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Dynamics of the Dicke laser model II: an example of a semi-classical theory in the presence of focal points

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Abstract. The nature of the time-dependent WKB–Maslov approximation is investigated for the quantum mechanical system which is equivalent to the Dicke Laser model. The local semi-classical solution and its continuation beyond the focal points are discussed.

1. Introduction

The previous paper [1] on the dynamics of the Dicke model was concerned with explicit calculations of the most important dynamical quantities of the system. The main tool in these calculations was the time-dependent WKB-Maslov method. In the present paper this method will be studied in detail; in particular, the peculiarities that appear in the application to the Dicke Hamiltonian will be analysed. This is a subject of its own interest because it gives an interesting example for the use of the WKB method. This example shows how miserable our imagination is, even for quantum systems with one degree of freedom, if trained only with the harmonic oscillator.

The quantum mechanical problem to be discussed is defined in the next section. Its relation to the Dicke Hamiltonian was discussed before [1] and is unimportant in the present context, so that the subject is fully understandable without any reference to quantum optics. The corresponding classical mechanical problem is then considered. The classical solution shows clearly the local nature of the Hamilton–Jacobi theory: The Cauchy problem for the Hamilton–Jacobi equation has a unique solution only for small times $t < T_0$. For larger times the solution becomes multivalued, the branch-points are the focal points. As a consequence, the semi-classical WKB solution can be derived only for $t < T_0$ (Section 3), and in fact, it diverges at the focal point T_0. The main problem is the continuation of this local semi-classical solution beyond the focal points. This can be done according to a method of Maslov [2] by going over to the momentum space. Then, beyond a focal point new branches $S_k(x, t)$ of the action contribute to the semi-classical solution. In this way, the global semi-classical solution is obtained (Section 4). It is valid on a much larger time interval, except in the neighborhood of the focal points. Since the number of branches $S_k(x, t)$ increases linearly with time t, the error in the semi-classical solution increases in the same way. This is the mechanism which finally leads to the breakdown of the global semi-classical solution for large times, where the system then shows an essentially quantum-mechanical behavior.

2. The quantum mechanical and the classical Cauchy problems

We will consider the following time-dependent Schrödinger equation

$$ih\frac{\partial y}{\partial t} = -\frac{h^2}{2}\left(x\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2}{\partial x^2}xy\right) + V(x)y \tag{2.1}$$

$$\stackrel{\text{def}}{=} Hy$$

with the potential energy

$$V(x) = \tfrac{1}{4}x^3 - \eta x + \frac{\lambda^2}{x}, \quad \eta, \lambda > 0. \tag{2.2}$$

For simplicity, a (non-resonant) term $\delta x^2/2$ in $V(x)$ [1] has been dropped. The whole analysis goes through with some minor changes in this non-resonant case too.

The Hamiltonian

$$H = -h^2\frac{\partial}{\partial x}x\frac{\partial}{\partial x} + V(x) \tag{2.3}$$

defines a symmetric differential operator H in $L^2(0, \infty)$. It is essential for the formulation of a Cauchy problem for equation (2.1) to extend H to a self-adjoint operator. Both end-points $x = 0$ and ∞ are singular points which belong, according to the classification of H. Weyl, either to the limit-point or to the limit-circle case [3]: H is in the limit-circle case at 0 (respectively at ∞) if all solutions y of

$$Hy = 0 \tag{2.4}$$

are square integrable at 0 (respectively at ∞), otherwise H is in the limit-point case. According to a simple criterion ([3], p. 231), H is always in the limit-point case at infinity. Since for $x \to 0$ the two fundamental solutions y_1, y_2 of (2.4) behave as

$$y_1(x) \sim x^{\lambda}, \qquad y_2(x) \sim x^{-\lambda}, \tag{2.5}$$

H is in the limit-point case at 0 for $\lambda \geqslant \frac{1}{2}$ and in the limit-circle case for $\lambda < \frac{1}{2}$ (the case $\lambda = \frac{1}{2}$ is just the superradiant case previously discussed [5]). According to the general theory [3, 4], there exists a unique self-adjoint extension of H in the limit-point case $\lambda \geqslant \frac{1}{2}$.

In the limit-circle case there are many self-adjoint extensions specified by a boundary condition at $x = 0$ of the form ([4], p. 187)

$$\lim_{x \to 0}\{y(x)\lambda(\alpha x^{\lambda} - \beta x^{-\lambda}) - y'(x)x(\alpha x^{\lambda} + \beta x^{-\lambda})\} = 0 \tag{2.6}$$

We know, however, from reference [1] that the physically important solution $y(x, t)$ corresponding to the quantum optical problem behaves as x^{λ} for $x \to 0$. That requires $\beta = 0$ in (2.6). We therefore take for $\lambda < \frac{1}{2}$ the self-adjoint extension defined by

$$\lim_{x \to 0}\{\lambda x^{\lambda}y(x) - x^{\lambda+1}y'(x)\} = 0 \tag{2.7}$$

In both cases, a self-adjoint Hamiltonian (denoted also by H) has been obtained. This operator H then generates a unitary group which gives the unique global solution $y(x, t)$ of the Cauchy problem for the Schrödinger equation (2.1) in $L^2(0, \infty)$.

Let us now turn to classical mechanics. The Hamiltonian function $H(p, x)$ corresponding to (2.3) is

$$H(p, x) = xp^2 + V(x), \tag{2.8}$$

and the corresponding Hamiltonian–Jacobi equation reads

$$\frac{\partial S}{\partial t} + x\left(\frac{\partial S}{\partial x}\right)^2 + V(x) = 0. \tag{2.9}$$

A complete integral S of this equation depending on two constants of integration x_0 and ε is given by the following elliptic integral

$$\tilde{S}(x, x_0, \varepsilon; t, 0) = \int_{x_0}^{x} dx' \left(\eta - \tfrac{1}{4}x'^2 - \frac{\varepsilon}{x'} - \frac{\lambda^2}{x'^2}\right)^{1/2} + \varepsilon t. \tag{2.10}$$

Then the Cauchy problem of determining a solution $S(x, t)$ of equation (2.9) which satisfies the initial condition

$$S(x, 0) = S_0(x) \tag{2.11}$$

at $t = 0$, is solved by forming envelopes of solutions (2.10)

$$S(x, t) = \tilde{S}(x, x_0, \varepsilon; t, 0) + S_0(x_0), \tag{2.12}$$

where the functions $x_0 = x_0(x, t)$ and $\varepsilon = \varepsilon(x, t)$ are implicitly given by the two relations

$$\frac{\partial \tilde{S}}{\partial \varepsilon} = 0 \tag{2.13}$$

$$\frac{\partial \tilde{S}}{\partial x_0} + \frac{\partial S_0}{\partial x_0} = 0. \tag{2.14}$$

The first relation (2.13) leads to

$$t = \int_{x_0}^{x} R(x')^{-1/2}\, dx' \tag{2.15}$$

with

$$R(x) = -x^4 + 4\eta x^2 - 4\varepsilon x - 4\lambda^2. \tag{2.16}$$

This equation (2.15) can be inverted [5], yielding x_0 (or x respectively)

$$x_0 = x + \frac{P'(t)R(x)^{1/2} + (-2x^3 + 4\eta x - 2\varepsilon)(P + \frac{1}{2}x^2) - xR(x)}{2(P + \frac{1}{2}x^2)^2 + \frac{1}{2}R(x)}. \tag{2.17}$$

Here

$$P(t) = p(t) - \tfrac{1}{3}\eta, \qquad P'(t) = \frac{dp(t)}{dt} \tag{2.18}$$

where

$$p = p(t; g_2, g_3)$$

is Weierstrass' p-function with invariants

$$g_2 = \tfrac{4}{3}\eta^2 + 4\lambda^2$$

$$g_3 = -\tfrac{8}{27}\eta^3 + \tfrac{8}{3}\eta\lambda^2 + \varepsilon^2. \tag{2.19}$$

The equation for x is obtained from (2.17) by the changes $x \leftrightarrow x_0$, $t \leftrightarrow -t$. These equations give the spatial trajectories leading from the point x to the point x_0 in time t (or reversed) for a given energy $-\varepsilon$.

In order to discuss the second relation (2.14), let us consider the following initial condition (2.11)

$$S_0(x) = -i(\lambda \log x - \tfrac{1}{4}x^2), \tag{2.20}$$

which will be of interest in the following. Since this S_0 is no longer real, the corresponding action $S(x, t)$ will be complex. The corresponding complex trajectories do not describe classical motions in the ordinary sense. However, if the quantum mechanical (time dependent) wave function is considered in a classically forbidden region, then the relevant classical trajectories are necessarily complex. Therefore, in a general semi-classical theory, all (real and complex) trajectories have to be used. This point is missed in the book by Maslov [2] (apart from a remark on the last page, 328), which restricts the usefulness of the general apparatus developed there. Inserting (2.20) into equation (2.14), we get the simple expression

$$\varepsilon = (\eta - \lambda)x_0 . \tag{2.21}$$

This has to be substituted for ε in (2.17) yielding a transcendental equation for $x_0 = x_0(x, t)$.

Let us consider this equation in some detail for the special value $x = 0$, which is of particular interest in the quantum optical problem [1]. In this case the equation for $x_0(0, t)$ becomes

$$x_0 = -\frac{i\lambda P'(t)}{P^2 - \lambda^2 + (\eta - \lambda)P}. \tag{2.22}$$

We emphasize that x_0 enters also in the invariant g_3 (2.19) of the p-function through ε (2.21). The p-function and its derivative (2.18) are periodic functions of t with real half-periods $\omega = \omega(g_2, g_3)$ depending on the invariants g_2, g_3. Consequently, if $x_0(t_1)$ is a solution of (2.22) for $t = t_1$, it is also a solution for $t = t_1 + \omega$. But the periods vary with the value of x_0. Therefore $x_0(t)$ is not periodic in t, but it can be constructed from the values in the interval $0 \leqslant t \leqslant \omega_0 = \omega(x_0 = 0)$ by a non-uniform shear transformation (see Figure 1). For small times, $x_0(t)$ is unique and shows an oscillatory behavior. Since the period $\omega(x_0)$ increases with increasing $|x_0|$, the larger values of $|x_0|$ are shifted more to the right than the smaller ones during the next oscillation. This leads to a fall over of the wave, which occurs already in the second oscillation, in general. In Figure 1 numerical values of $x_0(0, t)$ are shown for $\lambda = \frac{1}{2}$, $\eta = \frac{3}{2}$. We see that $x_0(t)$ necessarily becomes multivalued for larger t. The branch-points where new branches $x_{0k}(t)$ appear (or disappear), i.e. where

$$\frac{\partial x_0}{\partial t} = \pm\infty \tag{2.23}$$

are the focal points. The number of branches increases linearly with t.

In the general case $x \neq 0$ the situation is qualitatively the same. Since $x_0(x, t)$ is multivalued, $S(x, t)$ (2.12) becomes multivalued as well. We therefore get finitely many different solutions $S_k(x, t)$ of the Cauchy problem for the Hamilton–Jacobi equation. But there exists a finite time interval $[0, T(x))$ (larger than one oscillation period) where the solution $x_0(x, t)$ is unique. Moreover, the focal points $T(x)$ are

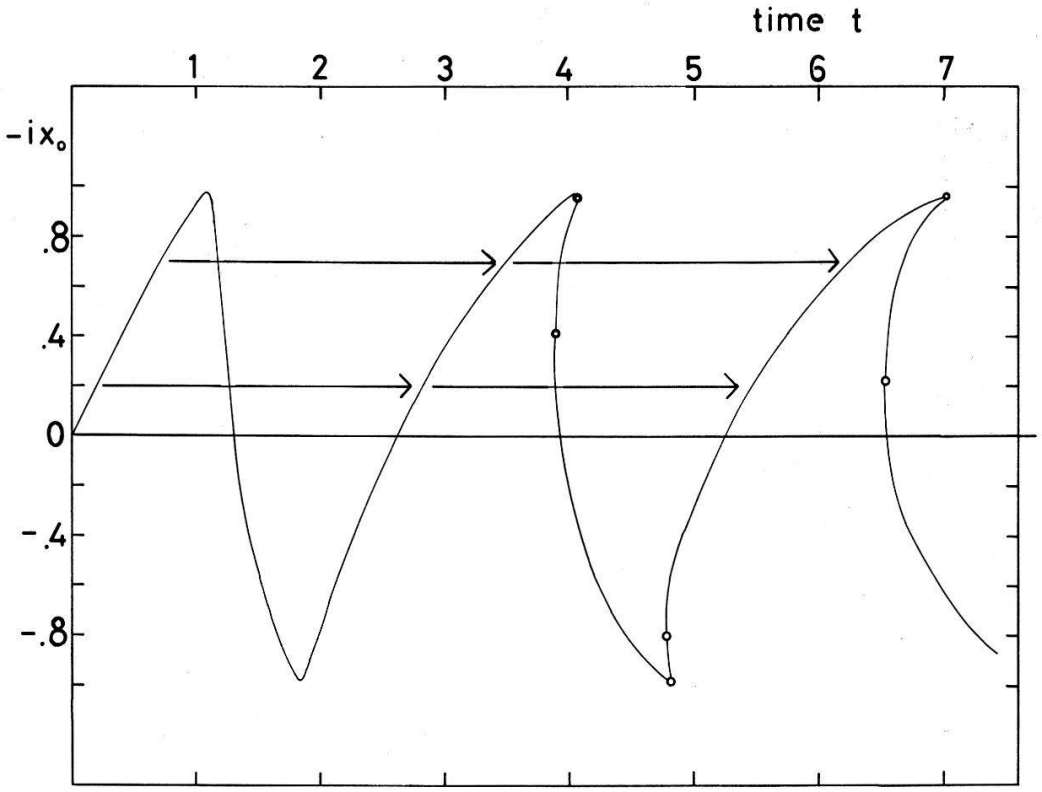

Figure 1:
$x_0(x, t)$ for $x = 0$, $\lambda = 0.5$, $\eta = 1.5$ as a function of t. The arrows illustrate the shear transformation. The focal points are marked by little circles.

bounded away from 0

$$T_0 = \inf_{0 \leqslant x < \infty} T(x) > 0. \tag{2.24}$$

As in the case of ordinary mechanical systems we could try to solve equation (2.15) for ε

$$\varepsilon = \varepsilon(x, x_0, t) \tag{2.25}$$

in order to get the action $\tilde{S}$ (2.10) as a function of the coordinates x_0, x and t. Equation (2.15) can indeed be inverted in the form ([5] equation 4.19)

$$\varepsilon = (x + x_0)(\tfrac{1}{2}xx_0 - P) + Q^{1/2} \tag{2.26}$$

with

$$Q = [(x + x_0)^2 - 4\eta - 4P][\tfrac{1}{4}x^2x_0^2 + \lambda^2 - Pxx_0]. \tag{2.27}$$

But this transcendental equation for ε has infinitely many solutions, in general, *even for arbitrarily small t!* This can be seen as follows: For fixed x, x_0, $\varepsilon(x, x_0, t)$ is a monotonely decreasing function of $t(\varepsilon(t = 0) = +\infty)$, following one branch. In the same manner as was discussed above, infinitely many new branches can be generated from a given one by non-uniform shear transformations. The crucial point is that the real period ω of the shear translation goes to 0 for $t \to 0$, which can be deduced from the expansion of the p-function for small t. Consequently, the branches approach each other for $t \to 0$. Then there are infinitely many different values of $\varepsilon(x, x_0, t)$ *for all* $t > 0$. In Figure 2, $\varepsilon(x, x_0, t)$ as a function of t is depicted for fixed x, x_0 in a typical case.

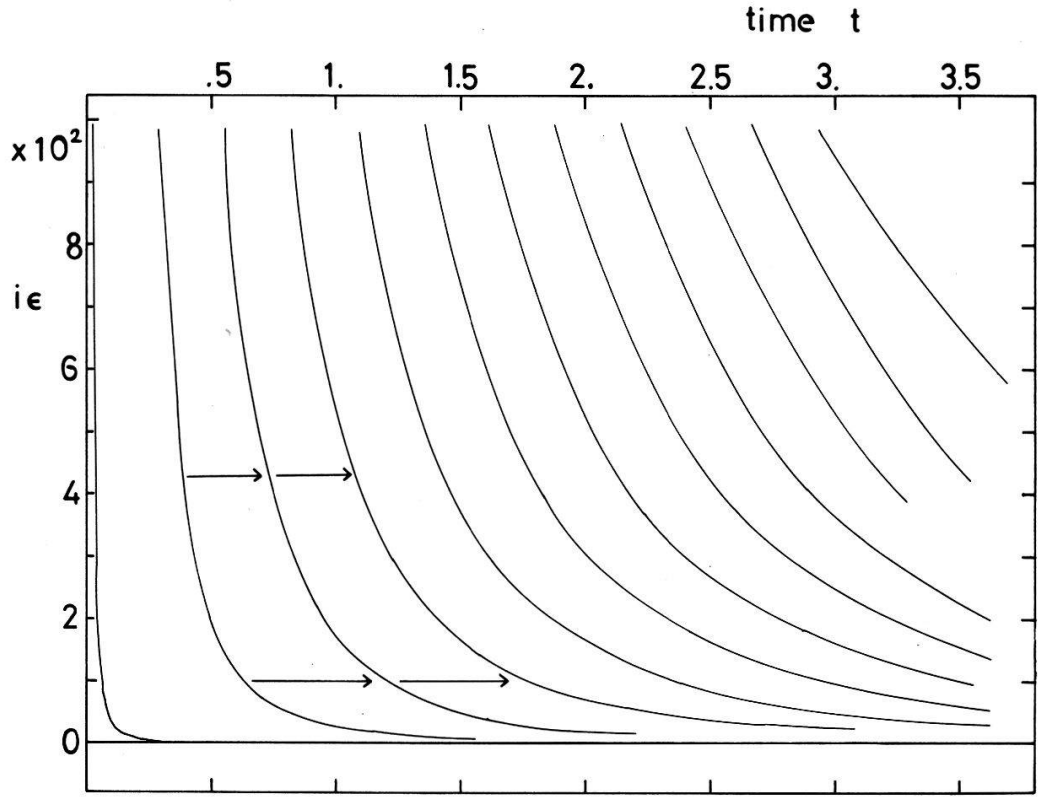

Figure 2:
Energy $\varepsilon(0, x_0, t)$ for $x_0 = 0.4i$, $\lambda = 0.25$, $\eta = 1.25$ as a function of t. The arrows illustrate the shear transformation.

This behavior of the energy ε has some unusual consequences. Given two points x_0 and x in space, then there exist infinitely many trajectories leading from x_0 to x in a given time t, and this remains true for arbitrarily small t. That is in sharp contrast to the behavior of a ordinary point mass (with kinetic energy $p^2/2m$). At least if the potential $V(x)$ is bounded, there is only one trajectory from x_0 to x for sufficiently small t ([7], theorem 3.1). At the same time the action $\tilde{S}_k(x, x_0, t)$ has infinitely many different branches for all $t > 0$. Then, it is quite complicated to derive a semi-classical approximation for the Green's function $K(x, x_0, t)$ of the Schrödinger equation (2.1) even for small t, as can be done in ordinary quantum mechanics [7]. Therefore, it seems to be rather difficult to convert the formal arguments of reference [1] into a proof. For this reason, we proceed a somewhat different way in the following sections.

3. The local semi-classical solution

It is our aim to construct an asymptotic approximation for $h \to 0$ to the solution $y(x, t)$ of the Cauchy problem for the Schrödinger equation (2.1). Following Maslov [6], we consider the initial condition

$$y(x, 0) = y_0(x) = \rho(x) \exp \frac{1}{h} \varphi(x). \tag{3.1}$$

In this section, we assume ρ to be in $L^2(0, \infty)$ and Re $\varphi(x)$ bounded from above, in addition ρ and φ are supposed to have analytic continuations into the complex x-plane. For the physically most interesting initial state (the so-called fully excited state of quantum optics) we have [1]

$$y_0(x) = \exp \frac{1}{h}(\lambda \log x - \tfrac{1}{4}x^2), \tag{3.2}$$

which is not precisely of the form (3.1). But if y_0 is normalized such that the $L^2(0, \infty)$-norm is 1

$$\| y_0 \|^2 = 1, \tag{3.3}$$

then in the limit $h \to 0$, $y_0^2(x)$ goes to

$$y_0^2(x) \to \delta(x - \sqrt{2\lambda}) \tag{3.4}$$

in the sense of distributions. Therefore, if we multiply by a continuous $\rho(x) \in L^2(0, \infty)$ with $\rho(\sqrt{2\lambda}) = 1$, the error is exponentially small for $h \to 0$.

For the solution $y(x, t)$ of the Cauchy problem, we now make the following ansatz

$$y(x, t) = u(x, t) \exp \frac{i}{h} S(x, t) \tag{3.5}$$

where $S(x, t)$ is the solution of the Hamilton–Jacobi equation (2.9) satisfying the initial condition

$$S(x, 0) = S_0(x) = -i\varphi(x). \tag{3.6}$$

As was pointed out in the last section, $S(x, t)$ is uniquely determined for small $t < T_0$ (2.24).

In particular, for the fully excited initial state (3.2) this has been investigated (2.20). If (3.5) is substituted into the Schrödinger equation (2.1), one gets the following equation for $u(x, t)$

$$\begin{aligned} ih \frac{\partial u}{\partial t} &= \left(\frac{\partial S}{\partial t} + x \left(\frac{\partial S}{\partial x} \right)^2 + V(x) \right) u \\ &- ih \left(x \frac{\partial^2 S}{\partial x^2} u + 2x \frac{\partial S}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial S}{\partial x} u \right) - h^2 \left(x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \right), \end{aligned} \tag{3.7}$$

where the first term on the right side vanishes due to the choice of $S(x, t)$. The corresponding initial condition is

$$u(x, 0) = \rho(x). \tag{3.8}$$

Let us consider the function $x_0(x, t)$ which is implicitly given in terms of the trajectory

$$x = x(x_0, t). \tag{3.9}$$

We calculate the time derivative of x_0 (with x fixed) from (3.9)

$$\frac{\partial x_0}{\partial t} = - \frac{\dfrac{\partial x}{\partial t}}{\dfrac{\partial x}{\partial x_0}} \tag{3.10}$$

and use

$$\frac{\partial x}{\partial t} = \frac{\partial H}{\partial p} = 2xp = 2x \frac{\partial S}{\partial x} \tag{3.11}$$

obtaining

$$\frac{\partial x_0}{\partial t} = -\frac{\partial x_0}{\partial x} 2x \frac{\partial S}{\partial x} \tag{3.12}$$

Differentiating this with respect to x, we get the relation

$$\frac{\partial}{\partial x}\frac{\partial x_0}{\partial t} = \frac{\partial}{\partial t}\frac{\partial x_0}{\partial x} = -\frac{\partial^2 x_0}{\partial x^2} 2x \frac{\partial S}{\partial x} - 2\frac{\partial x_0}{\partial x}\frac{\partial S}{\partial x} - 2x\frac{\partial x_0}{\partial x}\frac{\partial^2 S}{\partial x^2}. \tag{3.13}$$

For the solution $u(x, t)$ of (3.7), (3.8), we now make the ansatz

$$u(x, t) = \left(\frac{\partial x_0}{\partial x}\right)^{1/2} v_h(x, t). \tag{3.14}$$

Substituting into equation (3.7) and using (3.13), we find the following equation for $v_h(x, t)$

$$\frac{\partial v_h}{\partial t} + 2x\frac{\partial S}{\partial x}\frac{\partial v_h}{\partial x} = ih\left(\frac{\partial x_0}{\partial x}\right)^{-1/2} \frac{\partial}{\partial x} x \frac{\partial}{\partial x}\left(\frac{\partial x_0}{\partial x}\right)^{1/2} v_h. \tag{3.15}$$

The initial condition now reads

$$v_h(x, 0) = \rho(x). \tag{3.16}$$

For the left side of equation (3.15) we can write

$$\frac{\partial v_h}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial v_h}{\partial x} = \frac{d}{dt} v_h(x(x_0, t), t) \tag{3.17}$$

which is the time derivative along the trajectory (3.9). Then (3.15) and (3.16) are equivalent to the integral equation

$$v_h(x, t) = \rho(x_0) + ih\int_0^t dt_1 \left(\frac{\partial x_0}{\partial x}\right)^{-1/2} \frac{\partial}{\partial x} x \frac{\partial}{\partial x}\left(\frac{\partial x_0}{\partial x}\right)^{1/2} v_h(x, t_1), \tag{3.18}$$

where the integral is taken along the trajectory (3.9), that means x has to be substituted by $x(x_0, t_1)$. Since for $0 \leqslant t < T_0$ there exists a unique trajectory, it follows that

$$\frac{\partial x}{\partial x_0} \neq 0. \tag{3.19}$$

Therefore, the factors appearing in (3.14) and (3.18) are bounded. If the time t approaches a focal point, then according to (2.23) and (3.10)

$$\frac{\partial x}{\partial x_0} \to 0, \tag{3.20}$$

which is usually taken as the definition of a focal point [7].

We must now control the h-dependence of $v_h(x, t)$. For this purpose, let us perform a scaling transformation

$$\begin{aligned} x' &= xh^{-1/2}, & t' &= th^{1/2} \\ \lambda' &= \lambda/h, & \eta' &= \eta/h \end{aligned} \tag{3.21}$$

in equation (3.15), which leads to

$$\frac{\partial v}{\partial t'} + \frac{\partial x'}{\partial t'}\frac{\partial v}{\partial x'} = i\left(\frac{\partial x_0'}{\partial x'}\right)^{-1/2}\frac{\partial}{\partial x'}x'\frac{\partial}{\partial x'}\left(\frac{\partial x_0'}{\partial x'}\right)^{1/2} v. \tag{3.22}$$

Since this equation no longer contains the parameter h and the initial condition (3.16) is independent of h too, it follows that

$$v_h(x, t) = v(xh^{-1/2}, th^{1/2}). \tag{3.23}$$

Then, since $v(x', t')$ is bounded for $x' \to \infty$, $t' \to 0$, $v_h(x, t)$ is bounded with respect to h uniformly in x and $0 \leqslant t \leqslant t_1 < T_0$, together with its first and second x-derivatives. Consequently, we may conclude from the integral equation (3.18) that

$$v_h(x, t) = \rho(x_0(x, t)) + 0(h) \tag{3.24}$$

for $h \to 0$ and x fixed.

Summing up, we have shown that

$$y(x, t) = \left(\frac{\partial x_0}{\partial x}\right)^{1/2} e^{i/(h)\,S(x,t)}\,[\rho(x_0) + 0(h)] \tag{3.25}$$

for $t < T_0$. Since the error term is square integrable, we have also an approximation in the L^2 sense

$$\|y(x, t) - y_c(x, t)\| = 0(h) \tag{3.26}$$

where

$$y_c(x, t) = \rho(x_0)\left(\frac{\partial x_0}{\partial x}\right)^{1/2} \exp\left(\frac{i}{h}S(x, t)\right) \tag{3.27}$$

is the local semi-classical solution.

4. The global semi-classical solution

The local semi-classical solution (3.27) diverges at the first focal point according to (3.20). However, beyond the focal points the expression (3.27) is again finite but not unique. The exact solution $y(x, t)$, on the other hand, remains regular at the focal point. There rises the connection problem of continuing $y_c(x, t)$ in t through the focal points in such a way that it approximates $y(x, t)$ beyond the focal points.

For technical reasons, we now assume that $\rho(x)$ and $\varphi(x)$ appearing in the initial condition (3.1) are analytic functions of x, regular in a certain region Ω' of the complex x-plane which contains a part of the positive real axis and will be further specified below. Outside Ω' $\rho(x)$ is assumed to be C^2 with compact support $\Omega \supset \Omega'$, while $\varphi(x)$ is C^2 in $\Omega \backslash \Omega'$. We remark that these assumptions are not too restrictive, because a $\rho(x)$ with non-compact support can be decomposed by means of a decomposition of unity

$$1 = \sum_k \rho_k(x), \qquad \rho_k \in C_0^2.$$

The focal points $T(x)$ of the local semi-classical solution (3.27) lie in the time interval $[T_1, T_2]$

$$T_1 = \min_{x_0 \in \Omega} T(x)$$
$$T_2 = \max_{x_0 \in \Omega} T(x) \tag{4.1}$$

Following a method of Maslov ([2], p. 285), we bridge the gap between T_1 and T_2 by going over to the momentum space. Let us choose $t_1 < T_1$, so that the semi-classical solution $y_c(x, t_1)$ (3.27) is still valid at t_1. We compute the Fourier transform

$$\hat{y}_c(p, t_1) = (2\pi h)^{-1/2} \int dx\, \mathrm{e}^{-i/(h)px} \left(\frac{\partial x_0}{\partial x}\right)^{1/2} \mathrm{e}^{i/(h)S(x,t_1)}\, \rho(x_0), \tag{4.2}$$

where the integral actually runs only over the compact set Ω_x

$$\Omega_x = \{x(x_0, t_1) \in \mathbb{R}^1_+ \mid x_0 \in \Omega\}. \tag{4.3}$$

The integrand in (4.2) is analytic in

$$\Omega'_x = \{x(x_0, t_1) \mid x_0 \in \Omega'\}. \tag{4.4}$$

The integral (4.2) will be evaluated by the saddle-point method. A saddle-point x_1 is a solution of the equation

$$\frac{\partial S(x, t_1)}{\partial x} = p, \tag{4.5}$$

that means p must be the momentum at time t_1 corresponding to the unique trajectory (3.9)

$$p = p(x_0, t_1)$$
$$x_1 = x(x_0, t_1). \tag{4.6}$$

Let Λ be the set of momentum values

$$\Lambda = \{p(x_0, t_1) \mid x(x_0, t_1) \in \Omega_x\}. \tag{4.7}$$

Then, for $p \in \Lambda$ there exists a unique saddle-point x_1, generally complex. For $p \in \Lambda'$

$$\Lambda' = \{p(x_0, t_1) \in \Lambda \mid x_0 \in \Omega'\} \tag{4.8}$$

the saddle-point x_1 lies in the analyticity domain Ω'_x (4.4). The path of integration can then be deformed inside Ω'_x so that it runs through x_1. Now the integral inside Ω'_x can be evaluated by the complex saddle-point method ([8], p. 232) up to an exponentially small error, the integral in the rest of Ω_x is $0(h^2)$ because $\rho \in C^2$. This yields

$$\hat{y}_c(p, t_1) = \left(\frac{2\pi h}{\frac{\partial^2}{\partial x^2} S(x, t_1)}\right)^{1/2}_{x_1} (2\pi h)^{-1/2}\, \mathrm{e}^{-i/(h)px_1} \left(\frac{\partial x_0}{\partial x}\right)^{1/2}_{x_1}$$
$$\times\, \mathrm{e}^{i/(h)S(x_1,t_1)}\, \rho(x'_0) + 0(h^{3/2}), \tag{4.9}$$

with $x'_0 = x_0(x_1, t_1)$. The error term in (4.9) can be estimated in the sup-norm or in the L^2 norm. In the following, all estimates are understood in the L^2 sense. For $p \notin \Lambda$ there exists no saddle point, therefore

$$\hat{y}_c(p, t_1) = 0(h^{3/2}). \tag{4.10}$$

The contribution from $p \in \Lambda \backslash \Lambda'$ can be made arbitrarily small in the L^2 sense, say

$0(h^{3/2})$ also, by choosing $\Omega \backslash \Omega'$ small enough. The error $0(h^{3/2})$ can be neglected compared with the error $0(h)$ of the local semi-classical solution (3.27) (see (4.22) below). From (4.6) we get

$$x_0 = x_0(p, t_1) \tag{4.11}$$

and

$$x_1 = x_1(p, t_1). \tag{4.12}$$

Using (4.5) and the chain rule, we can write (4.9) as

$$\hat{y}_c(p, t_1) = \left(\frac{\partial x_0}{\partial p}\right)^{1/2} e^{i/(h)S(p, t_1)} \rho(x_0') + 0(h^{3/2}) \tag{4.13}$$

where

$$S(p, t_1) = S(x_1(p, t_1), t_1) - p x_1(p, t_1). \tag{4.14}$$

According to (4.5), this is a Legendre transformation. We have

$$\frac{\partial S(p, t_1)}{\partial p} = -x_1. \tag{4.15}$$

Let Λ_p be the compact set on the real p-axis which is in Λ' (4.8), and $\chi_p(p) \in C_0^\infty(\mathbb{R}^1)$

$$\chi_p(p) = \begin{cases} 1 & p \in \Lambda_p \\ <1 & p \notin \Lambda_p \\ 0 & p \notin \operatorname{supp} \chi_p \supset \Lambda_p, \end{cases} \tag{4.16}$$

and similarly

$$\chi_x(x) = \begin{cases} 1 & x \in \Omega_x \\ <1 & x \notin \Omega_x \\ 0 & x \notin \operatorname{supp} \chi_x \supset \Omega_x. \end{cases} \tag{4.17}$$

In addition, we define χ_x and $\chi_p = 1$ in some complex region of the p or x plane, respectively, specified below. The function

$$\tilde{y}_c(p, t_1) = \chi_p(p) \left(\frac{\partial x_0}{\partial p}\right)^{1/2} e^{i/(h)S(p, t_1)} \rho(x_0') \tag{4.18}$$

is approximately equal to $\hat{y}_c(p, t_1)$ in $L_2(\mathbb{R}^1)$ up to $0(h^{3/2})$. It then follows from the Fourier inversion theorem in L^2 that

$$y_c(x, t_1) = (2\pi h)^{-1/2} \int dp \tilde{y}_c(p, t_1) e^{i/(h)px} + 0(h^{3/2}). \tag{4.19}$$

Multiplying with χ_x (4.17), we have also

$$y_c(x, t_1) = \chi_x(x)(2\pi h)^{-1/2} \int dp \tilde{y}_c(p, t_1) e^{i/(h)px} + 0(h^{3/2}). \tag{4.20}$$

We now consider $t > t_1$. Let $y_p(x, t)$ be the solution of the Schrödinger equation (2.1) with the initial condition

$$y_p(x, t_1) = \chi_x(x) e^{i/(h)px} \tag{4.21}$$

at $t = t_1$. It follows from (3.27) and (4.20) that

$$y(x, t) = (2\pi h)^{-1/2} \int dp \tilde{y}_c(p, t_1) y_p(x, t) + 0(h) \tag{4.22}$$

for arbitrary $t \geqslant t_1$, due to the unitarity of the exact time evolution operator. Since $y_p(x, t_1)$ is of the form (3.1), we can apply the results of Section 3 and compute its local semi-classical approximation $y_{pc}(x, t)$. We have to consider the trajectory

$$x' = x'(x_2, p, t') \tag{4.23}$$

$$p' = p'(x_2, p, t'), \; t_1 \leqslant t' \leqslant t \tag{4.24}$$

with $x' = x_2$, $p' = p$ at $t = t_1$. For every given $x = x(t)$ and $p = p'(t_1) \in \text{supp } \chi_p$ there exists a unique initial coordinate x_2

$$x_2 = x_2(x, p, t), \tag{4.25}$$

if t is not too big, say

$$t - t_1 \leqslant \Delta t. \tag{4.26}$$

Then there also exists a unique action $S_p(x, t)$ given by (2.12)

$$S_p(x, t) = \tilde{S}(x, x_2; t, t_1) + p x_2 \tag{4.27}$$

and a corresponding local semi-classical solution

$$y_{pc}(x, t) = \chi_x(x_2) \left(\frac{\partial x_2}{\partial x} \right)^{1/2} \exp \left[\frac{i}{h} S_p(x, t) \right]. \tag{4.28}$$

The energy ε has been dropped in the argument of $\tilde{S}$, because it is now explicitly given by

$$-\varepsilon = x_2 p^2 + V(x_2). \tag{4.29}$$

Using

$$\frac{\partial \tilde{S}}{\partial x_2} = -p, \tag{4.30}$$

it follows from (4.27) that

$$\frac{\partial S_p}{\partial p} = \left(\frac{\partial \tilde{S}}{\partial x_2} + p \right) \frac{\partial x_2}{\partial p} + x_2 = x_2. \tag{4.31}$$

Finally, substituting y_{pc} (4.28) for y_p into (4.22), we obtain with (4.18)

$$\begin{aligned} y(x, t) = (2\pi h)^{-1/2} \int dp \chi_p(p) \left(\frac{\partial x_0}{\partial p} \right)^{1/2} e^{i/(h) S(p, t_1)} \rho(x_0') \\ \times \chi_x(x_2) \left(\frac{\partial x_2}{\partial x} \right)^{1/2} e^{i/(h) S_p(x, t)} + 0(h). \end{aligned} \tag{4.32}$$

This integral will again be computed by the saddle-point method. The saddle-points are solutions of

$$\frac{\partial S(p, t_1)}{\partial p} + \frac{\partial S_p(x, t)}{\partial p} = 0 \tag{4.33}$$

or with (4.15) and (4.31)

$$x_1(p, t_1) = x_2(x, p). \tag{4.34}$$

Consequently, p must be chosen in such a way that the two trajectories (3.9) from time 0 to t_1 and (4.23) from t_1 to t join together, forming one trajectory from 0 to t. Since the latter is no longer unique for $t > T_1$, there exists now more than one saddle point (the argument given in Maslov's book [2] p. 293 is wrong at this point). The exponent in (4.32) simplifies at the saddle-points p_k:

$$\begin{aligned} S(p_k, t_1) + S_{p_k}(x, t) &= -p_k x_1(p_k) + \tilde{S}(x_1, x_0; t_1, 0) \\ &\quad + S_0(x_0) + \tilde{S}(x, x_2; t, t_1) + p_k x_2(p_k) \\ &= \tilde{S}_k(x, x_0(p_k); t, 0) + S_0(x_0(p_k)) \\ &\stackrel{\text{def}}{=} S_k(x, t). \end{aligned} \tag{4.35}$$

The domains of analyticity for χ_p and χ_x must be chosen big enough such that all (complex) saddle-points p_k lie inside. Assuming $t_1 + \Delta t > T_2$, which can always be arranged by a suitable restriction of the support Ω of ρ, we can compute the integral (4.32) for $T_2 < t < t_1 + \Delta t$ by the saddle-point method

$$\begin{aligned} y(x, t) = \sum_k &\left(\frac{2\pi h}{\dfrac{\partial x_2}{\partial p} - \dfrac{\partial x_1}{\partial p}} \frac{1}{2\pi h} \frac{\partial x_0}{\partial p} \frac{\partial x_2}{\partial x} \right)^{1/2}_{p_k} \\ &\times e^{i/(h) S_k(x,t)} \rho(x_{0k}) + 0(h) \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} x_{0k} &= x_0(x_1(p_k, t_1), t_1) = x_0(x_2(x, p_k), t_1) \\ &\stackrel{\text{def}}{=} x_{0k}(x, t) \end{aligned} \tag{4.37}$$

Here p (respectively p_k) appears as a parameter which must be determined by x, so that everything on the right side of (4.36) depends ultimately on x only.

We set $t' = t_1$ in (4.24) and solve (4.25) for p to obtain

$$p = p'(x_2, p(x, x_2, t), t_1) \stackrel{\text{def}}{=} p(x_2, x) \tag{4.38}$$

or because of (4.34)

$$p - p(x_1(p), x) = 0, \tag{4.39}$$

omitting the time arguments which are now of no interest. This equation (4.39) gives p as an implicit function of x. The x-derivative then is

$$\frac{\partial p}{\partial x} = \frac{\left(\dfrac{\partial p}{\partial x}\right)_{x_2}}{1 - \left(\dfrac{\partial p}{\partial x_2}\right)_x \dfrac{\partial x_1}{\partial p}} = \frac{\left(\dfrac{\partial x_2}{\partial p}\right)_x \left(\dfrac{\partial p}{\partial x}\right)_{x_2}}{\left(\dfrac{\partial x_2}{\partial p}\right)_x - \dfrac{\partial x_1}{\partial p}}. \tag{4.40}$$

Using this expression in (4.36), we get the final result ([1], equation (2.31))

$$y(x, t) = \sum_k \left(\frac{\partial x_{0k}}{\partial x} \right)^{1/2} e^{i/(h) S_k(x,t)} \rho(x_{0k}) + 0(h). \tag{4.41}$$

This formula shows that the local semi-classical solution can be continued beyond the first focal points in the most plausible way by adding up the contributions

of the different branches. The same arguments can then be used at the following focal points, at which one considers each branch separately. In this way, the global semi-classical solution is established. As we have seen, at each focal point an additional error $0(h)$ comes in. Since the number of focal points increases linearly with t, the total error increases in the same way, if all error terms have the same order of magnitude. In the quantum optical application [1] h is proportional to N^{-1} (N = number of radiating atoms). The global semi-classical solution then breaks down after $0(N)$ focal points or after $0(N)$ oscillations. This is precisely what one finds in numerical calculations for the Dicke model (see also [5] p. 85).

A final remark is concerned with the nature of the semi-classical approximation. The semi-classical solution $y_c(x, t)$ approaches $y(x, t)$ for $h \to 0$ in the L^2 sense, but in the applications [1] the behavior of $y(x, t)$ in the neighborhood of $x = 0$ is required. However, going over to a subsequence $h_j \to 0$, we get convergence almost everywhere, and this is sufficient for the applications.

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