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Remarks on exponential interactions and the quantum sine-Gordon equation in two space-time dimensions

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Abstract. We prove best possible results concerning the convergence of the lattice approximation for the exponential interactions of Albeverio and Høegh-Krohn and the quantum sine-Gordon equation. These results are then used to construct infinite volume limit theories for these interactions that satisfy the Osterwalder-Schrader axioms.

1 Introduction, main results

In this note we study the existence of relativistic scalar Bose fields in two space-time dimensions with some non-polynomial self-interaction V given by either

$$\text{(exp)}\quad V(\Lambda, \Lambda') = \int_\Lambda d^2x \int d\nu_1(\varepsilon) :e^{\varepsilon\phi}: (x) + \int_{\Lambda'} d^2x \int d\nu_2(\varepsilon) :e^{\varepsilon\phi}: (x) \tag{1.1}$$

or

$$(s-G)\quad V(\Lambda, \Lambda') = \int_\Lambda d^2x \int d\nu(\varepsilon) :\cos \varepsilon(\phi + \theta): (x) - \frac{\sigma}{2}\int_{\Lambda'} d^2x :\phi^2: (x) \tag{1.2}$$

Here ϕ is a real, scalar Euclidean field on $\mathbb{R}^2$, $V(\Lambda, \Lambda')$ is the interacting Euclidean action with space-time cutoffs $\Lambda' \subseteq \Lambda$; the double colons denote Wick ordering with respect to some fixed bare mass $m_0 > 0$; $d\nu_1$, $d\nu_2$ are finite, positive measures with

$$\operatorname{supp} d\nu_1 \subset [0, 2\sqrt{\pi}), \operatorname{supp} d\nu_2 \subset (-2\sqrt{\pi}, 0], \tag{1.3}$$

$d\nu$ is a finite, real measure with

$$\operatorname{supp} d\nu \subset (-2\sqrt{\pi}, 2\sqrt{\pi}), \tag{1.4}$$

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θ is a real constant and σ some number in the interval $(-\infty, m_0^2)$; our main interest is in the case where $\sigma \geqslant 0$. It is no loss of generality to set $m_0 = 1$ throughout the following; (scaling!).

The (exp)-models have previously been studied by Albeverio and Høegh-Krohn [1] under the *additional assumptions* that

$$dv_1(\varepsilon) = dv_2(-\varepsilon) \text{ and supp } dv_{1/2} \subset \left(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}}\right).$$

See also [16]. In this paper we drop both restrictions. This requires an improved analysis of the lattice approximation [12, 14] and a construction of the infinite volume limit by means of FKG-inequalities rather than Griffiths-(GKS-)inequalities, (as in [1]). Our new construction is similar to the one of all $P(\phi)_2$-models recently proposed in [8].

The interest in the (exp)-models comes from the fact that they define relativistic quantum field theories (in the Wightman sense) in spite of the fact that perturbation theory is ultraviolet divergent in *all orders* beyond some finite order (that depends on supp dv_i).

Moreover, in space-time dimensions > 2, the (exp)-models retain such appealing, formal properties that one might conjecture they define relativistic quantum field theories. Our results cast some doubts upon whether any perturbation approach to these models is reasonable.

The (s-G)-model has been constructed in [7] under the assumptions that

$$\text{supp } dv \subset (-2\sqrt{\pi}, 2\sqrt{\pi})$$

$$\int |dv| \ll 1, \quad \sigma \leqslant 0,$$

by means of the cluster expansion of [11]. It has been shown in [7] that under these hypotheses *Feynman perturbation theory* for the Euclidean Green's functions *converges*. This situation is in a notable contrast to the one met in the (exp)-models.

In [2] the (s-G)-model has been constructed under the only assumption that

$$\text{supp } dv \subset \left(-\frac{4}{\sqrt{\pi}}, \frac{4}{\sqrt{\pi}}\right).$$

Here we present a construction of the model for arbitrary $\int |dv|$, with dv satisfying (1.4), and all $\sigma \leqslant m_0$. We make use of GKS inequalities, 'weak coupling boundary conditions' [10] and results of [7] and [8].

Admittedly this paper serves mainly cosmetic purposes.

1.1 Definitions and notations

Let $C^{(\Lambda, X)}(x, y)$ denote the kernel of the operator $(-\Delta^{(\Lambda, X)} + 1)^{-1}$, where $\Delta^{(\Lambda, X)}$ is the two dimensional Laplacean with X boundary conditions at $\partial\Lambda$. In this paper we shall only consider two cases:

(a) $X = F$ (free b.c.); in this case we also use the notation $C^F(x - y)$, as $C^{(\Lambda, F)}(x, y)$ is independent of Λ and translation invariant.

(b) $X = P$ (periodic b.c.); see [12, 16].

The Gaussian measure on $\mathscr{S}'(\Lambda)$ (where $\mathscr{S}'(\Lambda) = C_0^\infty(\Lambda)'$) with mean 0 and covariance $C^{(\Lambda, X)}$ is denoted $d\mu_0^{(\Lambda, X)}$, and $\langle - \rangle_0^{(\Lambda, X)}$ denotes expectations with respect to $d\mu_0^{(\Lambda, X)}$.

Let $:-:_{(\Lambda, X)}$ denote Wick ordering with respect to $d\mu_0^{(\Lambda, X)}$. We abbreviate (Λ, Λ', X) by I; $V(I)$ denotes the cutoff Euclidean action with ordinary Wick ordering replaced by $:-:_{(\Lambda, X)}$. Cutoff Euclidean Green's functions (EGF's) are defined by

$$S^I(f_1, \ldots, f_n) = Z_I^{-1} \left\langle \prod_{i=1}^{n} \phi(f_i)\, e^{-V(I)} \right\rangle_0^{(\Lambda, X)} \tag{1.5}$$

where $f_1, \ldots, f_n$ are functions in $\mathscr{S}(\Lambda)$ and

$$Z_I = \langle e^{-V(I)} \rangle_0^{(\Lambda, X)} \tag{1.6}$$

is the partition function.

Let $\mathbb{Z}_\delta^2$ be the lattice $\{n\delta : n \in \mathbb{Z}^2\}$ with δ some positive lattice constant. Following [12] we also consider a lattice field $\phi_\delta(x)$, a lattice action $V_\delta(I)$ and a lattice measure $d\mu_\delta^{(\Lambda, X)}$. (This is the Gaussian measure with mean 0 and covariance $C_\delta^{(\Lambda, X)}(x, y) =$ kernel of $(-\Delta_\delta^{(\Lambda, X)} + 1)^{-1}$, where $\Delta_\delta^{(\Lambda, X)}$ is the finite difference Laplacean with b.c. X at $\partial\Lambda$, Λ is assumed to be some rectangle in $\mathbb{Z}_\delta^2$).

We may then define (in the obvious way) lattice EGF's S_δ^I and a lattice partition function $Z_{\delta, I}$.

The existence of the cutoff EGF's S^I, S_δ^I and of Z_I and $Z_{\delta, I}$ has been established for the (exp) model in [1] and for the (s-G) model in [2, 3].

From [12] we know that ϕ_δ and $V_\delta(I)$ may, in a natural way, be regarded as random variables relative to $d\mu_0^{(\Lambda, X)}$.

It is shown there that, for all $f \in \mathscr{S}(\Lambda)$,

$$\phi_\delta(f) \to \phi(f), \quad \text{in } L^p(d\mu_0^{(\Lambda, X)}), \tag{1.7}$$

as $\delta \searrow 0$, for $X = F, P, \ldots$, $1 \leqslant p < \infty$.

Furthermore, with

$$:\phi_\delta^2:(\Lambda') = \int_{\Lambda'} d^2x :\phi_\delta^2:(x),$$

we have

$$e^{\sigma/2 :\phi_\delta^2:(\Lambda')} \to e^{\sigma/2 :\phi^2:(\Lambda')}, \quad \text{in } L^q(d\mu_0^{(\Lambda, X)}), \tag{1.7'}$$

as $\delta \searrow 0$, for $X = F, P, \ldots$, for all $1 \leqslant q < \infty$, provided $\sigma \leqslant 0$, and $1 \leqslant q < m_0^2/\sigma$, provided $\sigma \in (0, m_0^2)$, see [12]; (1.7') permits us to set $\sigma = 0$ throughout many parts of the following analysis.

1.2 The main results

In the following we always choose

$$\begin{aligned} X &= F, \quad \text{for the (exp)-models, and} \\ X &= P, \quad \text{for the (s-G)-models.} \end{aligned} \tag{1.8}$$

In Section 2 we prove

Theorem A: *With the convention (1.8) the following holds for the* (exp)*-models and the (s-G)-models with $\sigma = 0$ defined in (1.1)–(1.6):*

(1) $\lim_{\delta \downarrow 0} V_\delta(I) = V(I)$, *in* $L^2(d\mu_0^{(\Lambda, X)})$

(2) $\lim_{\delta \downarrow 0} e^{-V_\delta(I)} = e^{-V(I)}$, *in* $L^p(d\mu_0^{(\Lambda, X)})$,

for all $1 \leqslant p < \infty$.

Combining (1.7), (1.7′), (1.8) and Theorem A with Hölder's inequality (for $L^p(d\mu_0^{(\Lambda, X)})$, $1 \leqslant p < \infty$) we immediately obtain (see [12]):

Corollary B. *Under the hypotheses of Theorem A, but for all $\sigma \in (-\infty, m_0^2)$, and for $f_1, \ldots, f_n$ in $\mathscr{S}(\Lambda)$,*

(1) $\lim_{\delta \downarrow 0} S_\delta^I(f_1, \ldots, f_n) = S^I(f_1, \ldots, f_n)$

(2) $\lim_{\delta \downarrow 0} Z_{\delta, I} = Z_I$

(3) *The measure*

$$d\mu^I(\phi) = Z_I^{-1} e^{-V(I)} d\mu_0^{(\Lambda, X)}(\phi) \tag{1.9}$$

satisfies the FKG inequalities and, in the case of the (s-G) models with $\theta = 0$, the GKS inequalities; (see also [2, Section IV]).

In Section 3 we prove

Theorem C. *Under the hypotheses of Corollary B, for all n and for all $f_1, \ldots, f_n$ in $\mathscr{D}(\mathbb{R}^2)$,*

$$S(f_1, \ldots, f_n) = \lim_{\Lambda' \nearrow \mathbb{R}^2} \lim_{\Lambda \nearrow \mathbb{R}^2} S^I(f_1, \ldots, f_n)$$

exists. The limits are Schwinger functions satisfying all the Osterwalder-Schrader axioms with the possible exception of the cluster property.

In Section 4 we briefly review and detail the existence of some limiting theory, as $\sigma \nearrow m_0^2$, for the case of the (S-G) theory. (By [3], this proves existence of the correlation functions of the two dimensional, neutral classical Coulomb gas in the grand canonical ensemble.)

2. The lattice approximation: proof of Theorem A

We need some additional notation.

Let $\delta = \delta_m = 2^{-m}$, for some positive integer m. We assume that the regions Λ' and $\Lambda \supseteq \Lambda'$ are rectangles with sides of integer length:

$l_i, l_i' \leqslant l_i, \quad i = 1, 2.$

In the following $\Lambda^{(\prime)}$ denotes both, the rectangle in $\mathbb{R}^2$ and the set of all lattice points in $\Lambda^{(\prime)} \cap \mathbb{Z}_\delta^2$. We define

$$T_\Lambda = \frac{2\pi}{l_1}\mathbb{Z} \times \frac{2\pi}{l_2}\mathbb{Z}, \qquad (2.1)$$
$$T_{\Lambda,\delta} = T_\Lambda \cap \left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right] \times \left[-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right],$$

and $\mu(k)^2 = k^2 + 1$,

$$\mu_\delta(k)^2 = \delta^{-2}\left[4 - 2\sum_{i=1}^{2}\cos(\delta k^{(i)})\right] + 1, \qquad (2.2)$$

where $k^{(i)}$ is the ith component of k. We have

$$C^F(x) = (2\pi)^{-2}\int d^2k\, e^{ikx}\mu(k)^{-2} \qquad (2.3)$$
$$C_\delta^F(x') = (2\pi)^{-2}\int_{-\pi/\delta}^{\pi/\delta}\int_{-\pi/\delta}^{\pi/\delta} d^2k\, e^{ikx'}\mu_\delta(k)^{-2},$$

where $x' \in \mathbb{Z}_\delta^2$.

Furthermore

$$C^{(\Lambda,P)}(x) = \sum_{k\in T_\Lambda}\frac{1}{|\Lambda|}e^{ikx}\,\mu(k)^{-2}, \quad \text{and} \qquad (2.4)$$
$$C_\delta^{(\Lambda,P)}(x) = \sum_{k\in T_{\Lambda,\delta}}\frac{1}{|\Lambda|}e^{ikx}\,\mu_\delta(k)^{-2},$$

with $x \in \Lambda \cap \mathbb{R}^2$.

From [12, 16] we know that

$$\mu_\delta(k)^{-2} \leqslant \frac{\pi^2}{4}\mu(k)^{-2}. \qquad (2.5)$$

For the proof of convergence of the lattice approximation under the hypotheses (1.1)–(1.4) we must improve estimate (2.5).

It is obvious that, given $0 \leqslant \alpha \leqslant \pi$, there exists a constant $\hat{d}(\alpha)$ such that

$$1 - \cos y \geqslant \tfrac{1}{2}\hat{d}(\alpha)y^2, \quad \text{for all } y \in [-\alpha, \alpha],$$

with $\hat{d}(\alpha) \nearrow 1$, as $\alpha \searrow 0$.

This together with (2.2) readily yields

$$\mu_\delta(k)^{-2} \leqslant \frac{1}{d(\alpha)}\mu(k)^{-2} \qquad (2.6)$$

for all $k^{(i)} \in \left[-\frac{\alpha}{\delta}, \frac{\alpha}{\delta}\right], \quad i = 1, 2,$

with $d(\alpha) \nearrow 1$, as $\alpha \searrow 0$.

For the purpose of analyzing the local singularities of the covariances (as $\delta \searrow 0$) we decompose them into two pieces: We set

$$C_\delta^{F,\alpha}(x) = (2\pi)^{-2} \int_{-\alpha/\delta}^{\alpha/\delta} \int_{-\alpha/\delta}^{\alpha/\delta} d^2k\, e^{ikx}\, \mu_\delta(k)^{-2}, \tag{2.7}$$

and

$$\tilde{C}_\delta^{F,\alpha}(x) = C_\delta^F(x) - C_\delta^{F,\alpha}(x), \quad x \in \mathbb{Z}_\delta^2.$$

Let $T_{\Lambda,\delta}^\alpha = T_\Lambda \cap [-\alpha/\delta, \alpha/\delta] \times [-\alpha/\delta, \alpha/\delta]$.

For all $x \in \mathbb{R}^2 \cap \Lambda$ we define

$$\begin{aligned} C^\alpha(x) &\equiv C_\delta^{(\Lambda,P,\alpha)}(x) = \sum_{k \in T_{\Lambda,\delta}^\alpha} \frac{1}{|\Lambda|} e^{ikx}\, \mu_\delta(k)^{-2}, \\ \tilde{C}^\alpha(x) &\equiv \tilde{C}_\delta^{(\Lambda,P,\alpha)}(x) = C_\delta^{(\Lambda,P)}(x) - C^\alpha(x). \end{aligned} \tag{2.8}$$

By (2.6),

$$C^\alpha \leqslant \frac{1}{d(\alpha)} C^{(\Lambda,P)}, \tag{2.6'}$$

in the sense of *quadratic forms* on $L^2(\Lambda, d^2x)$.

Using (2.5) we obtain, for $0 < \alpha \leqslant \pi$,

$$\begin{aligned} |\tilde{C}_\delta^{F,\alpha}(x)| &\leqslant \frac{1}{16} \left(\int_{-\pi/\delta}^{\pi/\delta} \int_{-\pi/\delta}^{\pi/\delta} - \int_{-\alpha/\delta}^{\alpha/\delta} \int_{-\alpha/\delta}^{\alpha/\delta} \right) dk^{(1)}\, dk^{(2)} \mu(k)^{-2} \\ &\leqslant \text{const} \left(\frac{\alpha}{\delta}\right)^{-2} \int_{-\pi/\delta}^{\pi/\delta} \int_{-\pi/\delta}^{\pi/\delta} d^2k \\ &\leqslant k(\alpha), \text{ uniformly in } \delta. \end{aligned} \tag{2.9}$$

Clearly inequality (2.9) also holds for $|\tilde{C}^{(\Lambda,P,\alpha)}(x)|$.

We now proceed to prove Theorem A.

2.1. *Proof of Theorem A for the (exp)-models*

Proof of (1):

Let

$$V_\delta^{(m)}(I) = \sum_{x \in \Lambda} \delta^2 \int d\nu_1(\varepsilon) \frac{\varepsilon^m}{m!} :\phi_\delta^m: (x) + \sum_{x \in \Lambda'} \delta^2 \int d\nu_2(\varepsilon) \frac{\varepsilon^m}{m!} : \phi_\delta^m :(x).$$

Clearly $V_\delta(I) = \sum_{m=0}^\infty V_\delta^{(m)}(I)$.

From [12] we know that, for all $m < \infty$,

$$V_\delta^{(m)}(I) \to V^{(m)}(I), \quad \text{as } \delta \searrow 0, \tag{2.10}$$

in $L^p\ (d\mu_0^{(\Lambda,X)})$, for all $1 \leqslant p < \infty$, and $X = F, P, \ldots$.

Thus it suffices to show that $\langle V_\delta(I)^2 \rangle_0^{(\Lambda,X)}$ is bounded, uniformly in δ, for $X = F, (P)$:

$$\langle V_\delta(I)^2\rangle_0^{(\Lambda,X)} = \int d\nu_1(\varepsilon) \sum_{x,y\in\Lambda} \delta^4 \exp\left[\frac{\varepsilon^2}{2} C_\delta^{(\Lambda,X)}(x,y)\right]$$

$$+ \int d\nu_2(\varepsilon) \sum_{x,y\in\Lambda'} \delta^4 \exp\left[\frac{\varepsilon^2}{2} C_\delta^{(\Lambda,X)}(x,y)\right]$$

$$\leqslant \exp[2\pi k(\alpha)]\left\{\int d\nu_1(\varepsilon) \sum_{x,y\in\Lambda} \delta^4 \exp\left[\frac{\varepsilon^2}{2} C_\delta^{(\Lambda,X,\alpha)}(x-y)\right]\right.$$

$$\left.+ \int d\nu_2(\varepsilon) \sum_{x,y\in\Lambda'} \delta^4 \exp\left[\frac{\varepsilon^2}{2} C_\delta^{(\Lambda,X,\alpha)}(x-y)\right]\right\},$$

(where we have used (1.3) and (2.7)–(2.9))

$$\leqslant e^{2\pi k(\alpha)}\left\{\left(\int d\nu_1(\varepsilon) + d\nu_2(\varepsilon)\right)|\Lambda|^2\right.$$

$$\left.+ |\Lambda|\left(\int d\nu_1(\varepsilon) + d\nu_2(\varepsilon)\right)\left(\sum_{x\in\mathbf{Z}_\delta^2} \delta^2\left[\exp\left[\frac{\hat{\varepsilon}^2}{2} C_\delta^{(\Lambda,X,\alpha)}(x)\right] - 1\right]\right)\right\}$$

(and we have set $|\Lambda| = l_1 . l_2$ and used that $|\Lambda'| \leqslant |\Lambda|$)

$$\leqslant e^{2\pi k(\alpha)}\left\{\left(\int d\nu_1(\varepsilon) + d\nu_2(\varepsilon)\right)|\Lambda|^2 + |\Lambda|\left(\int d\nu_1(\varepsilon) + d\nu_2(\varepsilon)\right)\right.$$

$$\left.\times\left(\int d^2x\left[\exp\left[\frac{\hat{\varepsilon}^2}{2} d(\alpha)^{-1} C^{(\Lambda,X)}(x)\right] - 1\right]\right)\right\} < \infty, \qquad (2.11)$$

and

$$\hat{\varepsilon} \equiv \max\{\varepsilon : \varepsilon \in \operatorname{supp} d\nu_{1/2}\}, \text{ and } \alpha \text{ such that } \frac{\hat{\varepsilon}^2}{d(\alpha)} < 4\pi;$$

the second but last inequality follows by Fourier transformation and then applying (2.6), (2.6').

Since the r.h.s. of (2.11) is independent of δ, this completes the proof of (1).

The proof of Theorem A, (2) now follows from the fact that

$$V_\delta(I) \geqslant 0, \quad \text{i.e. } e^{-qV_\delta(I)} \leqslant 1,$$

for all $\delta \geqslant 0$, all $q \geqslant 0$, using

$$e^{-pV_\delta(I)} - e^{-pV_{\delta'}(I)} = p(V_\delta(I) - V_{\delta'}(I)) \int_0^1 ds\, e^{-spV_\delta(I)-(1-s)pV_{\delta'}(I)}. \qquad (2.12)$$

Q.E.D.

2.2. Proof of Theorem A for the (s-G) models

If we compute $\langle V_\delta(I)^2\rangle_0^{(\Lambda,P)}$ explicitly (see e.g. [2], equation (IV.1)) and in view of (2.10) and (2.11) we see that the proof of Theorem A, (1) is almost identical to the proof presented in 2.1 for the (exp)-models.

Proof of Theorem A, (2). As a consequence of equation (2.12) and Hölder's inequality for $L^r(d\mu_0^{(\Lambda,P)})$, $1 \leqslant r < \infty$, it suffices for the proof of (2) to derive an upper bound on

$$\langle e^{-qV_\delta(I)}\rangle_0^{(\Lambda,P)} \tag{2.13}$$

that is uniform in δ.

We need some more notation:

Let C be the kernel of some positive quadratic form defined and continuous on $\mathscr{S}(\Lambda) \times \mathscr{S}(\Lambda)$, $d\mu_C$ the Gaussian measure on $\mathscr{S}'(\Lambda)$ with mean 0 and covariance C, let $\langle - \rangle_C$ denote expectations and $:-:_C$ Wick ordering with respect to $d\mu_C$; see e.g. [16]. We set

$$\begin{aligned} V(C, \Lambda) &= \int d\nu(\varepsilon) \int_\Lambda d^2x :\cos \varepsilon(\phi + \theta):_C (x). \\ &\equiv \int d\nu(\varepsilon) :\cos \varepsilon(\phi + \theta):_C (\Lambda). \end{aligned} \tag{2.14}$$

Since we have chosen *periodic b.c.*, $V_\delta(I)$ and $\langle - \rangle_0^{(\Lambda,P)}$ are *translation invariant*. Therefore

$$\begin{aligned} &\langle e^{-qV_\delta(I)}\rangle_0^{(\Lambda,P)} = \langle \exp[-qV(C_\delta^{(\Lambda,P)}, \Lambda)]\rangle_{C_\delta^{(\Lambda,P)}} \\ &\leqslant 2\langle \operatorname{Cosh}(qV(C_\delta^{(\Lambda,P)}, \Lambda))\rangle_{C_\delta^{(\Lambda,P)}} \end{aligned} \tag{2.15}$$

(see also [2], equation (IV. 39)). A bound on (2.13) follows therefore from

$$\langle \operatorname{Cosh}(qV(C_\delta^{(\Lambda,P)}, \Lambda))\rangle_{C_\delta^{(\Lambda,P)}} \leqslant \text{const} < \infty, \tag{2.16}$$

uniformly in δ.

Next

$$\langle \operatorname{Cosh}(qV(C, \Lambda))\rangle_C = \sum_{m=0}^{\infty} \frac{q^{2m}}{(2m)!} \langle V(C, \Lambda)^{2m}\rangle_C \tag{2.17}$$

Let $d\tilde{\nu}(\varepsilon) = (\int |d\nu(\varepsilon)|)^{-1}|d\nu(\varepsilon)|$. Clearly $d\tilde{\nu}$ is a probability measure. Then

$$\begin{aligned} &|\langle V(C, \Lambda)^{2m}\rangle_C| \\ &\leqslant \left(\int |d\nu(\varepsilon)|\right)^{2m} \int \prod_{i=1}^{2m} d\tilde{\nu}(\varepsilon_i) \left| \left\langle \prod_{i=1}^{2m} :\cos \varepsilon_i(\phi + \theta):_C (\Lambda) \right\rangle_C \right| \\ &\leqslant \left(\int |d\nu(\varepsilon)|\right)^{2m} \int \prod_{i=1}^{2m} d\tilde{\nu}(\varepsilon_i) \prod_{j=1}^{2m} \left\langle (:\cos \varepsilon_j(\phi + \theta):_C (\Lambda))^{2m} \right\rangle_C^{1/2m} \\ &\leqslant \left(\int |d\nu(\varepsilon)|\right)^{2m} \int d\tilde{\nu}(\varepsilon)\langle (:\cos \varepsilon(\phi + \theta):_C (\Lambda))^{2m}\rangle_C \end{aligned}$$

(and we have used Hölder's inequality twice)

$$\leqslant \left(\int |d\nu(\varepsilon)|\right)^{2m} \int d\tilde{\nu}(\varepsilon)\langle (:\cos \varepsilon\phi:_C (\Lambda))^{2m}\rangle_C \tag{2.18}$$

(by Lemma IV. 1 of [2], or an explicit calculation).

Next, by Lemma IV. 1 and (IV. 39) of [2], (1.4), (2.8) and (2.9),

$$\begin{aligned}&\langle(:\cos \varepsilon\phi :_{C_\delta^{(\Lambda,P)}} (\Lambda))^{2m}\rangle_{C_\delta^{(\Lambda,P)}} \\ &\leqslant e^{2\pi k(\alpha)\cdot 2m}\langle(:\cos \varepsilon\phi :_{C^\alpha} (\Lambda))^{2m}\rangle_{C^\alpha}\end{aligned} \tag{2.19}$$

By inequality (2.6′) ($C^\alpha \leqslant d(\alpha)^{-1}C^{(\Lambda,P)}$) and Lemma IV. 3 of [2] ('conditioning'; see also [12, 16]) we have

$$\langle(:\cos \varepsilon\phi :_{C^\alpha} (\Lambda))^{2m}\rangle_{C^\alpha} \leqslant \langle(:\cos \varepsilon\phi :_{d(\alpha)^{-1}C^{(\Lambda,P)}} (\Lambda))^{2m}\rangle_{d(\alpha)^{-1}C^{(\Lambda,P)}} \tag{2.20}$$

for all $m = 0, 1, 2, \ldots$.

Next we note that by (1.4) we may choose $\alpha > 0$ so small that

$$\varepsilon^2\, d(\alpha)^{-1} < 4\pi, \tag{2.21}$$

for all $\varepsilon \in \operatorname{supp} d\tilde{v} = \operatorname{supp} dv$.

Inequality (2.21) permits us to apply Theorems IV.11 (and IV.6) of [2] which give

$$\left\langle \operatorname{Cosh}\left(e^{2\pi k(\alpha)} \int |dv(\varepsilon)|\, :\cos \hat{\varepsilon}\phi :_{d(\alpha)^{-1}C^{(\Lambda,P)}} (\Lambda)\right)\right\rangle_{d(\alpha)^{-1}C^{(\Lambda,P)}} \leqslant e^{\text{const.}\,|\Lambda|}, \tag{2.22}$$

with $\hat{\varepsilon} \equiv \max\{\varepsilon : \varepsilon \in \operatorname{supp} dv\}$.

The proof of (2.13) is now completed by combining (2.15) with (2.17)–(2.22).

Q.E.D.

This completes our discussion of the lattice approximation for the (exp) – and (s-G) – models under conditions (1.3), (1.4). In the following sections we present some applications.

3. The infinite volume limit: proof of Theorem C

3.1. The (exp)-models

Rather than proving Theorem C directly for the Schwinger functions we first consider their generating functionals

$$\langle e^{\zeta\phi(f)}\rangle^I \equiv \int d\mu^I(\phi)\, e^{\zeta\phi(f)}, \tag{3.1}$$

where $d\mu^I$ is the cutoff interacting measure defined in (1.9), f is e.g. in $\mathscr{D}(\mathbb{R}^2)$ and $\zeta \in \mathbb{C}$. For such f and ζ we prove the existence of a limit in (3.1), first as $\Lambda \nearrow \mathbb{R}^2$ (by inclusion) and then as $\Lambda' \nearrow \mathbb{R}^2$ (by inclusion). The existence of these limits is proven by using the FKG inequalities [12] to derive monotonicity properties in Λ and Λ' and proving uniform upper bounds. Our proof follows closely the methods developed in Section 4 of [8], exploiting one additional idea.

Schwinger functions S^I may be obtained from the generating functional $\langle e^{\zeta\phi(f)}\rangle^I$ by differentiation, and this will remain true in the limit $\Lambda = \Lambda' = \mathbb{R}^2$.

Since the Schwinger functions are multilinear, it suffices to consider *real* test functions f in (3.1). Given f we set $f_+ = \frac{1}{2}(|f| + f), f_- = \frac{1}{2}(|f| - f)$. Then

$$\begin{aligned}|\langle e^{\zeta\phi(f)}\rangle^I| &= |\langle e^{\zeta(\phi(f_+) - \phi(f_-))}\rangle^I| \\ &\leqslant \langle e^{\operatorname{Re}\zeta(\phi(f_+) - \phi(f_-))}\rangle^I \\ &\leqslant \{\langle e^{2\operatorname{Re}\zeta\phi(f_+)}\rangle^I\}^{1/2}\{\langle e^{-2\operatorname{Re}\zeta\phi(f_-)}\rangle^I\}^{1/2}\end{aligned} \tag{3.2}$$

Hence uniform upper bounds on $|\langle e^{\zeta\phi(f)}\rangle^{I}|$, (for f real, arbitrary $\zeta \in \mathbb{C}$) follow from uniform bounds on

$$\langle e^{\pm\phi(h)}\rangle^{I}, \qquad 0 \leqslant h \in \mathscr{D}(\mathbb{R}^2).$$

Next we note that

$$V(\Lambda, \Lambda') = F^1_{\Lambda} + F^2_{\Lambda'}, \tag{3.3}$$

where $F^1_{\Lambda} = \int_{\Lambda} d^2x \int dv_1(\varepsilon) :e^{\varepsilon\phi}: (x)$ is *monotone increasing* in ϕ, and

$$F^2_{\Lambda'} = \int_{\Lambda'} d^2x \int dv_2(\varepsilon) :e^{\varepsilon\phi}: (x) \tag{3.4}$$

is *monotone decreasing* in ϕ, a consequence of (1.3).

Applying the FKG inequalities we immediately conclude that $\langle e^{\pm\phi(h)}\rangle^{I}$ is

$$\text{monotone}\begin{pmatrix}\text{decreasing}\\ \text{increasing}\end{pmatrix}\text{ in } \Lambda, \quad \text{and} \tag{3.5}$$

$$\text{monotone}\begin{pmatrix}\text{increasing}\\ \text{decreasing}\end{pmatrix}\text{ in } \Lambda'. \tag{3.6}$$

We now fix Λ', $|\Lambda'| < \infty$, for the moment. Then, by (3.5), $\langle e^{\phi(h)}\rangle^{(\Lambda,\Lambda',F)}$ is decreasing in Λ and therefore uniformly bounded in Λ.

Let $l \times T$ denote the rectangle with sides of length l and T parallel to the coordinate axes and centered at the origin. Then

$$0 < \langle e^{-\phi(h)}\rangle^{(\Lambda,\Lambda',F)} \leqslant \langle e^{-\phi(h)}\rangle^{(l\times T,\Lambda',F)}, \tag{3.7}$$

whenever $l \times T \supset \Lambda$, a consequence of (3.5). Next, by (3.6),

$$\langle e^{-\phi(h)}\rangle^{(l\times T,\Lambda',F)} \leqslant \langle e^{-\phi(h)}\rangle^{(l\times T,\varnothing,F)} \tag{3.8}$$

As first $T \to \infty$ and then $l \to \infty$ the r.h.s. of (3.8) is bounded uniformly in l and T, by a standard exponential ϕ-bound, [4, 8], whence uniform boundedness of $\langle e^{-\phi(h)}\rangle^{(\Lambda,\Lambda' F)}$ in Λ.

Remark. The proof of exponential ϕ-bounds for the (exp)- and the (s-G)-models is identical to the well known ones given for the $P(\phi)_2$-models in [4, 8], and references given there. Indeed, almost all methods and results of these references apply to all two dimensional Bose self-interactions that are free of ultraviolet divergences.

Combining (3.5)–(3.8) we conclude that

$$\lim_{\Lambda\nearrow\mathbb{R}^2} \langle e^{\pm\phi(h)}\rangle^{(\Lambda,\Lambda',F)} \equiv \langle e^{\pm\phi(h)}\rangle^{\Lambda'}$$

exists. Uniform boundedness (see (3.2), (3.5), (3.7), (3.8)) and a standard application of Vitali's theorem (see e.g. [5]) yield

$$\lim_{\Lambda\nearrow\mathbb{R}^2} \langle e^{\zeta\phi(f)}\rangle^{(\Lambda,\Lambda',F)} \equiv \langle e^{\zeta\phi(f)}\rangle^{\Lambda'} \tag{3.9}$$

exists, for all $\zeta \in \mathbb{C}$ and all real $f \in \mathscr{D}(\mathbb{R}^2)$.

Next, for $h \geqslant 0$, $\langle e^{-\phi(h)}\rangle^{\Lambda'}$ is decreasing in Λ', by (3.6), hence uniformly bounded, and

$$0 \leqslant \langle e^{\phi(h)}\rangle^{\Lambda'} \leqslant \langle e^{\phi(h)}\rangle^{l\times T} \tag{3.10}$$

for all l and T with $l \times T \supset \Lambda'$.

As first $T \to \infty$ and then $l \to \infty$, the r.h.s. of (3.10) is bounded uniformly in T and l, by an extension of the exponential ϕ-bounds proven in Sections 2 and 7 of [8]; (see also [10]).

Repeating the previous arguments we conclude that

$$\lim_{\Lambda' \nearrow \mathbb{R}^2} \langle e^{\zeta\phi(f)}\rangle^{\Lambda'} \equiv \langle e^{\zeta\phi(f)}\rangle, \tag{3.11}$$

for all $\zeta \in \mathbb{C}$, $f \in \mathscr{D}(\mathbb{R}^2)$.

Euclidean invariance and Osterwalder-Schrader positivity of $\langle e^{\zeta\phi(f)}\rangle$ follow from the fact that the limits $\Lambda \nearrow \mathbb{R}^2$, $\Lambda' \nearrow \mathbb{R}^2$ can be taken by inclusion; see [12, 5].

The exponential ϕ-bounds show that $\langle e^{\zeta\phi(f)}\rangle$ has, for each $\zeta \in \mathbb{C}$, an extension to a functional continuous on $\mathscr{S}_{\text{real}}(\mathbb{R}^2)$.

The proof of Theorem C for the (exp)-models is now completed as in [5], (where the $P(\phi)_2$-models were considered). We remark that, using the methods of [8], it is easy to construct clustering infinite volume Schwinger functions for the (exp)-models, as well.

3.2. *Proof of Theorem C for the (s-G)-models*

Given Corollary B the proof is almost identical to the ones of [10, 6] for, resp., the ϕ_2^4–ϕ_3^4-models.

By introducing a new field

$$\tilde{\phi} = \text{sig}\,\theta[\phi + \theta]$$

(and a suitable redefinition of $V(\Lambda, \Lambda')$) we see that it suffices to consider the case where $\theta = 0$ in equation (1.2), but a term $-\mu \int_\Lambda \phi(x)\, d^2x$, $\mu > 0$, is added to the action $V(\Lambda, \Lambda')$ (see (1.2)).

Then the measure $d\mu^I(\phi)$ defined in (1.9) satisfies the first and the second Griffiths (GKS) inequality.

(1) Let $f_1, \ldots, f_n \in \mathscr{D}(\mathbb{R}_2)$. Applying the convergence of the cluster expansion of [11] for the (s-G) models, a result of [7], we conclude that

$$\lim_{\Lambda \nearrow \mathbb{R}^2} \left\langle \prod_{i=1}^n \phi(f_i) \right\rangle^{(\Lambda, \Lambda', P)} \equiv \left\langle \prod_{i=1}^n \phi(f_i) \right\rangle^{\Lambda'} \tag{3.12}$$

exists, (and the limit is 'independent of how $\Lambda \nearrow \mathbb{R}^2$').

(2) Applying the first and the second Griffiths inequality we obtain (see [12, 6])

$$\begin{aligned} &\left| \left\langle \prod_{i=1}^n \phi(f_i) \right\rangle^{(\Lambda, \Lambda', P)} \right| \\ &\leqslant \left\langle \prod_{i=1}^n \phi(|f_i|) \right\rangle^{(\Lambda, \Lambda', P)}, \text{ by GKSI} \\ &\leqslant \left\langle \prod_{i=1}^n \phi(|f_i|) \right\rangle^{(\Lambda, \Lambda, P)}, \text{ by GKSII.} \end{aligned} \tag{3.13}$$

Since we have chosen periodic boundary conditions, Λ is necessarily a rectangle of the form $l \times T$. As a consequence of exponential ϕ-bounds for *periodic* boundary conditions [15], (see also [8]) the r.h.s. of (3.13) remains uniformly bounded, as T and l tend to ∞, (in any order). This and (3.12) prove that

$$\left| \left\langle \prod_{i=1}^{n} \phi(f_i) \right\rangle^{\Lambda'} \right| \tag{3.14}$$

is bounded uniformly in Λ'.

(3) For positive test functions $f_1, \ldots, f_n$,

$$\left\langle \prod_{i=1}^{n} \phi(f_i) \right\rangle^{\Lambda'}$$

is *increasing* in Λ', by GKSII; see [10, 6]. This and multilinearity of $\langle \prod_{i=1}^{n} \phi(f_i) \rangle^{\Lambda'}$ complete the proof of Theorem C; (see also [5, 6, 10]).

(4) Applying the ϕ-bounds of [10, 8, 15] to the (s-G)-models we conclude that the limiting EGF's constructed in (2), (3) are the moments of a probability measure $d\mu^{\sigma}(\phi)$ on $\mathscr{S}_{\text{real}}(\mathbb{R}^2)'$, and, using [7], (2) and (3)

$$\int d\mu^{\sigma}(\phi)\, e^{\phi(f)} = \lim_{l' \to \infty} \lim_{T' \to \infty} \lim_{l \to \infty} \lim_{T \to \infty} \int d\mu^{(l \times T,\, l' \times T',\, P)}(\phi)\, e^{\phi(f)} \tag{3.15}$$

for arbitrary Schwartz space functions f.

Because of (3.15) we say, following [8], that

$$d\mu^{(\Lambda, \Lambda', P)}(\phi) \to d\mu^{\sigma}(\phi),$$

'by iteration'.

4. The zero bare mass limit ($\sigma \nearrow m_0^2$) for the (s-G) models

Let $V(\Lambda, \Lambda')$ be defined as in (1.2). For simplicity we consider the case where

$$dv(\varepsilon) = \lambda \delta(\varepsilon - \varepsilon_0)\, d\varepsilon, \tag{4.1}$$

for some ε_0, with $\varepsilon_0^2 < 4\pi$, and some real λ.

(This is the case of interest in the construction of 'soliton-sectors' for the quantum sine-Gordon equation; see [2].)

We define

$$\begin{aligned} c(f) &= \int d^2x : \cos \varepsilon_0(\phi + \theta) : (x) f(x) \\ s(f) &= \int d^2x : \sin \varepsilon_0(\phi + \theta) : (x) f(x), \end{aligned} \tag{4.2}$$

and Wick ordering is done with respect to bare mass m_0; (m_0 is chosen so big that the cluster expansion converges for dv such as in (4.1)).

We also define

$$\partial\phi(f) = \int d^2x\, \partial_\mu \phi(x) f^\mu(x) \tag{4.3}$$

Finally $\Lambda(x)$ denotes the characteristic function of the region $\Lambda \subset \mathbb{R}^2$ and $|\Lambda|$ its volume.

Let $d\mu_0^M$ be the Gaussian measure with bare mass M; (see Section 1).

Definition: For $\sigma \leqslant m_0^2$ we define the vacuum energy density (pressure)

$$p(\lambda_1, \lambda_2; \sigma) = \lim_{\Lambda \nearrow \mathbb{R}^2} \frac{1}{|\Lambda|}\Big\{\log \int d\mu_0^{m_0}(\phi) \times \exp[\lambda_1 c(\Lambda) + \lambda_2 s(\Lambda) + (\sigma/2):\phi^2:(\Lambda)] - \log \int d\mu_0^{m_0}(\phi) \exp[(\sigma/2):\phi^2:(\Lambda)]\Big\} \tag{4.4}$$

As a special case of Theorem 3.1 of [8] (applied to the (s-G)-model) we note that

$$p(\lambda_1, \lambda_2; \sigma) = \lim_{\Lambda \nearrow \mathbb{R}^2} \frac{1}{|\Lambda|}\Big\{\log \int d\mu_0^{\sqrt{m_0^2-\sigma}}(\phi) \exp[\lambda_1 c(\Lambda) + \lambda_2 s(\Lambda)] \tag{4.5}$$

We now quote a basic result of [2, 3]:

Theorem 4.1. *$|p(\lambda_1, \lambda_2; \sigma)|$ is bounded uniformly in σ on $[0, m_0^2]$, and*

$$|p(\lambda_1, \lambda_2; \sigma)| \leqslant K_r(|\lambda_1|^r + |\lambda_2|^r) \tag{4.6}$$

where K_r is a constant independent of σ that is finite for

$$r \in ((1 - \varepsilon_0^2/4\pi)^{-1}, \infty). \tag{4.7}$$

(For the proof see Section IV of [2] or Section 3 of [3].)

From Section 3.2 we know that the infinite volume interacting measure $d\mu^\sigma(\phi)$ of the (s-G)-model exists and that

$$d\mu^{(\Lambda, \Lambda', P)}(\phi) \to d\mu^\sigma(\phi), \quad \text{for} \quad \sigma < m_0^2, \tag{4.8}$$

by *iteration*; see (3.15).

Because of (4.6), (see also [3] Section 3), and (4.8), and for $\sigma < m_0^2$, we may apply Theorem 7.2 of Reference [8] which tells us that, for arbitrary Schwartz space functions f and g, $\exp[c(f) + s(g)]$ is a well defined random variable for $d\mu^\sigma$ and

$$\exp[c(f) + s(g)] \in \bigcap_{1 \leqslant r < \infty} L^r(\mathcal{S}', d\mu^\sigma). \tag{4.9}$$

Appealing now to Theorems 2.3 and 2.4 of Reference [8] we obtain the basic inequality

$$\left|\int d\mu^\sigma(\phi) \exp[c(f) + s(g)]\right| \leqslant \exp \int d^2x\{p(\lambda + \operatorname{Re} f(x), \operatorname{Re} g(x); \sigma) - p(\lambda, 0; \sigma)\} \tag{4.10}$$

Since $p(\lambda_1, \lambda_2; \sigma)$ is jointly *convex* in λ_1 and λ_2, see [3], the integral on the r.h.s. of (4.10) is well defined. Moreover, it is easy to show, using the bounds of [2, 3], that the *first* derivatives of $p(\lambda_1, \lambda_2; \sigma)$ in λ_1, resp. λ_2 are bounded *uniformly* in $\sigma \in [0, m_0^2]$

and λ_1, λ_2 in an arbitrary compact set. Therefore we may apply (4.6) to conclude that the r.h.s. of (4.10) is bounded by

$$\exp\left[\tilde{K}_r(\|\mathrm{Ref}\|_r^r + \|\mathrm{Ref}\|_1 + \|\mathrm{Reg}\|_r^r + \|\mathrm{Reg}\|_1)\right], \tag{4.11}$$

where $\tilde{K}_r$ is a constant *independent* of σ that is finite if r satisfies (4.7).

Finally we combine the methods of Section 2 of Reference [8] with the result – due to [13] – that the infinite volume limit pressure defined over some Gaussian measure with positive bare mass is *independent* of classical boundary conditions and with the exp $(\partial\phi)$-bounds proven in [9] for *periodic* (rather than weak coupling) b.c. to prove the inequality

$$\int d\mu^\sigma(\phi)\, e^{\partial\phi(f)} \leqslant e^{1/2\|f\|_2^2}. \tag{4.12}$$

(The details of the proof of (4.12) are very much based on the convergence of the lattice approximation established in Section 2, as the lattice approximation is used in [9]! We omit these details, because they amount to a lengthy repetition and rephrasing of known results.)

Since the bounds (4.11) and (4.12) are *independent* of $\sigma \in [0, m_0^2)$, we may now apply a standard *compactness argument* (Cantor's diagonal procedure) to conclude that there exists a sequence $\{\sigma_n\}_{n=0}^{\infty}$ converging to m_0^2 such that, for all f and g in $\mathscr{S}(\mathbb{R}^2)$ and h in $L^2(\mathbb{R}^2)$,

$$Z(f, g; h) \equiv \lim_{n\to\infty} \int d\mu^{\sigma_n}(\phi)\, e^{c(f)+s(g)}\, e^{\partial\phi(h)} \tag{4.13}$$

exists. Since the measures $d\mu^{\sigma_n}$, $n = 0, 1, 2 \ldots$ are all Euclidean invariant, so is the functional $Z(f, g; h)$. This fact combined with Theorem C and some standard arguments proves

Theorem 4.2. *The functional $Z(f, g; h)$ is the generating functional of Euclidean Green's functions that satisfy all the Osterwalder-Schrader axioms with the possible exception of the cluster properties.*

Remarks:

1. It is known (see [2] and references given there) that the field theory reconstructed from these EGF's has non-trivial superselection ('solition'-) sectors disjoint from the vacuum sector.
2. For $h = 0$, the functional derivatives of $Z(f, g; 0)$ in f and g are the correlation functions of the two dimensional, neutral, classical Coulomb gas; see [3]. For $\varepsilon_0^2 \ll 4\pi$ we expect exponential clustering (Debye screening).

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