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Systems of two quantal Einstein relativistic particles

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Abstract. We outline a method for the construction of quantal theories for the description of any number of Einstein relativistic particles with any spin, by applying it to the systems of two particles of spin 0 and the system of a spin 0 particle and a particle of spin s_0. The point of departure for the construction is the definition of an Einstein relativistic particle as being a physical system associated to an irreducible unitary projective representation of the restricted inhomogeneous Lorentz group.

1. Introduction

It is usual to define a free quantal Einstein relativistic particle of mass m' and spin s_0 as a physical system associated with a state-space carrying an irreducible unitary projective representation, IUPR, (m', s_0) of the restricted inhomogeneous Lorentz group $SO(3, 1) \times_s \mathbb{R}^4$. For example, the electron and the proton being particles of masses m_e and m_p and spin $\frac{1}{2}$, is described by the representations $(m_e, \frac{1}{2})$ and $(m_p, \frac{1}{2})$. Moreover, a hydrogen atom in a given internal stationary state of internal energy e and total internal angular momentum j, is a particle of mass $m_e + m_p + \Delta m(e)$ and spin j, and thus described by the representation $(m_e + m_p + \Delta m(e), j)$.

The problem being studied in this paper, and which is suggested by the above example, is how to put together two quantal Einstein relativistic particles in such a way that the center of mass of the system in a given internal stationary state appear as a quantal Einstein relativistic particle itself.

This problem was first considered for two classical Einstein relativistic particles. The construction of the present quantum theory is based on the results obtained in the classical case [1].

2. The quantal Einstein relativistic particle

(a) *The particle of 'spin 0'*

Definition 1. The quantal particle of 'spin 0' and kinematical mass $m > 0$ is a physical system associated with

(i) the space $\{H_t\}$, $t \in \mathbb{R}$, where each H_t is a Hilbert-space isomorphic to

$$L^2(M, d^4p) \tag{1}$$

with

$$M = \{p^4 \in \mathbb{R}^4 \mid (p^0 + mc)^2 - \mathbf{p}^2 > 0 \,\&\, p^0 > -mc\}.$$

(ii) the kinematical symmetry group $SO(3, 1) \times_s \mathbb{R}^4$, being represented by the unitary projective representation

$$(\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)f)_t(p^\mu) = \exp\left(-\frac{i}{\hbar} a^\mu p_\mu\right) \exp\left(\frac{i}{\hbar} t v^\mu(-\mathbf{u}) p_\mu\right) \times f_t(\Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu p^\nu + m v^\mu(-\mathbf{u})) \quad (2)$$

with $v^\mu(\mathbf{u}) = (c(\gamma - 1), \gamma \mathbf{u})$, $\gamma = (1 - \mathbf{u}^2/c^2)^{-1/2}$.

(iii) the observables energy[1]) $p^0 c$, massdefect Δm, momentum $\mathbf{p}$, position $\mathbf{x}$ and time t, being realized by the following self-adjoint operators

$$(\hat{p}^0 f)_t(p^\mu) = p^0 f_t(p^\mu)$$

$$(\Delta\hat{m} f)_t(p^\mu) = \left(\frac{1}{c}\sqrt{(p^0 + mc)^2 - \mathbf{p}^2} - m\right) f_f(p^\mu)$$

$$(\hat{\mathbf{p}} f)_t(p^\mu) = \mathbf{p} f_t(p^\mu)$$

$$(\hat{\mathbf{x}} f)_t(p^\mu) = i\hbar\left(\partial \mathbf{p} + \frac{\mathbf{p}}{p^0 + mc}\,\partial p^0 - \frac{1}{2}\frac{\mathbf{p}}{(p^0 + mc)}\right) f_t(p^\mu)$$

$$(\hat{t} f)_t(p^\mu) = t f_t(p^\mu). \quad (3)$$

It is then easy to verify that the operators $\hat{p}^\mu = (\hat{p}^0, \hat{\mathbf{p}})$, $\Delta\hat{m}$ and $\hat{t}$ transform according to

$$\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)\hat{p}^\mu \hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) = \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu \hat{p}^\nu + m v^\mu(\mathbf{u}) = \hat{p}'^\mu$$

$$\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)\, \Delta\hat{m}\, \hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) = \Delta\hat{m}$$

$$\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)\hat{t}\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) = \hat{t}; \quad (4)$$

while the transformation properties of $\hat{\mathbf{x}}$ is expressed by

$$\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}))i\hbar\left(\partial p_i + \frac{p^i}{p^0 + mc}\,\partial p^0 - \frac{1}{2}\frac{p^i}{(p^0 + mc)^2}\right)\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}))$$

$$= \sigma(\Delta m(p^0, \mathbf{p}), \mathbf{p}, \Lambda(\boldsymbol{\theta}, \mathbf{u}))^i_j i\hbar\left(\partial p_j + \frac{p^j}{p^0 + mc}\,\partial p^0\right)$$

$$\times \frac{i\hbar p'^i}{2(p'^0 + mc)^2} + \gamma \mathbf{u} t - \frac{\mathbf{p}'(\gamma - 1)c}{p'^0 + mc}\, t \quad (5)$$

and

$$\hat{U}^{-1}(a^\mu)\hat{\mathbf{x}}\hat{U}(a^\mu) = \hat{\mathbf{x}} + \mathbf{a} - \frac{\hat{\mathbf{p}}}{\hat{p}^0 + mc}\, a^0. \quad (6)$$

The explicit expressions for the matrix σ^i_j are found in [1, eq. 6].

Another useful representation of H_t is the one diagonalizing $\Delta\hat{m}$ and $\hat{\mathbf{p}}$. It is obtained by the isometry

$$\hat{V}_{\phi-1}: L^2(M, d^4p) \to L^2\left(M, \frac{(\Delta m + m)c^2\, d\,\Delta m\, d^3p)}{\sqrt{\mathbf{p}^2 + (\Delta m + m)^2 c^2}}\right. \quad (7)$$

---

[1]) The usual definition of energy in the Einstein relativistic case corresponds to $p^0 c + mc^2$ in our notation.

defined by

$$f(p^0, \mathbf{p}) \rightarrow (f \circ \phi)(\Delta m, \mathbf{p}) = \psi(\Delta m, \mathbf{p})$$

being induced by the diffeomorphism

$$\phi: M \rightarrow M, (p^0, \mathbf{p}) \rightarrow (\Delta m, \mathbf{p}) = \frac{1}{c}\sqrt{(p^0 + mc)^2 - \mathbf{p}^2} - m, \mathbf{p}).$$

The operators $\hat{p}^0$, $\hat{\mathbf{p}}$, $\Delta\hat{m}$ and $\hat{\mathbf{x}}$ are now represented as follows

$$(\hat{p}^0\psi)(\Delta m, \mathbf{p}) = (\sqrt{\mathbf{p}^2 + (\Delta m + m)^2c^2} - mc)\psi(\Delta \mathrm{m}, \mathbf{p})$$

$$(\hat{\mathbf{p}}\psi)(\Delta m, \mathbf{p}) = \mathbf{p}\psi(\Delta m, \mathbf{p})$$

$$(\Delta\hat{m}\psi)(\Delta m, \mathbf{p}) = \Delta m\, \psi(\Delta m, \mathbf{p})$$

$$(\hat{\mathbf{x}}\psi)(\Delta m, \mathbf{p}) = i\hbar\left(\partial\mathbf{p} - \frac{1}{2}\frac{\mathbf{p}}{\mathbf{p}^2 + (\Delta m + m)^2c^2}\right)\psi(\Delta m, \mathbf{p}).$$

(b) *The particle of spin s_0*

Definition 2. The quantal particle of spin $s_0(s_0 \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\})$[2]) and kinematical mass $m > 0$ is a physical system associated with

(i) the space H_t, $t \in \mathbb{R}$, with each H_t being a Hilbert-space isomorphic to $L^2(M, l^2; d^4p)$[3]) (8)

(ii) the kinematical symmetry group SO(3, 1) $\times_s \mathbb{R}^4$ being represented by the unitary projective representation

$$(\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)f)(p^\mu) = \exp\left(-\frac{i}{\hbar}a^\mu p_\mu\right)\exp\left(\frac{i}{\hbar}tv^\mu(-\mathbf{u})p_\mu\right)$$
$$\times\, \hat{D}(\Lambda(\boldsymbol{\phi}_w(p^\mu, \Lambda(\theta, \mathbf{u}))) f(\Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu p^\nu + mv^\mu(-\mathbf{u})) \quad (9)$$

where $\boldsymbol{\phi}_w(p^\mu, \Lambda(\theta, \mathbf{u}))$ is the angle defined by the rotation

$$\Lambda(\boldsymbol{\phi}_w) = L^{-1}(p^\mu)\Lambda(\boldsymbol{\theta}, \mathbf{u})L(\Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu p^a + mv^\mu(-\mathbf{u}))$$

$$L(p^\mu) = \Lambda\left(\frac{\mathbf{p}c}{p^0 + mc}\right). \quad (10)$$

$\hat{D}$ moreover, is a unitary projective representation of SO(3) in l^2, such that $\hat{D} = \sum^{\oplus}_{s=s_0, s_0+1, \ldots} \hat{D}^{(s)}$ where $\hat{D}^{(s)}$ is an IUPR of SO(3) in $\mathbb{C}^{2s+1}$. We denote by $\hat{s}^i$ $i = 1, 2, 3$ the self-adjoint generators of $\hat{D}$.

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[2]) Notice that for $s_0 = 0$ we get a notion of spin 0 different from the one discussed in the last section.

[3]) This is to be considered as the Hilbert space of functions on M with values in the Hilbert space $l^2 = \sum^{\oplus}_{s=s_0, s_0+1, \ldots} \mathbb{C}^{2s+1}$, such that

$$\int_{\mathrm{m}} (f(p^\mu), f(p^\mu))_{l^2}\, d^4p < \infty$$

where $(\ ,\ )_{l^2}$ is the scalarproduct on l^2.

(iii) the observables energy p^0, massdefect Δm, momentum $\mathbf{p}$, position $\mathbf{x}$, spin $\mathbf{S}$ and time t. Of these observables, p^0, $\mathbf{p}$, Δm, $\mathbf{x}$ and t are realized as in Definition 1(iii), while $\mathbf{S}$ is represented by the operator

$$(\hat{\mathbf{S}}f)_t(p^\mu) = \hbar(\hat{\mathbf{s}}f)_t(p^\mu).$$

It is easy to verify that the operators $\hat{p}^\mu$ and $\Delta\hat{m}$ transform as in (4); moreover, the transformation properties of $\hat{\mathbf{x}}$ are the same for the rotations $\{\boldsymbol{\theta}\}$ and the translations $\{a^\mu\}$. Finally, $\hat{\mathbf{S}}$ is invariant under the translations, and

$$\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}))\hat{S}^i\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u})) = \Lambda^{-1}(\boldsymbol{\phi}_w(\mathbf{p}, \Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u}))\,{}^i_j\hat{S}^j = \Lambda(\boldsymbol{\theta}_w(\hat{p}, \Lambda(\boldsymbol{\theta}, \mathbf{u}))\,{}^i_j\hat{S}^j.$$

As for the 'spin 0' case, we can define a representation like (7) diagonalizing $\Delta\hat{m}$ and $\hat{\mathbf{p}}$.

3. The dynamics of the one-particle system

The evolution of an Einstein particle is by assumption given by a family of unitary operators [2]

$$V_t(\tau): H_t \to H_{t+\tau}$$

induced by a permutation of the real line,

$$t \to t + \tau$$

i.e.

$$\hat{V}_{t+\tau_1}(\tau_2)\hat{V}_t(\tau_1) = \hat{V}_t(\tau_1 + \tau_2).$$

Under suitable technical conditions this is equivalent to the Schrödinger equation

$$i\hbar\, \partial_t f_t = \mathscr{H}_t f_t$$

where $\mathscr{H}_t$ is a self-adjoint operator on H_t, the Hamiltonian of the system.

(a) *The particle of 'spin 0'*

We will consider only the free case and postulate that the Hamiltonian $\mathscr{H}$ is represented by the operator

$$(\hat{\mathscr{H}}f)(p^\mu) = \left(\frac{p^\mu p_\mu}{2m} + \mathscr{k}\right)f(p^\mu),$$

on (1) [1], for $\mathscr{k}(>\frac{1}{2}mc^2)$ being a constant denoting the internal energy of the system. On (7) it thus takes the form

$$(\hat{\mathscr{H}}\psi)(\Delta m, \mathbf{p}) = \left(c\sqrt{\mathbf{p}^2 + (\Delta m + m)^2c^2} - (\Delta m + m)c^2 - \frac{\Delta m^2}{2m}c^2 + \mathscr{k}\right)\psi(\Delta m, \mathbf{p})$$

Let the operator $\hat{\alpha}$ be defined on (7) by

$$(\hat{\alpha}\psi)(\Delta m, \mathbf{p}) = \left(\Delta m + \frac{\Delta m^2}{2m} - \frac{\mathscr{k}}{c^2}\right)\psi(\Delta m, \mathbf{p}). \tag{11}$$

$\hat{\alpha}$ is evidently self-adjoint on a properly chosen domain and its spectrum is purely continuous. Thus, it determines a decomposition of $H = \int_{\mathrm{sp}(\alpha)} H^{(\alpha)}\, d\mu(\alpha)$. Furthermore, the Hamiltonian $\hat{\mathscr{H}}$ commuting with $\hat{\alpha}$, also decomposes according to

$$\hat{\mathscr{H}} = \int_{\mathrm{sp}(\hat{\alpha})} \hat{\mathscr{H}}^{(\alpha)}\, d\mu(\alpha).$$

By assumption, the energy-spectrum of the system is given by the spectrum of $\hat{\mathscr{H}}^{(0)}$ in $H^{(0)}$, i.e.

$$(\hat{\mathscr{H}}^{(0)}\psi^0)(\mathbf{p}) = \left(c\sqrt{\mathbf{p}^2 + m^2c^2\left(1 + \frac{2\hbar}{mc^2}\right)} - mc^2\right)\psi^0(\mathbf{p})$$

where

$$\psi^0 \in L^2\left(\mathbb{R}^3; \frac{mc\, d^3p}{\sqrt{\mathbf{p}^2 + m^2c^2(1 + 2\hbar/mc^2)}}\right) \cong H^{(0)}.$$

We will express this by saying that the system satisfies the Lorentz-invariant constraint '$\alpha = 0$'.

Accordingly, by applying the constraint '$\alpha = 0$', we recover a theory which is essentially equivalent to a Wigner theory of spin 0 and mass $m + \Delta m = m\sqrt{1 + 2\hbar/mc^2}$. In fact, also the representation $\hat{U}$ (2), decomposes according to $\hat{\alpha}$, $\hat{U} = \int_{\mathrm{sp}(\alpha)} \hat{U}^{(\alpha)}\, d\mu(\alpha)$ each $\hat{U}^{(\alpha)}$ being an IUPR of SO(3, 1) $\times_s \mathbb{R}^4$.

(b) *The particle of spin s_0*

Let $\hat{P}_{s_0}$ be the projection operator

$$\hat{P}_{s_0}: L^2(M, l^2; d^4p) \to L^2(M, \mathbb{C}^{2s_0+1}; d^4p)$$

defined by the Lorentz-invariant constraint

$$(\gamma\psi)(p^\mu) = ((\hat{\mathbf{s}}^2 - s_0(s_0 + 1))\psi)(p^\mu) = 0 \tag{12}$$

The Hamiltonian of the free particle of spin s_0 is then assumed to be given by

$$(\hat{\mathscr{H}}f)_t(p^\mu) = \left(\left(\frac{\hat{p}^\mu\hat{p}_\mu}{2m} + \hat{\hbar}_{s_0}\ f\right)f\right)_t(p^\mu)$$

where $\hat{\hbar}_{s_0} = \hbar\hat{P}_{s_0}$, and $\hbar$ denotes the internal energy of the system.

Let $\hat{\alpha}$ be the operator corresponding to (11), and let

$$H = \int_{\mathrm{sp}(\alpha,\gamma)} H^{(\alpha,\gamma)}\, d\mu(\alpha, \gamma) \quad \text{and} \quad \mathscr{H} = \int_{\mathrm{sp}(\alpha,\gamma)} \mathscr{H}^{(\alpha,\gamma)}\, d\mu(\alpha, \gamma)$$

be the decomposition of H and $\hat{\mathscr{H}}$ with respect to $\hat{\alpha}$ and $\hat{\gamma}$. By assumption, the energy-spectrum of the system is represented by the spectrum of $\hat{\mathscr{H}}^{(0,0)}$ in $H^{(0,0)}$. We express this by saying that the system satisfies the constraints '$\hat{\alpha} = 0$' and $\hat{\gamma} = 0$.

Again the constraints serve as irreducibility conditions, and we obtain a theory which is essentially equivalent to a Wigner theory of spin s_0 and mass $m + \Delta m = m\sqrt{1 + 2\hbar/mc^2}$.

4. The two-particle system

(a) *The system of two 'spin 0' particles*

Definition 3. The system of two particles of 'spin 0' is by assumption associated with

(i) the space $\{H_t \mid t \in \mathbb{R}\}$, where each H_t is a Hilbert-space isomorphic to

$$L^2(M, dp_1^4) \otimes L^2(M, dp_2^4) \cong L^2(M \times M, dp_1^4\, dp_2^4); \tag{13}$$

(ii) the kinematical symmetry group $\mathrm{SO}(3, 1) \times_s \mathbb{R}^4$ acting by $\hat{U} = \hat{U}_1 \otimes \hat{U}_2$, where $\hat{U}_1$ and $\hat{U}_2$ are the representations associated to the individual particles (2);

(iii) the observable p_i^0, Δm_i, $\mathbf{p}_i$, $\mathbf{x}_i$ $(i = 1, 2)$ and t realized in the same way as for the one-particle systems (3). In addition we will consider observables P^0, ΔM, $\mathbf{P}$ and $\mathbf{X}$ describing the center of mass and p^μ, q^μ describing the internal system.

For the purpose of realizing the second set of observables, we consider the representation

$$L^2(M \times M, d^4P\, d^4p)$$

which diagonalize the representatives of P^μ and p^μ. This representation is obtained from (12) by the isometry

$$(V_{\phi^{-1}} f)(p_1^\mu, p_2^\mu) = g(P^\mu, p^\mu) = \sqrt{J_{\phi^{-1}}(P^\mu, p^\mu)}\,(f \circ \phi)(P^\mu, p^\mu)$$

induced by the diffeomorphism $\phi: M \times M \to M \times M$ defined by

$$P^\mu = p_1^\mu + P_2^\mu$$

$$p^\mu = L^{-1}(p_1^\mu + p_2^\mu)^\mu_\nu \, \frac{m_1 p_2^\nu - m_2 p_1^\nu}{m_1 + m_2} \tag{14}$$

with $L(P^\mu) = \Lambda(\mathbf{P}c/(P^0 + Mc)$. $J_{\phi^{-1}}$ is the Jacobian determinant of ϕ^{-1}. Moreover, in terms of the coordinates (P^μ, p^μ), the manifold $M \times M$ is characterized by

$$\Big\{(P^\mu, p^\mu) \in \mathbb{R}^8 \mid (P^0 + Mc)^2 - \mathbf{P}^2 > 0,\ P^0 > Mc,$$

$$\frac{m_1}{M}\sqrt{(P^0 + Mc)^2 - \mathbf{P}^2} > p^0 > -\frac{m_2}{M}\sqrt{(P^0 + Mc)^2 - \mathbf{P}^2},$$

$$\frac{m_1((P^0 + Mc)^2 - \mathbf{P}^2) - M^2 p^\mu p_\mu}{2m_1 M\sqrt{(P^0 + Mc)^2 - \mathbf{P}^2}} > p^0 > -\frac{m_2((P^0 + Mc)^2 - \mathbf{P}^2) - M^2 p^\mu p_\mu}{2m_2 M\sqrt{(P^0 + Mc)^2 - \mathbf{P}^2}}\Big\}$$

Remark. There exists another choice of two-particle space which is preferable from a technical point of view. That is to take $L^2(N, d^4P\, d^4p)$, where N is the manifold

$$N = \Big\{(P^\mu, p^\mu) \in \mathbb{R}^8 \mid (P^0 + Mc)^2 - \mathbf{P}^2 > 0,\ P^0 > -Mc,$$

$$\frac{m_1}{M}\sqrt{(P^0 + Mc)^2 - \mathbf{P}^2} > p^0 > -\frac{m_2}{M}\sqrt{(P^0 + Mc)^2 > -\mathbf{P}^2}\Big\}.$$

Thus,

$$M \times M = \left\{(P^\mu, p^\mu) \in N \middle| \frac{m_1((P^0 + Mc)^2 - \mathbf{P}^2) - M^2 p^\mu p_\mu}{2m_1 M \sqrt{(P^0 + Mc)^2 - \mathbf{P}^2}} > p^0 > \frac{m_2((P^0 + Mc)^2 - \mathbf{P}^2) - M^2 p^\mu p_\mu}{2m_2 M \sqrt{(P^0 + Mc)^2 - \mathbf{P}^2}}\right\}$$

is a submanifold of N, and $L^2(M \times M, d^4P\, d^4p)$ is a subspace of $L^2(N, d^4P\, d^4p)$.

With the choice $L^2(N)$, the one-particle observable Δm_i and $x_i (i = 1, 2)$ cannot be represented on the whole space, but only on the subspace $L^2(M \times M)$. In the following we will consider the theory on $L^2(N)$.

In the representation $L^2(N; d^4P\, d^4p)$, the observables P^μ, Δm, $\mathbf{X}$, p^μ and q^μ are realized by the self-adjoint operators

$$\begin{aligned}
(\hat{P}^\mu g)(P^\mu, p^\mu) &= P^\mu g(P^\mu, p^\mu) \\
(\Delta\hat{M} g)(P^\mu, p^\mu) &= \left(\frac{1}{c}\sqrt{(P^0 + Mc)^2 - \mathbf{P}^2} - M\right) g(P^\mu, p^\mu) \\
(\hat{\mathbf{X}} g)(P^\mu, p^\mu) &= i\hbar\left(\partial \mathbf{p} + \frac{\mathbf{P}}{P^0 + Mc}\partial p^0 - \frac{1}{2}\frac{\mathbf{P}}{(P^0 + Mc)^2}\right) g(P^\mu, p^\mu) \\
(\hat{p}^\mu g)(P^\mu, p^\mu) &= p^\mu g(P^\mu, p^\mu) \\
(\hat{q}^\mu g)(P^\mu, p^\mu) &= i\hbar\, \partial \mathrm{p}_\mu g(P^\mu, p^\mu).
\end{aligned} \tag{15}$$

For a more complete discussion of the two sets of observables we refer to [1].

The representation $\hat{U}$ of $\mathrm{SO}(3, 1) \times_s \mathbb{R}^4$ is given by

$$\begin{aligned}
(\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) g)(P^\mu, p^\mu) &= \exp\left(-\frac{i}{\hbar} a^\mu P_\mu\right) \exp\left(\frac{i}{\hbar} t v^\mu(-\mathbf{u}) p_\mu\right) \\
&\quad \times g(\Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})\, {}^\mu_\nu P^\nu + M v^\mu(-\mathbf{u}), \Lambda(\boldsymbol{\theta}_w(P^\mu, \Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})))^\mu_\nu p^\nu)
\end{aligned}$$

where $\boldsymbol{\theta}_w$ is the angle of the rotation

$$\Lambda(\boldsymbol{\theta}_w) = L(\Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu P^\nu + M v^\mu(\mathbf{u}))\Lambda(\boldsymbol{\theta}, \mathbf{u}) L^{-1}(P). \tag{16}$$

Thus the operators $\hat{p}^\mu$, $\Delta\hat{M}$, $\hat{p}^\mu$ and $\hat{q}^\mu$ transform according to

$$\begin{aligned}
\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a\mu)\hat{P}^\mu \hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) &= \Lambda(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu P^\nu + M v^\mu(\mathbf{u}) \\
\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)\, \Delta\hat{M}\, \hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) &= \Delta\hat{M} \\
\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)\hat{p}^\mu \hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) &= \Lambda(\hat{\boldsymbol{\theta}}_w)^\mu_\nu \hat{p}^\nu \\
\hat{U}^{-1}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)\hat{q}^\mu \hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu) &= \Lambda(\hat{\boldsymbol{\theta}}_w)^\mu_\nu \hat{q}^\nu.
\end{aligned} \tag{17}$$

The transformed of $\hat{\mathbf{X}}$ is more complicated than (5). It depends on the internal degrees of freedom in the same way as the transformed of the position operator of a particle with spin.

Moreover, it is possible to define a representation like (7) for the center of mass.

(b) *The system of two particles, one of 'spin 0' and one of spin s_0*

Definition 4. The system of two particles, one of 'spin 0' and the other of spin s_0 is by assumption associated with

(i) the space $\{H_t \mid t \in \mathbb{R}\}$ where each H_t is a Hilbert-space isomorphic to

$$L^2(M, d^4p_1) \otimes L^2(M, l^2; d^4p_2) \cong L^2(M \times M, l^2; d^4p_1\, d^4p_2); \tag{18}$$

(ii) the kinematical symmetry group $SO(3, 1) \times_s \mathbb{R}^4$ being represented by the unitary projective representation $\hat{U} = \hat{U}_1 \otimes \hat{U}_2$, where $\hat{U}_1$ and $\hat{U}_2$ are defined by (2) and (9) respectively, i.e.

$$\begin{aligned}(\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)g)(p_1^\mu, p_2^\mu) &= \exp\left(-\frac{i}{\hbar} a^\mu p_{1\mu}\right) \exp\left(\frac{i}{\hbar} tv^\mu(-\mathbf{u})p_{1\mu}\right) \\ &\times \exp\left(-\frac{i}{\hbar} a^\mu p_{2\mu}\right) \exp\left(\frac{i}{\hbar} tv^\mu(-\mathbf{u})p_{2\mu}\right) \hat{D}(\Lambda(\boldsymbol{\phi}_w(p_2^\mu, \Lambda(\boldsymbol{\theta}, \mathbf{u})) \\ &\times g(\Gamma^{-1}(\boldsymbol{\theta}, \mathbf{u})^\mu{}_\nu p_1^\nu + m_1 v^\mu(-\mathbf{u}), \Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu p_2^\nu + m_2 v^\mu(-\mathbf{u}))\end{aligned}$$

with $\hat{D}$ as in (9) and $\boldsymbol{\phi}_w$ being the angle of the rotation defined by (10);

(iii) the observables $p_1^\mu, \Delta m, \mathbf{X}_1, p_2^\mu, \Delta m_2, \mathbf{X}_2, \mathbf{s}_2$ of the individual particles and t, which is realized as for the individual particles, and the observables P^μ, $\Delta M, \mathbf{X}$ describing the center of mass and the observables p^μ and q^μ describing the internal degrees of freedom.

To realize the second set of observables by operators on H_t, we will choose another representation than (18). To define this however, we need to extend the representation $\hat{D}$ of SO(3) in l^2 to a unitary representation $\hat{\hat{D}}$ of SO(3, 1).

Consider the operators $\hat{F}_3$, $\hat{F}_+$ and $\hat{F}_-$ on l^2, defined by

$$\begin{aligned}(\hat{F}_3\xi)^s_{m_s} &= C_s\sqrt{s^2 - m_s^2}\, \xi^{s-1}_{m_s} - A_s m_s \xi^s_{m_s} \\ &\quad - C_{s+1}\sqrt{(s+1)^2 - m_s^2}\, \xi^{s+1}_{m_s} \\ (\hat{F}_+\xi)^s_{m_s} &= C_s\sqrt{(s - m_s)(s - m_s - 1)}\, \xi^{s-1}_{m_s+1} \\ &\quad - A_s\sqrt{(s - m_s)(s + m_s + 1)}\, \xi^s_{m_s+1} \\ &\quad + C_{s+1}\sqrt{(s + m_s + 1)(s + m_s + 2)}\, \xi^{s+1}_{m_s+1} \\ (\hat{F}_-\xi)^s_{m_s} &= -C_s\sqrt{(s + m_s)(s + m_s - 1)}\, \xi^{s-1}_{m_s-1} \\ &\quad - A_s\sqrt{(s + m_s)(s - m_s + 1)}\, \xi^s_{m-1} \\ &\quad - C_{s+1}\sqrt{(s - m_s + 1)(s - m_s + 2)}\, \xi^{s+1}_{m_s-1}\end{aligned}$$

with

$$A_s = \frac{-s_0 k}{s(s+1)}$$

$$C_s = \frac{i}{s}\sqrt{\frac{(s^2 - s_0^2)(s^2 + k^2)}{4s^2 - 1}}, \quad (k \in \mathbb{R}).$$

$\{\xi^s_{m_s}\}$ is the canonical basis for the decomposition $D = \sum^{\oplus}_{s = s_0, s_0+1, \ldots} D^{(s)}$ diagonalizing $\mathbf{s}^2, s_z$.

The operators $\{s^i, k^i \mid i = 1, 2, 3\}$ with

$$\hat{k}^1 = \tfrac{1}{2}(\hat{F}_+ + \hat{F}_-)$$
$$\hat{k}^2 = \tfrac{1}{2}(\hat{F}_- - \hat{F}_+)$$
$$\hat{k}^3 = \hat{F}_3 \tag{20}$$

then form a basis for an irreducible representation of the Lie algebra so(3, 1) of SO(3, 1). By integrating the representation of so(3, 1), we obtain an IUPR $\hat{\bar{D}}$ of SO(3, 1) on l^2. $\hat{\bar{D}}$ is a representation in the principal series characterized by the two numbers (s_0, ik) [3].

We then denote by $\hat{V}_{\phi-1}$ the isometry defined by

$$\begin{aligned}(\hat{V}_{\phi-1}f)(p_1^\mu, p_2^\mu) &= \sqrt{J_{\phi-1}(P^\mu, p^\mu)}\,\hat{D}^{-1}(L(P^\mu)) \\ &\quad \times \hat{D}(L(p_2^\mu)(P^\mu, p^\mu))(f \circ \phi)(P^\mu, p^\mu) \\ &= g(P^\mu, p^\mu)\end{aligned} \tag{19}$$

$L^2(M \times M, l^2; d^4p_1\, d^4p_2) \to L^2(M \times M, l^2; d^4P\, d^4p)$, ϕ is defined as in (14).

In this new representation, the observables $P^\mu, \Delta M, \mathbf{X}, p^\mu, q^\mu$ are supposed to be realized as in (15); moreover, the representation $\hat{U}$ of SO(3, 1) reads

$$\begin{aligned}(\hat{U}(\Lambda(\boldsymbol{\theta}, \mathbf{u}), a^\mu)g)_t(P^\mu, p^\mu) &= \exp\left(-\frac{i}{\hbar}a^\mu P_\mu\right)\exp\left(\frac{i}{\hbar}tv^\mu(-\mathbf{u})P_\mu\right) \\ &\quad \times \hat{D}(\Lambda(\boldsymbol{\phi}_w(P^\mu, \Lambda(\boldsymbol{\theta}, \mathbf{u}))))g_t(\Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})^\mu_\nu P^\nu \\ &\quad + Mv^\mu(-\mathbf{u}), \Lambda(\boldsymbol{\theta}_w(P^\mu, \Lambda^{-1}(\boldsymbol{\theta}, \mathbf{u})))^\mu_\nu p^\nu)\end{aligned} \tag{21}$$

where $\boldsymbol{\phi}_w$ and $\boldsymbol{\theta}_w$ are defined in (10) and (16) respectively. Notice that in this representation the Wigner rotations $\hat{D}(\Lambda(\boldsymbol{\phi}_w))$ depends on $\hat{P}^\mu$ only, not p^μ. Accordingly $\hat{P}^\mu$, $\hat{p}^\mu$ and $\hat{q}^\mu$ transforms as in (17).

5. The dynamics of the two-particle system

The dynamics of the systems of two particles is assumed to be described by the Schrödinger and the Heisenberg equations.

In the following, we will consider the situation of two particles in mutual interaction, the center of mass being a free particle.

(a) *The system of two particles of 'spin 0'*

The Hamiltonian for the system of two particles of 'spin 0' is by assumption of the form

$$\hat{\mathscr{H}} = \frac{\hat{P}^\mu \hat{P}_\mu}{2M} + \hat{h}$$

where $\hat{h} = h(\hat{p}^\mu, \hat{q}^\mu, \hat{t})$, the internal Hamiltonian, acts nontrivially on the internal space only.

Moreover, consider the self-adjoint operators $\hat{\alpha}$ and $\hat{\beta}$ defined by

$$\hat{\alpha} = \Delta\hat{M} + \frac{\Delta\hat{M}}{2M} - \frac{1}{c^2}\hat{h}$$

$$\hat{\beta} = \hat{p}^0 - \frac{m_1 - m_2}{Mc}\frac{\hat{h}}{\sqrt{1 + (2/Mc^2)\hat{h}}}. \tag{22}$$

For a conservative system, the internal Hamiltonian $\hat{h}$ is independent of $\hat{q}^0$ and $\hat{t}$, and $\hat{\alpha}$, $\hat{\beta}$ and $\hat{h}$ are mutually commuting operators and can be diagonalized simultaneously. Denote the decomposition of H, $\hat{\mathscr{H}}$ and $\hat{h}$ with respect to $\hat{\alpha}$ and $\hat{\beta}$ by

$$H = \int_{\mathrm{sp}(\hat{\alpha},\hat{\beta})} H^{(\alpha,\beta)}\, d\mu(\alpha, \beta)$$
$$\hat{\mathscr{H}} = \int_{\mathrm{sp}(\hat{\alpha},\hat{\beta})} \hat{\mathscr{H}}^{(\alpha,\beta)}\, d\mu(\alpha, \beta)$$
$$\hat{h} = \int_{\mathrm{sp}(\hat{\alpha},\hat{\beta})} \hat{h}^{(\alpha,\beta)}\, d\mu(\alpha\ \beta).$$

Then we assume that the total and internal energies of the system is represented by the spectra of $\hat{\mathscr{H}}^{(0,0)}$ and $\hat{h}^{(0,0)}$ in $H^{(0,0)}$.

The only *a priori* conditions we can put on the form of $\hat{h}$, is that it should be invariant under rotations, and that in the case of no interaction (coupling constants $\rightarrow 0$)

$$\hat{h} = \frac{\hat{p}^\mu \hat{p}_\mu}{2m}$$

with

$$m = \frac{m_1 m_2}{m_1 + m_2}.$$

By interpreting the operators $\hat{\alpha}$, $\hat{\beta}$, $\hat{P}^\mu$ and $\hat{p}^\mu$ as operators on $L^2(M \times M)$, we can write

$$\hat{\mathscr{H}} = \frac{\hat{P}^\mu \hat{P}_\mu}{2M} + \frac{\hat{p}^\mu \hat{p}_\mu}{2m}$$
$$= \frac{\hat{p}_1^\mu \hat{p}_{1\mu}}{2m} + \frac{\hat{p}_2^\mu \hat{p}_{2\mu}}{2m_2}.$$

Moreover, the constraints '$\hat{\alpha} = 0$' & '$\hat{\beta} = 0$' implies the constraints '$\hat{\alpha}_1 = 0$' & '$\hat{\alpha}_2 = 0$' where $\hat{\alpha}_i (i = 1, 2)$ is defined by

$$\hat{\alpha}_i = \Delta\hat{m}_i + \frac{\Delta\hat{m}_i^2}{2m_i}.$$

Thus we recover the system of two free Einstein relativistic particles.

(b) *The system of two particles, one of 'spin 0' and one of spin s_0*

Let

$$\hat{P}_{s_0} : L^2(M \times M, l^2, d^4p_1\, d^4p_2) \rightarrow L^2(M \times M, \mathbb{C}^{2_0+1}, d^4p_1\, d^4p_2)$$

be the projection operator defined by the constraint

$$(\hat{\gamma}f)(p_1^\mu, p_2^\mu) = ((\hat{\mathbf{s}}^2 - s_0(s_0 + 1)\hat{I})f)(p_1^\mu, p_2^\mu) = 0. \tag{23}$$

We denote by $\hat{\mathbf{\Pi}}_{so}$ the image of $\hat{P}_{so}$ under the isometry (19); i.e.

$$(\hat{\Pi}_{so}g)(P^\mu, p^\mu) = (\hat{\hat{D}}^{-1}(L(\hat{P}^\mu)\hat{\hat{D}}(L(\hat{p}_2^\mu))\hat{P}_{so}\hat{\hat{D}}^{-1}(L(\hat{p}_2^\mu))\hat{\hat{D}}(L(\hat{P}^\mu))g)(P^\mu, p^\mu).$$

Using the facts that $\hat{\hat{D}}$ is a representation of SO(3, 1), and that L is a Lorentz transformation, we can write

$$\begin{aligned}\hat{\hat{D}}^{-1}(L(P^\mu))\hat{\hat{D}}(L((p_2^\mu)(P^\mu, p^\mu))) &= \hat{\hat{D}}(L^{-1}(P^\mu)L((p_1^\mu)(P^\mu, p^\mu)))\\ &= \hat{\hat{D}}(L(L^{-1}(P^\mu)^\mu{}_\nu p_2^\nu(P^\mu, p^\mu)\\ &\quad + m_2 w^\mu(P^0, -\mathbf{P})\Lambda^{-1}(\phi_w(p_2^\mu(P^\mu, p^\mu), L(P^\mu)))\end{aligned}$$

for $w^\mu(P^\mu) = v^\mu(\mathbf{P}c/(P^0 + Mc)$. Now

$$L^{-1}(P^\mu)^\mu_\nu p_2^\nu(P^\mu, p^\mu) + m_2 w^\mu(P^0, -\mathbf{P}) = \left(\frac{m_2}{M}\Delta Mc + p^0, \mathbf{p}\right);$$

moreover, $\hat{P}_{so}$ is invariant under rotations, thus

$$\begin{aligned}(\hat{\Pi}_{so}g)(P^\mu, p^\mu) = \Bigg(\hat{D}\left(\Lambda\left(\frac{\mathbf{p}c}{(m_2/M)\,\Delta Mc + p^0 + m_2 c}\right)\right)\hat{P}_{so}\\ \times \hat{D}^{-1}\left(\Lambda\left(\frac{\mathbf{p}c}{(m_2/M)\,\Delta Mc + p^0 + m_2 c}\right)\right)g\Bigg)(P^\mu, p^\mu).\end{aligned}$$

The Hamiltonian describing the dynamics of the system is supposed to be of the form[4])

$$\hat{\mathscr{H}} = \frac{\hat{P}^\mu\hat{P}_\mu}{2M} + \hat{h}'_{so}, \qquad \hat{h}'_{so} = \hat{h}'\hat{\Pi}_{so}$$

such that $[\hat{h}', \hat{\Pi}_{so}] = 0$. Moreover, we impose constraints of the form (22) and the constraint (23).

As for the form of the internal Hamiltonian, the only *a priori* conditions we can impose are those of invariance under rotations and reduction to free case, i.e.

$$\hat{h}'_{so} = \left(\frac{\hat{p}^\mu\hat{p}_\mu}{2m}\right)\hat{\Pi}_{so}$$

when coupling constants $\rightarrow 0$.

To study the spectrum of $\hat{h}'_{so}$ and the associated energy spectrum it is worthwhile to choose another representation. Thus, consider the unitary transformation defined by

$$(\hat{F}g)(P^\mu, p^\mu) = \hat{\hat{D}}^{-1}\left(\Lambda\left(\frac{\mathbf{p}c}{(m_2/M)\,\Delta Mc + p^0 + m_2 c}\right)\right)g(P^\mu, p^\mu).$$

If we denote by $\hat{h}_{so}$ and $\hat{h}$ the image of $\hat{h}'_{so}$ and $\hat{h}'$ under $\hat{F}$, we find that

$$\hat{h}_{so} = \hat{h}\hat{P}_{so} = \hat{P}_{so}\hat{h}.$$

Notice also that since $\hat{F}$ is invariant under the rotations, the representation $\hat{U}$ (21) is invariant under $\hat{F}$, i.e. $\hat{F}\hat{U}\hat{F}^{-1} = \hat{U}$.

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[4]) From now on we assume the relevant operators to be defined on the space $L^2(N, l^2)$ of which $L^2(M \times M, l^2)$ is a subspace.

Remark. In the treatment of the two-particle system we have worked with the convention $m_1 > m_2$. Moreover, for simplicity we have assumed that the internal energies of the individual particles $\hat{h}_1 = \hat{h}_2 = 0$.

We also notice that although it is necessary to use $L^2(N)$ to define the two-particle system in terms of its observables, it is natural to consider the dynamics only on the subspace on which

$$\frac{m_1(\Delta M + M)^2 c^2 - 2m_2 M\hat{h}}{2M(\Delta M + M)c} > p^0 > \frac{m_2(\Delta M + M)^2 c^2 - 2m_1 M\hat{h}}{2M(\Delta M + M)c}.$$

For a more complete discussion of these aspects of the theory, and for an application, we refer to [4].

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