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The conservation laws of nonrelativistic classical and quantum mechanics for a system of interacting particles

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(1. II. 1978)

Abstract. The various classical or quantum mechanical equations describing a system of N particles with central two-body interactions are invariant under the 10 transformations of the Galilei group, and for interaction potentials inversely proportional to the squares of the particle separations also under two further transformations. From the invariance of the corresponding classical and quantum mechanical variational principles under this 12-parameter conformal extension of the Galilei group, the 'Jacobi–Schrödinger group', the 12 well-known conservation laws of Newtonian dynamics as well as 12 local conservation laws implied by the Schrödinger equation are obtained via Noether's theorem. Under appropriate conditions on the wave functions, these local laws yield 12 global conservation laws which are analogous to the Newtonian ones. The Hamilton–Jacobi equation implies a classical equation differing from the Schrödinger equation only by a potential-like term involving the Van Vleck determinant, from which 12 local balance equations and the corresponding global equations are obtained, which under certain conditions reduce to true conservation laws. The various sets of conservation laws are compared with each other as well as with the conservation laws implied by the conformal invariance of Fierz's classical equations for fields of zero rest mass and arbitrary spin.

I. Introduction

For a system of N particles with central two-body interactions described by the Hamiltonian

$$H = \sum_{K=1}^{N} \frac{\mathbf{p}_K^2}{2m_K} + V,$$
$$V = \tfrac{1}{2} \sum_{\substack{K,L=1 \\ K \neq L}}^{N}\sum V_{KL}(r_{KL}), \quad r_{KL} \equiv |\mathbf{r}_K - \mathbf{r}_L|, \tag{1}$$

where the V_{KL} are arbitrary functions of their arguments, the invariance of the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = H\psi, \qquad \mathbf{p}_K \equiv \frac{\hbar}{i}\nabla_K \equiv \frac{\hbar}{i}\frac{\partial}{\partial \mathbf{r}_K}, \tag{2}$$

as well as of the corresponding classical Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H = 0, \qquad \mathbf{p}_K \equiv \nabla_K S, \tag{3}$$

under the 10-parameter Galilei group has been known for a long time (for a detailed discussion of this group see [1]). However, it was realized only recently that in the absence of interactions equation (2) is invariant under a wider group of transformations, the 12-parameter 'Schrödinger group' [2], and equation (3) under a 13-parameter group containing the Schrödinger group as a subgroup [3],[1]) and that the invariance under the Schrödinger group also holds for both equations for interactions of the form (1) with [3]

$$V_{KL} = C_{KL} r_{KL}^{-2}. \tag{4}$$

As noted in [3], it was already known to Jacobi [5] that Newton's equations of motion for an N-body system with interactions of the form (4) are invariant under this 12-parameter group, and therefore it was suggested that it would be more appropriate to call it the 'Jacobi–Schrödinger group'.

This group has been the subject of a number of recent investigations [6–14], most of which are concerned with representations of the group.[2]) Its relation to the conformal group was studied in [10] and [11] by comparing the Jacobi–Schrödinger group to the nonrelativistic limit of the conformal extension of the Poincaré group. This is a 15-parameter Lie group, as is its nonrelativistic limit; however, it does not contain the Jacobi–Schrödinger group as a subgroup, but has a more complicated relation to it. In [3], instead, various conformal extensions of the Galilei group were defined in physical analogy to that of the Poincaré group. They are much wider than the latter, being gauge groups (i.e. containing arbitrary functions), and do contain the Jacobi–Schrödinger group as a subgroup.

The invariance of equation (2) under the Jacobi–Schrödinger group was established in [3] from that of the action integral

$$I = \int \left[\frac{\hbar}{2i} \left(\psi \frac{\partial \psi^*}{\partial t} - \psi^* \frac{\partial \psi}{\partial t} \right) - \sum_K \frac{\hbar^2}{2m_K} \frac{\partial \psi^*}{\partial x_K^l} \frac{\partial \psi}{\partial x_K^l} - \psi^* \psi V \right] d^4x \tag{5}$$

(where here and in the following summation over the range of any repeated index denoting components – but not particle labels – is understood), where $\delta I = 0$ implies (2). This not only simplifies the calculations considerably, but, as noted in [3], the invariance of (5) under the infinitesimal transformations of the 12-parameter group, by Noether's theorem [15, 16], implies the existence of 12 local conservation laws (conserved currents), in addition to the well-known law of conservation of probability

$$\frac{\partial}{\partial t}(\psi^* \psi) + \sum_K \nabla_K \cdot \left[\frac{\hbar}{i m_K} (\psi^* \nabla_K \psi - \psi \nabla_K \psi^*) \right] = 0, \tag{6}$$

which follows from the invariance of (5) under changes of phase.

No corresponding variational principle exists for the Hamilton–Jacobi equation, and therefore its invariance does not guarantee the existence of conservation laws. However, it has been shown by Van Vleck [17] (for generalizations see [18–20]) that the ensembles of particles described by this equation satisfy a conservation law analogous to (6)

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[1]) For a different approach to the symmetries of the Hamilton–Jacobi equation see [4].

[2]) The studies concerned with the kinematic invariance groups for arbitrary external potentials not describing interactions [9, 12, 13] use a different approach from that of this paper. From the point of view taken here only the potentials (2) with (4) are invariant under all the transformations of the Jacobi–Schrödinger group.

$$\frac{\partial \bar{D}}{\partial t} + \sum_K \nabla_K \cdot (\bar{D}\mathbf{v}_K) = 0, \tag{7}$$

where

$$\bar{D} \equiv \det \frac{\partial^2 S}{\partial q^i \, \partial \alpha^j} \tag{8}$$

is the Van Vleck determinant formed from a 'complete integral' of (3) containing $3N$ arbitrary constants α^j, q^i stands for the $3N$ components x_K^l of the N $\mathbf{r}'_{KS}$,[3]) and

$$\mathbf{v}_K \equiv \frac{\partial H}{\partial \nabla_K S}. \tag{9}$$

This is the consequence of a very deep rooted relation between equations (2) and (3), which allows us to derive from the classical Hamilton–Jacobi equation (3) a modified classical Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial \psi_c}{\partial t} = H\psi_c + \frac{\hbar^2}{2}\frac{1}{R}\sum_K \frac{1}{m_K}\nabla_K^2 R\,\psi_c, \tag{10}$$

where

$$R \equiv |\bar{D}|^{1/2}, \qquad \psi_c = R\,e^{iS/\hbar}, \tag{11}$$

and $\hbar$ has the dimensions of action, but does not necessarily have the same numerical value as Planck's constant [18–20].

Although equation (10) can not be derived from a simple variational principle, it is shown in Section V that it is possible to exploit the similarity of equations (2) and (10) to obtain local balance equations from equation (10) which are analogous to the 12 quantum mechanical conservation laws of equation (3), and which reduce to true local conservation laws under certain conditions. Illustrations of these laws and equations for a system of free particles are given in the Appendix.

The 12 conservation laws obtained from equation (2) are those following directly from Noether's theorem. However, since the integrand in the action integral (5) is homogeneous quadratic in ψ and its derivatives, it follows from a theorem of Steudel [21] that there exists an infinity of conserved currents, which can be obtained from the original conserved currents by a procedure established in [21]. Correspondingly, there exists an infinity of balance equations for equation (3).

The physical interpretation of the local conservation laws obtained, and especially of the global ones following from them under appropriate conditions (to be discussed in Section VI), is facilitated by comparison with the conservation laws of Newtonian point mechanics. These are most easily obtained from the classical action integral appropriate to the Hamiltonian (1) [and corresponding to the quantum mechanical

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[3]) Although the entire theory can be developed in generalized coordinates [18, 19], for our purposes it is more convenient to work with Cartesian coordinates except as noted otherwise. Also, some authors include a factor $(-1)^{3N}$ in the definition (8), but the difference is irrelevant for our considerations.

action integral (5)]

$$I = \int \left(\sum_K \tfrac{1}{2} m_K \mathbf{v}_K^2 - V \right) dt, \qquad \mathbf{v}_K \equiv \frac{d\mathbf{r}_K}{dt}, \tag{12}$$

where V is given in (1). This integral is invariant (up to a divergence) under the Galilei group for arbitrary V_{KL}, and under the additional transformations of the Jacobi–Schrödinger group for the special interactions (4), and thus with these interactions there exist 12 conservation laws, which are derived in Section III.

II. Noether's theorem and the Jacobi–Schrödinger group

The fundamental connection between the invariance properties of a variational principle and the conservation laws has been expressed in two theorems by E. Noether, referring to the invariance under a finite and an infinite continuous group, respectively.[4]) We consider an integral

$$I = \int L\left(q^\mu, \varphi_a, \frac{\partial \varphi}{\partial q^\mu}\right) d^n q, \quad \mu = 0 \ldots n-1, \quad a = 1 \ldots A, \tag{13}$$

with the Lagrangean derivatives

$$L_a \equiv \frac{\partial L}{\partial \varphi_a} - \frac{\partial}{\partial q^\mu} \frac{\partial L}{\partial \dfrac{\partial \varphi_a}{\partial q^\mu}}, \tag{14}$$

and the infinitesimal transformations

$$q'^\mu = q^\mu + \delta q^\mu, \qquad \varphi'_a = \varphi_a + \delta \varphi_a, \tag{15}$$

where δq^μ and $\delta \varphi_a$ are functions of the q^μ, φ_a, $\partial \varphi_a / \partial q^\mu$ depending linearly on the p infinitesimal parameters of a group G_p. Then, according to Noether's first theorem, if I is invariant under the infinitesimal transformations of G_p up to a divergence, i.e. if

$$L' \, d^n q' = L d^n q + \frac{\partial C^\mu}{\partial q^\mu} d^n q, \tag{16}$$

where L' is the same function of the transformed variables (14) as L is of the untransformed ones, there exist precisely p linearly independent combinations of the

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[4]) The theorems were given in [15], and with a slight generalization, also credited to Noether, in [16]. Many statements of the theorem in the literature of physics are incomplete if not incorrect; furthermore, in quantum field theory usually only the special example of Poincaré invariance is considered (compare e.g. [22]). For further references, with some comments, see [23]. The connection between conservation laws and invariance groups is discussed in some detail in [24].

Lagrangean derivatives which are divergences; they have the form

$$\frac{\partial}{\partial q^\mu}\left[\left(\frac{\partial L}{\partial \frac{\partial \varphi_a}{\partial q^\mu}}\frac{\partial \varphi_a}{\partial q^\nu} - L\,\delta^\mu_\nu\right)\delta q^\nu - \frac{\partial L}{\partial \frac{\partial \varphi_a}{\partial q^\mu}}\delta\varphi_a + C^\mu\right] = L_a\left(\delta\varphi_a - \frac{\partial \varphi_a}{\partial q^\mu}\delta q^\mu\right), \quad (17)$$

and thus imply p local conservation laws if the Euler–Lagrange equations

$$L_a = 0 \quad (18)$$

are satisfied.[5])

If L is homogeneous quadratic in the φ_a and the $\partial\varphi_a/\partial q^\mu$, and A_1, A_2, A_3 are three (not necessarily distinct) elements of the Lie algebra of the invariance group of (12), then, according to a theorem by Steudel [21],

$$\bar{A} \equiv \tfrac{1}{2}(A_1\,A_2\,A_3 + A_3\,A_2\,A_1) \quad (19)$$

is also an element of this algebra, and via Noether's theorem implies a local conservation law. Thus for such an L there exist infinitely many conservation laws.

The action integral (5) is of the form (13), with

$$\begin{aligned} &n = 3N + 1, \qquad A = 2, \qquad q^0 = t,\\ &q^1 \cdots q^{n-1} \equiv x^1_1, x^2_1, x^3_1 \cdots x^1_N, x^2_N, x^3_N\\ &\qquad\qquad \equiv x^l_K \quad (K = 1 \ldots N, \quad l = 1, 2, 3). \end{aligned} \quad (20)$$

The action integral (12) is also of the form (13), with

$$\begin{aligned} &n = 1, \qquad A = 3N, \qquad q^0 = t,\\ &\varphi_1 \cdots \varphi_{3N} \equiv x^1_1, x^2_1, x^3_1 \cdots x^1_N, x^2_N, x^3_N\\ &\qquad\qquad \equiv x^l_K \quad (K = 1 \ldots N, \quad l = 1, 2, 3). \end{aligned} \quad (21)$$

We take the ten independent infinitesimal transformations of the Galilei group in the customary form

$$\begin{aligned} &\text{(I)} \quad \delta t = \tau, \qquad \delta x^l_K = 0,\\ &\text{(II)} \quad \delta t = 0, \qquad \delta x^l_K = \alpha^l,\\ &\text{(III)} \quad \delta t = 0, \qquad \delta x^l_K = \beta^l_j x^j_K, \quad \beta^l_j = -\beta^j_l,\\ &\text{(IV)} \quad \delta t = 0, \qquad \delta x^l_\mu = \varepsilon^l t, \end{aligned}$$

corresponding to time translations, space translations, rotations, and Galilei 'boosts', respectively, and the additional infinitesimal transformations of the Jacobi–Schrödinger group as

$$\begin{aligned} &\text{(V)} \quad \delta t = 2\gamma t, \qquad \delta x^l_K = \gamma x^l_K,\\ &\text{(VI)} \quad \delta t = \lambda t^2, \qquad \delta x^l_K = \lambda t x^l_K, \end{aligned}$$

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[5]) We are not concerned here with the converse of this theorem, or with Noether's second theorem. Also, it has been shown in [21] that the transformations (16) are equivalent to transformations of the φ_a *alone* as far as the conservation laws resulting from (18) are concerned, but no use will be made of this here.

corresponding to the Schrödinger dilations and expansions, respectively. To assure the invariance of (5) under the transformations (I)–(VI), we must take

$$\delta\psi = \delta\psi^* = 0 \tag{22}$$

for the cases (I)–(III), and

$$\begin{aligned} \delta\psi &= \frac{i}{\hbar}\sum_K m_K \varepsilon^l x_K^l \psi, \\ \delta\psi^* &= -\frac{i}{\hbar}\sum_K m_K \varepsilon^l x_K^l \psi^*, \end{aligned} \tag{23}$$

$$\delta\psi = -\tfrac{3}{2}\gamma\psi, \qquad \delta\psi^* = -\tfrac{3}{2}\gamma\psi^*, \tag{24}$$

$$\begin{aligned} \delta\psi &= -\frac{\lambda}{2}\left(3t - \frac{i}{\hbar}\sum_K m_K x_K^l x_K^l\right)\psi, \\ \delta\psi^* &= -\frac{\lambda}{2}\left(3t + \frac{i}{\hbar}\sum_K m_K x_K^l x_K^l\right)\psi^*, \end{aligned} \tag{25}$$

for the cases (IV), (V), and (VI), respectively, as well as $C^\mu = 0$ in all cases. The form of the $\delta\psi$ in equations (23) and (24) follows from the finite transformations (36) and (40) given in [3], and that of equation (25) from the infinitesimal transformations (44) of [3] (correcting a misprint in the sign of the last term), if due account is taken of the fact that the action integral (5) refers to N particles, whereas the equations of [3] quoted refer to the case of a single particle.

Unlike (5), for some of the transformations considered the action integral (12) is only invariant up to a divergence, i.e. because of the identifications (20), up to a time derivative. While $C^0 = 0$ for the cases (I)–(III) and (V), we have

$$C^0 = \sum_K m_K \varepsilon^l x_K^l \tag{26}$$

and

$$C^0 = \tfrac{1}{2}\sum_K m_K \lambda x_K^l x_K^l \tag{27}$$

for the cases (IV) and (VI), respectively. The form (26) is well known [16, 23]; the form (27) follows immediately by evaluation of (16) for the transformation (VI).

The local conservation laws following from (5), (17), and (19) are of the form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0, \quad \nabla \equiv (\nabla_1 \ldots \nabla_N). \tag{28}$$

Thus 'local' refers to localization in configuration rather than in physical space. A local law of the form (28) leads to a global conservation law

$$\frac{d}{dt}\int_v \rho\, d^{3N}x = 0 \tag{29}$$

only if

$$\int_v \nabla\cdot\mathbf{j}\, d^{3N}x = \int_{\mathscr{S}} \mathbf{j}\cdot\mathbf{n}\, d\varphi = 0, \tag{30}$$

where v is the entire configuration space and $\mathscr{S}$ the surface bounding it. It should be noted that equation (30) is *not* satisfied in general. It is, however, satisfied for the conservation law (6); indeed, a wave function is only acceptable if this is the case, as otherwise $\psi^*\psi$ could not be interpreted as a probability density.

The existence of local conservation laws which do not necessarily imply a global conservation law is of course well known from electrodynamics, where energy can be 'radiated away' to infinity. However, even for time-independent fields some of the surface integrals may not vanish in many cases. In particular, this may occur for the conservation laws obtained from the invariance of Maxwell's theory under conformal transformations by Bessel-Hagen [16], which are discussed in Section VI, and we shall encounter similar cases here.

The conservation laws following from (12), (17), and (20) are of the form

$$\frac{dB}{dt} = 0, \tag{31}$$

where B, unlike the conserved quantity (29), is a function of the x_K^l and their derivatives. Nevertheless, there exists a close analogy between the conservation laws (29) and (31), which will be exhibited in Section IV.

III. The conservation laws of the Newtonian N-body system

The conservation laws for the Newtonian N-body system described by the action integral (12) follow from the transformation (I)–(VI) and equations (17), (18), (21), (26), and (27) without any difficulty. Written in vector notation, they are

$$\text{(I N)} \quad \frac{dH}{dt} = 0, \quad H \equiv \sum_K \frac{\mathbf{p}_K^2}{2m_K} + V, \quad \mathbf{p}_K \equiv \frac{\partial L}{\partial \mathbf{v}_K} = m_K \mathbf{v}_K,$$

$$\text{(II N)} \quad \frac{d\mathbf{P}}{dt} = 0, \quad \mathbf{P} \equiv \sum_K \mathbf{p}_K,$$

$$\text{(III N)} \quad \frac{d\mathbf{J}}{dt} = 0, \quad \mathbf{J} \equiv \sum_K \mathbf{r}_K \times \mathbf{p}_K,$$

$$\text{(IV N)} \quad \frac{d\mathbf{G}}{dt} = 0, \quad \mathbf{G} \equiv \sum_K m_K \mathbf{r}_K - \mathbf{P}t,$$

$$\text{(V N)} \quad \frac{dD}{dt} = 0, \quad D \equiv 2Ht - \sum_K \mathbf{r}_K \cdot \mathbf{p}_K,$$

$$\text{(VI N)} \quad \frac{dA}{dt} = 0, \quad A \equiv Ht^2 - \sum_K (\mathbf{r}_K \cdot \mathbf{p}_K t - \tfrac{1}{2} m_K \mathbf{r}_K^2).$$

The first four have of course been known since the last century, and were first derived via Noether's theorem by Bessel-Hagen [16]. The last two [valid only for free particles or interactions of the form (4)] are due to Wintner [25][6]) but seem not to have been derived previously via Noether's theorem.

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[6]) Wintner credits them to Jacobi [25], who did consider the transformations (V) and (VI), but did not explicitly write his results in the form of conservation laws.

For the validity of the conservation laws (I N)–(VI N) it is of course irrelevant whether they are written in terms of canonical variables (positions and momenta) as above, or of positions and velocities. But if expressed in terms of canonical variables, H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{G}$, D, and A form a representation of the classical Lie algebra of the Jacobi–Schrödinger group, which is

$$\begin{aligned}
&[J_i, J_j] = \varepsilon_{ijk} J_k, \qquad [G_i, G_j] = 0, \qquad [P_i, P_j] = 0,\\
&[J_i, P_j] = \varepsilon_{ijk} P_k, \qquad [G_i, P_j] = \delta_{ij} \sum_K m_K, \qquad [P_i, H] = 0,\\
&[J_i, G_j] = \varepsilon_{ijk} G_k, \qquad [G_i, H] = P_i, \qquad [J_i, H] = 0,\\
&[D, J_i] = 0, \qquad [D, G_i] = G_i, \qquad [D, P_i] = -P_i,\\
&[A, J_i] = 0, \qquad [A, G_i] = 0, \qquad [A, P_i] = -G_i,\\
&[D, H] = -2H, \qquad [A, H] = D, \qquad [D, A] = 2A;
\end{aligned} \tag{32}$$

this differs from the quantum mechanical Lie algebra [2] only by omission of the factors i and replacement of commutators by Poisson brackets. The first nine brackets are those of the Lie algebra of the Galilei group. The last nine involve the expressions (V N) and (VI N) and are easily verified to hold for the conserved quantities (I N)–(VI N). The structure of the algebra as well as various isomorphisms of the Jacobi–Schrödinger group have been discussed by a number of authors [2, 6, 7].

IV. The conservation laws of the Schrödinger equation

The local conservation laws for the Schrödinger equation follow by applying the transformations (I)–(VI), to the action integral (5) and using equations (17), (18), (20), and (22)–(25). A straightforward calculation yields

$$\text{(I S)} \quad \frac{\partial}{\partial t}\left[\sum_K \frac{\hbar^2}{2m_K} \nabla_K \psi^* \cdot \nabla_K \psi + V\psi^*\psi\right] + \sum_K \nabla_K \cdot \left[-\frac{\hbar^2}{2m_K}\left(\nabla_K \psi^* \frac{\partial \psi}{\partial t} + \nabla_K \psi \frac{\partial \psi^*}{\partial t}\right)\right] = 0,$$

$$\begin{aligned}
\text{(II S)} \quad & \frac{\partial}{\partial t}\left[\frac{\hbar}{2i}\sum_K \left(\psi^* \frac{\partial \psi}{\partial x_K^l} - \psi \frac{\partial \psi^*}{\partial x_K^l}\right)\right]\\
&+ \sum_K \frac{\partial}{\partial x_K^r}\left\{-\left[\frac{\hbar^2}{2m_K} \nabla_K \psi^* \cdot \nabla_K \psi + \frac{\hbar}{2i}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right.\right.\\
&\left.\left. + V\psi^*\psi\right]\delta^{lr} + \frac{\hbar^2}{2m_K}\left(\frac{\partial \psi^*}{\partial x_K^r}\frac{\partial \psi}{\partial x_K^l} + \frac{\partial \psi}{\partial x_K^r}\frac{\partial \psi^*}{\partial x_K^l}\right)\right\} = 0, \quad l = 1, 2, 3,
\end{aligned}$$

$$\begin{aligned}
\text{(III S)} \quad & \frac{\partial}{\partial t}\left\{\frac{\hbar}{2i}\sum_K \left[\psi^*\left(\frac{\partial \psi}{\partial x_K^n} x_K^l - \frac{\partial \psi}{\partial x_K^l} x_K^n\right) - \psi\left(\frac{\partial \psi^*}{\partial x_K^n} x_K^l - \frac{\partial \psi^*}{\partial x_K^l} x_K^n\right)\right]\right.\\
&+ \sum_K \frac{\partial}{\partial x_K^r}\left\{-\left[\frac{\hbar^2}{2m_K} \nabla_K \psi^* \cdot \nabla_K \psi + \frac{\hbar}{2i}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right.\right.\\
&\left.\left. + V\psi^*\psi\right](\delta^{rn} x_K^l - \delta^{rl} x_K^n)\right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\hbar^2}{2m_K}\left[\frac{\partial\psi^*}{\partial x_K^n}\left(\frac{\partial\psi}{\partial x_K^n}x_K^l - \frac{\partial\psi}{\partial x_K^l}x_K^n\right)\right.\\
&\left.\left.+ \frac{\partial\psi}{\partial x_K^r}\left(\frac{\partial\psi^*}{\partial x_K^n}x_K^l - \frac{\partial\psi^*}{\partial x_K^l}x_K^n\right)\right]\right\} = 0, \quad l, n = 1, 2, 3,
\end{aligned}
$$

$$
\begin{aligned}
\text{(IV S)}\quad &\frac{\partial}{\partial t}\sum_K\left[-\frac{\hbar}{2i}\left(\psi^*\frac{\partial\psi}{\partial x_K^l} - \psi\frac{\partial\psi^*}{\partial x_K^l}\right)t + m_K\psi^*\psi\, x_K^l\right]\\
&+ \sum_K\frac{\partial}{\partial x_K^r}\left\{\left[\frac{\hbar^2}{2m_K}\nabla_K\psi^*\cdot\nabla_K\psi + \frac{\hbar}{2i}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right.\right.\\
&\left.+ V\psi^*\psi\right]\delta^{lr}t + \frac{\hbar}{2im_K}\left(\psi^*\frac{\partial\psi}{\partial x_K^r} - \psi\frac{\partial\psi^*}{\partial x_K^r}\right)\sum_L m_L x_L^l\\
&\left.+ \frac{\hbar^2}{2m_K}\left(\frac{\partial\psi^*}{\partial x_K^r}\frac{\partial\psi}{\partial x_K^l} + \frac{\partial\psi}{\partial x_K^r}\frac{\partial\psi^*}{\partial x_K^l}\right)t\right\} = 0, \quad l = 1, 2, 3,
\end{aligned}
$$

$$
\begin{aligned}
\text{(V S)}\quad &\frac{\partial}{\partial t}\left\{\sum_K\left[-\frac{\hbar}{2i}(\psi^*\nabla_K\psi - \psi\nabla_K\psi^*)\cdot\mathbf{r}_K + \frac{\hbar^2}{m_K}\nabla_K\psi^*\cdot\nabla_K\psi t\right.\right.\\
&\left.+ 2V\psi^*\psi t\right\} + \sum_K\nabla_K\cdot\left\{\frac{\hbar}{2i}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\mathbf{r}_K\right.\\
&+ \frac{\hbar^2}{m_K}\left(\nabla_K\psi^*\frac{\partial\psi}{\partial t} + \nabla_K\psi\frac{\partial\psi^*}{\partial t}\right)t + V\psi^*\psi\mathbf{r}_K\\
&+ \frac{\hbar^2}{2m_K}[(\nabla_K\psi^*\cdot\nabla_K\psi)\mathbf{r}_K - \nabla_K\psi^*(\mathbf{r}_K\cdot\nabla_K\psi)\\
&\left.\left.- \nabla_K\psi(\mathbf{r}_K\cdot\nabla_K\psi^*) - \tfrac{3}{2}(\psi\nabla_K\psi^* + \psi^*\nabla_K\psi)]\right\} = 0,
\end{aligned}
$$

$$
\begin{aligned}
\text{(VI S)}\quad &\frac{\partial}{\partial t}\left\{\sum_K\left[-\frac{\hbar}{2i}(\psi^*\nabla_K\psi - \psi\nabla_K\psi^*)\cdot\mathbf{r}_K t\right.\right.\\
&\left.\left.+ \psi^*\psi\frac{m_K}{2}\mathbf{r}_K^2 + \frac{\hbar^2}{2m_K}\nabla_K\psi^*\cdot\nabla_K\psi t^2\right] + V\psi^*\psi t^2\right\}\\
&+ \sum_K\nabla_K\cdot\left\{\frac{\hbar}{2i}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\mathbf{r}_K t\right.\\
&+ \frac{\hbar}{4im_K}(\psi^*\nabla_K\psi - \psi\nabla_K\psi^*)\sum_L m_L\mathbf{r}_L^2 + V\psi^*\psi\mathbf{r}_K t\\
&- \frac{\hbar^2}{2m_K}\left[\nabla_K\psi^*(\mathbf{r}_K\cdot\nabla_K\psi)t + \nabla_K\psi(\mathbf{r}_K\cdot\nabla_K\psi^*)t\right.\\
&+ \left(\nabla_K\psi^*\frac{\partial\psi}{\partial t} + \nabla_K\psi\frac{\partial\psi^*}{\partial t}\right)t^2 - \nabla_K\psi^*\cdot\nabla_K\psi\mathbf{r}_K t\\
&\left.\left.+ \tfrac{3}{2}(\psi\nabla_K\psi^* + \psi^*\nabla_K\psi)t\right]\right\} = 0.
\end{aligned}
$$

From these local laws of the form (28) we can obtain global conservation laws of the form (29), provided that the surface integrals (30) vanish. This clearly imposes conditions on the wave functions that are much stronger than those following from

the standard requirement that equation (6) should lead to conservation of probability. Assuming that these conditions are satisfied, transforming the various volume integrals by integrations by part and expressing the spatial derivatives of ψ and ψ^* in terms of the momentum operators defined in (2), we obtain

$$\text{(I S}')\quad \frac{d}{dt}\int_{\mathscr{V}} \psi^* H\psi \, d^{3N}x = 0,$$

$$\text{(II S}')\quad \frac{d}{dt}\int_{\mathscr{V}} \psi^* \mathbf{P}\psi \, d^{3N}x = 0,$$

$$\text{(III S}')\quad \frac{d}{dt}\int_{\mathscr{V}} \psi^* \mathbf{J}\psi \, d^{3N}x = 0,$$

$$\text{(IV S}')\quad \frac{d}{dt}\int_{\mathscr{V}} \psi^* \mathbf{G}\psi \, d^{3N}x = 0,$$

$$\text{(V S}')\quad \frac{d}{dt}\int_{\mathscr{V}} \psi^* D\psi \, d^{3N}x = 0,$$

$$\text{(VI S}')\quad \frac{d}{dt}\int_{\mathscr{V}} \psi^* A\psi \, d^{3N}x = 0,$$

where the various operators in the integrands are given in terms of the individual position and momentum operators as in equations (I N)–(VI N), except that in the expressions for $\mathbf{J}$, D, and A we have to apply the usual rule of symmetrization in the ordering of the factors of any product of $\mathbf{r}_K$ and $\mathbf{p}_K$ [26].

V. The balance equations of the Hamilton–Jacobi equation

To obtain local conservation laws for the Hamilton–Jacobi equation, one can not make use of Noether's theorem, since no variational principle of the form (13) is available. Instead, we can exploit the similarity of equations (2) and (10). They differ only in the last term of (10), which enters the equation just like a potential. Thus it might appear at first sight that one could write down conservation laws of the form (I S)–(VI S), but with $V\psi$ replaced by

$$\left(V + \frac{\hbar^2}{2}\frac{1}{R}\sum_K \frac{1}{m_K}\nabla_K^2 R\right)\psi_c. \tag{33}$$

[Of course, if non-Cartesian coordinates are used, $\sum_K \nabla_K^2$ must be understood to be the operator

$$g^{-1/2}\frac{\partial}{\partial q^i}\left(g^{1/2}\, g^{ik}\frac{\partial}{\partial q^k}\right), \quad i, k = 1\cdots 3N \tag{34}$$

(where g^{ik} is the contravariant metric tensor of configuration space, with determinant g^{-1}), both in (33) and in the Schrödinger equation, with corresponding changes in its action integral (2) and in the Hamilton–Jacobi equation (3).] However, the validity of several of these laws depends on the fact that V is a function only of the r_{KL} rather than of the individual coordinates and that it is independent of the time. No such statements can be made about $R^{-1}\sum_K m_K^{-1}\nabla_K^2 R$ in general. The form of this term

depends on the particular form chosen for the complete integral of the Hamilton–Jacobi equation. Even in the simplest case of free particles and Cartesian coordinates it does not necessarily vanish, although solutions exist for which it does.

If we substitute the expression (33) for $V\psi$ in the conservation laws (I S)–(VI S) (with ψ replaced by ψ_c everywhere) and evaluate their left hand sides, the terms in which $R^{-1}\sum_K m_K^{-1}\nabla_K^2 R$ is *not* differentiated will provide the additional contributions needed to take into account the difference between equations (2) and (10). However, there will be further terms obtained by differentiation of $R^{-1}\sum_K m_K^{-1}\nabla_K^2 R$ which, unlike the analogous terms following from differentiation of V, do not necessarily vanish, and thus must be introduced as source terms on the right hand side. We then obtain

$$\text{(I H-J)}\quad \frac{\partial}{\partial t}\left[\sum_K \frac{\hbar^2}{2m_K}\nabla_K\psi_c^*\cdot\nabla_K\psi_c + \left(V + \frac{\hbar^2}{2}\frac{1}{R}\sum_K\frac{1}{m_K}\nabla_K^2 R\right)\psi_c^*\psi_c\right] + \sum_K \nabla_K\cdot\left[-\frac{\hbar^2}{2m_K}\left(\nabla_K\psi_c^*\frac{\partial\psi_c}{\partial t} + \nabla_K\psi_c\frac{\partial\psi_c^*}{\partial t}\right)\right] = \psi_c^*\psi_c\frac{\hbar^2}{2}\sum_K\frac{1}{m_K}\frac{\partial}{\partial t}\frac{\nabla_K^2 R}{R},$$

$$\text{(II H-J)}\quad \frac{\partial}{\partial t}\left[\frac{\hbar^2}{2i}\sum_K\left(\psi_c^*\frac{\partial\psi_c}{\partial x_K^l} - \psi_c\frac{\partial\psi_c^*}{\partial x_K^l}\right)\right] + \sum_K\frac{\partial}{\partial x_K^r}\left\{-\left[\frac{\hbar^2}{2m_K}\nabla_K\psi_c^*\cdot\nabla_K\psi_c + \frac{\hbar}{2i}\left(\psi_c^*\frac{\partial\psi_c}{\partial t} - \psi\frac{\partial\psi_c^*}{\partial t}\right) + \left(V + \frac{\hbar^2}{2m_K}\frac{1}{R}\nabla_K^2 R\right)\psi_c^*\psi_c\right]\delta^{lr} + \frac{\hbar^2}{2m_K}\left(\frac{\partial\psi_c^*}{\partial x_K^r}\frac{\partial\psi_c}{\partial x_K^l} + \frac{\partial\psi_c}{\partial x_K^r}\frac{\partial\psi_c^*}{\partial x_K^l}\right)\right\} = \psi_c^*\psi_c\frac{\hbar^2}{2}\sum_K\frac{1}{m_K}\frac{\partial}{\partial x_K^l}\frac{\nabla_K^2 R}{R},\quad l = 1, 2, 3,$$

$$\text{(III H-J)}\quad \frac{\partial}{\partial t}\left\{\frac{\hbar}{2i}\sum_K\left[\psi_c^*\left(\frac{\partial\psi_c}{\partial x_K^n}x_K^l - \frac{\partial\psi_c}{\partial x_K^l}x_K^n\right) - \psi_c\left(\frac{\partial\psi_c^*}{\partial x_K^n}x_K^l - \frac{\partial\psi_c}{\partial x_K^l}x_K^n\right)\right]\right\} + \sum_K\frac{\partial}{\partial x_K^r}\left\{-\left[\frac{\hbar^2}{2m_K}\nabla_K\psi_c^*\cdot\nabla_K\psi_c + \frac{\hbar}{2i}\left(\psi_c^*\frac{\partial\psi_c}{\partial t} - \psi_c\frac{\partial\psi_c^*}{\partial t}\right) + \left(V + \frac{\hbar^2}{2m_K}\frac{1}{R}\nabla_K^2 R\right)\psi_c^*\psi_c\right](\delta^{rn}x_K^l - \delta^{rl}x_K^n) - \frac{\hbar^2}{2m_K}\left[\frac{\partial\psi_c^*}{\partial x_K^r}\left(\frac{\partial\psi_c}{\partial x_K^n}x_K^l - \frac{\partial\psi_c}{\partial x_K^l}x_K^n\right) + \frac{\partial\psi_c}{\partial x_K^r}\left(\frac{\partial\psi_c^*}{\partial x_K^n}x_K^l - \frac{\partial\psi_c^*}{\partial x_K^l}x_K^n\right)\right] = \psi_c^*\psi_c\frac{\hbar^2}{2}\sum_K\frac{1}{m_K}\left(x_K^l\frac{\partial}{\partial x_K^n} - x_K^n\frac{\partial}{\partial x_K^l}\right)\frac{\nabla_K^2 R}{R},\quad l, n = 1, 2, 3,$$

$$\text{(IV H-J)}\quad \frac{\partial}{\partial t}\sum_K\left[-\frac{\hbar}{2i}\left(\psi_c^*\frac{\partial\psi_c}{\partial x_K^l} - \psi_c\frac{\partial\psi_c^*}{\partial x_K^l}\right)t + m_K\psi_c^*\psi_c x_K^l\right]$$

$$
\begin{aligned}
&+ \sum_K \frac{\partial}{\partial x_K^r} \left\{ \left[\frac{\hbar^2}{2m_K} \nabla_K \psi_c^* \cdot \nabla_K \psi_c + \frac{\hbar}{2i} \left(\psi_c^* \frac{\partial \psi_c}{\partial t} - \psi_c \frac{\partial \psi_c}{\partial t} \right) \right.\right. \\
&+ \left. \left(V + \frac{\hbar^2}{2m_K} \frac{1}{R} \nabla_K^2 R \right) \psi_c^* \psi_c \right] \delta^{lr} t \\
&+ \frac{\hbar}{2im_K} \left(\psi_c^* \frac{\partial \psi_c}{\partial x_K^r} - \psi_c \frac{\partial \psi_c^*}{\partial x_K^r} \right) \sum_L m_L x_L^l \\
&+ \left. \frac{\hbar^2}{2m_K} \left(\frac{\partial \psi_c^*}{\partial x_K^r} \frac{\partial \psi_c}{\partial x_K^l} + \frac{\partial \psi}{\partial x_K^r} \frac{\partial \psi_c^*}{\partial x_K^l} \right) t \right\} \\
&= \psi_c^* \psi_c \frac{\hbar^2}{2} t \sum_K \frac{1}{m_K} \frac{\partial}{\partial x_K^l} \frac{\nabla_K^2 R}{R}, \quad l = 1, 2, 3,
\end{aligned}
$$

$$
\begin{aligned}
\text{(V H-J)} \quad & \frac{\partial}{\partial t} \left\{ \sum_K \left[-\frac{\hbar}{2i} (\psi_c^* \nabla_K \psi_c - \psi_c \nabla_K \psi_c^*) \cdot \mathbf{r}_K + \frac{\hbar^2}{m_K} \nabla_K \psi_c^* \cdot \nabla_K \psi_c t \right] \right. \\
&+ \left. 2 \left(V + \frac{\hbar^2}{2} \frac{1}{R} \sum_K \frac{1}{m_K} \nabla_K^2 R \right) \psi_c^* \psi_c t \right\} \\
&+ \sum_K \nabla_K \cdot \left\{ \frac{\hbar}{2i} \left(\psi_c^* \frac{\partial \psi_c}{\partial t} - \psi_c \frac{\partial \psi_c^*}{\partial t} \right) \mathbf{r}_K \right. \\
&+ \frac{\hbar^2}{m_K} \left(\nabla_K \psi_c^* \frac{\partial \psi_c}{\partial t} + \nabla_K \psi_c \frac{\partial \psi_c^*}{\partial t} \right) t + \left(V + \frac{\hbar^2}{2m_K} \frac{1}{R} \nabla_K^2 R \right) \psi_c^* \psi_c \mathbf{r}_K \\
&+ \frac{\hbar^2}{2m_K} \left[(\nabla_K \psi_c^* \cdot \nabla_K \psi_c) \mathbf{r}_K - \nabla_K \psi_c^* (\mathbf{r}_K \cdot \nabla_K \psi_c) \right. \\
&\left. \left. - \nabla_K \psi_c (\mathbf{r}_K \cdot \nabla_K \psi_c^*) - \tfrac{3}{2} (\psi_c \nabla_K \psi_c^* + \psi_c^* \nabla_K \psi_c) \right] \right\} \\
&= \psi_c^* \psi_c \hbar^2 \sum_K \left\{ \frac{1}{m_K} \left[\nabla_K \cdot \left(\tfrac{1}{2} \frac{\mathbf{r}_K}{R} \nabla_K^2 R \right) + t \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
\text{(VI H-J)} \quad & \frac{\partial}{\partial t} \left\{ \sum_K \left[-\frac{\hbar}{2i} (\psi_c^* \nabla_K \psi_c - \psi_c \nabla_K \psi_c^*) \cdot \mathbf{r}_K t + \psi_c^* \psi_c \frac{m_K}{2} \mathbf{r}_K \right. \right. \\
&\left. \left. + \frac{\hbar^2}{2m_K} \nabla_K \psi_c^* \cdot \nabla_K \psi_c t^2 \right] + \left(V + \frac{\hbar^2}{2} \frac{1}{R} \sum_K \frac{1}{m_K} \nabla_K^2 R \right) \psi_c^* \psi_c t^2 \right\} \\
&+ \sum_K \nabla_K \cdot \left\{ \frac{\hbar}{2i} \left(\psi_c^* \frac{\partial \psi_c}{\partial t} - \psi_c \frac{\partial \psi_c^*}{\partial t} \right) \mathbf{r}_K t \right. \\
&\left. + \frac{\hbar}{4im_K} (\psi_c^* \nabla_K \psi_c - \psi_c \nabla_K \psi_c^*) \sum_L m_L \mathbf{r}_L^2 \right] + \left(V + \frac{\hbar^2}{2m_K} \frac{1}{R} \nabla_K^2 R \right) \\
&\times \psi_c^* \psi_c \mathbf{r}_K - \frac{\hbar^2}{2m_K} \left[\nabla_K \psi_c^* (\mathbf{r}_K \cdot \nabla_K \psi_c) t + \nabla_K \psi_c (\mathbf{r}_K \cdot \nabla_K \psi_c^*) \right. \\
&+ \left(\nabla_K \psi_c^* \frac{\partial \psi_c}{\partial t} + \nabla_K \psi_c \frac{\partial \psi_c^*}{\partial t} \right) t^2 - \nabla_K \psi_c^* \cdot \nabla_K \psi_c \mathbf{r}_K \\
&\left. \left. + \tfrac{3}{2} (\psi_c \nabla_K \psi_c^* + \psi_c^* \nabla_K \psi_c) t \right] \right\}
\end{aligned}
$$

$$= \psi_c^* \psi_c \frac{\hbar^2}{2} \sum_K \left\{ \frac{1}{m_K} \left[t \nabla_K \cdot \left(\frac{\mathbf{r}_K}{R} \nabla_K^2 R \right) + t^2 \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \right] \right\},$$

where ψ_c and R are defined through equation (11) with (8).

Thus we obtain true local conservation laws only if

$$\sum_K \frac{1}{m_K} \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} = 0, \tag{35.1}$$

$$\sum_K \frac{1}{m_K} \frac{\partial}{\partial x_K^l} \frac{\nabla_K^2 R}{R} = 0, \quad l = 1, 2, 3, \tag{35.2}$$

$$\sum_K \frac{1}{m_K} \left(x_K^l \frac{\partial}{\partial x_K^n} - x_K^n \frac{\partial}{\partial x_K^l} \right) \frac{\nabla_K^2 R}{R} = 0, \quad l, n = 1, 2, 3, \tag{35.3}$$

$$\sum_K \frac{1}{m_K} \frac{\partial}{\partial x_K^l} \frac{\nabla_K^2 R}{R} = 0, \quad l = 1, 2, 3, \tag{35.4}$$

$$\sum_K \left\{ \frac{1}{m_K} \left[\nabla_K \cdot \left(\tfrac{1}{2} \frac{\mathbf{r}_K}{R} \nabla_K^2 R \right) + t \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \right] \right\} = 0, \tag{35.5}$$

$$\sum_K \left\{ \frac{1}{m_K} \left[t \nabla_K \cdot \left(\frac{\mathbf{r}_K}{R} \nabla_K^2 R \right) + t^2 \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \right] \right\} = 0, \tag{35.6}$$

for equations (I H-J)–(VI H-J), respectively. These equations are of course satisfied if $\nabla_K^2 R$ vanishes, but are less restrictive than this condition.

Provided that the various surface integrals vanish, equations (I H-J)–(VI H-J) imply global balance equations [which reduce to global conservation laws if the expressions (35) – or at least their volume integrals – vanish]. They are

$$\text{(I H-J}') \quad \frac{d}{dt} \int_{\mathscr{V}} \psi_c^* H \psi_c \, d^{3N}x = \int_{\mathscr{V}} \psi_c^* \frac{\hbar^2}{2} \sum_K \frac{1}{m_K} \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \psi_c \, d^{3N}x,$$

$$\text{(II H-J}') \quad \frac{d}{dt} \int_{\mathscr{V}} \psi_c^* \mathbf{P} \psi_c \, d^{3N}x = \int_{\mathscr{V}} \psi_c^* \frac{\hbar^2}{2} \sum_K \frac{1}{m_K} \nabla_K \frac{\nabla_K^2 R}{R} \psi_c \, d^{3N}x,$$

$$\text{(III H-J}') \quad \frac{d}{dt} \int_{\mathscr{V}} \psi_c^* \mathbf{J} \psi_c \, d^{3N}x = \int_{\mathscr{V}} \psi_c^* \frac{\hbar^2}{2} \sum_K \frac{\mathbf{r}_K}{m_K} \times \nabla_K \left(\frac{\nabla_K^2 R}{R} \right) \psi_c \, d^{3N}x,$$

$$\text{(IV H-J}') \quad \frac{d}{dt} \int_{\mathscr{V}} \psi_c^* \mathbf{G} \psi_c \, d^{3N}x = \int_{\mathscr{V}} \psi_c^* \frac{\hbar^2}{2} t \sum_K \frac{1}{m_K} \nabla_K \left(\frac{\nabla_K^2 R}{R} \right) \psi_c \, d^{3N}x,$$

$$\text{(V H-J}') \quad \frac{d}{dt} \int_{\mathscr{V}} \psi_c^* \left(D + \hbar^2 t \frac{1}{R} \sum_K \nabla_K^2 R \right) \psi_c \, d^{3N}x$$

$$= \int_{\mathscr{V}} \psi_c^* \hbar^2 \sum_K \left\{ \frac{1}{m_K} \left[\nabla_K \cdot \left(\tfrac{1}{2} \frac{\mathbf{r}_K}{R} \nabla_K^2 R \right) + t \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \right] \right\} \psi_c \, d^{3N}x,$$

$$\text{(VI H-J}') \quad \frac{d}{dt} \int_{\mathscr{V}} \psi_c^* \left(A + \frac{\hbar^2}{2} t^2 \frac{1}{R} \sum_K \nabla_K^2 R \right) \psi_c \, d^{3N}x$$

$$= \int_{\mathscr{V}} \psi_c^* \frac{\hbar^2}{2} \sum_K \left\{ \frac{1}{m_K} \left[t \nabla_K \cdot \left(\frac{\mathbf{r}_K}{R} \nabla_K^2 R \right) + t^2 \frac{\partial}{\partial t} \frac{\nabla_K^2 R}{R} \right] \right\} \psi_c \, d^{3N}x,$$

where the various operators in the integrands of the expressions on the left hand sides are given as for the corresponding expressions in Section IV, including the definitions of the individual momentum operators $\mathbf{p}_K = -i\hbar\nabla_K$. On the right hand side, of course, the factors $\psi_c^*\psi_c$ in all the integrals reduce to R^2 from the definitions (11).

It should also be noted that solutions to the classical Schrödinger equation have a significance in classical mechanics only if S and R are real. For bound states this is true only for limited ranges of the separations r_{KL}. Therefore once the center-of-mass motion is separated out, the remaining part of the density $\bar{D}$ and thus of R vanishes outside a limited region of configuration space, and one can always choose the surface $\mathscr{S}$ such that all surface integrals vanish apart from the contribution of the center-of-mass motion. The latter depends on the particular form of the complete integral of the Hamilton–Jacobi equation.

Illustrations of the balance equations (I H-J)–(VI H-J) for a system of free particles are given in the Appendix.

VI. Discussion

The various equations describing a system of N particles with central two-body interactions either classically or quantum mechanically are invariant under the ten transformations of the Galilei group, and for interactions of the form (4) also under two further transformations; the infinitesimal form of the transformations of this 12-parameter Jacobi–Schrödinger group is given by equations (I)–(VI) of Section II. The corresponding 12 well-known conservation laws of Newtonian dynamics (I N)–(VI N) are rederived in Section III via Noether's theorem. In Section IV the corresponding local conservation laws of the Schrödinger equation (I S)–(VI S) are derived by the same method. Under appropriate conditions on the wave functions these yield the 12 global conservation laws (I S′)–(VI S′); the conserved quantities are the quantum mechanical averages of the operators corresponding to the Newtonian conserved quantities.[7])

The Hamilton–Jacobi equation (3) implies a classical Schrödinger-type equation (10) that differs from the Schrödinger equation (2) only by a potential-like term involving the Van Vleck determinant (8), but due to the presence of this term, equation (10), in contrast of equation (2), is not derivable from a variational principle of the form (13) and thus no local conservation laws are implied by its invariance properties. However, exploiting the similarity of equations (2) and (10), it is possible to obtain 12 local balance equations (I H-J)–(VI H-J) for the ensembles described by the Hamilton–Jacobi equation analogous to the local conservation laws of the Schrödinger equation, and 12 corresponding global balance equations (I H-J′)–(VI H-J′). These contain source terms which vanish only under special conditions. For the special case of gravitational interactions, the first four of these balance equations may be of use in the study of problems of celestial mechanics.

It is instructive to compare the global conservation laws obtained in Sections III–V (which are implied by invariance under the Jacobi–Schrödinger group that is

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[7]) Although the local conservation laws (I S)–(IV S) following from Galilei invariance most likely have been obtained before, I have not been able to find them in the literature; the global ones are well known consequences of the general principles of quantum mechanics. The laws (V S) and (VI S) are new.

one of the conformal extensions of the Galilei group [3]) with the global conservation laws of electrodynamics (which are implied by invariance under the conformal extension of the Poincaré group [16]).[8]) The ten Maxwellian conservation laws implied by the Poincaré invariance are {compare equations (27a, b) and (28) of [16]}

$$\text{(I M)} \quad \frac{d}{dt}\int_{\mathscr{V}} h\, d^3x = 0,$$

$$\text{(II M)} \quad \frac{d}{dt}\int_{\mathscr{V}} \mathbf{p}\, d^3x = 0,$$

$$\text{(III M)} \quad \frac{d}{dt}\int_{\mathscr{V}} \mathbf{r} \times \mathbf{p}\, d^3x = 0,$$

$$\text{(IV M)} \quad \frac{d}{dt}\int_{\mathscr{V}} \left(\frac{h}{c^2}\mathbf{r} - \mathbf{p}t\right) d^3x = 0,$$

where h and $\mathbf{p}$ are the electromagnetic energy and momentum densities; furthermore, h/c^2 is the mass density μ of the electromagnetic field. Therefore these conservation laws are completely analogous to the Newtonian conservation laws (I N)–(IV N) (as already noted in [16]) as well as to the quantum mechanical conservation laws (I S′)–(IV S′) implied by Galilei invariance.

The additional Maxwellian conservation laws implied by conformal invariance in the absence of ponderable matter are {compare equations (27c, d) of [16]}

$$\text{(V M)} \quad \frac{d}{dt}\int_{\mathscr{V}} (2ht - 2\mathbf{r}\cdot\mathbf{p})\, d^3x = 0,$$

$$\text{(VI M)} \quad \frac{d}{dt}\int_{\mathscr{V}} \left(ht^2 - 2\mathbf{r}\cdot\mathbf{p}t + \frac{h}{c^2}\mathbf{r}^2\right) d^3x = 0,$$

$$\text{(VII M)} \quad \frac{d}{dt}\int_{\mathscr{V}} [2h\mathbf{r}t - c^2t^2\mathbf{p} - \mathbf{r}(\mathbf{r}\cdot\mathbf{p}) - \mathbf{r} \times (\mathbf{r} \times \mathbf{p})]\, d^3x = 0.$$

The last of these conservation laws has no Newtonian analogue. The laws (V M) and (VI M), on the other hand, are analogous to the Newtonian conservation laws (V N) and (VI N), except that in each of these laws the terms involving $\mathbf{r}\cdot\mathbf{p}$ appear with a factor two missing from the Newtonian laws. This extra contribution arises from the specifically relativistic effect of the equality of the momentum density and the energy flux divided by c^2 (which is responsible for the symmetry of the energy-momentum tensor $T^{\mu\nu}$, compare, e.g. [27]).

Maxwell's equations correspond to a classical rest mass zero, spin one field. All fields of rest mass zero (i.e. implying d'Alembert's equation) which satisfy Poincaré-invariant field equations corresponding to integral or half-integral spin [28–31] are also invariant under the conformal extension of the Poincaré group [32, 33]. For all spins s other than zero they satisfy conservation laws that are precisely of the form (I M)–(VII M) [32], where now h and $\mathbf{p}$ are the energy and momentum densities of the field of spin s. For spin zero (scalar field U), the conservation

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[8]) These laws are of a very simple form if written in four-dimensional notation, but will be given in three-dimensional notation to emphasize the analogy to the Newtonian conservation laws.

laws implied by Poincaré invariance are still of the form (I M)–(IV M), but the additional conservation laws implied by conformal invariance {equations (10) and (11) of [32]} take the form

$$\text{(V Z)} \quad \frac{d}{dt}\int_{\mathscr{V}} 2\left(ht - \mathbf{r}\cdot\mathbf{p} + U\frac{\partial U}{dt}\right) d^3x = 0,$$

$$\text{(VI Z)} \quad \frac{d}{dt}\int_{\mathscr{V}} \left(ht^2 - 2\mathbf{r}\cdot\mathbf{p}t + \frac{h}{c^2}\mathbf{r}^2 - U^2 + \frac{\partial U^2}{\partial t}t\right) d^3x = 0,$$

$$\text{(VII Z)} \quad \frac{d}{dt}\int_{\mathscr{V}} \left[2h\mathbf{r}t - c^2t^2\mathbf{p} - \mathbf{r}(\mathbf{r}\cdot\mathbf{p}) - \mathbf{r}\times(\mathbf{r}\times\mathbf{p}) + \frac{\partial U^2}{\partial t}\mathbf{r}\right] d^3x = 0,$$

which, because of the appearance of quadratic combinations of U and its derivatives other than h and $\mathbf{p}$, are not analogous to any Newtonian laws.

This paper was concerned with a study of the conservation laws for the case of particle interactions with potentials of the form (1). However, the methods used here are equally applicable for the case of external potentials if they admit appropriate symmetries.

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Appendix

As noted in Section V, in general the balance equations (I H-J)–(VI H-J) of the Hamilton–Jacobi equation (3) do not reduce to the local conservation laws; whether they do depends on the particular form of the complete integral of (3), i.e. on the particular ensemble described by this integral (for a discussion of such classical ensembles see [17–20, 34, 35]). Here this will be illustrated for the case of a system of free particles.

Then one possible complete integral of (3) is

$$S = \sum_K \mathbf{r}_K\cdot\mathbf{p}_K - Et, \quad E = \sum_K \frac{\mathbf{p}_K^2}{2m_K}, \tag{A.1}$$

where the $\mathbf{p}_K$'s are arbitrary constant vectors. Thus from equations (8) and (11) we have

$$\bar{D} = 1, \qquad R = \pm 1, \tag{A.2}$$

and therefore

$$\nabla_K^2 R = 0 \tag{A.3}$$

and equations (35) are trivially satisfied. On using the definition (11) of $\psi_c^{\cdot}$, equations (I H-J)–(VI H-J) can then be written

$$\text{(I H-J}'') \quad \frac{\partial}{\partial t}\sum_K \frac{\mathbf{p}^2{}_K}{2m_K} + \sum_K \nabla_K\cdot\left(\frac{\mathbf{p}_K}{m_K}E\right) = 0,$$

$$\text{(II H-J}'') \quad \frac{\partial}{\partial t}\sum_K \mathbf{p}_K^l + \sum_K \frac{\partial}{\partial x_K^r}\left(\frac{\mathbf{p}_K^l\mathbf{p}_K^r}{m_K}\right) = 0, \quad l = 1, 2, 3,$$

$$\text{(III H-J}'') \quad \frac{\partial}{\partial t}\sum_K (x_K^n\mathbf{p}_K^l - x_K^l\mathbf{p}_K^n) + \sum_K \frac{\partial}{\partial x_K^r}\left[\frac{\mathbf{p}_K^r}{m_K}(x_K^n\mathbf{p}_K^l - x_K^l\mathbf{p}_K^n)\right] = 0, \quad l, n = 1, 2, 3,$$

$$\text{(IV H-J}'') \quad \frac{\partial}{\partial t}\sum_K (m_K x_K^l - \mathbf{p}_K^l t) + \sum_K \frac{\partial}{\partial x_K^r}\left(\mathbf{p}_K^r x_K^l + \frac{\mathbf{p}_K^l\mathbf{p}_K^r}{m_K}t\right) = 0, \quad l = 1, 2, 3,$$

$$\text{(V H-J}'') \quad \frac{\partial}{\partial t}\sum_K \left(\frac{\mathbf{p}_K^2}{m_K}t - \mathbf{r}_K\cdot\mathbf{p}_K\right) + \sum_K \nabla_K\cdot\left\{-E\mathbf{r}_K - 4\frac{\mathbf{p}_K}{m_K}Et + \frac{1}{2m_K}[\mathbf{p}_K^2\mathbf{r}_K - 2(\mathbf{r}_K\cdot\mathbf{p}_K)\mathbf{p}_K]\right\} = 0,$$

$$\text{(VI H-J}'') \quad \frac{\partial}{\partial t}\sum_K \left(-\mathbf{r}_K\cdot\mathbf{p}_K t + \tfrac{1}{2}m_K\mathbf{r}_K^2 + \frac{\mathbf{p}_K^2}{2m_K}t^2\right) + \sum_K \nabla_K\cdot\left\{-E\mathbf{r}_K t + \tfrac{1}{2}\mathbf{r}_K^2\mathbf{p}_K - \frac{1}{m_K}[(\mathbf{r}_K\cdot\mathbf{p}_K)\mathbf{p}_K t - E\mathbf{p}_K t^2 - \tfrac{1}{2}\mathbf{p}_K^2\mathbf{r}_K t]\right\} = 0.$$

Precisely the same expressions follow from the quantum mechanical conservation laws (I S)–(VI S) on substitution of a wave function for free particles of definite momenta

$$\psi = \exp\left(\sum_K \mathbf{r}_K\cdot\mathbf{p}_K - Et\right)/\hbar, \qquad E = \sum_K \frac{\mathbf{p}_K^2}{2m_K}, \tag{A.4}$$

as is obvious from equation (11) with (A.1) and (A.2). Although these expressions are all of the form (28), the specific form of the densities has little physical content, since many of the time derivatives and divergences vanish separately because various terms in the densities do not depend on the time and the coordinates, respectively. Furthermore, they do not lead to any global conservation laws, since the integrals of all the densities diverge; but just as it is possible in quantum mechanics to use free-particle wave packets instead of (A.4) which yield finite integrals, one can use ensembles other than the simple one given by Eqs. (11), (A.1), and (A.2) to describe a free-

particle system which yield finite integrals in the case (I H-J′)–(VI H-J′), a point which will not be pursued here.

Instead, we shall consider a complete integral

$$S = \sum_{K=1}^{N} \left[-\frac{Ct}{2m_K} \left(\frac{C^2t^2}{2m_K^2} + \sum_{n=1}^{3} C|x_K^n - \xi_K^n| \right) + \frac{2}{3} \sum_{n=1}^{3} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{3/2} \right] C^{-1} \tag{A.5}$$

(where the ξ_K^n are arbitrary constants and C is a constant needed for dimensional reasons), for which equation (A.3) does not hold. It can easily be verified that (A.5) satisfies equation (3) in the absence of interactions, and that

$$\begin{aligned} \bar{D} &= -\left(\frac{C}{2}\right)^{3N} \prod_{K,n} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{-1/2}, \\ R &= \left(\frac{C}{2}\right)^{3N/2} \prod_{K,n} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{-1/4}, \end{aligned} \tag{A.6}$$

and therefore

$$\sum_K \frac{1}{m_K} \frac{\nabla_K^2 R}{R} = \frac{5}{16} \sum_{K,n} \frac{1}{m_K} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{-2} C^2. \tag{A.7}$$

Substitution of these expressions together with the definition (11) into equations (I H-J)–(VI H-J) yields very lengthy expressions; therefore, only the first one of these will be given explicitly. We obtain

$$\begin{aligned} \text{(I H-J}''') \quad & \frac{\partial}{\partial t} \left\{ \left[\sum_K \frac{3C^2t^2}{4m_K^3} + \sum_{K,n} \frac{C}{2m_K} |x_K^n - \xi_K^n| \right.\right. \\ & \left.\left. + \frac{\hbar^2}{2} \frac{3}{8} \sum_{K,n} \frac{C}{m_K} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{-2} \right] R^2 \right\} \\ & + \sum_{K,n} \frac{\partial}{\partial x_K^n} \left\{ -\frac{\hbar^2}{m_K} \frac{1}{16} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{-1} \right. \\ & \times \operatorname{sign}(x_K^n - \xi_K^n) R^2 \sum_{L,r} \left(\frac{C^2t^2}{4m_L^2} + C|x_L^r - \xi_L^r| \right)^{-1} \frac{tC^3}{2m_L^2} \\ & + \frac{1}{m_K} \left[-\frac{Ct}{2m_K} + \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{1/2} \right] \operatorname{sign}(x_K^n - \xi_K^n) \\ & \times R^2 \sum_L \left[\frac{3C^2t^2}{4m_L^3} + \frac{C}{2m_L} \sum_{r=1}^{3} |x_L^r - \xi_L^r| \right. \\ & \left.\left. - \sum_{r=1}^{3} \left(\frac{3C^2t^2}{4m_L^2} + C|x_L^r - \xi_L^r| \right)^{1/2} \frac{Ct}{2m_L^2} \right] \right\} \\ & = \frac{\hbar^2}{2} R^2 \frac{\partial}{\partial t} \left[\frac{5}{16} \sum_{K,n} \frac{1}{m_K} \left(\frac{C^2t^2}{4m_K^2} + C|x_K^n - \xi_K^n| \right)^{-2} \right]. \end{aligned}$$

Clearly, because of equation (A.7), the classical ψ_c determined by equations (A.5) and (A.6) only satisfies the modified 'Schrödinger equation' (10), but not the quantum mechanical equation (2).

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