

Zeitschrift: Helvetica Physica Acta
Band: 50 (1977)
Heft: 3

Artikel: On the description of free relativistic particles
Autor: Aaberge, T.
DOI: <https://doi.org/10.5169/seals-114860>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

On the description of free relativistic particles

by **T. Aaberge**

Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

(3, XI. 1976)

Abstract. Starting with a manifestly covariant classical symplectic structure, we construct the *unique* classical counterpart to Wigner's theory for free relativistic quantal particles of spin O . It is argued that the Wigner theories for quantal particles with spin different from O can not be considered as quantizations of this classical theory, however, that there exist theories for particles with 'spin'. We show how these 'spin' theories can be constructed, and that a certain class of them 'contains' the Wigner theories of spin.

1. Construction of the classical theory

Let (Ω, ω) be the symplectic manifold given by

$$\Omega = \{(p^\mu, q^\mu) \in \mathbb{R}^8 \mid p^{02}c^2 - \mathbf{p}^2 > 0 \text{ \& } p^0 > 0\}$$

$$\omega = dp_\mu \wedge dq^\mu = \sum_i dp^i \wedge dq^i - c^2 dp^0 \wedge dq^0$$

and let a symplectic action of the inhomogeneous Lorentz group \mathcal{P} be defined by

$$(p^\mu, q^\mu) \mapsto (\Lambda^\mu{}_\nu p^\nu, \Lambda^\mu{}_\nu q^\nu + a^\mu)$$

for Λ being the usual representation of the homogeneous Lorentz group $SO(1, 3)$ on \mathbb{R}^4 , isometric with respect to the Minkowski-metric.

Let us then consider the following change of coordinates¹⁾

$$(p^\mu, q^\mu) \mapsto (y^\mu, x^\mu)$$

$$\begin{aligned} y^\mu &= \begin{pmatrix} m \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \frac{1}{c} \sqrt{p^{02}c^2 - \mathbf{p}^2} \\ \mathbf{p} \end{pmatrix} \\ x^\mu &= \begin{pmatrix} x^0 \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{p^{02}c^2 - \mathbf{p}^2}}{p^0 c} q^0 \\ \mathbf{q} - \frac{\mathbf{p}}{p^0} q^0 \end{pmatrix} \end{aligned} \tag{1}$$

One verifies without difficulties that it is a canonical transformation,

$$\sum_i dp^i \wedge dq^i - c^2 dp^0 \wedge dq^0 = \sum_i dp^i \wedge dx^i - c^2 dm \wedge dx^0.$$

¹⁾ The observable \mathbf{x} has been considered in [1].

The action of \mathcal{P} on these new coordinates is easily determined,

$$m \mapsto m$$

$$p^i \mapsto \Lambda^i_v \left(\frac{1}{c} \sqrt{\mathbf{p}^2 + m^2 c^2}, \mathbf{p} \right)^v$$

(We will in the following use the notation $\Lambda \mathbf{p}$ and \mathbf{p}' for the transformed of \mathbf{p} .)

$$x^\mu \mapsto \sigma(m, \mathbf{p}, \Lambda)^\mu_\nu x^\nu$$

where

$$\sigma(m, \mathbf{p}, \Lambda(\boldsymbol{\theta})) = \Lambda(\boldsymbol{\theta}) \quad (\Lambda(\boldsymbol{\theta}) \text{ rotation})$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^0_0 = 1$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^0_i = \frac{m}{\sqrt{\mathbf{p}'^2 + m^2 c^2}} \gamma \frac{u_i}{c}$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^i_0 = 0$$

$$\sigma(m, \mathbf{p}, \Lambda(\mathbf{u}))^i_j = \delta^i_j + \frac{\gamma^2}{\gamma + 1} \frac{u_i u_j}{c^2} - \frac{\gamma p_i'^2 u_j}{c \sqrt{\mathbf{p}'^2 + m^2 c^2}}$$

$$\gamma = \left(1 - \frac{\mathbf{u}^2}{c^2} \right)^{-1/2}.$$

Moreover, for a translation we find

$$x^0 \mapsto x^0 + \frac{mc}{\sqrt{\mathbf{p}^2 + m^2 c^2}} a^0$$

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{a} - \frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2 + m^2 c^2}} a^0.$$

Let \sim be defined as

$$(y^\mu, x^\mu) \sim (y'^\mu, x'^\mu) \Leftrightarrow$$

$$y'^\mu = y^\mu, (x'^\mu - x^\mu) = (b, \mathbf{o}), \quad b \in \mathbb{R}.$$

It is a Lorentz-invariant equivalence relation; $x^\mu - x'^\mu$ is invariant under the translations, and moreover $x^\mu - x'^\mu = (b, \mathbf{o})$ implies $(\sigma(x - x'))^\mu = (b', \mathbf{o})$.

$$\Omega \xrightarrow{\Pi_0} \Omega_0 = \Omega / \sim$$

is thus a Lorentz-invariant fibration of Ω , i.e. \mathcal{P} acts by morphisms on $\Omega \xrightarrow{\Pi_0} \Omega_0$,

$$\Pi_0(m, \Lambda \mathbf{p}, (\sigma(m, \mathbf{p}, \Lambda)x)^\mu) = (m, \Lambda \mathbf{p}, \bar{\sigma}(m, \mathbf{p}, \Lambda)\mathbf{x})$$

where $\bar{\sigma}$ is given by

$$\bar{\sigma}^i_j = \sigma^i_j.$$

We proceed by defining a fibration of Ω_0 ,

$$\Omega_0 \xrightarrow{\Pi} \mathcal{B} = \{m\}$$

for

$$\Pi: (m, \mathbf{p}, \mathbf{x}) \mapsto m.$$

It obviously has the property of being invariant, and moreover each of the fibres $\Gamma_m = \Pi^{-1}(m)$ is a surface of transitivity for the given action of \mathcal{P} . Moreover, each Γ_m inherits a symplectic structure from Ω given by the form

$$\bar{\omega} = \sum_i dp^i \wedge dx^i.$$

One should notice that $\bar{\omega}$ is invariant under \mathcal{P} .

The free classical relativistic particle of rest-mass m is by definition described by the phase space $(\Gamma_m, \bar{\omega})$, on which the observables momentum and position are represented by the functions

$$p^i(\mathbf{p}, \mathbf{q}) = p^i$$

$$x^i(\mathbf{p}, \mathbf{q}) = x^i$$

We assume that the evolution of such a particle leaves invariant the symplectic structure. It is thus governed by the Hamilton equations. As usual, we let the associated Hamiltonian be of the form

$$\kappa = c\sqrt{\mathbf{p}^2 + m^2 c^2}$$

the equations of motion are then

$$\dot{p}^i = -\frac{\partial \kappa}{\partial x^i} = 0$$

$$\dot{x}^i = \frac{\partial \kappa}{\partial p^i} = \frac{p^i c}{\sqrt{\mathbf{p}^2 + m^2 c^2}} \quad (1)$$

If we denote by \mathbf{v} the velocity of the particle, we can write down the following solutions

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

$$\mathbf{x} = \mathbf{v}t + \mathbf{a}, \quad (2)$$

and on the orbit,

$$\kappa = \frac{mc^2}{\sqrt{1 - \mathbf{v}^2/c^2}}$$

This is, by construction, a covariant description of the free relativistic particle. To complete the interpretation, we notice that for $\mathbf{v} = \mathbf{0}$, \mathbf{x} transforms according to

$$\mathbf{x} \mapsto \mathbf{x} - \gamma \frac{\mathbf{u} \cdot \mathbf{x}}{c^2} \mathbf{u} + \frac{\gamma^2}{\gamma + 1} \frac{\mathbf{u} \cdot \mathbf{x}}{c^2} \mathbf{u}$$

for $\gamma = (1 - \mathbf{u}^2/c^2)^{-1/2}$, or

$$\mathbf{x}_\perp \mapsto \mathbf{x}_\perp, \quad \mathbf{x}_\parallel \mapsto \mathbf{x}_\parallel \sqrt{1 - \mathbf{u}^2/c^2}$$

under a special Lorentz transformation.

If one looks at the corresponding dynamics on Ω , one finds in addition to (1) the equations of motion

$$\dot{m} = 0$$

$$\dot{x}^0 = -\frac{mc}{\sqrt{\mathbf{p}^2 + m^2 c^2}}$$

with the solutions

$$m = \text{const}$$

$$x^0 = -\frac{1}{\gamma} t, \quad \gamma = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-1/2} \quad (3)$$

In the original coordinates these solutions (2) and (3) read

$$p^\mu = m(\gamma, \gamma \mathbf{v})$$

$$q^\mu = (-t, \mathbf{a}).$$

This should give a certain basis for judging the theory; in particular, we notice that the above classical theory is different from the 'usual' one [2].

2. The quantal particle of spin 0

Before considering the quantum model we give the abstract definition of a free relativistic particle.

Definition. A free relativistic particle of mass m is an irreducible realisation of the following 'system of imprimitivity'

$$\begin{aligned} S(\Lambda, a^\mu) \bar{\mathbf{p}}(\Delta) &= \bar{\mathbf{p}}(\Lambda \cdot \Delta) \\ S(\Lambda, a^\mu) \bar{\mathbf{x}}(\Delta) &= \bar{\mathbf{x}} \left(\bar{\sigma}(\mathbf{p}, \Lambda) \cdot \Delta + \mathbf{a} - \frac{\mathbf{p}c}{\sqrt{\mathbf{p}^2 + m^2 c^2}} a^0 \right) \end{aligned} \quad (4)$$

for $\Lambda \cdot \Delta = \{\Lambda \mathbf{p} / \mathbf{p} \in \Delta\}$ etc.,

for $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ denoting the observables momentum and position. $\Delta \in \mathcal{P}(\mathbb{R}^3)$ or $\mathcal{B}(\mathbb{R}^3)$, i.e. is a subset or a borel set of \mathbb{R}^3 representing the points of the spectra of $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$.

The classical particle described above is clearly an irreducible realization of the systems of imprimitivity. Moreover, a quantum particle is easily constructed; in fact, the following realization satisfies the 'imprimitivity systems'.

$$(U(\Lambda, a^\mu) f)(\mathbf{p}) = e^{(i/\hbar) p^\mu a_\mu} f(\Lambda^{-1} \mathbf{p})$$

in

$$\begin{aligned} &L^2\left(\mathbb{R}^3, \frac{d^3 p}{\sqrt{\mathbf{p}^2 + m^2 c^2}}\right), \\ &p^\mu = \left(\frac{1}{c} \sqrt{\mathbf{p}^2 + m^2 c^2}, \mathbf{p}\right); \end{aligned}$$

and the observables $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ are represented by

$$\begin{aligned}(\mathbf{p}f)(\mathbf{p}) &= \mathbf{p}f(\mathbf{p}) \\ (\mathbf{x}f)(\mathbf{p}) &= i\hbar\left(\partial_{\mathbf{p}} - \frac{1}{2} \frac{\mathbf{p}}{\mathbf{p}^2 + m^2c^2}\right)f(\mathbf{p}) \\ &= i\hbar(\mathbf{p}^2 + m^2c^2)^{1/4} \partial_{\mathbf{p}}(\mathbf{p}^2 + m^2c^2)^{-1/4}.\end{aligned}$$

It is well known that the 'imprimitivity system' for \mathbf{p} is satisfied, and also that for \mathbf{x} under the rotations and translations. What is left to be verified is that

$$\begin{aligned}U^{-1}(\Lambda(\mathbf{u}))i\hbar(\mathbf{p}^2 + m^2c^2)^{1/4} \partial_{\mathbf{p}}(\mathbf{p}^2 + m^2c^2)^{-1/4} U(\Lambda(\mathbf{u})) \\ = i\hbar(\mathbf{p}'^2 + m^2c^2)^{1/4} \sigma(\mathbf{p}, \Lambda(\mathbf{u})) \cdot \partial_{\mathbf{p}}(\mathbf{p}'^2 + m^2c^2)^{-1/4}\end{aligned}$$

which is done by a small calculation.

The theory thus constructed is what might be called the Wigner theory for free relativistic particles of spin O . The representation U of \mathcal{P} is the Wigner representation for spin O [3], while \mathbf{x} is what is usually referred to as the Newton–Wigner position operator [4]. The definition of \mathbf{x} given in terms of the above 'systems of imprimitivity' supposes the postulates of localizability for relativistic particles [4, 5], in fact, its representative is uniquely determined by these postulates. However, our definition contains one more condition, that the position transforms in a particular way under a special Lorentz transformation. Though this additional condition is satisfied for the Wigner theory of spin O , one verifies easily that it is *not* satisfied for a Wigner theory of spin $\neq O$. Thus a Wigner theory of spin $\neq O$ does not satisfy the above definition of a relativistic particle.

A reason for this is the following. A realization of (4) is obtained by a double application of the inducing construction to obtain realizations of the 'imprimitivity systems' for $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ and then glue these together by a unitary transformation. Since the action of \mathcal{P} on $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ has as stability groups $S(3) \ltimes \mathbb{R}^4$ and $S(1, 3)$ respectively, it follows that for this unitary transformation to exist, the representation transforming $\bar{\mathbf{p}}$ and $\bar{\mathbf{x}}$ must be induced from representations of $S(1, 3) \ltimes \mathbb{R}^4$ respectively. The Wigner representations however, are induced from representation of $S(3) \ltimes \mathbb{R}^4$, this being the reason for the non-existence of a position operator satisfying our definition in the Wigner theory, except for the spin O case when the representations from which we induce are trivial.

3. The quantal particle of 'spin' $\neq O$

The inducing construction tells why a Wigner theory of spin $\neq O$ is not a realization of (4). But it does also tell how a notion of 'spin', compatible with the above definition of a relativistic particle, can be introduced; in fact, one should induce from representations of $S(3, 1) \ltimes \mathbb{R}^4$.

Thus let D be an irreducible unitary representation of $S(3, 1)$ on l^2 , then U defined by

$$(U(\Lambda, a^\mu)f)(\mathbf{p}) = e^{(i/\hbar)p^\mu a_\mu} D(\Lambda)f(\Lambda^{-1}\mathbf{p})$$

is a unitary representation of \mathcal{P} on

$$L^2\left(\mathbb{R}^3, \frac{d^3p}{\sqrt{\mathbf{p}^2 + m^2c^2}}\right) \otimes l^2$$

induced from D . If one defines the operators representing the momentum and position as for the spin O case, the theory obtained satisfies the imprimitivity systems. Moreover, the additional degrees of freedom, the 'spins' are described by the six operators $S^{\mu\nu}$ on l^2 forming the representation of the Lie algebra of D . The $S^{\mu\nu}$ thus transform according to

$$U^{-1}(\Lambda, a^\mu) S^{\mu\nu} U(\Lambda, a^\mu) = (\Lambda \Lambda S)^{\mu\nu}.$$

In the case D is a representation in the principal series we can relate this definition of spin to that of Wigner in the following way. Let F be defined by

$$(Ff)(\mathbf{p}) = D^{-1}(L(\mathbf{p}))f(\mathbf{p}) = \psi(\mathbf{p})$$

where $L(\mathbf{p}) = \Lambda(\mathbf{p}/\sqrt{\mathbf{p}^2 + m^2 c^2})$. It is a unitary transformation which takes U into $V = FUF^{-1}$,

$$(V(\Lambda, a^\mu)\psi)(\mathbf{p}) = e^{(i/\hbar)p^\mu a_\mu} D(R(\mathbf{p}, \Lambda))\psi(\Lambda^{-1}\mathbf{p})$$

where

$$R(\mathbf{p}, \Lambda) = L^{-1}(\mathbf{p})\Lambda L(\Lambda^{-1}\mathbf{p})$$

is a Wigner rotation.

This form of the representation has the property that it decomposes into a direct sum of Wigner representations [6, 7]

$$(V(\Lambda, a^\mu)\psi)(\mathbf{p}) = \sum_{s=s', s'+1, \dots}^{\oplus} e^{(i/\hbar)p^\mu a_\mu} D^{(s)}(R(\mathbf{p}, \Lambda))\psi^{(s)}(\Lambda^{-1}\mathbf{p})$$

where $D^{(s)}$ denotes a unitary representation of $SU(2)$ in \mathbb{C}^{2s+1} , and s' is one of the two numbers characterizing the D .

The momentum \mathbf{p} commutes with F and is thus represented by the same operator in the new representation. The position operator \mathbf{x} however, does not commute with F , and moreover, its new representative does not decompose according to the above decomposition.

Acknowledgement

I am indebted to C. Piron and H. Fierz for numerous discussions on this subject.

REFERENCES

- [1] L. P. HORWITZ and C. PIRON, *Helv. Phys. Acta* 46, 316 (1973).
- [2] L. LANDAU and E. LIFCHITZ, *Théorie des Champs*, (Editions Mir, Moscou 1970). Ch. III.
- [3] E. P. WIGNER, *Ann. Math.* 40, 149 (1939).
- [4] T. D. NEWTON and E. P. WIGNER, *Rev. Mod. Phys.* 21, 400 (1949).
- [5] A. S. WIGHTMAN, *Rev. Mod. Phys.* 34, 845 (1962).
- [6] I. M. GEL'FAND, R. A. MINLOS and Z. Ya. SHAPIROS, *Representations of the Rotation and Lorentz Groups and Their Applications*. (Pergamon Press, Oxford, 1963).
- [7] T. AABERGE, *Helv. Phys. Acta* 48, 173 (1975).