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Relativistic dynamics for the spin $\frac{1}{2}$ particle

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Abstract. We develop a model for the interacting spin $\frac{1}{2}$ particle in the context of Relativistic Dynamics, a theory for which quantum states are given by rays in L^2 ($\mathbb{R}^4$, d^4x) and the evolution is labelled by an invariant parameter. We propose a new definition of spin $\frac{1}{2}$ in Relativity coming directly from the information given by the Stern–Gerlach apparatus. This leads us to introduce a continuous superselection rule and a family of Hilbert spaces isomorphic to $\mathbb{C}^2 \otimes L^2$ ($\mathbb{R}^4$, d^4x). We exhibit a representation for which the state transforms like a Dirac four-spinor and we prove the existence of covariant position and time self-adjoint operators. An explicit equation for the evolution of an interacting particle with external electromagnetic field is proposed. We state the results of this model applied to the hydrogen atom. Finally the Bargmann, Michel and Teledgi equation for the precession of the spin is shown to follow from this evolution equation as a quasi classical approximation.

1. Introduction

In a previous article 'Relativistic Dynamics' [1] we proposed a model for the spinless relativistic particle and a quantization method free from the usual difficulties associated with the Klein–Gordon equation. This article is the continuation of [1]. We formulate here our model for the spin $\frac{1}{2}$ case.

At first let us recall our point of view. Each particle as a classical object is assimilated to a point in the four-dimensional space-time. So in our theory space-time plays the same role as the three-dimensional space in Newton's theory. To be able to describe the motion of this point we introduce a new variable τ which is just a parameter labelling events during the evolution. It is not an observable like the fourth coordinate t, which is an observable associated with a clock. This parameter τ plays the role of the Newtonian time; we therefore call τ the historical time in contrast to t, a geometrical time.

Our quantum model for a spinless particle proposed in [1] is the following.

The states are described by the Hilbert space of the four-dimensional square-integrable functions $\psi(x) = \psi(\mathbf{x}, t)$ with the scalar product

$$\langle\psi, \varphi\rangle = \int_{\mathbb{R}^4} d^3x\, dt \psi^*(\mathbf{x}, t)\varphi(\mathbf{x}, t).$$

The Lorentz transformations Λ act on the states by unitary transformations $U(\Lambda)$

$$(U(\Lambda)\psi)(x) = \psi(\Lambda^{-1}x)$$

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where

$$(\Lambda x)^\mu = \Lambda^\mu_\nu x^\nu, \quad \mu, \nu = 1, 2, 3, 4,$$

$$g_{\mu\nu}\Lambda^\mu_{\mu'}\Lambda^\nu_{\nu'} = g_{\mu'\nu'} \quad \text{and} \quad g_{\mu\nu} = (1, 1, 1, -c^2).$$

The observables are the position-time $q^\mu = (\mathbf{q}, t)$ given by the multiplication operators

$$q^\mu \psi(x) = x^\mu \psi(x)$$

and the momentum-energy $p^\mu = (\mathbf{p}, E/c^2)$ given by

$$p^\mu \psi(x) = i\hbar g^{\mu\nu}\partial_\nu \psi(x),$$

where ∂_ν denotes the partial derivative with respect to x^ν.

The evolution of this particle is described by the Schrödinger equation

$$i\hbar\, \partial_\tau \psi_\tau = K\psi_\tau$$

and for a charged particle of charge e in an electromagnetic field we have proposed [1]

$$K = \frac{1}{2M} g_{\mu\nu}(p^\mu - eA^\mu(q))(p^\nu - eA^\nu(q)) \tag{1}$$

where the constant M denotes the mass of the particle. By virtue of the generalized Ehrenfest theorem, a four-dimensional wave packet, sufficiently small, follows the trajectory of a classical charged particle (this is the correspondence principle). The wave packet must be small enough in such a way that the electromagnetic field is more or less homogeneous in the region where $\psi(x)$ is large.

2. The spin ½ particle

A particle with spin is characterized by new observables compatible with p^μ and q^μ. In the usual formalism this is done by replacing the one-component wave function $\psi(x)$ by an n-component one

$$\psi(x) = \begin{bmatrix} \psi_1(x) \\ \cdots \\ \psi_n(x) \end{bmatrix}.$$

The scalar product is given by

$$\langle \psi, \varphi \rangle = \int_{\mathbb{R}^4} d^3x\, dt \psi^+(\mathbf{x}, t)\varphi(\mathbf{x}, t)$$

$$= \int_{\mathbb{R}^4} d^3x\, dt \sum_{i=1}^{n} \psi_i^*(\mathbf{x}, t)\varphi_i(\mathbf{x}, t)$$

and the Lorentz covariance by

$$(U(\Lambda)\psi)(x) = D(\Lambda)\psi(\Lambda^{-1}x)$$

where $D(\Lambda)$ is an irreducible unitary representation of the Lorentz group (more exactly of $SL(2, \mathbb{C})$, on the n-component vector space.

But such a model leads to the following difficulty: since each such non-trivial representation is infinite-dimensional, n is infinite and the corresponding spin also.

This result is in disagreement with experiments and the well-established degeneracy of order two for electrons in atoms or metals.

To overcome that difficulty we must introduce a super-selection rule, i.e. a family of Hilbert spaces [2]. Let us consider the Stern–Gerlach apparatus defining the spin observable and more precisely the symmetry of the magnetic field of this apparatus. Such a magnetic field is characterized by a strong gradient. It defines not only the space direction of the spin but also a unique time-like direction, the direction of the time given by the frame where the field is purely magnetic. Then the state of the spin $\frac{1}{2}$ particle is characterized by a direction in space (the spin) and a time-like four-vector n^μ with $n^4 > 0$ and $g_{\mu\nu}n^\mu n^\nu = -c^2$.

We postulate that this four-vector n^μ is a super-selection rule and we introduce a family of Hilbert spaces H_n indexed by n^μ. The Lorentz group acts on this family according to the representation defined by [3]:

$$(U(\Lambda)\psi)_n(x) = D(n, \Lambda)\psi_{\Lambda^{-1}n}(\Lambda^{-1}x) \tag{2}$$

where $D(n, \Lambda) = D(L^{-1}(n)\Lambda L(\Lambda^{-1}n))$, $L(n)$ are boosts such that

$$L(n)n_0 = n \quad \text{with} \quad n_0^\mu = (0, 0, 0, 1)$$

and D the usual 2×2 unitary projective representation of the rotation group corresponding to a spin $\frac{1}{2}$.

In other words, to every time-like unit vector n^μ is associated a Hilbert space H_n of two-component wave functions which is identical to $\mathbb{C}^2 \otimes L^2(\mathbb{R}^4, d^3x\, dt)$ and the Lorentz transformations applies H_n onto $H_{\Lambda n}$ by (2).

For $n^\mu = n_0^\mu$ the spin observable is given by the four matrices

$$W^i_{n_0} = \tfrac{1}{2}\sigma^i, \quad i = 1, 2, 3 \quad \text{and} \quad W^4_{n_0} = 0$$

where σ^i are the Pauli matrices. For any n^μ we can define

$$W^\mu_n = L(n)^\mu_\nu\, W^\nu_{n_0} \tag{3}$$

and so $n_\mu W^\mu_n = 0$. This definition is justified by the following physical considerations. Suppose a localised state is given in the usual representation of the Pauli matrices as

$$\psi_n(x) = \begin{bmatrix} e^{-i\varphi/2}\cos\theta/2 \\ \\ e^{i\varphi/2}\sin\theta/2 \end{bmatrix} f(\mathbf{x}, t).$$

One can easily verify that such a state is an eigenstate with eigenvalue $+\frac{1}{2}$ for the observable

$$\tfrac{1}{2}\mathbf{s}\boldsymbol{\sigma} = s_\mu W^\mu_{n_0} = s'_\mu W^\mu_n$$

where $s^\mu = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta, 0)$ and $s'^\mu = L(n)^\mu_\nu s^\nu$. In other words the observable $s'_\mu W^\mu_n$ corresponds to a measurement of the spin with a Stern–Gerlach apparatus whose time direction is given by n^μ and the (space-like) direction of the magnetic field given by s'^μ.

Let us now check the covariance of W^μ_n. We have

$$\begin{aligned} D(n, \Lambda)W^\mu_{\Lambda^{-1}n}D^{-1}(n, \Lambda) &= L(\Lambda^{-1}n)^\mu_\nu D(n, \Lambda)W^\nu_{n_0}D^{-1}(n, \Lambda) \\ &= L(\Lambda^{-1}n)^\mu_\nu (L^{-1}(n)\Lambda L(\Lambda^{-1}n))^{-1\nu}_{\rho} W^\rho_{n_0} \\ &= \Lambda^{-1\mu}_\nu L(n)^\nu_\rho W^\rho_{n_0} = \Lambda^{-1\mu}_\nu W^\nu_n \end{aligned} \tag{4}$$

since for any rotation R

$$D(R)\sigma^i D^{-1}(R) = R_j^{-1i}\sigma^j.$$

Then we get the expected transformation law for such spin observables as it is easy to see from our physical interpretation.

On the other hand we have the following commutation rules:

$$\begin{aligned}[W_n^\mu, W_n^\nu] &= L(n)^\mu_{\mu'}L(n)^\nu_{\nu'}[W_{n_0}^{\mu'}, W_{n_0}^{\nu'}] \\ &= L(n)^\mu_{\mu'}L(n)^\nu_{\nu'}ig^{\mu'\rho}g^{\nu'\lambda}\varepsilon_{\rho\lambda\sigma\tau}W_{n_0}^\sigma n_0^\tau \\ &= ig^{\mu\mu'}g^{\nu\nu'}\varepsilon_{\mu'\nu'\rho\lambda}W_n^\rho n^\lambda \end{aligned} \tag{5}$$

where $\varepsilon_{\mu\nu\rho\lambda} = \pm 1$ for $\mu\nu\rho\lambda$ an even or odd permutation of 1, 2, 3, 4 and 0 otherwise.

The spin observables can be defined in another way which is in fact the usual one. For an infinitesimal Lorentz transformation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^{\mu\rho}g_{\rho\nu}, \quad \omega_{\rho\nu} = -\omega_{\nu\rho}$$

let us write the corresponding $D(n, \Lambda)$ in the form

$$D(n, \Lambda) = I - i\tfrac{1}{2}S_n^{\rho\nu}\omega_{\rho\nu}.$$

We have

$$W_{n,\mu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}S_n^{\nu\rho}n^\lambda. \tag{6}$$

To prove this formula we first remark that it is compatible with the law of transformation under the Lorentz group. In fact we have [4]

$$\tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}S_n^{\nu\rho}n^\lambda = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\Sigma_n^{\nu\rho}n^\lambda$$

where

$$\Sigma_n^{\nu\rho} = S_n^{\nu\rho} + i\left(n^\nu\frac{\partial}{\partial n_\rho} - n^\rho\frac{\partial}{\partial n_\nu}\right)$$

and

$$D(n, \Lambda)\Sigma^{\nu\rho}_{\Lambda^{-1}n}D^{-1}(n, \Lambda) = \Lambda_\mu^{-1\nu}\Lambda_\lambda^{-1\rho}\Sigma_n^{\mu\lambda}$$

Then we have just to check the equality in (6) for $n^\mu = n_0^\mu$ and this is obvious since

$$S_{n_0}^{ij} = \tfrac{1}{2}\sigma^k$$

where i, j, k are cyclic permutations of 1, 2, 3.

To compare our model with that of Dirac, we want to reformulate it as closely as possible to Dirac's theory. More precisely we want to change our two-component representation in a four-component representation in such a way that the new wave function transforms under the Lorentz group as a Dirac four-spinor. Let us realize the Lorentz group as elements of $SL(2\mathbb{C})$, and write Λ and $L(n)$ for the corresponding 2×2 matrices. In this notation the law of transformation (2) can be written as

$$(U(\Lambda)\psi)_{\Lambda n}(x) = L^{-1}(\Lambda n)\Lambda L(n)\psi_n(\Lambda^{-1}x).$$

Alternatively we can choose the adjoint representation $\Lambda^\sim = (\Lambda^{-1})^\dagger$ and write:

$$(U(\Lambda)\psi)_{\Lambda n}(x) = L^{\sim -1}(\Lambda n)\Lambda^\sim L^\sim(n)\psi_n(\Lambda^{-1}x)$$

since

$$L^{\sim -1}(\Lambda n)\Lambda^{\sim} L^{\sim}(n) = L^{-1}(\Lambda n)\Lambda L(n) \in SU(2).$$

Given $\psi_n(x)$, we define a four-component wave function as follows:

$$\hat{\psi}_n(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} L(n) \\ L^{\sim}(n) \end{bmatrix} \psi_n(x). \tag{7}$$

The transformation law of such an object is then

$$(\hat{U}(\Lambda)\hat{\psi})_n(x) = \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{\sim} \end{bmatrix} \hat{\psi}_{\Lambda^{-1}n}(\Lambda^{-1}x)$$

and the scalar product becomes

$$\langle \psi_n, \varphi_n \rangle = \int_{\mathbb{R}^4} d^3x\, dt \hat{\psi}_n^{\dagger}(\mathbf{x}, t) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \hat{\varphi}_n(\mathbf{x}, t).$$

We recognize $\hat{\psi}_n(x)$ as a Dirac four-spinor. Performing an additional transformation given by

$$\psi_n^D(x) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \hat{\psi}_n(x), \tag{8}$$

we obtain what corresponds to the usual Pauli–Dirac representation of large and small components The scalar product is then given by

$$\begin{aligned} \langle \psi_n, \varphi_n \rangle &= \int_{\mathbb{R}^4} d^3x\, dt \psi_n^{\dagger}(\mathbf{x}, t)\varphi_n(\mathbf{x}, t) \\ &= \int_{\mathbb{R}^4} d^3x\, dt \psi_n^{D\dagger}(\mathbf{x}, t)\beta\varphi_n^D(\mathbf{x}, t) \end{aligned} \tag{9}$$

where

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

and the transformation law under the Lorentz group is

$$(U^D(\Lambda)\psi^D)_n(x) = S(\Lambda)\psi^D_{\Lambda^{-1}n}(\Lambda^{-1}x) \tag{10}$$

with

$$S(\Lambda) = \tfrac{1}{2} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{\sim} \end{bmatrix} \begin{bmatrix} I & -I \\ I & I \end{bmatrix}. \tag{11}$$

Introducing further

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{bmatrix}, \qquad \boldsymbol{\gamma} = \beta\boldsymbol{\alpha} \quad \text{and} \quad \gamma^4 = \beta/c \tag{12}$$

we find the usual relations for Dirac γ matrices, i.e.

$$[\gamma^\mu, \gamma^\nu]_+ = -2g^{\mu\nu}I,$$
$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu_\nu \gamma^\nu$$

and

$$S^\dagger(\Lambda)\beta = \beta S^{-1}(\Lambda).$$

Then the 'current density' is given by

$$\bar{\psi}_n^D(x)\gamma^\mu\psi_n^D(x)$$

where $\bar{\psi}_n^D(x) = \psi_n^{D\dagger}(x)\beta$ transforms like a four-vector:

$$\overline{(U^D(\Lambda)\psi^D)}_n(x)\gamma^\mu(U^D(\Lambda)\psi^D)_n(x)$$
$$= \Lambda^\mu_\nu \bar{\psi}^D_{\Lambda^{-1}n}(\Lambda^{-1}x)\gamma^\nu\psi^D_{\Lambda^{-1}n}(\Lambda^{-1}x).$$

In this Pauli–Dirac representation the observables q^μ and p^μ are defined by self-adjoint operators

$$q^\mu\psi_n^D(x) = x^\mu\psi_n^D(x)$$

and

$$p^\mu\psi_n^D(x) = -i\hbar g^{\mu\nu}\partial_\nu\psi_n^D(x)$$

which are the same as those, only formally defined, in the usual Dirac theory. Moreover according to (5) the spin observables $W^D_{n,\mu}$ are defined by

$$W^D_{n,\mu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\lambda}S^{\nu\rho}_D n^\lambda = \frac{i}{4}\varepsilon_{\mu\nu\rho\lambda}\gamma^\nu\gamma^\rho n^\lambda$$

since

$$S^{\nu\rho}_D \equiv \frac{i}{4}[\gamma^\nu, \gamma^\rho]$$

are the well-known generators of the Dirac four-spinor transformation $S(\Lambda)$. Explicitly we have

$$\mathbf{W}^D_n = \tfrac{1}{2}(n^4\mathbf{\Sigma} - i\mathbf{n} \wedge \boldsymbol{\alpha}/c)$$

and

$$W^D_{n,4} = -\tfrac{1}{2}\mathbf{n}\cdot\mathbf{\Sigma} \quad \text{with} \quad \mathbf{\Sigma} = \begin{bmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{bmatrix}. \tag{13}$$

Finally choosing as in the canonical formalism a pure Lorentz transformation for the boost $L(n)$ we successively find

$$L(n) = \frac{(n^4 + 1)I - \mathbf{n}\cdot\boldsymbol{\sigma}/c}{\sqrt{2(n^4 + 1)}}$$

and

$$\psi_n^D(x) = (n^4 + 1)^{-1/2} \begin{bmatrix} (n^4 + 1)\psi_n(x) \\ (\mathbf{n}\cdot\boldsymbol{\sigma}/c)\psi_n(x) \end{bmatrix} . \tag{14}$$

Let us now discuss the dynamics. In the case of the free particle of spin $\frac{1}{2}$ the dynamics is the same as the spinless particle. Then

$$K_0 = \frac{1}{2M} g_{\mu\nu} p^\mu p^\nu = -\frac{\hbar^2}{2M} \Box \tag{15}$$

and the four-vector n^μ is a constant of motion. For the spin 0 case, the usual theory, more precisely the Klein–Gordon equation, is interpreted here as an eigenstate equation

$$K_0 \psi(x) = -\tfrac{1}{2} M c^2 \psi(x).$$

This fact suggests a change of representation: the four-dimensional Fourier transform followed by the change of variables in the subspace $K_0 < 0$ given by

$$\psi(p) = (2\pi\hbar)^{-2} c \int_{\mathbb{R}^4} \psi(x) \exp(-i g_{\mu\nu} p^\mu x^\nu/\hbar)\, d^3x\, dt$$

$$\psi(p) \mapsto f(\mathbf{p}, m) = \frac{|m|^{1/2} c}{(\mathbf{p}^2 + m^2c^2)^{1/2}} \psi(\mathbf{p}, \operatorname{sign}(m)(\mathbf{p}^2 + m^2c^2)^{1/2}/c).$$

The corresponding scalar product is

$$\langle g, f\rangle = \int_{-\infty}^{+\infty} dm \int_{\mathbb{R}^3} d^3p\, g^*(\mathbf{p}, m) f(\mathbf{p}, m) = \langle \varphi, \psi\rangle.$$

The constant of motion

$$\mathbf{q}_r = \mathbf{q} - \tfrac{1}{2}\left(\frac{\mathbf{p}}{p^4} t + t \frac{\mathbf{p}}{p^4}\right)$$

decomposed in this new representation turns out to be just $i\hbar\, \partial_{\mathbf{p}}$, i.e. the Newton–Wigner position operator [1].

The same is true for the spin $\frac{1}{2}$ case. We can define through the same way the new representation in the subspace $K_0 < 0$. The corresponding scalar product is

$$\langle g_n, f_n\rangle = \int_{-\infty}^{\infty} dm \int_{\mathbb{R}^3} d^3p\, g_n^\dagger(\mathbf{p}, m) f_n(\mathbf{p}, m) = \langle \varphi_n, \psi_n\rangle.$$

Then the Newton–Wigner position operator is given by the same formula as before and corresponds to the same constant of motion $\mathbf{q}_r$.

The comparison of our theory with the Wigner theory where the states of the free particle are described by an irreducible unitary projective representation of the Poincaré group is easy. A Wigner state for a particle of mass M defined for given momentum and spin corresponds in our model to an eigenstate of K_0 for the eigenvalue $-\frac{1}{2}Mc^2$ and the same given momentum and spin and a time direction n^μ parallel to p^μ. Strictly speaking, the two theories are completely equivalent only for such special states.

To compare more closely the two theories let us consider in the canonical formalism a Wigner state $\varphi(\mathbf{p})$ corresponding to a momentum sharply defined (with an uncertainty or mean square deviation $\sigma(\mathbf{p}^2) \ll M^2c^2$). According to the Foldy–Wouthuysen transformation such a state corresponds in the Pauli–Dirac representation to the four-spinor

$$M^{-1/2}(p^4 + M)^{-1/2}\begin{bmatrix}(p^4 + M)\varphi(\mathbf{p})\\ (\boldsymbol{\sigma}\cdot\mathbf{p}/c)\varphi(\mathbf{p})\end{bmatrix}$$

where $p^4 = (\mathbf{p}^2 + M^2c^2)^{1/2}/c > 0$. This Dirac spinor can be approximatively identified in the Pauli–Dirac formulation of our model (14) with

$$\psi_n^D(p) = (n^4 + 1)^{-1/2}\begin{bmatrix}(n^4 + 1)\varphi(\mathbf{p})\\ (\boldsymbol{\sigma}\cdot\mathbf{n}/c)\varphi(\mathbf{p})\end{bmatrix}$$

where for n^μ we have chosen

$$n^\mu = \int_{\mathbb{R}^3} \frac{d^3p}{p^4}\,\varphi^\dagger(\mathbf{p})\,\frac{p^\mu}{M}\,\varphi(\mathbf{p}).$$

In other words such a sufficiently sharp state of the Wigner theory can be identified with a corresponding eigenstate of K_0 for the eigenvalue $-\frac{1}{2}Mc^2$ in our theory. For $n^\mu = n_0^\mu$ such a state can be identified with the 'large' component of the usual Dirac theory.

3. The spin $\frac{1}{2}$ particle interacting with an external electromagnetic field

Let us consider now the case of the spin $\frac{1}{2}$ particle in an external electromagnetic field $A_\mu(x) = (\mathbf{A}(x), -V(x))$. The Schrödinger operator is given by the corresponding one for the spin 0 case [1] modified by the terms due to the spin interaction with the electromagnetic field. Especially for the electron (or positron) we propose

$$\begin{aligned}K = {}&\frac{1}{2M}g_{\mu\nu}(p^\mu - eA^\mu(q))(p^\nu - eA^\nu(q))\\ &- \frac{g_1\mu_0}{Mc^2}(p^\mu - eA^\mu(q))\tilde{F}_{\mu\nu}(q)W_n^\nu\\ &+ \frac{g_2^2\mu_0^2}{8Mc^4}F_{\mu\nu}(q)n^\nu F_\rho^\mu(q)n^\rho - \frac{g_3\mu_0}{c^2}n^\mu\tilde{F}_{\mu\nu}(q)W_n^\nu\end{aligned} \tag{16}$$

where $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ and $F_{\mu\nu}(x) = \frac{1}{2}\varepsilon_{\mu\nu\rho\lambda}\tilde{F}^{\rho\lambda}(x)$, i.e.:

$$\boldsymbol{B} = (F_{23}, F_{31}, F_{12}) = (\tilde{F}^{14}, \tilde{F}^{24}, \tilde{F}^{34})$$
$$\mathbf{E} = (F_{14}, F_{24}, F_{34}) = (\tilde{F}^{23}, \tilde{F}^{31}, \tilde{F}^{12}).$$

The electric charge of the particle being e, μ_0 denotes the usual Bohr magneton $e\hbar/2M$ and g_1, g_2 and g_3 are dimensionless phenomenological constants.

Let us now discuss this expression of K. Since K is a Lorentz invariant we may consider only the case $n^\mu = n_0^\mu$. Clearly the first term is responsible for the Lorentz forces on the charged particle and the others are just perturbations. The second term, which can be written as

$$-\frac{g_1\mu_0}{Mc^2}((\mathbf{p}-e\mathbf{A}(q))\mathbf{W}_{n_0}\wedge\mathbf{E}(q)+(E-eV(q))\mathbf{W}_{n_0}\boldsymbol{B}(q)) \tag{17}$$

describes a spin-orbit coupling. It is formally self-adjoint since

$$[p_\mu - eA_\mu(q), \tilde{F}^{\mu\nu}(q)] = -i\hbar\,\partial_\mu\tilde{F}^{\mu\nu}(q) \equiv 0$$

by virtue of the homogeneous Maxwell equations. The third term

$$\frac{g_2^2\mu_0^2}{8Mc^2}\mathbf{E}^2(q) \tag{18}$$

is a contribution to K due to the spin-orbit coupling. Finally the fourth term

$$-g_3\mu_0\mathbf{W}_{n_0}\boldsymbol{B}(q) \tag{19}$$

is the only possible Lorentz invariant term coupling the magnetic field and the spin variables.

In view of the expressions (17) and (19) it is clear that for the particle at rest in a purely magnetic field, one has

$$K \simeq -\tfrac{1}{2}Mc^2 - (g_1+g_3)\mathbf{W}_{n_0}\boldsymbol{B}$$

and by definition $g_1 + g_3$ is the g-factor of the electron magnetic moment. We have chosen factors in (17), (18) and (19) in such a way that g_1, g_2 and g_3 are approximately equal to 1 for the electron. In fact application of our model in the hydrogen atom case [5] leads for $g_1 = g_2 = 1$ to the following energy spectrum which coincides with the results of Dirac [6] up to terms of order α^4

$$E_{n,l,j} = Mc^2\left(1-\frac{\alpha^2}{2n^2}-\frac{\alpha^4}{2n^4}\left(\frac{n}{j+1/2}-\frac{3}{4}\right)+\cdots\right).$$

n, l and j are the usual quantum numbers for energy levels in hydrogen atom and α is the fine structure constant. For g_1 and g_2 different from 1 we find

$$E_{n,0,1/2} - E_{n,1,1/2} = Mc^2\frac{\alpha^4}{n^3}\left(\frac{g_1-1}{6}+\frac{g_2-1}{2}\right),\ n \geq 2$$

i.e. the Lamb shift [6].

To completely determine the evolution of the state we still need an equation for $\dot{n}^\mu$. If the evolution is induced by automorphisms such an equation is of the form

$$\dot{n}^\mu = f^\mu(n) \tag{20}$$

with the condition $f^\mu(n)n_\mu = 0$ due to the constraint $n^\mu n_\mu = -c^2$. Physically our comparison with Dirac's theory suggests that the evolution is such that n^μ tends to be parallel to $\langle p^\mu\rangle$. In many cases the direction is practically the same as the one given by $\langle\dot{q}^\mu\rangle$ since we have:

$$\dot{q}^\mu = \frac{i}{\hbar}[K, q^\mu] = \frac{1}{M}\left(p^\mu - eA^\mu(q) - \frac{g_1\mu_0}{c^2}\tilde{F}^\mu_\nu(q)W^\nu_n\right). \tag{21}$$

In general such an evolution is not of the form (20) and gives rise to some irreversible process like radiation.

As an application we consider the evolution of the spin in an electromagnetic field which is not necessary homogeneous. More precisely from the expression (16) of K and from the definition of W_n^μ we can write an expression for the derivative $\dot{W}_n^\mu$. According to the formalism of continuous superselection rules [7] we have

$$\dot{W}_n^\mu = \frac{i}{\hbar}[K, W_n^\mu] + \partial_\tau W_n^\mu \tag{22}$$

where

$$\partial_\tau W_n^\mu = \lim \frac{W_{n(\tau+\delta\tau)}^\mu - W_{n(\tau)}^\mu}{\delta\tau} = \partial_\tau L(n(\tau))_\nu^\mu W_{n0}^\nu$$

is the usual derivative with respect to τ if the isomorphism between the Hilbert spaces $H_{n(\tau)}$ and $H_{n(\tau+\delta\tau)}$ is the identity. This is the case for $n^\mu = n_0^\mu$ if $L(n_0 + \delta n)$ is a pure Lorentz transformation, i.e.

$$L(n_0 + \delta n)_\nu^\mu = \delta_\nu^\mu + (n_0^\mu \delta n_\nu - \delta n^\mu n_{0\nu})/c^2$$

as is easy to check (since $n_0^\mu \delta n_\mu = 0$). Then we successively obtain

$$\partial_\tau L(n(\tau))_\nu^\mu|_{n(\tau)=n_0} = (n_0^\mu \dot{n}_{0\nu} - \dot{n}_0^\mu n_{0\nu})/c^2$$

and

$$\partial_\tau W_{n0}^\mu = n_0^\mu(\dot{n}_{0\nu} W_{n0}^\nu)/c^2 \tag{23}$$

since $n_{0\mu} W_{n_0}^\mu = 0$.

Performing a Lorentz transformation $L(n)_\nu^\mu$ on

$$\dot{W}_{n_0}^\mu = \frac{i}{\hbar}[K, W_{n_0}^\mu] + n_0^\mu(\dot{n}_{0\nu} W_{n_0}^\mu)/c^2 \tag{24}$$

we find

$$\dot{W}_n^\mu = \frac{i}{\hbar}[K, W_n^\mu] + n^\mu(\dot{n}_\nu W_n^\nu)/c^2. \tag{25}$$

Finally a straightforward calculation of the commutator in (25) leads to the following equation

$$\begin{aligned}\dot{W}_n^\mu = {} & \frac{g_1 \mu_0}{\hbar M c^2}\{n^\rho F_{\rho\lambda}(q) W_n^\lambda (p^\mu - eA^\mu(q)) \\ & - F_\lambda^\mu(q) W_n^\lambda n_\rho (p^\rho - eA^\rho(q)) + F_\lambda^\mu(q) n^\lambda W_{n\rho}(p^\rho - eA^\rho(q))\} \\ & + \frac{g_3 \mu_0}{\hbar c^2}\{n^\rho F_{\rho\lambda}(q) W_n^\lambda n^\mu + c^2 F_\lambda^\mu(q) W_n^\lambda\} \\ & + n^\mu(\dot{n}_\nu W_n^\nu)/c^2.\end{aligned} \tag{26}$$

From this general expression of $\dot{W}_n^\mu$ we are able to justify the BMT equation [8] as a semi-classical approximation.

Let us consider a wave packet around the mass shell and sharply defined in space-time, momentum-energy and spin. For such a state we can approximate

$\langle F_{\mu\nu}(q)\rangle$ by $F_{\mu\nu} = F_{\mu\nu}(\langle q\rangle)$ and then write:

$$n^\mu \simeq \frac{1}{M}\langle p^\mu - eA^\mu(q)\rangle \simeq \langle \dot{q}^\mu\rangle \tag{27}$$

and

$$\dot{n}^\mu \simeq \langle \ddot{q}^\mu\rangle \simeq \langle \frac{i}{\hbar}[K, \dot{q}^\mu]\rangle \simeq \frac{e}{M} F^\mu_\nu n^\nu. \tag{28}$$

Performing these approximations in (26) and rearranging the terms, we find the following equation which is the BMT equation for the precession of the polarisation of particles moving in an electromagnetic field

$$\dot{W}^\mu_n = \frac{\mu_0}{\hbar}\{gF^\mu_\nu W^\nu_n + (g - 2)n^\mu n^\rho F_{\rho\nu} W^\nu_n\}. \tag{29}$$

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