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Schrödinger invariant generalized heat equations

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Abstract. Invariance under the five-parameter Schrödinger group of coordinate transformations is investigated in the context of the generalized heat equations $u_t - \kappa u_{xx} + F(u, u_x) = 0$. There are four classes of invariant equations, among them Burgers' equation and other nonlinear equations used in fluid dynamics. The Schrödinger invariance is explained by the fact that all invariant equations can be converted from the heat equation by simple transformations of u. A larger number of generalized heat equations is shown to be invariant if a more general concept of Schrödinger invariance is used, and again they are simple conversions of the heat equation. The use of the Schrödinger group in the search for solutions to invariant equations is illustrated by two applications: first the similarity method is generalized to arbitrary one-parameter subgroups and ordinary differential equations are obtained for the invariant solutions, and then, Schrödinger transformations are applied to trivial solutions to produce new, non-trivial solutions.

1. Introduction

It has been recognized some time ago [1] that the Schrödinger equation of the free particle is invariant not only under the Galilei group but under a larger group of coordinate transformations, the so-called Schrödinger group. More equations exhibiting Schrödinger symmetry have since been found [2] and it is the purpose of the present paper to increase their number by the inclusion of equations which are generalizations of the heat equation. The heat equation itself is Schrödinger invariant because it formally is a free particle Schrödinger equation with imaginary mass and the results of [1] are valid whether or not the mass is real. Actually, the Schrödinger invariance of the heat equation has been known for a long time as it was first observed by S. Lie in 1882 [3]. The definition of the Schrödinger group and the implementation of Schrödinger invariance for the heat equation are summarized in Appendix A. For simplicity the discussion is restricted to one-dimensional space throughout.

The generalized heat equations considered in this paper are partial differential equations of the form

$$u_t - \kappa u_{xx} + F(u, u_x) = 0, \tag{1.1}$$

where κ is a real constant; (1.1) for $F = 0$ is the heat equation. An equation (1.1) is called Schrödinger invariant if, for each element g of the Schrödinger group, there

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exist companion functions $f_g(t, x)$ and $h_g(t, x)$, with $f_e = 1$, $h_e = 0$, such that the function $T_g u$, defined by

$$(T_g u)(t, x) = f_g[g^{-1}(t, x)]u[g^{-1}(t, x)] + h_g[g^{-1}(t, x)], \tag{1.2}$$

is a solution of (1.1) whenever u is a solution. The additive term h_g in (1.2) is included for greater flexibility; without it Schrödinger invariance would be restricted to equations homogeneous in u and u_x. Notice that f_g is not *a priori* specified to be the function $\mathring{f}_g$ for the heat equation, nor has it to be the same function for all generalized heat equations. This in contrast to some recently published work [4] in which symmetry breaking interactions are studied in the context of Schrödinger equations; in [4] the Schrödinger invariance, including the companion function $\mathring{f}_g$, is required to be that of the free particle. A further difference to [4] is that we allow F to depend on u_x as well as on u; on the other hand we require F to have no explicit dependence on the coordinates, a requirement which will be very crucial in the subsequent analysis. Thus a direct comparison between the results of [4] and of the present work is not possible. Our reason to use, as starting point, the heat equation rather than the free particle Schrödinger equation is simply that some of the resulting equations (1.1) are then recognized as equations of classical physics, notably of fluid dynamics.

The problem of finding all Schrödinger invariant generalized heat equations is solved in Section 2 and Appendix B. The result is that there are four classes of equations, with the functions F given by

$$\begin{aligned}
&\textit{Class 1:}\ F = k + cu_x, \\
&\textit{Class 2:}\ F = k + bu_x + cu_x^2, && (c \neq 0) \\
&\textit{Class 3:}\ F = b(u + q) + cu_x + \kappa(1 - \lambda)\frac{u_x^2}{u + q}, && (\lambda \neq 0) \\
&\textit{Class 4:}\ F = bu_x + cuu_x, && (c \neq 0)
\end{aligned} \tag{1.3}$$

where k, b, c, q, λ are arbitrary constants. All of these equations are more or less frequently used in fluid dynamics and we just quote some examples. The one-dimensional Navier–Stokes equation with homogeneous pressure is either an equation of Class 4 with u the velocity or an equation of Class 2 with u the velocity potential. An equation of Class 2 may be used to describe the burning of gas in a rocket [5, 6]. The Class 4 equation with $b = 0$ is well known in the literature as Burgers' equation [6, 7, 8]; it has been used time and again and is very popular as a model equation because it is the simplest equation combining dissipation $(-\kappa u_{xx})$ and nonlinear propagation $(u_t + cuu_x)$.

The joint Schrödinger invariance of all equations of the Classes (1.3) naturally leads to the question whether there is any deeper relation between these equations. An affirmative answer is suggested by the existence of the Cole–Hopf transformation [7, 9] which reduces Burgers' equation to the heat equation. A systematic analysis indeed shows that the equations of all four classes can be obtained from the heat equation by a simple transformation of the solutions u; such transformations, to distinguish them from Schrödinger transformations, will be called conversions in the following. In Section 3, all equations of type (1.1) which are obtained from the heat equation by conversions not involving derivatives are determined, and it is shown that those for which the Schrödinger invariance inherited from the heat equation has the form (1.2) are exactly the equations of the Classes 1–3. The equations of Class 4 are related to the heat equation through conversions involving the first derivative u_x.

In Section 4 the conversion formulas are used to calculate the companion functions f_g and h_g for each of the four classes. We then discuss the nature of the Schrödinger action (1.2) and show that T, in general, is a projective realization of the Schrödinger group; in some cases, T is a true realization or is a representation. The realizations T commute with the conversions from the heat equation. The conversion formulas, the functions f_g and h_g, and the factor systems of the projective realizations are collected at the end of Section 4.

Up to now we have restricted Schrödinger invariance to linear actions of the form (1.2). However, since already non-homogeneity has been allowed to enter (1.2) through the term h_g, there is no reason why more general actions should not be considered; in fact, the conversion analysis of Section 3 indicates that there definitely are generalized heat equations beyond (1.3) which are Schrödinger invariant only if nonlinear actions are allowed. This problem is taken up in Section 5 where all Schrödinger invariant generalized heat equations are determined for general action. It is shown that they can be obtained from equations (1.3) by a simple transformation of the variable u and thus are again convertible from the heat equation. The technical details of this section are contained in Appendix C.

The purpose of the remaining two sections is to illustrate possible applications of Schrödinger invariance to partial differential equations. To fix ideas we begin by quoting W. F. Ames [6]: 'While there is no existing general theory for nonlinear partial differential equations many special cases have yielded to appropriate changes of variables. In fact transformations are perhaps the most powerful general analytic tool currently available in this area'. These transformations, Ames continues, are mainly of two types: (i) transformations of the dependent variables which reduce the equation to a linear equation, (ii) transformations of the independent variables which reduce the equation to an ordinary differential equation. In the present context the transformations of the first type are what was called conversions in Section 3; they reduce the generalized heat equations to the well known, linear heat equation. Examples for transformations of the second type are provided by the similarity method [10, 11] where solutions invariant under a one-parameter group of transformations are expressed as functions of a similarity variable. The method is discussed in Section 6 for general one-parameter subgroups of the Schrödinger group and examples are given for selected subgroups. Finally, a third type of transformation is considered in Section 7 where we illustrate how the Schrödinger invariance of the heat equation can be used directly to find new solutions from old by applying arbitrary Schrödinger transformations to the trivial constant solutions. The set of solutions obtained in this way is then shown to be isomorphic, as a G-space, to the coordinate space.

2. The generalized heat equations

Our goal is to find all functions $F(u, u_x)$ for which the generalized heat equations (1.1) are invariant under the Schrödinger group, the Schrödinger action having the linear form (1.2). Taking (1.2) at the point $(t', x') = g(t, x)$ we may formulate the invariance condition as

$$\partial_{t'}(fu + h) - \kappa\, \partial_{x'x'}(fu + h) + F[fu + h, \partial_{x'}(fu + h)] = 0, \tag{2.1}$$

where the subscripts g on f and h have been dropped. The derivations $\partial_{t'}$, $\partial_{x'}$, for the Schrödinger transformation (A.2), are given by

$$\partial_{t'} = d^2\,\partial_t + d(\gamma x + \gamma a - \delta v)\,\partial_x, \qquad \partial_{x'} = d\,\partial_x. \tag{2.2}$$

Inserting (2.2) into (2.1) and replacing, for the solution u of (1.1), u_t by $\kappa u_{xx} - F(u, u_x)$ we obtain the invariance condition

$$d^2(f_t + h_t) + d(\gamma x + \gamma a - \delta v)(fu_x + f_x u + h_x) - \kappa\, d^2(2f_x u_x + f_{xx}u + h_{xx})$$
$$-d^2 fF(u, u_x) + F[fu + h, d(fu_x + f_x u + h_x)] = 0. \tag{2.3}$$

It must hold for all elements of the Schrödinger group and for all functions u.

Rather than attempt to solve the invariance condition in its global form (2.3), we solve it for infinitesimal Schrödinger transformations. To find the infinitesimal form of (2.3), we write the generic infinitesimal element of the Schrödinger group and the corresponding companion functions as

$$g = \left(\begin{pmatrix} 1 + \varepsilon\sigma & \varepsilon\beta \\ \varepsilon\gamma & 1 - \varepsilon\sigma \end{pmatrix}, \varepsilon a, \varepsilon v\right), \qquad f_g(t, x) = 1 + \varepsilon\varphi(t, x),$$
$$h_g(t, x) = \varepsilon\chi(t, x). \tag{2.4}$$

Condition (2.3) then takes its infinitesimal form

$$(\varphi u + \chi)F_u + [(\varphi + \gamma t - \sigma)u_x + \varphi_x u + \chi_x]F_{u_x} = (\varphi + 2\gamma t - 2\sigma)F$$
$$+ (2\kappa\varphi_x - \gamma x + v)u_x + (\kappa\varphi_{xx} - \varphi_t)u + \kappa\chi_{xx} - \chi_t, \tag{2.5}$$

where $F_u = \partial F/\partial u$, $F_{u_x} = \partial F/\partial_{u_x}$. Condition (2.5) is a quasilinear first-order partial differential equation for F and it can be solved by standard methods [12] even though its coefficients contain the unknown functions φ and χ. However, the general solution F of (2.5) will usually depend on (t, x) and we then have to impose the requirement that F should not explicitly depend on (t, x), i.e. $F_t = F_x = 0$ at constant u and u_x.

The analysis of (2.5) entails a cumbersome enumeration of various cases and is therefore deferred to an appendix (B); the result are the four classes (1.3) of solutions F. Inserting these functions, F, into the global invariance condition (2.3) we may determine the companion functions, f_g and h_g, either directly or through differential equations; they are given for each class at the end of Section 4 where they will be derived by a simpler method. As a final check it is straightforward to verify that the quantities F, f_g, h_g satisfy condition (2.3). The only reason for solving (2.5) is to show that the set of functions F in (1.3) is exhaustive.

3. Conversion from the heat equation

In this section it is shown that all generalized heat equations of the classes (1.3) are conversions from the heat equation itself, i.e. they are obtained from the heat equation by a transformation of the dependent variable u. The following *definition* is used: the equation

$$u_t - \kappa u_{xx} + F(u, u_x) = 0 \tag{3.1}$$

is said to be convertible from the heat equation if there exists a function $\Theta(t, x; w, w_x)$ such that $u = \Theta$ is a solution of (3.1) whenever w is a solution of the heat equation; the conversion is called zero order or first order according to the occurrence of w_x in Θ.

We begin with zero order conversions and we want to find all equations of the form (3.1) which are zero order convertible from the heat equation. Assuming Θ to be invertible with respect to w we write $w = \Lambda(t, x; u)$ and applying the heat equation to w we obtain equation (3.1) with

$$F = (\Lambda_t - \kappa\Lambda_{xx} - 2\kappa u_x\Lambda_{xu} - \kappa u_x^2\Lambda_{uu})/\Lambda_u, \tag{3.2}$$

where Λ_t, Λ_x denote the derivatives with respect to the explicit (t, x)-dependence. For arbitrary Λ the function F of (3.2) will in general explicitly depend on (t, x) and the next requirement is thus $F_t = F_x = 0$, i.e.

$$\Lambda_u(\Lambda_{tt} - \kappa\Lambda_{txx} - 2\kappa u_x\Lambda_{txu} - \kappa u_x^2\Lambda_{tuu}) = \Lambda_{tu}(\Lambda_t - \kappa\Lambda_{xx} - 2\kappa u_x\Lambda_{xu} - \kappa u_x^2\Lambda_{uu}) \tag{3.3}$$

and a similar condition for the x-derivative. Since Λ is independent of u_x we may compare the coefficients of different powers in u_x and extract the conditions

$$\begin{aligned} &\Lambda_u\Lambda_{tuu} = \Lambda_{tu}\Lambda_{uu}, && \Lambda_u\Lambda_{xuu} = \Lambda_{xu}\Lambda_{uu}, \\ &\Lambda_u\Lambda_{txu} = \Lambda_{tu}\Lambda_{xu}, && \Lambda_u\Lambda_{xxu} = \Lambda_{xu}^2, \\ &\Lambda_u(\Lambda_{tt} - \kappa\Lambda_{txx}) = \Lambda_{tu}(\Lambda_t - \kappa\Lambda_{xx}), && \Lambda_u(\Lambda_{tx} - \kappa\Lambda_{xxx}) = \Lambda_{xu}(\Lambda_t - \kappa\Lambda_{xx}). \end{aligned} \tag{3.4}$$

The first pair of these conditions implies that Λ is of the form

$$\Lambda = B(t, x) + C(t, x)S(u), \tag{3.5}$$

where S is an arbitrary nonconstant function, and the remaining conditions (3.4) are then differential equations for the functions B and C:

$$\begin{aligned} &CC_{tt} = C_t^2, \qquad CC_{xx} = C_x^2, \qquad CC_{tx} = C_tC_x, \\ &C(B_{tt} - \kappa B_{txx}) = C_t(B_t - \kappa B_{xx}), \qquad C(B_{tx} - \kappa B_{xxx}) = C_x(B_t - \kappa B_{xx}). \end{aligned} \tag{3.6}$$

They are solved by

$$C = k_1\, e^{k_2t + k_3x}, \qquad B_t - \kappa B_{xx} = cC, \tag{3.7}$$

where $k_1 \neq 0$, k_2, k_3, c are arbitrary constants. Using a slightly different notation for the constants of integration we may summarize the result as follows:

Lemma. *All equations* (3.1) *which are zero order converts from the heat equation* $w_t - \kappa w_{xx} = 0$ *are given by*

$$F(u, u_x) = Q(u) + bu_x + R(u)u_x^2, \tag{3.8}$$

$$w = B(t, x) + C(t, x)S(u), \qquad u = S^{-1}\left(\frac{w - B}{C}\right), \tag{3.9}$$

where

$$Q(u) = (c + aS)/S', \qquad R(u) = -\kappa S''/S',$$
$$C(t, x) = k \exp\left[\left(a + \frac{b^2}{4\kappa}\right)t - \frac{b}{2\kappa}x\right], \tag{3.10}$$

$S(u)$ is an arbitrary function with $S' \neq 0$, $B(t, x)$ is any function satisfying

$$B_t - \kappa B_{xx} = cC, \tag{3.11}$$

and a, b, c, $k \neq 0$ are arbitrary constants.

We next investigate how the Schrödinger invariance of the heat equation is converted into the corresponding invariance of the equation (3.1). For any function u, given in terms of a solution w of the heat equation by $u = \Theta(t, x; w)$, we may define a Schrödinger transformed function $T_g u$ by

$$(T_g u)(t, x) = \Theta[t, x; (T_g w)(t, x)], \tag{3.12}$$

and if u satisfies (3.1) then so will $T_g u$. For the conversion Θ given by (3.9) the Schrödinger action (3.12) can be written as

$$(T_g u)[g(t, x)] = S^{-1}[\tilde{f}_g(t, x)S(u(t, x)) + \tilde{h}_g(t, x)], \tag{3.13}$$

where $\tilde{f}_g$ and $\tilde{h}_g$ are defined by

$$\tilde{f}_g(t, x) = \frac{C(t, x)}{C[g(t, x)]}\underset{\circ}{f}_g(t, x), \qquad \tilde{h}_g(t, x) = \frac{\underset{\circ}{f}_g(t, x)B(t, x) - B[g(t, x)]}{C[g(t, x)]}, \tag{3.14}$$

and $\underset{\circ}{f}_g$ is the companion function (A.6) of the heat equation. In (1.2) the Schrödinger action T_g was restricted to be linear and we now ask which of the functions $F(u, u_x)$ of the lemma will admit of the linear action (1.2). The condition clearly is that there exist functions f_g, h_g such that

$$\tilde{f}_g S(u) + \tilde{h}_g = S(f_g u + h_g) \tag{3.15}$$

for all Schrödinger transformations g. It is sufficient to consider boosts; deriving (3.15) with respect to boosts v at $g = \mathrm{e}$ we obtain the differential equation

$$a_1 S(u) + a_2 = (b_1 u + b_2)S'(u), \tag{3.16}$$

where $a_1 = \partial\tilde{f}_g/\partial v$, $a_2 = \partial\tilde{h}_g/\partial v$, $b_1 = \partial f_g/\partial v$, $b_2 = \partial h_g/\partial v$, at $g = e$. The solutions of (3.16) are different according as b_1 does or does not vanish. If $b_1 \neq 0$ then S is of the form

$$S = (u + q)^\lambda \quad (\lambda \neq 0), \tag{3.17}$$

where some constants have been absorbed in the constants a and c of (3.10). For $\lambda = 1$ this S leads to an equation of Class 1 or 3. For $\lambda \neq 1$ insertion of (3.17) into (3.15) gives the conditions

$$f_g = \tilde{f}_g^{1/\lambda}, \qquad h_g = q(f_g - 1), \qquad \tilde{h}_g = 0. \tag{3.18}$$

(3.14) then implies

$$B[g(t, x)] = \underset{\circ}{f}_g(t, x)B(t, x), \tag{3.19}$$

and this condition can be satisfied for all g only if $B = 0$. Hence the constant c in

(3.11) vanishes and (3.8) leads to an equation of Class 3. If $b_1 = 0$ the function S is of the form

$$S = e^{\lambda u} \quad (\lambda \neq 0), \tag{3.20}$$

and similar considerations as in the previous case lead to equations of Class 2. We may thus summarize our analysis of zero order conversions as follows:

Theorem. *The equations* (3.1) *which are zero order convertible from the heat equation and for which the Schrödinger action is of the linear form* (1.2) *are exactly the generalized heat equations of the Classes* 1–3 *of* (1.3).

The equations (3.1, 3.8) with the nonlinear Schrödinger action (3.13) will be further discussed in Section 5 and we now turn to first-order conversions. It is possible to determine all equations (3.1) which are first-order convertible from the heat equation but, since $u = \Theta(t, x; w, w_x)$ is no longer algebraically invertible, the analysis is more complicated than in the zero order case and we confine ourselves to give the conversions for the equations of Class 4. It is easily seen that the functions

$$u = -\frac{2\kappa}{c}\frac{w_x}{w} - \frac{b}{c} \tag{3.21}$$

satisfy the equation

$$u_t - \kappa u_{xx} + bu_x + cuu_x = 0 \tag{3.22}$$

for all solutions w of the heat equation. The generalized heat equations of Class 4 are thus first-order converts of the heat equation. For $b = 0$ (3.22) is Burgers' equation and the conversion (3.21) is well known under the name of Cole–Hopf transformation [7, 9]. The Schrödinger action is defined by

$$T_g u = -\frac{2\kappa}{c}\frac{(T_g w)_x}{T_g w} - \frac{b}{c} \tag{3.23}$$

and is of the linear form (1.2).

4. The Schrödinger action

In Appendix A we have seen that the Schrödinger actions for the heat equation form a projective representation of the Schrödinger group on the vector space of solutions. The projective representation cannot be lifted to a true representation since lifting is already impossible for the Galilei subgroup [13]. Most of the generalized heat equations (1.3) are nonlinear, or nonhomogeneous, and their solutions do not form a linear space, thus T will no longer be a representation of the Schrödinger group. However, T_g still sends solutions into solutions hence the set of all solutions is a G-space and T may be called a realization of the Schrödinger group on this G-space. Again the realization is projective, i.e. it satisfies

$$(T_{g_2}(T_{g_1}u))(t, x) = \omega(g_2, g_1)(T_{g_2g_1}u)(t, x) + \vartheta(g_2, g_1; t, x), \tag{4.1}$$

with ω and ϑ given below.

The summary at the end of this section contains, for each class separately, (i) the conversion formula, i.e. the relation between solutions w of the heat equation and solutions u of the generalized heat equations, (ii) the companion functions f_g and h_g, and (iii) the quantities ω and ϑ characterizing the projective realization (4.1). The functions f_g and h_g are determined by conversion from $\overset{\circ}{f}_g$ of (A.6), which means that the slight degree of freedom left by the invariance condition (2.3) is thereby fixed. This determination of f_g and h_g has the advantage that conversion ($w \to u$, $T_g w \to T_g u$) and Schrödinger action ($w \to T_g w$, $u \to T_g u$) commute. The quantities ω and ϑ can be calculated either by conversion or directly from the given f_g and h_g by using (A.7, A.8, A.9), and the respective results agree; $\zeta(g_2, g_1)$ is defined in (A.10).

Notice that $\omega = 1$ and $\vartheta = 0$ for the equations of Class 4, hence the map T is a homomorphism of the Schrödinger group and thus a true realization. If the generalized heat equation is linear and homogeneous (Class 1 for $k = 0$ and Class 3 for $\lambda = 1$, $q = 0$), then $h_g = \vartheta = 0$ and T is a (projective) representation on the vector space of solutions.

Summary: Schrödinger symmetry of generalized heat equations

Class 1: $F = k + cu_x$

$$u = \exp\left[-\frac{c^2}{4\kappa}t + \frac{c}{2\kappa}x\right]w - kt,$$

$$f_g(t,x) = (\gamma t + \delta)^{1/2}\exp\left\{\frac{1}{4\kappa}A_g(t,x) + \frac{c^2}{4\kappa}[t - g(t)] - \frac{c}{2\kappa}[x - g(x)]\right\}, \tag{4.2}$$

$$h_g(t,x) = k[tf_g(t,x) - g(t)],$$

$$\omega(g_2, g_1) = \exp\left[\frac{1}{4\kappa}\zeta(g_2, g_1)\right], \qquad \vartheta(g_2, g_1; t, x) = kt[\omega(g_2, g_1) - 1].$$

Class 2: $F = k + bu_x + cu_x^2 \quad (c \neq 0)$

$$u = -\frac{\kappa}{c}\ln w - \left(k - \frac{b^2}{4c}\right)t - \frac{b}{2c}x, \qquad f_g(t,x) = 1,$$

$$h_g(t,x) = -\frac{\kappa}{2c}\ln(\gamma t + \delta) - \frac{1}{4c}A_g(t,x) + \left(k - \frac{b^2}{4c}\right)[t - g(t)] + \frac{b}{2c}[x - g(x)],$$

$$\omega(g_2, g_1) = 1, \qquad \vartheta(g_2, g_1; t, x) = -\frac{1}{4c}\zeta(g_2, g_1). \tag{4.3}$$

Class 3: $F = b(u + q) + cu_x + \kappa(1 - \lambda)u_x^2/(u + q) \quad (\lambda \neq 0)$

$$u = \exp\left[-\left(b + \frac{c^2}{4\kappa\lambda}\right)t + \frac{c}{2\kappa\lambda}x\right]w^{1/\lambda} - q,$$

$$f_g(t,x) = (\gamma t + \delta)^{1/2\lambda}\exp\left\{\frac{1}{4\kappa\lambda}A_g(t,x) + \left(b + \frac{c^2}{4\kappa\lambda}\right)[t - g(t)] - \frac{c}{2\kappa\lambda}[x - g(x)]\right\},$$

$$h_g(t,x) = q[f_g(t,x) - 1],$$

$$\omega(g_2, g_1) = \exp\left[\frac{1}{4\kappa\lambda}\zeta(g_2, g_1)\right],$$

$$\vartheta(g_2, g_1; t, x) = q[\omega(g_2, g_1) - 1]. \tag{4.4}$$

Class 4: $F = bu_x + cuu_x \quad (c \neq 0)$

$$u = -\frac{2\kappa}{c}\frac{w_x}{w} - \frac{b}{c},$$

$$f_g(t, x) = \gamma t + \delta, \qquad h_g(t, x) = -\frac{1}{c}[\gamma x + \gamma a - \delta v + b - (\gamma t + \delta)b], \tag{4.5}$$

$$\omega(g_2, g_1) = 1, \qquad \vartheta(g_2, g_1; t, x) = 0.$$

5. General Schrödinger actions

Hitherto the Schrödinger actions were restricted to the linear form (1.2) and we now generalize (1.2) by letting the right-hand side be an arbitrary function of g, t, x, and u; more precisely, we allow actions of the form

$$(T_g u)(t, x) = Q[g; g^{-1}(t, x); u(g^{-1}(t, x))], \quad Q_u \neq 0, \tag{5.1}$$

with $Q = u$ for $g = e$. The problem is to find all functions $F(u, u_x)$ for which the generalized heat equations (1.1) are Schrödinger invariant with the action (5.1).

We first present a simple example of a nonlinear Schrödinger action. Let $\psi_t - \kappa\psi_{xx} + F_0(\psi, \psi_x) = 0$ be a generalized heat equation with F_0 selected from among the functions (1.3) and define $u = S^{-1}(\psi)$ where S is any invertible function. Then u satisfies the generalized heat equation

$$u_t - \kappa u_{xx} + F(u, u_x) = 0, \qquad F(u, u_x) = \frac{1}{S'(u)}F_0[S(u), u_x S'(u)] - \kappa\frac{S''(u)}{S'(u)}u_x^2, \tag{5.2}$$

and the equation is Schrödinger invariant with the action

$$\begin{aligned}(T_g u)(t, x) &\equiv S^{-1}[(T_g\psi)(t, x)] \\ &= S^{-1}[f_g(g^{-1}(t, x))S(u(g^{-1}(t, x))) + h_g(g^{-1}(t, x))],\end{aligned} \tag{5.3}$$

where f_g and h_g are the companion functions belonging to F_0. An action of this type has already been encountered in (3.13) for zero order converts of the heat equation and, in fact, all zero order converts of the lemma in Section 3 are contained in (5.2) if F_0 runs through Classes 1–3 of (1.3). If F_0 in (5.2) is of Class 4 then

$$F(u, u_x) = [b + cS(u)]u_x - \kappa\frac{S''(u)}{S'(u)}u_x^2, \tag{5.4}$$

and there exists the first-order conversion

$$u = S^{-1}\left[-\frac{2\kappa}{c}\frac{w_x}{w} - \frac{b}{c}\right]. \tag{5.5}$$

Thus all generalized heat equations (5.2) are convertible from the heat equation.

Returning to the general problem we proceed exactly as in Section 2 and obtain the invariance condition

$$d^2Q_t + d(\gamma x + \gamma a - \delta v)(u_x Q_u + Q_x) - \kappa d^2(u_x^2 Q_{uu} + 2u_x Q_{xu} + Q_{xx}) - d^2 F(u, u_x)Q_u + F[Q, d(u_x Q_u + Q_x)] = 0, \quad (5.6)$$

where $Q = Q[g; t, x; u(t, x)]$; (5.6) is the generalization of (2.3). The analysis of (5.6) is rather long-winded and is sketched in Appendix C. The result is that equations (5.2), with F_0 running through all classes of (1.3), already exhaustively describe all Schrödinger invariant generalized heat equations with the general action (5.1). The actions necessarily have the form (5.3) and the equations are all connected to the equations of the Classes (1.3) through the transformation $u = S^{-1}(\psi)$ and are thus convertible from the heat equation.

6. The similarity method

The similarity method has its origin in hydrodynamics [10] where, say, the Navier–Stokes equation is invariant under an appropriate scaling of the involved quantities. The method is by no means restricted to scale transformations and in the present section we apply it to the one-parameter subgroups of the Schrödinger group. Our approach is through the concept of invariant solutions.

Let a one-parameter subgroup H of the Schrödinger group be parametrized by the additive parameter s. We replace the coordinates (t, x) by variables (η, ρ) which are better adapted to H in that η is invariant under H and ρ has a simple transformation behaviour,

$$s\colon (\eta, \rho) \to g_s(\eta, \rho) = (\eta, \rho + s) \quad (\forall s \in H). \quad (6.1)$$

η is called an absolute invariant or a similarity variable and its importance lies in the fact that any function satisfying $S[g_s(t, x)] = S(t, x)$ for all $s \in H$ has the general form $S = S(\eta)$. The variable ρ, though not necessary for the similarity method, will be convenient later. It is usually not difficult to find a pair (η, ρ) for given H and some examples may be found below (6.9ff).

We first consider the heat equation $w_t - \kappa w_{xx} = 0$. A solution is called invariant, or a similarity solution, with respect to the one-parameter subgroup H if it satisfies the invariance condition

$$(T_s w)(t, x) \equiv f_s[g_s^{-1}(t, x)]w[g_s^{-1}(t, x)] = B(s)w(t, x) \quad (\forall s \in H), \quad (6.2)$$

where f_s is given in (A.6). The constant factor B has to form a one-dimensional projective representation of H,

$$B(s_2)B(s_1) = \omega(s_2, s_1)B(s_2 + s_1), \quad (6.3)$$

where ω is the factor system of the heat equation. It is easy to see that all functions which are invariant in the sense of (6.2) can be written in the form

$$w(t, x) = p(t, x)S(\eta), \quad (6.4)$$

where $p \neq 0$ is any fixed function satisfying (6.2) and S is arbitrary. If w is to be a solution of the heat equation then $S(\eta)$ must be determined from an ordinary differential equation obtained by insertion of (6.4) into the heat equation; thus, for solutions invariant under H, a partial differential equation is reduced to an ordinary differential equation, and this is the very idea behind the similarity method. Our next problem is to find the particular invariant function $p(t, x)$ and we claim that it can be taken as

$$p(t, x) = B^{-1}(\rho - r) f_{\rho - r}(\eta, r), \tag{6.5}$$

where r is any real number and $f_{\rho-r}(\eta, r)$ is understood as $f_s(t, x)$ for $\rho(t, x) = r$ and $s = \rho - r$. The invariance condition (6.2) for p, written in the variables (η, ρ), is

$$f_s(\eta, \rho - s) p(\eta, \rho - s) = B(s) p(\eta, \rho), \tag{6.6}$$

and is easily seen to be satisfied for (6.5) if one makes use of (6.3) and the relation

$$f_s(\eta, \rho - s) f_{\rho - s - r}(\eta, r) = \omega(s, \rho - s - r) f_{\rho - r}(\eta, r) \tag{6.7}$$

obtained from (A.9). Once η and p are chosen for a given one-parameter subgroup H all invariant solutions can be found in the form (6.4).

In the following, we list the results of the similarity method as applied to the five most conspicuous one-parameter subgroups. In all five cases $\omega = 1$ and thus the (true) representation B has the form

$$B(s) = e^{\mu s}, \tag{6.8}$$

where μ is some number. For each of the subgroups we give the one-parameter transformations $g_s(t, x)$, a choice of the variables (η, ρ), the invariant solutions $w = p(t, x) S(\eta)$ with p taken from (6.5) but sometimes modified by a function of η, and the ordinary differential equation for $S(\eta)$.

Time-translations:

$$g_s(t, x) = (t + s, x), \qquad (\eta, \rho) = (x, t),$$
$$w = e^{-\mu t} S(\eta), \qquad \kappa S'' + \mu S = 0. \tag{6.9}$$

Space-translations:

$$g_s(t, x) = (t, x + s), \qquad (\eta, \rho) = (t, x),$$
$$w = e^{-\mu x} S(\eta), \qquad S' - \kappa \mu^2 S = 0. \tag{6.10}$$

Boosts:

$$g_s(t, x) = (t, x + st), \qquad (\eta, \rho) = (t, t^{-1} x),$$
$$w = \exp\left[-\frac{(x + 2\mu\kappa)^2}{4\kappa t}\right] S(\eta), \qquad 2\eta S' + S = 0. \tag{6.11}$$

Dilations:

$$g_s(t, x) = (e^{2s} t, e^s x), \qquad (\eta, \rho) = (t^{-1/2} x, \tfrac{1}{2} \ln t),$$
$$w = t^{-(2\mu + 1)/4} S(\eta), \qquad 4\kappa S'' + 2\eta S' + (2\mu + 1) S = 0. \tag{6.12}$$

'*Expansions*':

$$g_s(t, x) = \left(\frac{t}{1 + st}, \frac{x}{1 + st}\right), \qquad (\eta, \rho) = (t^{-1}x, t^{-1}),$$

$$w = t^{-1/2} \exp\left[-\frac{(x - v)^2}{4\kappa t}\right] S(\eta), \qquad \kappa S'' + vS' = 0 \quad (v^2 \equiv 4\mu\kappa). \tag{6.13}$$

Invariant solutions for the generalized heat equations (1.3) can be obtained by conversion from the heat equation but such solutions are generally no longer invariant in the sense of (6.2). Rather they satisfy an invariance condition of the form

$$(T_s u)(t, x) = B(s)u(t, x) + C(s; tx) \quad (\forall s \in H), \tag{6.14}$$

where B and C, as calculated by conversion, are restricted by the relations

$$\begin{aligned} &\text{Class 1:} \quad C = kt(B - 1),\\ &\text{Class 2:} \quad B = 1, C = \text{constant},\\ &\text{Class 3:} \quad C = q(B - 1),\\ &\text{Class 4:} \quad B = 1, C = 0. \end{aligned} \tag{6.15}$$

These quantities are compatible with the requirement that the function $T_s u$ of (6.14) is, together with u, a solution of the corresponding generalized heat equation. Notice that Class 4 is particularly interesting because $B = 1$, $C = 0$ and thus all invariant solutions are automatically absolutely invariant.

7. The orbit of trivial solutions

In this section we want to demonstrate the direct use of Schrödinger symmetry in the search for solutions of a Schrödinger invariant equation. We mainly work with the heat equation $w_t - \kappa w_{xx} = 0$. Applying the Schrödinger transformation (A.5) to the trivial solutions $w = K$ = constant we obtain the transformed solutions

$$(T_g K)(t, x) = K\sqrt{-\frac{\kappa}{\gamma}} \exp\left[\frac{1}{4\kappa}\left(av - \frac{\delta}{\gamma} v^2\right)\right] w_{\tau,\xi}(t, x) \quad (\gamma \neq 0), \tag{7.1}$$

where

$$\tau = \frac{\alpha}{\gamma}, \qquad \xi = \frac{v}{\gamma}, \qquad w_{\tau,\xi}(t, x) = [\kappa(t - \tau)]^{-1/2} \exp\left[-\frac{(x - \xi)^2}{4\kappa(t - \tau)}\right]. \tag{7.2}$$

If $\gamma = 0$ (and thus $\alpha \neq 0$) we have

$$(T_g K)(t, x) = K\sqrt{\frac{1}{\alpha}} \exp\left(\frac{\alpha a - \beta v}{4\kappa\alpha} v\right) w_\mu(t, x) \quad (\gamma = 0), \tag{7.3}$$

where

$$\mu = \frac{v}{2\kappa\alpha}, \qquad w_\mu(t, x) = e^{\kappa\mu^2 t - \mu x}. \tag{7.4}$$

It can be shown that (7.3) is the limit of (7.1) as $\gamma \to 0$; thus in the following the functions (7.3) will be considered as special cases of the functions (7.1). The point to be made here is that by applying a Schrödinger transformation to the trivial solutions we have obtained the solution $w_{\tau,\xi}$ which is probably the single most important solution of the heat equation.

We next apply a Schrödinger transformation to $w_{\tau,\xi}$ itself and obtain, after a straightforward calculation, the result

$$(T_g w_{\tau,\xi})(t, x) = (\gamma\tau + \delta)^{-1} \mathring{f}_g(\tau, \xi) w_{g(\tau,\xi)}, \tag{7.5}$$

where $g(\tau, \xi)$ is the same transformation (A.2) as for the coordinates. Let $\Phi_{\tau,\xi}$ denote the set of all multiples of the solution $w_{\tau,\xi}$ and $\Omega = \{\Phi_{\tau,\xi} \mid \tau, \xi \text{ real}\}$. What was shown above is, (i) that Ω is closed under Schrödinger transformations because the factor $(\gamma\tau + \delta)^{-1}\mathring{f}_g(\tau, \xi)$ in (7.5) is a constant, (ii) that each $\Phi_{\tau,\xi}$ is generated from the trivial solutions by (7.1) and thus Ω is the Schrödinger orbit of the trivial solutions, and (iii) that the set Ω, as a G-space, is isomorphic to the coordinate space.

The preceding analysis can easily be extended to the generalized heat equations by conversion. As an example we only consider the somewhat exceptional case of the Class 4 equations. The analog of $w_{\tau,\xi}$ is

$$u_{\tau,\xi}(t, x) = \frac{1}{c}\frac{x - \xi}{t - \tau} - \frac{b}{c}, \tag{7.6}$$

which is obtained either by conversion from $w_{\tau,\xi}$ or by applying T_g to the trivial solution u = constant. The Schrödinger transformations of $u_{\tau,\xi}$ are given by

$$T_g u_{\tau,\xi} = u_{g(\tau,\xi)} \tag{7.7}$$

without any additional factors. Thus the set of solutions $u_{\tau,\xi}$ is the orbit of the trivial solutions and is isomorphic to coordinate space. The absence of any factors in (7.7) and the isomorphism of coordinate space to the set of solutions $u_{\tau,\xi}$ rather than the set of multiples of some kind are related to the fact, mentioned in Section 4, that the equations of Class 4 permit a true, as opposed to projective, realization of the Schrödinger group.

Appendix A: The Schrödinger group

The elements of the one-dimensional Schrödinger group G are denoted by

$$g = \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, a, v\right), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R}), a \in \mathbb{R}, v \in \mathbb{R}, \tag{A.1}$$

and defined by the coordinate transformations

$$g\colon \quad (t, x) \to g(t, x) = \left(\frac{\alpha t + \beta}{\gamma t + \delta}, \frac{x + vt + a}{\gamma t + \delta}\right). \tag{A.2}$$

The group is a 5-parameter Lie group and may be considered as a generalized Galilei group with the time-translations replaced by the 3-parameter group $SL(2, \mathbb{R})$

of projective transformations. Product and inverse are given by

$$g_3 = g_2 g_1: \quad \begin{pmatrix} \alpha_3 & \beta_3 \\ \gamma_3 & \delta_3 \end{pmatrix} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_2 \end{pmatrix}, \quad \begin{array}{l} a_3 = a_1 + \delta_1 a_2 + \beta_1 v_2, \\ v_3 = v_1 + \alpha_1 v_2 + \gamma_1 a_2, \end{array} \tag{A.3}$$

$$g' = g^{-1}: \quad \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}, \quad \begin{array}{l} a' = -\alpha a + \beta v, \\ v' = -\delta v + \gamma a. \end{array} \tag{A.4}$$

The heat equation $u_t - \kappa u_{xx} = 0$ is invariant under the Schrödinger group in the sense that for each solution u and each $g \in G$ there exists a transformed solution $T_g u$ defined by

$$(T_g u)(t, x) = \mathring{f}_g[g^{-1}(t, x)]u[g^{-1}(t, x)], \tag{A.5}$$

where $\mathring{f}_g$ is given by

$$\mathring{f}_g(t, x) = [d_g(t)]^{1/2} \exp\left[\frac{1}{4\kappa} A_g(t, x)\right], \quad d_g(t) = \gamma t + \delta,$$

$$A_g(t, x) = \gamma \frac{(x + vt + a)^2}{\gamma t + \delta} - 2vx - v(vt + a). \tag{A.6}$$

(The condition that $T_g u$ be a solution determines $\mathring{f}_g$ up to a constant; such a constant has been inserted in the definition (A.6) as compared to the definition of $\mathring{f}_g$ in [1].)

The functions (A.6) satisfy the relations

$$d_{g_2}[g_1(t)]d_{g_1}(t) = d_{g_2 g_1}(t), \tag{A.7}$$

$$A_{g_2}[g_1(t, x)] + A_{g_1}(t, x) = A_{g_2 g_1}(t, x) + \zeta(g_2, g_1), \tag{A.8}$$

$$\mathring{f}_{g_2}[g_1(t, x)]\mathring{f}_{g_1}(t, x) = \exp\left[\frac{1}{4\kappa}\zeta(g_2, g_1)\right]\mathring{f}_{g_2 g_1}(t, x), \tag{A.9}$$

where ζ is given by

$$\zeta(g_2, g_1) = -\alpha_1 a_1 v_2 + \beta_1 v_1 v_2 - \gamma_1 a_1 a_2 + \delta_1 v_1 a_2. \tag{A.10}$$

The quantity $\exp \zeta/4\kappa$ has all the properties of a factor system [13]. The operators T_g themselves satisfy the relation

$$T_{g_2} T_{g_1} = \exp\left[\frac{1}{4\kappa}\zeta(g_2, g_1)\right] T_{g_2 g_1}, \tag{A.11}$$

and thus they constitute a projective representation of the Schrödinger group.

Appendix B: The differential equation for F

We solve the differential equation (2.5) for the function $F(u, u_x)$ with $F_t = F_x = 0$. Since the functions $\varphi(t, x)$ and $\chi(t, x)$ are not known we have to distinguish several cases. Throughout this appendix the notation $y = u$, $z = u_x$ is used.

Case 1: $\varphi \neq 0$.

Equation (2.5) can be written as

$$(y + q)F_y + [A(y + q) + Bz + C]F_z = DF + E(y + q) + Gz + H, \tag{B.1}$$

where

$$q = \frac{\chi}{\varphi}, \quad p = \frac{\gamma t - \sigma}{\varphi}, \quad A = \frac{\varphi_x}{\varphi}, \quad B = 1 + p, \quad C = q_x,$$

$$D = 1 + 2p, \quad E = \frac{\kappa\varphi_{xx} - \varphi_t}{\varphi}, \quad G = \frac{2\kappa\varphi_x - \gamma x + v}{\varphi},$$

$$H = \kappa q_{xx} - q_t + 2\kappa q_x \frac{\varphi_x}{\varphi}. \tag{B.2}$$

The characteristic equations of the differential equation (B.1), with parameter s, are

$$\frac{dy}{ds} = y + q, \quad \frac{dz}{ds} = Bz + A(y + q) + C,$$

$$\frac{dF}{ds} = DF + E(y + q) + Gz + H, \tag{B.3}$$

the first of which is solved by

$$\eta \equiv y + q = e^s. \tag{B.4}$$

The solution of the characteristic equation for z is different according as B does or does not vanish.

Case 1.1: $\varphi \neq -\gamma t + \sigma$.
In this case $B \neq 0$ and the second equation (B.3) is solved by

$$z = k_1\, e^{Bs} - \frac{A}{p} e^s - \frac{C}{B}, \tag{B.5}$$

where k_1 is the constant of integration. A first integral, i.e. a function constant along characteristic curves, is therefore given by

$$\tau = \eta^{-B}\left(z + \frac{A}{p}\eta + \frac{C}{B}\right). \tag{B.6}$$

To solve the third characteristic equation, we again have to distinguish between $D \neq 0$ and $D = 0$.

Case 1.1.1: $\varphi \neq -2(\gamma t - \sigma)$.
The characteristic equation for F is solved by

$$F = k_2\eta^D - \frac{AG + pE}{2p^2}\eta - \frac{G}{p}z - \frac{CG}{pD} - \frac{H}{D}. \tag{B.7}$$

Thus k_2 is another first integral and the general solution of (B.1) is

$$F = \eta^D\Phi(\tau) - \frac{AG + pE}{2p^2}\eta - \frac{G}{p}z - \frac{CG}{pD} - \frac{H}{D}$$

$$= \eta^D\Phi(\tau) + \frac{AG - pE}{2p^2}\eta - \frac{G}{p}\eta^B\tau + \frac{CG - BH}{BD}, \tag{B.8}$$

where Φ is an arbitrary function. Of course, once this solution is known we may easily verify it by replacing (y, z) by the new variables (η, τ), using (B.8) with $\Phi = \Phi(\eta, \tau)$ as an Ansatz, and inserting (B.8) into (B.1) to obtain $\Phi_\eta = 0$.

The solutions (B.8) will in general, through their coefficients, depend on (t, x) and we have to impose the further conditions $F_t = F_x = 0$. Thus

$$\begin{aligned} 0 = F_x = &\left(D_x \ln \eta + D\frac{q_x}{\eta}\right)\eta^D\Phi + \eta^D\Phi_x + \eta^D\tau_x\Phi_\tau + \left(\frac{AG - pE}{2p^2}\right)_x \eta \\ &+ \frac{AG - pE}{2p^2} q_x - \left(\frac{G}{p}\right)_x \eta^B\tau - \frac{G}{p}\left[\left(\frac{A}{p}\right)_x \eta + \frac{A}{p} q_x + \left(\frac{C}{B}\right)_x\right] \\ &+ \left(\frac{CG - BH}{BD}\right)_x, \end{aligned} \tag{B.9}$$

$$\tau_x = -\left(B_x \ln \eta + B\frac{q_x}{\eta}\right)\tau + \eta^{-B}\left[\left(\frac{A}{p}\right)_x \eta + \frac{A}{p} q_x + \left(\frac{C}{B}\right)_x\right],$$

and similarly for $F_t = 0$. Condition (B.9) must hold for any values of (η, τ); in particular, the coefficient of $\ln \eta$ must vanish, hence

$$p_t(\tau\Phi_\tau - 2\Phi) = p_x(\tau\Phi_\tau - 2\Phi) = 0, \tag{B.10}$$

and we have to distinguish as to whether p is constant or not.

Case 1.1.1.1: $p \neq$ constant.

The differential equation (B.10) has the general solution

$$\Phi = a(t, x)\tau^2. \tag{B.11}$$

Inserting (B.11) into (B.9) and collecting the terms quadratic in τ we obtain the conditions

$$a = \text{constant}, \qquad aq_x = aq_t = 0. \tag{B.12}$$

For $a = 0$ we have

$$F = -\frac{CG}{pD} - \frac{H}{D} - \frac{q}{2p^2}(AG + pE) - \frac{AG + pE}{2p^2} y - \frac{G}{p} z. \tag{B.13}$$

If $a \neq 0$ then q is constant, $C = H = 0$, and

$$F = \frac{(2aA - G)A - pE}{2p^2}(y + q) + \frac{2aA - G}{p} z + a\frac{z^2}{y + q}. \tag{B.14}$$

The remainder of (B.9) is the condition that the coefficients in (B.13) and (B.14) are constants. In particular, the condition that $(2aA - G)/p$ is constant implies $(a - \kappa)\varphi_{xx} + \gamma = 0$ and thus $a \neq \kappa$. Since we are not so much interested in the auxiliary functions φ and χ we end the analysis here. The two solutions (B.13) and (B.14), with $a \neq \kappa$, cover Classes 1 and 3 of (1.3).

Case 1.1.1.2: $p =$ constant ($\neq 0$).

Condition (B.9) now reads

$$\eta^{2p+1}\Phi_x + q_x\eta^{2p}(D\Phi - B\tau\Phi_\tau) + \frac{C_x}{B}\eta^p\Phi_\tau - \frac{G_x}{p}\eta^{p+1}\tau - \frac{E_x}{2p}\eta$$
$$- \frac{E}{2p}q_x - \frac{C_xG}{pB} + \frac{(CG)_x}{BD} - \frac{H_x}{D} = 0. \quad \text{(B.15)}$$

Comparing the various powers of η one finally obtains the general solution

$$F = bz + cyz \quad (c \neq 0), \quad \text{(B.16)}$$

where b and c are constants. This is Class 4 of (1.3).

Case 1.1.2: $\varphi = -2(\gamma t - \sigma)$.

With $D = 0$ the characteristic equation for F is easily solved and the general solution of (B.1) is

$$F = \Phi(\tau) + 2G\eta^{1/2}\tau + E\eta + (H\text{-}2CG)\ln\eta,$$
$$\tau = \eta^{-1/2}(z + 2C). \quad \text{(B.17)}$$

Since $E = -\gamma/(\gamma t - \sigma)$ neither vanishes nor is t-independent, the function F cannot satisfy $F_t = 0$ and we conclude that the present case has no solution.

Case 1.2: $\varphi = -\gamma t + \sigma$.

The characteristic equations for z and F are

$$\frac{dz}{ds} = C, \qquad \frac{dF}{ds} = -F + E\eta + Gz + H, \quad \text{(B.18)}$$

and the general solution of (B.1) is

$$F = \eta^{-1}\Phi(\tau) + G\tau + \tfrac{1}{2}E\eta + CG\ln\eta + H - CG,$$
$$\tau = z - C\ln\eta. \quad \text{(B.19)}$$

As in the previous case, we conclude that there is no t-independent solution F.

Case 2: $\varphi = 0, \chi \neq 0$

The differential equation (2.5) can be written as

$$F_y + (Az + B)F_z = 2AF + Cz + D, \quad \text{(B.20)}$$

where

$$A = \frac{\gamma t - \sigma}{\chi}, \qquad B = \frac{\chi_x}{\chi}, \qquad C = \frac{-\gamma x + v}{\chi}, \qquad D = \frac{\kappa\chi_{xx} - \chi_t}{\chi}. \quad \text{(B.21)}$$

The general solution has the form

$$F = e^{2Ay}\Phi(\tau) - \frac{C}{A}z - \frac{AD + BC}{2A^2}, \qquad \tau = e^{-Ay}\left(z + \frac{B}{A}\right). \quad \text{(B.22)}$$

The requirement $F_t = F_x = 0$ then leads, after a short analysis, to the solutions

$$F = k + bz + cz^2 \quad (c \neq 0), \quad \text{(B.23)}$$

which cover Class 2 of (1.3).

Case 3: $\varphi = \chi = 0$.

The differential equation (2.5) is

$$zF_z = 2F - qz, \quad q = -(\gamma x - v)/(\gamma t - \sigma) \tag{B.24}$$

and has no solution for (t, x)-independent F.

Appendix C: Schrödinger invariance with general action

We want to prove that all generalized heat equations which are Schrödinger invariant with the general action (5.1) are of the form (5.2). Applying the invariance condition (5.6) to the infinitesimal Schrödinger transformations (2.4) with the corresponding action

$$Q[g;t,x;u] = u + \varepsilon R[t,x;u] \tag{C.1}$$

we obtain the infinitesimal invariance condition

$$RF_u + [(R_u + \gamma t - \sigma)u_x + R_x]F_{u_x} = (R_u + 2\gamma t - 2\sigma)F + \kappa(u_x^2 R_{uu} + 2u_x R_{xu} + R_{xx}) - (\gamma x - v)u_x - R_t, \tag{C.2}$$

which is the analog of (2.5) and goes into (2.5) for $R = \varphi u + \chi$.

The first step in the proof consists in showing that R must necessarily be of the form

$$R[t,x;u] = [\varphi(t,x)S(u) + \chi(t,x)]/S'(u), \tag{C.3}$$

where the function S does not explicitly depend on (t, x). We frequently use the following lemma: The general solution of the differential equation

$$R_u + P(u)R = \varphi(t,x) \tag{C.4}$$

is (C.3) where S is any function with $S''/S' = P$ and $\chi(t, x)$ is arbitrary. If F is a polynomial in u_x of at most degree two, it is easily seen that the terms of (C.2) proportional to u_x^2 imply a relation of the form (C.4) and thus (C.3) holds. We therefore assume that F is not quadratic in u_x and, by deriving (C.2) three times with respect to u_x, we obtain the differential equation

$$L_u + (A + Bu_x)L_{u_x} + C = 0, \tag{C.5}$$

where $L = \ln \partial^3 F/\partial u_x^3$ and

$$A(t,x;u) = \frac{R_x}{R}, \quad B(t,x;u) = \frac{R_u + \gamma t - \sigma}{R}, \quad C(t,x;u) = \frac{2R_u + \gamma t - \sigma}{R} \tag{C.6}$$

($R = 0$ is excluded by the x-dependence of the term $\gamma x u_x$ in (C.2)). If B is independent of (t, x) the relation (C.4) and thus (C.3) holds, hence we assume that B_t and B_x do not both vanish. (C.5) then implies

$$L_{u_x} = -\frac{\hat{C}}{\hat{A} + \hat{B}u_x}, \tag{C.7}$$

where the caret ($\hat{}$) denotes partial differentiation with respect to one of the variables (t, x) for which $\hat{B} \neq 0$, and where $\hat{L}_{u_x} = 0$ has been used. Inserting (C.7) into (C.5) and invoking the integrability condition $L_{uu_x} = L_{u_xu}$ we find that $\hat{C}/\hat{B}$ is independent of u, thus

$$\hat{C} = \lambda \hat{B}, \tag{C.8}$$

where λ may yet depend on (t, x). The latter, however, is impossible if L_{u_x} in (C.7) is to be (t, x)-independent. Integrating (C.8), we have

$$C = \lambda B + k(u), \tag{C.9}$$

where λ is a constant and k is an arbitrary function. (C.9) is a relation of the form (C.4) and thus R is given by (C.3). This completes the first step of the analysis of (C.2).

The second step consists in showing that the invariance condition (C.2) can be brought into the invariance condition (2.5) by a transformation of u, u_x, F. Indeed, using (C.3) in (C.2) and defining the quantities

$$y = S(u), \qquad z = S'u_x, \qquad F_0 = S'F + \kappa S''u_x^2, \tag{C.10}$$

we obtain the invariance condition

$$\begin{aligned}(\varphi y + \chi)F_{0y} &+ [(\varphi + \gamma t - \sigma)z + \varphi_x y + \chi_x]F_{0z} \\ &= (\varphi + 2\gamma t - 2\sigma)F_0 + (2\kappa\varphi_x - \gamma x + v)z + (\kappa\varphi_{xx} - \varphi_t)y + \kappa\chi_{xx} - \chi_t,\end{aligned} \tag{C.11}$$

which is the condition (2.5) except that (u, u_x, F) in (2.5) are now replaced by (y, z, F_0) of (C.10). Since S is a function of u alone, the transformations (C.10) do not introduce any new explicit (t, x)-dependence, thus the solving of the differential equation (C.11) proceeds exactly as in Appendix B and the solutions are the same.

The outcome of the foregoing analysis is that the most general solutions of condition (C.2) are

$$F(u, u_x) = \frac{1}{S'(u)}\{F_0[S(u), S'(u)u_x] - \kappa S''(u)u_x^2\}, \tag{C.12}$$

where S in an arbitrary function with $S' \neq 0$ and F_0 is any one of the functions (1.3). This concludes the proof.

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