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Group-theoretical aspects of the Wigner–Weyl isomorphism

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Abstract. We consider the group $E^{(2)}$ of symmetries with respect to points ('inversions') and displacements of phase space. A Wigner–Weyl system is defined as a projective representation of this group; it is a proper extension of a Weyl system. We derive the basic properties of Wigner–Weyl systems and show:

(i) Their use clarifies the role of symplectic Fourier transform in the Weyl correspondence.
(ii) The quasiprobability density of Wigner can be written in an intrinsic, symplectically covariant way as a matrix element of Wigner operators.

1. Introduction

In the study of the Wigner quasiprobability density [1–4] there appears naturally a certain family of operators labelled by points in phase space. One of us has shown [5] that these operators are best associated to symmetries (i.e. inversions with respect to points) of phase space. On the other hand, the same family of 'displaced parity operators' proved useful in the study of Weyl systems [6].

Here we bring together the two points of view. We consider the group $E^{(2)}$ of symmetries and displacements of phase space, and define a Wigner–Weyl system as a projective representation of this group. The operator corresponding to a symmetry is called a Wigner operator. The product of two Wigner operators corresponds to a displacement; it is a Weyl operator. For this and other reasons, we claim that Wigner operators are the more natural building blocks for quantum mechanics.

After defining Wigner–Weyl systems, we look at questions of uniqueness (Section 4), at symplectic Fourier transforms (Section 5), relationships with the Wigner quasiprobability function (Section 6) and applications to quantization (Section 7).

2. Phase space (notations)

Let E be the vector space $E = R^{2n}$ of $(2n)$-tuples of real numbers. An element $a \in E$ will be written as a pair of n-tuples: $a = \{q_a, p_a\}$. Given a positive number $\hbar$, define on E the symplectic (i.e. bilinear, antisymmetric, nondegenerate) form

$$\sigma(a, b) = \tfrac{2}{\hbar}(q_a p_b - p_a q_b) = -\tfrac{4}{\hbar}\int p\, dq;$$

it is the action along the triangular path $\Delta(0, a, b)$, in units of $\hbar/4$. The (Lebesgue) volume element in E will be normalized as follows:

$$d^*a = \frac{d^{2n}a}{(\pi\hbar)^n} = d^n q\, d^*p$$

where $d^*p = d^n p/(\pi\hbar)^n$.

Writing $a = \{q_a, 0\} + \{0, p_a\}$, we obtain a decomposition of E into two maximal isotropic subspaces. More generally, let T be a linear and such that:

$$T^2 = 1, \qquad \sigma(Ta, Tb) = \sigma(b, a)$$

(an involutive antimetry of E; [7]). Consider the two complementary subspaces $V^{\pm} = \frac{1}{2}(1 \pm T)E$, and decompose $a \in E$:

$$a = \xi_a + \pi_a$$

with $\xi_a = \frac{1}{2}(1 + T)a$, and $\pi_a = \frac{1}{2}(1 - T)a$. One verifies then that $\sigma(\xi_a, \xi_b) = \sigma(\pi_a, \pi_b) = 0$, for all $a \in E$, $b \in E$.

With an appropriate choice of coordinates, we can write the volume element

$$d^*a = d^n\xi_a\, d^*\pi_a, \quad \text{where} \quad d^*\pi_a = d^n\pi_a/(\pi\hbar)^n.$$

If we choose T time reversal ($T\{q, p\} = \{q, -p\}$), then V_+ is coordinate space and V_- momentum space.

The reader will notice that the specific choice $E = R^{2n}$ is not important. What we are using is:

(i) an even-dimensional real vector space with a symplectic inner product, and
(ii) appropriate normalizations of the invariant volume element in E and in its isotropic subspaces.

On suitable spaces of (generalized) functions on E, we can consider the *left* and *right* symplectic Fourier transform defined, respectively, by

$$(F_l f)(a) = \int f(b)\, e^{i\sigma(b, a)}\, d^*b$$

$$(F_r f)(a) = \int e^{i\sigma(a, b)} f(b)\, d^*b$$

which are simply related to each other:

$$F_l f = F_r \check{f} = (F_r f)^{\check{}}$$
$$F_r f = (F_l f)^{\check{}} = \check{F_l} f$$

where $\check{f}(a) = f(-a)$.

With the above conventions on volume elements, they are both involutory:

$$F_l^2 = 1, \qquad F_r^2 = 1.$$

3. The Group $E^{(2)}$

Consider in E the transformation of symmetry (i.e. parity) around a point a. It is the map that sends any $b \in E$ into

$$\Pi_a b = 2a - b.$$

The product of two such transformations is a displacement τ: define

$$\tau_a(b) = -a + b.$$

Endowed with their natural product, the symmetries around points and displacements form a group $E^{(2)}$ which has two connected components, each homeomorphic to E. Denote by $E^{(+)}$ the subgroup of displacements and by $E^{(-)}$ the coset of symmetries. It is convenient to introduce the group Z_2 consisting of the numbers $+1$ and -1, and to write the elements of $E^{(2)}$ as pairs:

$$\{+1, a\} = \tau_{2a} \quad \text{(notice the factor of 2)}$$

$$\{-1, a\} = \Pi_a \quad \text{(the factor 2 is absent).}$$

Then the (non-abelian) group law in $E^{(2)}$ can be written as

$$\{\varepsilon, a\}\{\eta, b\} = \{\varepsilon\eta, b + \eta a\} \quad (a, b \in E, \varepsilon, \eta \in Z_2).$$

We see that $E^{(2)}$ is a semi-direct product of E and of Z_2. The identity in $E^{(2)}$ is

$$e = \{+, 0\} = \{\varepsilon, a\}\{\varepsilon, -\varepsilon a\}.$$

Any displacement is a product of two symmetries; so e.g.

$$\{+, a\} = \{+, -a\}^{-1} = \{-, 0\}\{-, a\}.$$

The inner automorphisms of $E^{(2)}$ are given by

$$\{\eta, b\}^{-1}\{\varepsilon, a\}\{\eta, b\} = \{\varepsilon, \eta a + (1 - \varepsilon)b\}.$$

4. Wigner–Weyl systems

We define a Wigner–Weyl system as a unitary irreducible projective representation of $E^{(2)}$, with multiplier $e^{-i\sigma(a, \eta b)}$ [1]). In other words, a Wigner–Weyl system is a strongly continuous correspondence $\{\varepsilon, a\} \to W(\varepsilon; a)$ from $E^{(2)}$ to unitary operators in a separable Hilbert space $\mathscr{H}$, such that:

$$W(\varepsilon; a)W(\eta; b) = e^{-i\sigma(a, \eta b)}\, W(\varepsilon\eta; b + \eta a) \quad (\varepsilon, \eta \in Z_2; a, b \in E). \tag{1}$$

If we denote by $W^*(\eta, b)$ the adjoint of $W(\eta, b)$, this can also be written as

$$W(\varepsilon; a)W^*(\eta; b) = e^{i\sigma(a, b)}\, W(\varepsilon\eta; \eta(a - b)). \tag{2}$$

We assume irreducibility, i.e. non-existence of non-trivial closed subspaces of $\mathscr{H}$ stable under all operators $W(\varepsilon; a)$.

The restriction of a Wigner–Weyl system to $E^{(+)}$ is a Weyl system, familiar from the study of canonical commutation relations. The uniqueness theorem of von Neumann and Stone [8] states that there exists, given E and σ, exactly one irreducible

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[1]) Theorem 9.4 of [10] (together with known facts on multipliers of Z_2 and E) shows that this is not an arbitrary choice.

Weyl system up to unitary equivalence. The study of Wigner–Weyl systems can be based on this theorem, together with the generalization, to the non-compact case, of a theorem by Clifford [9]. All the necessary results, in much greater generality than what is required here, can be found in a paper by Mackey [10]. They can be specialized as follows.

The restriction of an (irreducible) Wigner–Weyl system to $E^{(+)}$ is an irreducible Weyl system.

Given E, σ, there exist, up to unitary equivalence, exactly *two* (irreducible) Wigner–Weyl systems. If $W(\varepsilon; a)$ is a Wigner–Weyl system, then $\varepsilon W(\varepsilon; a)$ is another such system, not equivalent to $W(\varepsilon; a)$. Later in this paper, we shall give a criterion for choosing the overall sign of $W(-; a)$, given $W(+; a)$.

Products of two Wigner–Weyl operators are given by the defining equation (1). Products of three Wigner operators $W(-; a)$ are particularly interesting:

$$W(-; a)W(-; b)W(-; c) = e^{i\varphi(a,b,c)}\, W(-; a - b + c)$$

where the exponent is proportional to the action integral over the triangle $\Delta(a, b, c)$ with vertices a, b, c.

$$\varphi(a, b, c) = \sigma(a - c, b - c) = -\sigma(a, c) - \sigma(c, b) - \sigma(b, a)$$
$$= \frac{4}{\hbar}\int p\, dq.$$

The function φ has the following obvious properties:

$$\varphi(a, b, c) = \varphi(b, c, a) = -\varphi(a, c, b) = -\varphi(c, b, a) = -\varphi(b, a, c)$$
$$= \varphi(a + x, b + x, c + x) = \lambda^{-2}\varphi(\lambda a, \lambda b, \lambda c) = \varphi(La, Lb, Lc)$$

where λ is a nonzero real number, and where L is a symplectic transformation of E: $L \in \mathrm{Sp}\,(2n)$, i.e. $\sigma(La, Lb) = \sigma(a, b)$.

The notation $\varphi(a, b, c)$ is related through $\varphi(a, b, c) = -\frac{1}{\hbar}(a, b, c)$ to the symbol (a, b, c) introduced in [4].

The product of any three Wigner–Weyl operators can be expressed, though less elegantly, with the help of the function φ:

$$W(\varepsilon; a)W(\eta; b)W(\zeta; c) = e^{i\varphi(-\eta a, b, -\zeta c)}\, W(\varepsilon\eta\zeta; c + \xi(b + \eta a)). \tag{3}$$

The inner automorphisms are, again, simpler:

$$W^*(\eta; b)W(\varepsilon; a)W(\eta; b) = e^{i\eta(1+\varepsilon)\sigma(a,b)}\, W(\varepsilon; \eta a + (1 - \varepsilon)b).$$

The set of Weyl operator $W(+; .)$ is stable under inner automorphisms.

Every Wigner–Weyl operator is, by definition, unitary. The Wigner operators $W(-; \varepsilon)$ have the additional properties of being self-adjoint and involutive

$$W(-; a)^* = W(-; a) \qquad W(-; a)^2 = 1. \tag{4}$$

Consequently, the spectrum of any Wigner operator consists of the numbers $+1$ and -1. It is easy to find the corresponding eigenspaces [6].

5. Symplectic Fourier transform of Wigner–Weyl systems

Theorem. (i) *Let $W(\varepsilon; a)$ be any (irreducible) Wigner–Weyl system. Then the subfamilies $W(+; a)$ and $W(-; a)$ are, up to a sign, the right symplectic Fourier transforms of each other:*

$$W(\varepsilon; a) = \pm \int e^{i\sigma(a,b)} W(-\varepsilon; b)\, d^*b. \tag{5}$$

(ii) *If $W(+; a)$ is any irreducible Weyl system and if we define $W(-; a)$ through (5) (with $\varepsilon = -1$), then $W(+; a)$ and $W(-; a)$ form a Wigner–Weyl system.*

Remarks. (1) Equation (5) will give the promised criterion for choosing the overall sign of $W(-; a)$.

(2) If $W(+; a)$ is any irreducible Weyl system acting in the Hilbert space $\mathscr{H}$, then every matrix element $\omega_{\varphi\psi}(a) = (\varphi, W(+; a)\psi)$ $(\varphi, \psi \in \mathscr{H}$ is square integrable over $E)$. This can be shown directly or deduced from general results [11, p. 349]; it gives meaning to the Fourier transform (5). Furthermore, there exists a dense domain $\mathscr{D} \subset \mathscr{H}$ such that the matrix elements $\omega_{\varphi\psi}$ $(\varphi, \psi \in \mathscr{D})$ are all absolutely integrable; this will justify the operator calculations below.

Proof. Let $W(\varepsilon; a)$ be an (irreducible) Wigner–Weyl system. Consider the two operators

$$A^{(\varepsilon)} = \int W(\varepsilon; a)\, d^*a \quad \varepsilon \in Z_2.$$

The square of both of them is the identity:

$$\begin{aligned}(A^{(\varepsilon)})^2 &= \int\int W(\varepsilon; a) W(\varepsilon; b) d^*a\, d^*b \\ &= \int\int e^{-i\sigma(a,\varepsilon b)}\, W(+; b + \varepsilon a)\, d^*a\, d^*b \\ &= \int\int e^{-i\sigma(a,\varepsilon b')}\, W(+; b') d^*a\, d^*b' = W(+; 0) = 1.\end{aligned}$$

A crucial distinction between $A^{(+)}$ and $A^{(-)}$ appears when we look at their commutation properties with the operators $W(\varepsilon; b)$. A trivial computation shows that

$$A^{(-)} W(\varepsilon; b) = W(\varepsilon; b) A^{(-)} \quad (\varepsilon \in Z_2, b \in E)$$

while $A^{(+)}$ satisfies

$$A^{(+)} W(\varepsilon; b) = W(\varepsilon; -b) A^{(+)}$$

by Schur's lemma, $A^{(-)}$ is either $+1$ or -1 (the identity) while $A^{(+)}$ is the parity operator, determined up to a sign. Having chosen the sign of $W(-; 0)$ so that

$$1 = \int W(-; b)\, d^*b \tag{6}$$

we obtain

$$W(\varepsilon; a) = \int W(\varepsilon; a)W(-\,; b)\, d^*b = \int e^{i\sigma(a,b)}\, W(-\varepsilon; b)\, d^*b \tag{7}$$

which is the required relationship.

The uniqueness theorem of von Neumann and Stone gives now:

Let $W_1(\varepsilon; a)$ and $W_2(\varepsilon; a)$ be two (irreducible) Wigner–Weyl systems, both satisfying (7). Then they are unitarily equivalent.

From now on we shall assume that (7) holds.

6. Wigner–Weyl systems over phase space with a real polarization; Wigner function

Let $E = E_\xi + E_\pi$ be any decomposition of E into maximal isotropic subspaces (a real polarization) as discussed in Section 2. It is sometimes convenient to realize the abstract irreducible Wigner–Weyl system in a Hilbert space adapted to that decomposition (the 'ξ-representation'). The construction of such a space is a straightforward example of induced representation ([12]; see also [13]). It runs as follows:

Let H_T consist of (measurable) complex-valued functions on E that satisfy

(i) the covariance condition along E_π:

$$\Phi(a + \pi_b) = e^{i\sigma(\pi_b,\, a)}\, \Phi(a)$$

for all $\pi_b \in E_\pi$, $a \in E$.

(ii) the condition of square integrability on E/E_π

$$\int_{E_\xi} |\Phi(a)|^2\, d^n\xi < \infty.$$

The last condition is meaningful because, by (i), $|\Phi(a)|^2$ depends only on ξ_a and not on π_a. With the corresponding definition of inner product, H_T becomes a Hilbert space.

The Wigner–Weyl operators in H_T are defined by

$$(W(\varepsilon; a)\Phi)(b) = e^{-2i\sigma(a,b)}\, \Phi(\varepsilon(b - 2a)).$$

If we identify every $\Phi \in H_T$ with its restriction to E_ξ, i.e. if we define on E_ξ the functions $\psi(\xi_a) = \Phi(\xi_a)$, then H_T is identified with $L^2(E_\xi)$ and the Weyl operators take the familiar form

$$(W(+\,; \xi_a)\psi)(\xi_b) = \psi(\xi_b - 2\xi_a)$$

$$(W(+\,; \pi_a)\psi)(\xi_b) = e^{(8i/\hbar)\pi_a\xi_b}\, \psi(\xi_b) = e^{4i\sigma(\xi_b,\, \pi_a)}\, \psi(\xi_b).$$

In order to use Dirac notation concerning generalized eigenfunctions, we embed $L^2(E_\xi)$ into a suitable larger space (say the space $S'(E_\xi)$ of tempered distributions). The dyadics $|\xi_a\rangle\langle\xi_b|$ can then be defined either as maps from $S(E_\xi)$ to $S'(E_\xi)$, or as unbounded quadratic forms, or else as operators in a partial inner product space [14].

Let now $\mathscr{H}$ be an arbitrary irreducible representation space for Weyl–Wigner operators over E. The uniqueness theorem discussed in Section 4 enables us to transport unambiguously the above definitions from H_T to $\mathscr{H}$, with the help of the intertwining operator which exists, and is determined up to a phase factor. The phase factor drops out from the definition of operators $|\xi_a\rangle\langle\xi_b|$.

A straightforward calculation gives then

Theorem. *Let $W(\varepsilon; a)$ be an (irreducible) Wigner–Weyl system acting in a Hilbert space $\mathscr{H}$. Let $E = E_\xi + E_\pi$ be a decomposition of E into maximal isotropic subspaces. Then the dyadics $|\xi_a\rangle\langle\xi_b|$ can be expressed with the help of Wigner operators as follows:*

$$|\xi_a\rangle\langle\xi_b| = \int e^{i\sigma(\pi_c, (\xi_a - \xi_b)/2)}\, W\left(-; \frac{\xi_a + \xi_b}{2} + \pi_c\right) d^*\pi_c.$$

In particular

$$|\xi_a\rangle\langle\xi_a| = \int W(-; \xi_a + \pi_a)\, d^*\pi_a.$$

Conversely, the Wigner operator $W(-; a)$ can be written with the help of dyadics as

$$W(-; a) = \int |\xi_a - \xi_b\rangle\, d^n\xi_b\, e^{i\sigma(\pi_a, \xi_b)} \langle\xi_a + \xi_b|. \tag{8}$$

Remarks. (1) Similar relationships hold for Weyl operators (which are products of Wigner operators and parity). They are, however, less symmetric, since Weyl operators are not selfadjoint.

(2) It is sometimes convenient to consider the operators $2^n W(-; a)$, normalized so as to have generalized trace equal to 1. (See Section 7 [2], [15].) Then, by (8),

$$2^n W(-; b) = \int |\xi_b - \tfrac{1}{2}\xi_a\rangle\, d^n\xi_a\, e^{(i/2)\sigma(\pi_b, \xi_a)} \langle\xi_b + \tfrac{1}{2}\xi_a|.$$

The expectation value of the operator (8) in a pure state, written in the ξ-representation, is the famous expression of Wigner quasiprobability density:

$$2^n\langle\psi|W(-; b)|\psi\rangle = \int \psi^*(\xi_b - \tfrac{1}{2}\xi_a)\, e^{(i/2)\sigma(\pi_b, \xi_a)}\, \psi(\xi_b + \tfrac{1}{2}\xi_a)\, d^n\xi_a.$$

7. Quantization and its inverse (Weyl–Wigner correspondence)

We now turn to the relationships between operators in Hilbert space and the corresponding classical functions. The results of this section have been known for a long time [16, 17, 22, 23], but they take now a particularly transparent form.

In order to avoid technicalities, we shall first state the result for a very restricted class of operators. (This procedure is, roughly speaking, equivalent to restricting the discussion of Fourier transforms to absolutely integrable functions.)

Theorem. *Let $\mathscr{H}$ be a Hilbert space carrying an irreducible Wigner–Weyl system $W(\varepsilon; a)$. If H is any operator of trace class on $\mathscr{H}$, define on $E^{(2)}$ the function*

$$h(\varepsilon; a) = 2^n \operatorname{tr}(W^*(\varepsilon; a)H).$$

Then

(A) *The functions $h(+;a)$ and $h(-;a)$ on E are left symplectic Fourier transforms of each other:*

$$h(-\varepsilon;a)=\int h(\varepsilon;b)\,e^{i\sigma(b,a)}\,d^*b\quad(\varepsilon\in Z_2,a\in E).$$

(B) *The correspondence $H\to h$ is inverted as follows:*

$$H=\int h(+;a)W(+;a)\,d^*a=\int h(-;a)W(-;a)\,d^*a,$$

i.e. H can be reconstructed either with the help of $h(+,\cdot)$ or with the help of $h(-,\cdot)$.

(C) *If a function h on $E^{(2)}$ satisfies*

$$\int h(+;a)W(+;a)\,d^*a=\int h(-;a)W(-;a)\,d^*a$$

then $h(+;\cdot)$ and $h(-;\cdot)$ are left symplectic Fourier transforms of each other.

(D) *The trace of H is given by*

$$\operatorname{tr}H=2^{-n}\int h(-;a)\,d^*a=2^{-n}h(+;0).$$

(E) *The correspondence between H and h is unitary in the following sense. If H_1 and H_2 are as above, and if we denote by $(H_1,H_2)_{\mathrm{HS}}$ their Hilbert–Schmidt scalar product* $\operatorname{tr}(H_1^*H_2)$ *then we have*

$$\begin{aligned}2^n((H_1,H_2))_{\mathrm{HS}}&=\int \bar{h}_1(-;a)h_2(-;a)\,d^*a\\&=\int \bar{h}_1(+;a)h_2(+;a)\,d^*a.\end{aligned}$$

(F) *The function h^* corresponding to the operator H^* (adjoint of H) is*

$$h^*(\varepsilon;a)=\bar{h}(\varepsilon;-\varepsilon a)$$

where the bar denotes complex conjugation. In particular, $h^*(-;a)=\bar{h}(-;a)$.

(G) *If $K=H_1H_2$ (operator product) then*

$$k(-;a)=\int e^{i\varphi(a,b,c)}\,h_1(-;b)h_2(-;c)\,d^*b\,d^*c \tag{9}$$

(where φ is the function introduced in Section 4) and

$$k(+;a)=\int h_1(+;b)h_2(+;a-b)\,e^{i\sigma(a,b)}\,d^*b. \tag{10}$$

(H) *If H is of trace class on $\mathscr{H}$, $h(\pm,\cdot)$ are bounded, continuous and square integrable functions on E.*

Remarks. (1) Formula (9) defines the 'twisted product' [18–21]. The commutator with respect to this product is the Moyal bracket [22]. Formula (10) defines 'twisted convolution'.

(2) With the help of (D), (F), and (G), we can obtain alternative expressions for the Hilbert–Schmidt scalar product of operators. Denote the r.h.s. of (9) by $(h_1^{(-)} \circ h_2^{(-)})(a)$, and the r.h.s. of (10) by $(h_1^{(+)} * h_2^{(+)})(a)$.

$$\begin{aligned}((H_1, H_2))_{\mathrm{HS}} &= \mathrm{Tr}\,(H_1 H_2) \\ &= 2^{-n} \int (\bar{h}_1^{(-)} \circ h_2^{(-)})(a)\, d^*a \\ &= 2^{-n}(h_1^{(+)*} * h_2^{(+)})(0).\end{aligned}$$

It is possible to extend the correspondence between H and $h^{(\cdot)}(\cdot)$ to wider classes of operators and (generalized) functions (say by duality, or continuity). We shall not do it here, but shall give some examples:

H	$h^{(-)}(a)$	$h^{(+)}(a)$
$1 = W(+\,;0)$	1	$(\pi\hbar)^n\,\delta(a)$
$W(+\,;a_0)$	$e^{i\sigma(a_0,\,a)}$	$(\pi\hbar)^n\,\delta(a - a_0)$
$\sigma(a_0, X) \equiv \dfrac{1}{i}\left(\dfrac{d}{d\lambda}\, W(+\,;\lambda a_0)\right)_{\lambda=0}$	$\sigma(a_0, a)$	$(\pi\hbar)^n(a_0\cdot\nabla)\,\delta(a)$
$\Pi = W(-\,;0)$	$(\pi\hbar)^n\,\delta(a)$	1
$W(-\,;a_0)$	$(\pi\hbar)^n\,\delta(a - a_0)$	$e^{i\sigma(a_0,\,a)}$

Here $a_0\cdot\nabla$ denotes the derivative in the direction a_0.

For the sake of simplicity, consider now a system of one degree of freedom, and define the Fourier transform by

$$\hat{g}(p) = \int g(q)\, e^{(2i/\hbar)qp}\, dq.$$

Let $|0\rangle$ be the ground state of the harmonic oscillator of mass m and frequency ω. We have then:

H	$h^{(-)}(a)$	$h^{(+)}(a)$
$g(Q)$	$g(p)$	$\delta(q)\hat{g}(p)$
$g(P)$	$g(p)$	$\delta(p)\hat{g}(q)$
$\lvert 0\rangle\,\langle 0\rvert$	$2\,e^{-(m\omega^2q^2 + p^2/m\omega^2)/\hbar}$	$2\,e^{-(m\omega^2q^2 + p^2/m\omega^2)/\hbar}$

The examples $g(Q)$ and $g(P)$ show again that the correspondence between H and $h^{(-)}(a)$ is simpler than the correspondence between H and $h^{(-)}(a)$ is simpler than the correspondence between H and $h^{(+)}(a)$.

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