Zeitschrift: Helvetica Physica Acta
Band: 51 (1978)
Heft: 2

Artikel: The number of states bound by non-central potentials
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DOI: https://doi.org/10.5169/seals-114950

Helvetica Physica Acta, Vol. 51 (1978), Birkhäuser Verlag, Basel

The number of states bound by non-central potentials

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(23.I.1978)

Abstract. Levinson's theorem on the relation between the number of bound states and the scattering phase shift in any given angular momentum sector can be generalized to abstract scattering systems [5]. In the present paper this abstract result is applied to a maximally large class of potential scattering systems in three dimensions. In the special case of a spherically symmetric potential Levinson's theorem is recovered.

1. Introduction

It is the aim of the present paper to give a mathematically rigorous generalization of the result commonly known as 'Levinson's theorem' to a maximally large class of non-relativistic quantum mechanical scattering systems in three dimensions. In particular, no symmetry requirements will be imposed on the potential.

While working on the inverse scattering problem, N. Levinson proved that for the differential equation $-y'' + v(x)y = \lambda y$ on the half axis $[0, \infty)$ the relation

$$\pi N = \delta(0) - \delta(\infty) \tag{1}$$

holds between the number $N < \infty$ of eigenvalues and the asymptotic phase shift $\delta(\lambda)$ appearing in the asymptotic behaviour of the solutions [10]. The obvious generalization of (1) to the Schrödinger equation with a spherically symmetric potential became known in the physics literature as 'Levinson's theorem' [11, § 12.1.3; 16, § 12.e]. Levinson's theorem turned out to provide not only some deep theoretical insight into the scattering process but also a considerable potential for applications; for a review, see [1]. Therefore it appeared worthwhile to discover analogous relations for more general scattering systems and in particular for scattering by three-dimensional non-central potentials. Such a generalization is *a priori* non-trivial since on the one hand the asymptotic phase shift is not a well defined quantity

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for a partial differential equation and on the other hand summation of (1) over all partial waves leads in general to a divergent right-hand side.

Substantial progress was first achieved by Buslaev [2], who proved a Levinson type relation for Schrödinger equations with rapidly decreasing infinitely differentiable potentials. Most realistic potentials cannot be assumed to have these properties. In the present paper a similar relation is shown to hold for all integrable potentials satisfying the Rollnik condition. This class contains most potentials for which a non-modified scattering theory is known to exist. There are strong indications that for any larger class a Levinson type relation cannot be expected to hold.

The method employed in the present paper is based on abstract stationary scattering theory. The results of the abstract part have been presented in [5] and will be reviewed in Section 2 (Theorem 1). Section 3 is devoted to the proof of the main result (Theorem 2). In Section 4, finally, Levinson's result will be obtained as a corollary (Theorem 3). Applications which go beyond potential scattering may be found in [6] and [7].

This paper is based to a large extent on part of the author's PhD thesis [8]. Recently, R. G. Newton [12], as well as T. A. Osborn and D. Bollé [13] have independently obtained related results. Newton's results differ from the present ones insofar as they use more restrictive methods and apply to a considerably smaller class of potentials; in return, they provide some more detail. The results of Osborn and Bollé, on the other hand, cannot be considered complete. In fact, they express their extended Levinson's theorem in terms of the trace of a 'time delay operator'. Such an operator has, however, never been shown to exist as a bona fide operator on a Hilbert space, and much less even as a trace class operator. A careful analysis of [14], and of all papers about time delay cited in [14], reveals that the notion of time delay in the full configuration space exists only as a numerical valued function $T(\phi)$, which associates to each state ϕ the limit as $R \to \infty$ of its time delay $T_R(\phi)$ in a finite sphere of radius R. Moreover, even the function $T(\phi)$ has not been shown to be well defined for all scattering systems with potentials in $L^1 \cap L^2$, for which results are claimed in [13, 14], but only for a considerably smaller class of scattering systems. Therefore, a further exposition without the above drawbacks on a generalization of Levinson's relation appears necessary, whereas a satisfactory treatment of time delay for quantum scattering systems is still outstanding.

2. The abstract Levinson's theorem

In this section the abstract version of Levinson's theorem presented in [5] will be recalled. Consider two selfadjoint operators acting on a separable Hilbert space $\mathscr{H}$: The 'free Hamiltonian' H_0 with spectrum $\sigma(H_0)$, resolvent set $\rho(H_0)$ and resolvent operator $R_0(z) = (H_0 - z)^{-1}$ for $z \in \rho(H_0)$; and the 'perturbation' V, factorized according to $V = AB$, with $B = |V|^{1/2} = (V^*V)^{1/4}$ and $A = DB$. Note that $V = DB^2 = D|V|$ is the polar decomposition of V; in particular D is selfadjoint and of norm one. V is a perturbation of H_0 in the sense specified by Assumption A:

(A) $\mathscr{D}(H_0) \subset \mathscr{D}(B)$. The densely defined operator $BR_0(z)A$ has a compact extension for all $z \varepsilon \rho(H_0)$. There is a $z \varepsilon \rho(H_0)$ such that $I + \overline{BR_0(z)A}$ is invertible. The boundary values $\lim_{\delta \downarrow 0} BR_0(\lambda \pm i\delta)A$ exist in norm for all $\lambda \varepsilon \mathbb{R}$, uniformly on each compact subset of $\mathbb{R}$.

The following notations will be used throughout:

$$B(z) = BR_0(z) \quad \text{for } z \in \rho(H_0),$$
$$Q(z) = I + \overline{BR_0(z)A} \quad \text{for } z \in \rho(H_0),$$
$$Q_\pm(\lambda) = \lim_{\delta \downarrow 0} Q(\lambda \pm i\delta) \quad \text{for } \lambda \in \mathbb{R},$$
$$\rho = \{z \in \rho(H_0)/Q(z) \text{ has a bounded inverse}\},$$
$$\Gamma = \{\lambda \in \mathbb{R}/Q_\pm(\lambda) \text{ have bounded inverses}\},$$
$$Y(z) = Q(z)^{-1}Q(\bar{z}) - I \quad \text{for } z \in \rho,$$
$$Y(\lambda) = \lim_{\delta \downarrow 0} Y(\lambda \pm i\delta) \quad \text{for } \lambda \in \Gamma. \tag{2}$$

Furthermore, β will denote the border of $\sigma(H_0)$; this notion is precisely defined in [5]. Here a partly intuitive description may suffice: β consists of the boundary points of the complement in $\mathbb{R}$ of $\sigma(H_0)$; e.g. if $\sigma(H_0) = [a, a + 1] \cup [a + 2, \infty)$ then $\beta = \{-\infty, a, a + 1, a + 2\}$. The value of a function $f(\lambda)$ on β will be taken to mean the sum of the signed values of f at the points of β, the sign being $+$ for upper and $-$ for lower endpoints. For the above example:

$$f(\lambda)\big|_\beta = f(a + 2) - f(a + 1) + f(a) - \lim_{\lambda \to -\infty} f(\lambda).$$

Several supplementary assumptions are now introduced (Assumption B has been dropped; see [5, Remark a]):

(C) (i) $Y(z)$ is a continuous traceclass valued function on ρ.
(ii) For $\delta > 0$, $Y(\lambda + i\delta)$ is continuously differentiable with respect to λ in tracenorm topology.
(iii) The limit in (2) exists in tracenorm; the convergence is uniform on each compact subset of Γ.

(D) $\lambda \in \beta,\ |\lambda| < \infty \Rightarrow \lambda \in \Gamma$.

(E) $B(z)$ is a continuous Hilbert Schmidt operator valued function on $\rho(H_0)$. Furthermore, there exist $c > 0$ and $\eta < 1$ such that, for $\mu \in \beta$, $\|B(z)\|_2^2 < c|z - \mu|^{-\eta}$ when z varies over a small semicircle with center μ and such that Re z does not belong to $\sigma(H_0)$.

Formula (1) can now be given the following abstract generalization:

Theorem 1. *Assume A, C, D and E. Then a suitable resolvent equation determines a selfadjoint operator H on $\mathscr{H}$ with the following properties:*

- *H is an extension of $H_0 + V$.*
- *$\rho = \rho(H_0) \cap \rho(H)$.*
- *The number N of isolated eigenvalues of H is finite.*
- *The wave operators $\Omega_\pm = s - \lim_{t \to \pm\infty} e^{iHt} e^{-iH_0t}$ exist and are complete.*

Moreover, there exists a spectral representation with respect to H_0, given by a unitary operator U from $\mathscr{H}$ to a direct integral Hilbert space $\mathscr{G} = \int^\oplus \mathscr{G}(\lambda)\, d\lambda$ such that the scattering operator $S = \Omega_+^*\Omega_-$ is diagonal:

$$USU^{-1} = \int^{\oplus} S(\lambda)\, d\lambda$$

and for $\lambda \in \Gamma$

– $S(\lambda)$ *is unitary on* $\mathscr{G}(\lambda)$,
– $S(\lambda) - I(\lambda)$ *is of traceclass on* $\mathscr{G}(\lambda)$,
– $\det S(\lambda) = \det [I + Y(\lambda)]$. (3)

Finally, $\det S(\lambda)$ *is continuous on* Γ *and satisfies the generalized Levinson relation*

$$\ln \det S(\lambda)\big|_{\beta} = 2\pi i N. \tag{4}$$

Theorem 1 is proved in [5]. Here we only recall that the logarithm in (4) is defined as the boundary value of a continuous logarithm of $\det [I + Y(z)]$, $\operatorname{Im} z > 0$. This definition of the logarithm of $\det S(\lambda)$ is justified by (3).

3. Application to non-central potentials

A concrete case in which Theorem 1 is applicable is non-relativistic quantum mechanical scattering in three dimensions by a potential in $L^1 \cap R$. More precisely, let $\mathscr{H}$ be realized as $L^2(\mathbb{R}^3)$, H_0 as the unique selfadjoint extension of the Laplacean $-\Delta$ acting, say, on C_0^∞ and V as the maximal multiplication operator with a real-valued measurable function $v(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$ satisfying

$$\int |v(\mathbf{x})|\, d^3x < \infty$$

and

$$\int |v(\mathbf{x})|\, |\mathbf{x} - y|^{-2} |v(\mathbf{y})|\, d^3x\, d^3y < \infty.$$

It will be shown in this section that for the above scattering system Assumptions A, C and E are satisfied. This is known for A [15] and easily verified for E. For the validity of C, in particular C(iii), a detailed proof will be given. It is based on an analysis of the integral kernel of $Y(z)$ and makes use of a criterion by E. B. Davies for convergence in tracenorm of positive operators (see Lemma 2 of the Appendix).

Proposition 1. *Suppose* $\mathscr{H}$, H^0 *and* V *as above. Then Assumption A is satisfied. Moreover,* $Q(z)$ *is analytic in* ρ *and tends in norm to the unit operator* I *as* $|z|$ *tends to* ∞. *The convergence is uniform in* $0 \leq \arg z \leq \pi$.

Proof. It is sufficient to assume $V \in R$. In [15, § 1.4] it is shown that $\mathscr{D}(H_0) \subset \mathscr{D}(B)$, that $BR_0(z)A$ has an extension $Q(z) - I$ which is a Hilbert Schmidt operator, that this operator valued family is analytic and that it has boundary values in Hilbert Schmidt norm. A slight extension of the proof of Theorem I.23 in [15] yields

$$\|Q(\lambda + i\delta) - I\|^4 \leq \int e^{i \operatorname{Re}\sqrt{\lambda + i\delta}\, s}\, e^{-|\operatorname{Im}\sqrt{\lambda + i\delta}\, s|} f(s)\, ds.$$

Using the Riemann Lebesgue Lemma one concludes that

$$\|Q(\lambda + i\delta) - I\| \to 0 \quad \text{uniformly in } \delta \text{ as } \lambda \to \pm\infty$$

and uniformly in λ as $\delta \to +\infty$. It follows that $Q(z) \to I$ uniformly in $\arg z$ as $|z| \to \infty$. In particular it follows that $Q(z)$ is invertible for $|z|$ sufficiently large. This completes the proof of the proposition. ▮

Proposition 2. *Suppose $\mathscr{H}$, H_0 and V as above. Then Assumption E is satisfied. Moreover, $B(z)$ is analytic in ρ.*

Proof. It is sufficient to assume $V \in L^1$. Then

$$\begin{aligned}\|B(z)\|_2^2 &= \int |v(\mathbf{x})|\, e^{-2\,\mathrm{Im}\sqrt{z}\,|\mathbf{x}-\mathbf{y}|}\,(4\pi|\mathbf{x}-\mathbf{y}|)^{-2}\, d^3x\, d^3y \\ &= \|v(\mathbf{x})\|_1 \int e^{-2\,\mathrm{Im}\sqrt{z}\,u}\,(4\pi u)^{-2} 4\pi u^2\, du \\ &= \|v(\mathbf{x})\|_1 (8\pi\, \mathrm{Im}\sqrt{z})^{-1}.\end{aligned}$$

The estimate in Assumption E now follows from the fact that $\mathrm{Im}\sqrt{z} \geq |z|^{1/2}$ for $\mathrm{Re}\, z \leq 0$. Analyticity follows from a similar estimate of the Hilbert Schmidt norm of $(B(z') - B(z))/(z' - z)$ and an application of the dominated convergence theorem. ▮

Proposition 3. *Suppose $\mathscr{H}$, H_0 and V as above. Then Assumption C is satisfied.*

Proof. Observe that

$$Y(z) = -Q(z)^{-1}[Q(z) - Q(\bar{z})] = Q(z)^{-1}(z - \bar{z})B(z)B(z)^* D \tag{5}$$

by the first resolvent equation. The traceclass property of Assumption C(i) now follows from the Hilbert Schmidt property of $B(z)$ proved in Proposition 2. The differentiability in Assumption C(ii) follows from the analyticity of $Q(z)$ and $B(z)$ proved in Propositions 1 and 2.

Now turn to the verification of Assumption C(iii). In view of (5) and Proposition 1 you know that, for $\lambda \in \Gamma$, $Y(\lambda + i\delta)$ converges in Hilbert Schmidt norm to $-Q_+(\lambda)^{-1}[Q_+(\lambda) - Q_-(\lambda)]$ as $\delta \downarrow 0$. It is then sufficient to show that the sequence of positive Hilbert Schmidt operators

$$K(\delta) = (2i)^{-1}(z - \bar{z})B(z)B(z)^*, \quad \delta = \mathrm{Im}\, z$$

converges in tracenorm as $\delta \downarrow 0$.

This will first be shown for the case where the potential is a continuous function of compact support, $V \in C_0$. In this case the integral kernel

$$k_\delta(\mathbf{x}, \mathbf{y}) = |v(\mathbf{x})|^{1/2}\left(e^{i\sqrt{\lambda + i\delta}\,|\mathbf{x}-\mathbf{y}|} - e^{-i\sqrt{\lambda - i\delta}\,|\mathbf{x}-\mathbf{y}|}\right)(8\pi i|\mathbf{x}-\mathbf{y}|)^{-1}|v(\mathbf{y})|^{1/2}$$

of $K(\delta)$ and the kernel

$$k(\mathbf{x}, \mathbf{y}) = |v(\mathbf{x})|^{1/2} \sin\sqrt{\lambda}\,|\mathbf{x}-\mathbf{y}|\,(4\pi|\mathbf{x}-\mathbf{y}|)^{-1}|v(\mathbf{y})|^{1/2}$$

of the limit operator $K = \lim_{\delta \downarrow 0} K(\delta)$ are in C_0 as well. You may then conclude from Lemma 1 (Appendix) that the operators $K(\delta)$, K are in traceclass with traces

$$\mathrm{tr}\, K(\delta) = (4\pi)^{-1}\|v(\mathbf{x})\|_1 2^{-1/2}((\lambda^2 + \delta^2)^{1/2} + \lambda)^{1/2}$$

and tr $K = (4\pi)^{-1}\|v(\mathbf{x})\|_1\lambda^{1/2}$, respectively. Thus $\lim_{\delta\downarrow 0}$ tr $K(\delta) =$ tr K and Lemma 2 (Appendix) implies that the convergence takes place in tracenorm.

Returning to the general case $V \in L^1 \cap R$, choose $\varepsilon > 0$ and $V' \in C_0$ such that $\| |v(\mathbf{x})|^{1/2} - v'(\mathbf{x})^{1/2}\|_2 < \varepsilon$. The following argument then shows that $K(\delta)$ is a Cauchy sequence in tracenorm:

$$\begin{aligned}
&\|K(\gamma) - K(\delta)\|_1 \leq \\
&\quad \leq \|\gamma B(\lambda + i\gamma)B(\lambda + i\gamma)^* - \gamma B'(\lambda + i\gamma)B(\lambda + i\gamma)^*\|_1 \\
&\quad + \|\gamma B'(\lambda + i\gamma)B(\lambda + i\gamma)^* - \gamma B'(\lambda + i\gamma)B'(\lambda + i\gamma)^*\|_1 \\
&\quad + \|\gamma B'(\lambda + i\gamma)B'(\lambda + i\gamma)^* - \delta B'(\lambda + i\delta)B'(\lambda + i\delta)^*\|_1 \\
&\quad + \|\delta B'(\lambda + i\delta)B'(\lambda + i\delta)^* - \delta B'(\lambda + i\delta)B(\lambda + i\delta)^*\|_1 \\
&\quad + \|\delta B'(\lambda + i\delta)B(\lambda + i\delta)^* - \delta B(\lambda + i\delta)B(\lambda + i\delta)^*\|_1.
\end{aligned}$$

In this sum the third term is small according to the first part of the proof because $V' \in C_0$. The other terms are bounded as follows:

$$\begin{aligned}
&\|\delta B'(\lambda + i\delta)B(\lambda + i\delta)^* - \delta B(\lambda + i\delta)B(\lambda + i\delta)^*\|_1 \\
&\quad \leq \delta\|B'(\lambda + i\delta) - B(\lambda + i\delta)\|_2\|B(\lambda + i\delta)^*\|_2 \\
&\quad = \delta(8\pi \operatorname{Im}(\lambda + i\delta)^{1/2})^{-1/2} \\
&\quad \|v'(\mathbf{x})^{1/2} - |v(\mathbf{x})|^{1/2}\|_2(8\pi \operatorname{Im}(\lambda - i\delta)^{1/2})^{-1/2}\| \, |v(\mathbf{x})|^{1/2}\|_2
\end{aligned}$$

where the Hilbert Schmidt norms have been evaluated as in the proof of Proposition 2. Since $\operatorname{Im}(\lambda \pm i\delta)^{1/2} \sim \delta$ for small δ the above expression can be made small by choosing ε small. The three remaining terms are estimated in much the same way. This implies that $K(\delta)$ converges in tracenorm uniformly on any compact subset of $\mathbb{R}$ and $Y(\lambda + i\delta)$ converges in tracenorm as $\delta \downarrow 0$ uniformly on any compact subset of Γ. ▮

We are now ready to state and prove the main result of the paper.

Theorem 2. *Consider a non-relativistic quantum mechanical potential scattering system with a potential* $V \in L^1 \cap R$. *Assume that* 0 *is not an exceptional point, i.e. assume* $0 \in \Gamma$. *Then the relation*

$$2\pi i N = \ln \det S(0) \tag{6}$$

holds between the number N *of bound states and the determinant of the scattering matrix. Hereby the logarithm is given as the boundary value of the continuous function* $\ln \det [I + Y(z)]$, *normalized by* $\ln \det S(-\infty) = 0$.

Proof. From $\sigma(H_0) = [0, \infty)$ one has $\beta = \{-\infty, 0\}$. Since $0 \in \Gamma$, Assumption D is satisfied. So are Assumptions A, C and E according to Propositions 1, 2 and 3. Theorem 1 may thus be applied. In particular, the determinant of the scattering matrix $S(\lambda)$ is well defined and, from (4), $2\pi i N = \ln \det S(0) - \ln \det S(-\infty)$. For details about the definition of the logarithm the reader is referred to [5]. Here we only recall that for λ smaller than the lower bound of the spectrum of H one has $\lambda \in \rho$, $Q_+(\lambda) = Q_-(\lambda)$, $Y(\lambda) = 0$ and $\det S(\lambda) = \det I = 1$. This allows for the normalization $\ln \det S(-\infty) = 0$. ▮

There is no general class of potentials for which assumption D can be verified. It is well known [11, § 11.2.2] that even for a square well potential there are isolated values of the coupling constant for which extraordinary phenomena occur: the cross section is infinite, the phase shift does not tend to an integer multiple of π at energy zero and there exist states which remain infinitely long in a bounded region without being proper bound states. For these values of the coupling constant 0 is an exceptional point. For many potentials it is possible to show that such coupling constants are isolated. An argument may be found in [9].

3. Levinson's relation

If the potential is not only in $L^1 \cap R$ but also spherically symmetric, relation (1) follows from Theorem 2.

Theorem 3. *In addition to the assumption made in Theorem 2, assume that $v(\mathbf{x})$ depends on $|\mathbf{x}|$ only. Then there exists a family of projection operators P_l, $l \in \mathbb{N}$ (projections on subspaces of fixed angular momentum), commuting among themselves as well as with H_0 and H such that for U as in Theorem 1 and for every $l \in \mathbb{N}$*

$$\begin{aligned} UP_l U^{-1} &= \int^{\oplus} P_l(\lambda)\, d\lambda, \\ P_l(\lambda)S(\lambda) &= \exp(2i\delta_l(\lambda))I_l(\lambda), \qquad (7) \\ \delta_l(\infty) &= \lim_{\lambda\to\infty} \delta_l(\lambda) \textit{ exists} \end{aligned}$$

and

$$\pi N_l = \delta_l(0) - \delta_l(\infty),$$

where N_l is the number of bound states of angular momentum l.

Proof. Since both, H_0 and H, commute with P_l, all operators of interest in the argument commute with P_l and you may restrict attention to $P_l\mathscr{H}$. Formula (7) is well known to hold in the $2l + 1$ dimensional space $P_l(\lambda)\mathscr{G}(\lambda)$ [16, § 6c]. An application of Theorem 2 in $P_l\mathscr{H}$ then yields

$$\begin{aligned} &2\pi i(2l + 1)N_l \\ &\quad = \ln\det P_l(0)S(0) - \ln\det P_l(-\infty)S(-\infty) \\ &\quad = 2i(2l + 1)[\delta_l(0) - \delta_l(-\infty)]. \end{aligned}$$

The factor $2l + 1$ on the lefthand side of the equation expresses the fact that N_l is defined so as not to count the $(2l + 1)$-fold degeneracy due to symmetry.

It remains to show that $\delta_l(+\infty) = \delta_l(-\infty)$. Recall from Proposition 2 that $Q(z)$ tends to I uniformly in $0 \le \arg z \le \pi$ as $|z| \to \infty$. With

$$\|Y(z)\| < 2\|I - Q(z)\|\,\|Q(z)\|^{-1}$$

it follows that $Y(z)$ tends to zero uniformly in $0 \le \arg z \le \pi$ as $|z| \to \infty$. Now, $P_l Y(z)$ is an operator of rank $2l + 1$, at most. (From [5], equation (18) one has $K(\lambda)P_l Y(\lambda) = P_l(\lambda)[S(\lambda) - I(\lambda)]K(\lambda)$, where $[P_l, Y(\lambda)] = 0$ and $K(\lambda)$ is isometric on the range of $Y(\lambda)$. Together with equation (7), this implies that the range of $P_l Y(\lambda)$ is of dimension $2l + 1$, at most.) Thus the convergence $Y(z) \to 0$ takes place in trace norm. It follows that $\det[I + P_l Y(z)]$ tends to 1 uniformly in $0 \le \arg z \le \pi$

as $|z| \to \infty$ and that ln det $[I + P_l Y(z)]$ takes identical values at $z = +\infty$ and $z = -\infty$. Using (3) and (7), you find the required equality $\delta_l(+\infty) = \delta_l(-\infty)$. ∎

Both, Theorems 2 and 3, give strong indications that $L^1 \cap R$ is the best class of potentials for which a Levinson type relation is to be expected. If you examine the 'power law behaviour' of potentials then $L^1 \cap R$ just admits potentials behaving like $r^{-3-\varepsilon}$ at infinity and like $r^{-2+\varepsilon}$ at the origin. But this corresponds exactly to the range of validity of Levinson's theorem [10, 12, 16], namely spherically symmetric potentials satisfying $\int_0 r|v(r)|dr < \infty$ and $\int^\infty r^2|v(r)|dr < \infty$. Moreover, if one were to drop the requirement $V \in L^1$ then the determinant in (6) would cease to be well defined, whereas dropping $V \in R$ will render the choice of the Hamiltonian H a very delicate affair. This substantiates the claim made in the introduction that a suitable generalization of Levinson's relation (1) has been proved for a maximally large class of potential scattering systems.

Acknowledgements

I would like to thank Werner O. Amrein and E. Brian Davies for useful discussions.

Appendix

Lemma 1. *Let* $\mathscr{H} = L^2(\mathbb{R}^n, d^n x)$, $K > 0$ *a Hilbert Schmidt operator with kernel* $k(\mathbf{x}, \mathbf{y})$, *continuous and of compact support. Then K is in trace class and*

$$\|K\|_1 = \operatorname{tr} K = \int k(\mathbf{x}, \mathbf{x})\, d^n x < \infty.$$

Proof. Let $K = \sum_{i=1}^{\infty} \lambda_i |\psi_i\rangle\langle\psi_i|$ be the canonical expansion of K. K being positive and $k(\mathbf{x}, \mathbf{y})$ continuous and of compact support, Mercer's theorem [3, § III 5.4] applies so that $k(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \lambda_i \psi_i(\mathbf{x})\overline{\psi_i(\mathbf{y})}$ pointwise with $\psi_i(\mathbf{x})$ continuous and the sum converging absolutely and uniformly in $\mathbf{x}$ and $\mathbf{y}$. From the monotone convergence theorem it follows that

$$\infty > \int k(\mathbf{x}, \mathbf{x})\, d^n x = \int \lim_{J \to \infty} \sum_{i=1}^{J} \lambda_i \psi_i(\mathbf{x})\overline{\psi_i(\mathbf{x})}\, d^n x$$
$$= \lim_{J \to \infty} \sum_{i=1}^{J} \lambda_i \int |\psi_i(\mathbf{x})|^2\, d^n x = \sum_{i=i}^{\infty} \lambda_i = \operatorname{tr} T. \quad ∎$$

The following criterion due to E. B. Davies is proved in [4, lemma 4.3].

Lemma 2. *Let* K_n, $n > 1$ *and K be positive traceclass operators. Suppose* $K_n \to K$ *weakly and* tr $K_n \to$ tr K. *Then* $K_n \to K$ *in tracenorm topology.*

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