X = \mathbb{1}_n \text{ if and only if } Y^*Y = \mathbb{1}_n.$$

Proof. It follows directly from the definition (A.1) that

$$\begin{aligned}\mathbb{1}_n - Y^*Y = (\mathbb{1}_n - A^2)^{1/2}(\mathbb{1}_n - X^*A)^{-1} \\ \times [(\mathbb{1}_n - X^*A)(\mathbb{1}_n - A^2)^{-1}(\mathbb{1}_n - AX) - (X^* - A) \\ \times (\mathbb{1}_n - A^2)^{-1}(X - A)](\mathbb{1}_n - AX)^{-1}(\mathbb{1}_n - A^2)^{1/2}.\end{aligned}$$

An explicit calculation of the expression within the square brackets shows that it is just $(\mathbb{1}_n - X^*X)$, which completes the proof. The calculation uses the result

$$\mathbb{1}_n - AX = (\mathbb{1}_n - A^2) - A(X - A). \quad \square$$

Lemma A.2. *If $X^T = X$, then $Y^T = Y$.*

Proof. From (A.1),

$$Y^T = (\mathbb{1}_n - A^2)^{1/2}(\mathbb{1}_n - XA)^{-1}(X - A)(\mathbb{1}_n - A^2)^{-1/2}.$$

Thus $Y^T = Y$ is equivalent to

$$(X - A)(\mathbb{1}_n - AX)^{-1}(\mathbb{1}_n - A^2) = (\mathbb{1}_n - A^2)(\mathbb{1}_n - XA)^{-1}(X - A).$$

But

$$\begin{aligned}\mathbb{1}_n - A^2 &= \mathbb{1}_n - AX + A(X - A) \\ &= \mathbb{1}_n - XA + (X - A)A,\end{aligned}$$

so that $Y^T = Y$ is equivalent to

$$(X - A)(\mathbb{1}_n - AX)^{-1}A(X - A) = (X - A)A(\mathbb{1}_n - XA)^{-1}(X - A).$$

Since

$$(\mathbb{1}_n - AX)A = A(\mathbb{1}_n - XA),$$

the result follows. $\square$

There is one result from the theory of analytic functions of a complex variable which we used constantly in the paper. This is the extension to analytic matrix functions of the maximum modulus principle for the usual complex-valued analytic functions. We refer to it as the *maximum principle* and give it now in the form we require.

Maximum principle. Let $F(z)$ be a matrix function analytic in $\Delta(0; 1)$ and continuous on $\overline{\Delta(0; 1)}$. (This is equivalent to requiring that each element of $F(z)$ possess these properties.) Then, if $M = \sup_\theta \|F(e^{i\theta})\|$, we have

$$\|F(z)\| \leq M, \quad z \in \Delta(0; 1).$$

Further, if $\|F(z_0)\| = M$ for some $z_0 \in \Delta(0; 1)$, then $\|F(z)\| = M$ for all z in $\overline{\Delta(0; 1)}$.

For a proof of the maximum principle we refer to Hille and Phillips [22]. We now apply it to prove two more lemmas.

Lemma A.3. *Let $F(z)$ be a matrix function analytic in $\Delta(0; 1)$ and continuous on $\overline{\Delta(0; 1)}$. Further, let $F(z)$ be unitary in $C(0; 1)$ and let $F(z_0)$ be unitary for some z_0 in $\Delta(0; 1)$. Then $F(z)$ is a constant (unitary) matrix for $z \in \overline{\Delta(0; 1)}$.*

Proof. Since $F(z_0)$ is unitary it may be diagonalized by a unitary matrix U:

$$F(z_0) = U^*DU, \qquad U^*U = \mathbb{1}_n.$$

where D is diagonal and $|D_{ii}| = 1$, $i = 1, \ldots, n$. If

$$G(z) = UF(z)U^*,$$

then $G(z)$ has the same properties as $F(z)$ and $|G_{ii}(z_0)| = 1$. Thus by the maximum principle for ordinary analytic functions, $G_{ii}(z)$ is a constant, of modulus 1, on $\overline{\Delta(0; 1)}$. Since $G(z)$ is unitary in $C(0; 1)$, $G_{ij}(z) = 0$ in $C(0; 1)$ for $i \neq j$. By the maximum principle again, $G_{ij}(z) = 0$ on $\overline{\Delta(0; 1)}$. Thus $G(z) = D$, $F(z) = U^*DU$, a constant unitary matrix, for $z \in \overline{\Delta(0; 1)}$. □

Lemma A.4. *Let $F(z)$ be a matrix function analytic in $\Delta(0; 1)$ and continuous on $\overline{\Delta(0; 1)}$. Further, suppose that $F(z)$ is unitary in $C(0; 1)$ and that, for some z_0 in $\Delta(0; 1)$, $F(z_0)$ is normal and $|\det F(z_0)| = 1$. Then $F(z)$ is a constant unitary matrix for $z \in \overline{\Delta(0; 1)}$.*

Proof. Since $F(z_0)$ is normal, it may be diagonalized by a unitary matrix U:

$$F(z_0) = U^*DU, \qquad U^*U = \mathbb{1}_n,$$

where D is diagonal and $|\det F(z_0)| = \prod_{i=1}^{n} |D_{ii}| = 1$. On the other hand, by the maximum principle,

$$\|F(z_0)\| = \max_i |D_{ii}| \leq 1.$$

Thus $|D_{ii}| = 1$, $i = 1, \ldots, n$ and $F(z_0)$ is unitary so that Lemma A.3 applies. □

The next Lemmas (A.5–A.7) concern results about the number of zeros of functions analytic in $\Delta(0; 1)$.

Lemma A.5. *If $f(z)$ (not identically zero) is analytic in $\Delta(0; 1)$ and continuous on $\overline{\Delta(0; 1)}$ and if $f(z)$ does not vanish in $C(0; 1)$, then $f(z)$ has a finite number of zeros in $\Delta(0; 1)$.*

Proof. If $f(z)$ has an infinite number of zeros in $\Delta(0; 1)$, these zeros have an accumulation point. This point cannot be in $\Delta(0; 1)$, for then $f(z)$ would be identically zero. On the other hand, this point cannot be in $C(0; 1)$, for this would contradict the continuity of $f(z)$ on $\overline{\Delta(0; 1)}$. □

Note that if, further, $|f(z)| = 1$ in $C(0; 1)$, then $f(z)$ is a Blaschke product with a finite number of zeros:

$$f(z) = \prod_{k=1}^{p} \frac{z - a_k}{1 - \bar{a}_k z}, \tag{A.2}$$

where we may choose the a_k so that $0 \leq |a_1| \leq \cdots \leq |a_p| < 1$.

Rouché's theorem. If $f(z)$, $g(z)$ are analytic in $\Delta(0; 1)$ and continuous on $\overline{\Delta(0; 1)}$ and if $|g(e^{i\theta}) - f(e^{i\theta})| < |f(e^{i\theta})|$ for all θ, then $f(z)$ and $g(z)$ have the same number of zeros in $\Delta(0; 1)$.

For the proof, see Rudin [23], Theorem 10.36 and Ex. 10, page 266.

Lemma A.6. *Let $f_\lambda(z)$ be analytic in $\Delta(0; 1)$, continuous on $\overline{\Delta(0; 1)}$ for each λ in $[0, 1]$, and continuous in λ on $[0, 1]$, uniformly with respect to z in $C(0; 1)$. If further*

$$\inf_\theta |f_\lambda(e^{i\theta})| > 0 \quad \text{for each } \lambda \text{ in } [0, 1],$$

then $f_\lambda(z)$ has the same number of zeros in $\Delta(0; 1)$ for all λ in $[0, 1]$.

Proof. Let $p(\lambda)$ be the number of zeros of $f_\lambda(z)$ in $\Delta(0; 1)$ and let $\lambda_0 \in [0, 1]$. Then we can find $\delta > 0$ such that, for all θ,

$$|f_\lambda(e^{i\theta}) - f_{\lambda_0}(e^{i\theta})| < \inf_\theta |f_{\lambda_0}(e^{i\theta})|, \quad \lambda \in [0, 1] \cap \{|\lambda - \lambda_0| < \delta\}.$$

By Rouché's theorem, $p(\lambda) = p(\lambda_0)$ for this set of values of λ, and so $p(\lambda)$ is constant on $[0, 1]$. □

Corollary. *Let $f(z)$ be a Blaschke product of the form* (A.2) *and let $g(z)$ be the function*

$$g(z) = (f(z) - a)(1 - \bar{a}f(z))^{-1}, \quad |a| < 1.$$

Then $g(z)$ also has p zeros in $\Delta(0; 1)$.

Proof. Define

$$f_\lambda(z) = (f(z) - \lambda a)(1 - \lambda\bar{a}f(z))^{-1}, \quad |a| < 1, 0 \leq \lambda \leq 1,$$

and use Lemma A.6. □

There is a natural extension of this result to the 'matrix Blaschke transform' of (A.1).

Lemma A.7. *Let $F(z)$ be a matrix function analytic in $\Delta(0; 1)$, continuous on $\overline{\Delta(0; 1)}$ and unitary in $C(0; 1)$. Let A be a real symmetric matrix with $\|A\| < 1$ and define the matrix function $G(z)$ by*

$$G(z) = (\mathbb{1}_n - A^2)^{-1/2}(F(z) - A)(\mathbb{1}_n - AF(z))^{-1}(\mathbb{1}_n - A^2)^{1/2}.$$

Then det $F(z)$ *and* det $G(z)$ *have the same number of zeros in $\Delta(0; 1)$.*

Proof. Define

$$F_\lambda(z) = (\mathbb{1}_n - \lambda^2 A^2)^{-1/2}(F(z) - \lambda A)(\mathbb{1}_n - \lambda A F(z))^{-1}(\mathbb{1}_n - \lambda^2 A^2)^{1/2}, \qquad 0 \leq \lambda \leq 1.$$

Then $F_\lambda(z)$ for each λ in $[0, 1]$ has the same properties as $F(z)$ and

$$F_0(z) = F(z), \qquad F_1(z) = G(z).$$

Moreover, it is clear that $F_\lambda(z)$ is continuous in λ on $[0, 1]$, uniformly with respect to z in $C(0; 1)$. This implies the continuity in λ of each of the elements of $F_\lambda(z)$ and thus of det $F_\lambda(z)$ ·(in the absolute value norm). Now apply Lemma A.6 to $f_\lambda(z) = \det F_\lambda(z)$. $\square$