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The Scattering Matrix is Non-Trivial for Weakly Coupled $P(\varphi)_2$ Models

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Abstract. We show that for sufficiently small coupling constant λ, the $\lambda P(\varphi)_2$ quantum field theory models have a scattering matrix which is different from $\mathbb{1}$. Our method is to write the scattering matrix elements as polynomials in λ, whose coefficients, though themselves functions of λ, are uniformly bounded for λ sufficiently small. The first order term in that expansion is the one given by perturbation theory.

I. Introduction

Weakly coupled $P(\varphi)_2$ quantum field theory models have an isolated one-particle hyperboloid [8, 9]. Hence by the Haag–Ruelle theory [10, 16, 12] they also possess a well defined scattering matrix, at least for non-overlapping in-going and out-going velocities, see Ref. [11]. In this paper we show that this scattering matrix describes a non-trivial collision process. This result is of course strongly suggested by perturbation theory and also by the results of Dimock [1] and Eckmann, Magnen, and Sénéor [2] who showed that the Schwinger functions have a perturbation expansion which is asymptotic and even Borel summable. A further indication that these models are not trivial is Fröhlich's proof [3] that their algebras of local observables are not in the same Borchers class as the free field algebras.

For a $\lambda P(\varphi)_2$ model, $P(\xi) = \xi^{2n}$ + lower order terms, we consider the S-matrix elements between k in-going and l out-going particles, and show that they can be written as polynomials in λ (of order $k + l$) whose coefficients are themselves functions of λ, bounded uniformly in λ for λ sufficiently small. If we choose $k + l = 2n$, then the first order term in that expansion is the one given by perturbation theory. This shows that for λ sufficiently small, the scattering matrix is different from $\mathbb{1}$.

To obtain this expansion for the S-matrix we start from a similar expansion for the Euclidean Green's functions and obtain the expansion for the time ordered Green's functions through analytic continuation. From there we use the LSZ reduction formulas [13] in the version of Hepp [11] to get the S-matrix elements by

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amputating the time ordered Green's functions and then restricting them to the mass shell.

In Section II we derive the expansion for the Euclidean Green's functions. In Section III we discuss the analytic continuation of this expansion to real times and the expansion for time ordered Green's functions. Finally in Section IV we prove that amputation and restriction to the mass shell can be carried through term by term in the expansion of the time ordered Green's functions. This leads to the expansion for the S-matrix elements.

II. Expanding the Euclidean Green's Functions

Euclidean Green's functions for a $\lambda P(\varphi)_2$ model are defined by

$$\mathfrak{S}_n(x_1 \cdots x_n) = \left\langle \prod_{i=1}^{n} \Phi_i \right\rangle$$

$$= \lim_{\Lambda \to \mathbb{R}^2} \frac{\int \prod_{i=1}^{n} \Phi_i \exp\left(-\lambda \int_\Lambda :P(\Phi): (x)\, d^2x\right) d\mu_{m_0}}{\int \exp\left(-\lambda \int_\Lambda :P(\Phi): (x)\, d^2x\right) d\mu_{m_0}} \tag{1}$$

see e.g. Ref. [8]. Here $\Phi_i = \Phi(x_i)$ and $d\mu_{m_0}$ is the free Gaussian measure on $\mathscr{S}'(\mathbb{R}^2)$ with mean 0 and covariance $C = C_{m_0} = (-\Delta + m_0^2)^{-1}$. Glimm, Jaffe and Spencer have shown [8, 9] that for $m_0 > 0$ fixed, there exists a $\lambda_0 > 0$ such that for all $\lambda \in [0, \lambda_0]$ the mass operator $M = (H^2 - \vec{P}^2)^{1/2}$ has eigenvalues 0, m and no other spectrum in $[0, m']$. Here $m' = 2m_0 - \epsilon$ and $m \in (m_0 - \epsilon, m_0 + \epsilon)$ for some $\epsilon = \epsilon(\lambda_0)$ which tends to 0 as λ_0 gets small. Once we have determined the physical mass m we may perform a 'mass shift' (in the finite volume theory, preferably with periodic boundary conditions) [5], such that without changing the Euclidean Green's functions we replace in (1) m_0 by m and simultaneously replace $:P(\Phi):$ by a new polynomial $:\hat{P}(\Phi):_m$ where $: :_m$ means Wick ordering with respect to C_m. The coefficients of $\hat{P}$ will in general depend on m and thus on λ (the coefficient of the highest order term is of course always equal to 1), but they are uniformly bounded as λ varies in the interval $[0, \lambda_0]$. From now on we will always assume that m_0 has been replaced by m and we write again P instead of $\hat{P}$.

We get an expansion for the Euclidean Green's functions by repeated application of the standard integration by parts formula (see Ref. [7])

$$\langle \Phi_1 A \rangle = C_1 \langle \delta_1 A \rangle - \lambda C_1 \langle V^1 A \rangle. \tag{2}$$

Here A is some product of Wick monomials, $C_i \equiv \int dy_i C(x_i - y_i), \ldots, \delta_i = \delta/\delta\Phi(y_i)$, and with $P^{(l)}(\xi) \equiv (d^l/d\xi^l)P(\xi), V^l(y) = :P^{(l)}(\Phi(y)):$.

Lemma 1. For $k > 2$,

$$\left\langle \prod_{i=1}^{k} \Phi_i \right\rangle^T = \sum_{\pi} (-\lambda)^{|\pi|} \int \prod_{\sigma \in \pi} \left(dy_\sigma \prod_{i \in \sigma} C(x_i - y_\sigma) \right) \left\langle \prod_{\sigma \in \pi} V^{|\sigma|}(y_\sigma) \right\rangle^T. \tag{3}$$

Here the sum runs over all partitions π of $\{1, 2, \ldots, k\}$ into $|\pi|$ nonempty mutually disjoint sets σ, with $|\pi|$ taking all values $1, \ldots, k$. $\langle\ \rangle^T$ means the truncated vacuum expectation value. $|\sigma|$ = the number of elements in σ.

We prove this lemma in Appendix I.

Glimm and Jaffe [6] have shown that the generalized Euclidean Green's functions

$$\mathfrak{S}_\pi(\mathbf{y})^T \equiv \left\langle \prod_{\sigma\in\pi} V^{|\sigma|}(y_\sigma) \right\rangle^T, \tag{4}$$

which are real analytic at points of noncoinciding arguments ($y_i \neq y_j$ for all $i \neq j$) have the same local singularities as the corresponding expressions in a free theory. (This actually holds not only for $P(\varphi)_2$ but also for $(\varphi^4)_3$ and probably for all super-renormalizable models.) This shows that the y_σ-integrations in (3) are well defined (as principal values) and that $\langle\prod_{i=1}^k \Phi_i\rangle^T$ is regular for all values of the variables x_i. Notice also that due to the truncation, (4) decreases exponentially at large separation of the arguments [2].

III. Time Ordered Green's Functions

The Euclidean Green's functions $\langle\prod_{i=1}^k \Phi_i\rangle = \mathfrak{S}_k(\mathbf{x})$ and the generalized Euclidean Green's functions $\mathfrak{S}_\pi(\mathbf{y}) = \langle\prod_{\sigma\in\pi} V^{|\sigma|}(y_\sigma)\rangle$, $\mathbf{x} = (x_1, \ldots, x_k)$, $\mathbf{y} = (y_1, \ldots, y_{|\pi|})$, satisfy all the axioms of Ref. [15] and can therefore be analytically continued in their time variables x_i^0 and y_i^0 respectively to all of $\mathbb{C}^k$ or $\mathbb{C}^{|\pi|}$ minus points where $\operatorname{Re} x_i^0 = \operatorname{Re} x_j^0$ or $\operatorname{Re} y_i^0 = \operatorname{Re} y_j^0$ for some $i \neq j$. We can therefore define time ordered (generalized) Green's functions by

$$\left\langle T\prod_{i=1}^k \varphi_i \right\rangle = \lim_{\mu\to i} \mathfrak{S}_k(\mu\mathbf{x}) \tag{5}$$

and

$$\left\langle T\prod_{\sigma\in\pi} v^{|\sigma|}(y_\sigma) \right\rangle = \lim_{\mu\to i} \mathfrak{S}_\pi(\mu\mathbf{y}), \tag{6}$$

where $\mu\mathbf{x} = (\mu x_1^0, \vec{x}_1, \ldots, \mu x_k^0, \vec{x}_k)$, $(x^0, \vec{x}) \in \mathbb{R}^2$; $\varphi_i = \varphi(x_i)$ is the relativistic field corresponding to the Euclidean field $\Phi(x_i)$ and $v^l(y) = {:}P^{(l)}(\varphi(y)){:}$. T means time ordering and $\mu \to i$ means $\mu = i - \delta$ and $\delta \downarrow 0$. More precisely we have

Lemma 2. For $f \in \mathscr{S}(\mathbb{R}^2)$, $G_\epsilon = \{\mathbf{x} \in \mathbb{R}^{2k} \mid |x_i^0 - x_j^0| > \epsilon \text{ for all } i \neq j\}$

$$\tau(f) = \lim_{\mu\to i}\lim_{\epsilon\downarrow 0} \int_{G_\epsilon} \mathfrak{S}_k(\mu\mathbf{x}) f(\mathbf{x})\, d^{2k}x \tag{7}$$

$$\tau_\pi(f) = \lim_{\mu\to i}\lim_{\epsilon\downarrow 0} \int_{G_\epsilon} \mathfrak{S}_\pi(\mu\mathbf{y}) f(\mathbf{y})\, d^{2k}y \tag{8}$$

define tempered distributions which are Lorentz invariant and which coincide with the (generalized) Wightman distributions $\langle\prod \varphi(x_{i_s})\rangle$ and $\langle\prod_s v^{l_{i_s}}(y_{i_s})\rangle$ in each sector where $x_{i_1}^0 > x_{i_2}^0 \cdots x_{i_k}^0$ and $y_{i_1}^0 > y_{i_2}^0 > \cdots$, $(i_1, \ldots, i_k)$ a permutation of $(1, \ldots, k)$. They are bounded uniformly in λ for $\lambda \in [0, \lambda_0]$, in the sense of distributions.

A proof of this lemma is given in Appendix II. Truncated time ordered Green's functions are defined in the same way from truncated Euclidean Green's functions. Notice that the existence of time ordered Green's functions has been first proven by Nelson [14], see also Ref. [4].

Next we want to 'analytically continue' the expansion (3). The analytic continuation of the free propagators C is given (for $x \neq 0$) by

$$C_\mu(x) = C(\mu x^0, \vec{x}) = \frac{\mu}{(2\pi)^2} \int \frac{e^{ipx}}{p_0^2 + \mu^2 \omega^2} \, d^2p, \tag{9}$$

where $px = p^0x^0 - \vec{p}\vec{x}$ and $\omega = \omega(\vec{p}) = (\vec{p}^2 + m^2)^{1/2}$. For later purposes we also define $C_\mu^{-1}(x)$ by

$$(C_\mu * C_\mu^{-1})(x) = \delta^2(x). \tag{10}$$

Because the singularities at points of equal times are weak (namely logarithmic), the function

$$\sum_\pi (-\lambda\mu)^{|\pi|} \int_{G_\epsilon} \prod_{\sigma\in\pi} \left(dy_\sigma \prod_{i\in\sigma} C_\mu(x_i - y_\sigma) \right) \mathfrak{S}_\pi(\mu \mathbf{y})^T \tag{11}$$

is analytic in $\mu \in M = \{\mu \mid |\arg \mu| < \pi/2, \frac{1}{2} < |\mu| < \frac{3}{2}\}$ and bounded uniformly in ϵ for μ in any compact subset of M, $\mathbf{x}$ fixed and $G_\epsilon = \{\mathbf{y} \mid |y_\sigma^0 - y_{\sigma'}^0| > \epsilon, |y_\sigma^0 - x_i^0| > \epsilon$ for all $\sigma \neq \sigma'$, $i \in \sigma\}$. By Vitali's theorem we may therefore take the limit $\epsilon \downarrow 0$ in (11) and obtain an analytic function which at $\text{Im}\,\mu = 0$ equals (3) (with all the variables scaled by μ). By the uniqueness of analytic continuations we thus find that for $\mu \in M$,

$$\mathfrak{S}_k(\mu \mathbf{x})^T = \sum_\pi (-\lambda\mu)^{|\pi|} \int \prod_{\sigma\in\pi} \left(dy_\sigma \prod_{i\in\sigma} C_\mu(x_i - y_\sigma) \right) \mathfrak{S}_\pi(\mu \mathbf{y})^T \tag{12}$$

(with $\int \cdots \equiv \lim_{\epsilon\downarrow 0} \int_{G_\epsilon} \cdots$). By (5) the limit $\mu \to i$ of (12) yields the time ordered Green's functions. It remains to be seen that this limit exists for every term in the sum on the r.h.s. of (12) *separately*, at least when tested with appropriate test functions. Fortunately we will only be interested in amputated Green's functions. Amputation will eliminate the factors C_μ from (12) and hence the discussion of the product of distributions in (12) (as $\mu \to i$) can be avoided.

IV. *S*-Matrix Elements

By the LSZ reduction formulas [13, 11] we obtain the scattering matrix elements by amputating the time ordered Green's functions and restricting them to the mass shell. In [11] Hepp has shown how this can be done rigorously in the framework of Wightman's axioms, if the in-going and out-going particles have all different velocities. Following Hepp we let $\mathscr{S}(G)$ be the space of test functions $\tilde{f} \in \mathscr{S}(\mathbb{R}^2)$ with $\text{supp}\,\tilde{f} \in \{p^0 > 0 \mid 0 < p^2 < m'^2\}$. For $\tilde{f} \in \mathscr{S}(G)$, and $\omega = \omega(\vec{p}) = (\vec{p}^2 + m^2)^{1/2}$ we define

$$\begin{aligned} f(x, t) &= (2\pi)^{-1} \int e^{ipx} \tilde{f}(p) e^{i(p^0 - \omega)t} \, d^2p \\ \hat{f}(\vec{p}) &= \tilde{f}(\omega, \vec{p}); \quad \dot{f}(x, t) = \frac{\partial f}{\partial t}(x, t). \end{aligned} \tag{13}$$

We call $\{\tilde{f}_i\} \in \mathscr{S}(G)$ non-overlapping if for all $p_i \in \text{supp}\,\tilde{f}_i$, $\omega_i = \omega(\vec{p}_i)$,

$$\omega_i^{-1}\vec{p}_i \neq \omega_j^{-1}\vec{p}_j \qquad \text{for all } i \neq j.$$

Then by Hepp's beautiful analysis [11], for non-overlapping $\{\tilde{f}_i\}$ the connected part of the S-matrix element

$$\langle \hat{f}_1^{\text{out}} \cdots \hat{f}_n^{\text{out}} \mid \hat{f}_{n+1}^{\text{in}} \cdots \hat{f}_{n+m}^{\text{in}} \rangle$$

is given (up to constant factors) by

$$\int \prod_{i=1}^{n+m} dt_i \int \prod_{i=1}^{n} dx_i \bar{\hat{f}}_i(x_i, t_i) \prod_{i=n+1}^{m} dx_i \hat{f}_i(x_i, t_i) \left\langle T \prod_{i=1}^{n+m} \varphi_i \right\rangle^T . \tag{14}$$

Writing $\hat{f}(x, t)$ as

$$\hat{f}(x, t) = \frac{i}{2\pi} \int d^2 p e^{ipx} \left(\frac{\tilde{f}(p)}{p^0 + \omega} \right) e^{i(p^0 - \omega)t} [p^2 - m^2] \tag{15}$$

one sees that the factor $[p^2 - m^2]$ provides for the amputation while the t-integration restricts the momenta to the mass shell. We set

$$\tilde{g}_i(p) = -\frac{\tilde{f}_i(p)}{p^0 + \omega} \in \mathscr{S}(G),$$

and

$$g_i(x, t) = \int e^{ipx} \tilde{g}_i(p) e^{i(p^0 - \omega)t} \, d^2 p. \tag{16}$$

Notice that the $\{\tilde{g}_i\}$ are non-overlapping if the $\{\tilde{f}_i\}$ are.

From (9), (10) and (15)

$$\hat{f}_i = \lim_{\mu \to i} g_i * C_\mu^{-1}$$

in the topology of $\mathscr{S}$. Here $(g_i * C_\mu^{-1})(x, t) = \int g(y, t) C_\mu^{-1}(y - x) \, dy$. Now we substitute (5) and (12) in (14), to find that (14) is equal to

$$\begin{aligned}
&\lim_{\mu \to i} \int \prod_{i=1}^{n} dx_i (-\bar{g}_i * C_\mu^{-1})(x_i, t_i) \prod_{i=n+1}^{n+m} dx_i (g_i * C_\mu^{-1})(x_i, t_i) \mathfrak{S}_{n+m}(\mu \mathbf{x})^T \\
&\quad = \lim_{\mu \to i} \sum_\pi (-\lambda\mu)^{|\pi|} \int \prod_{\sigma \in \pi} dy_\sigma \prod_{i=1}^{n} (-\bar{g}_i)(y_i, t_i) \prod_{i=n+1}^{n+m} g_i(y_i, t_i) \mathfrak{S}_\pi(\mu \mathbf{y})^T \\
&\quad = \sum_\pi (-i\lambda)^{|\pi|} \int \prod_{\sigma \in \pi} dy_\sigma \prod_{i=1}^{n} (-\bar{g}_i)(y_i, t_i) \prod_{i=n+1}^{n+m} g_i(y_i, t_i) \left\langle T \prod_{\sigma \in \pi} v^{|\sigma|}(y_\sigma) \right\rangle^T
\end{aligned} \tag{17}$$

when integrated over all the t_i-variables. We have used the convention that $y_i \equiv y_\sigma$ for $i \in \sigma$. The following lemma shows that the t_i-integrations can be done separately on every term in the sum $\sum_\pi$ in (17).

Lemma 3. For $\{\tilde{g}_i\} \in \mathscr{S}(G)$ non-overlapping, and any partition π of $\{1, \ldots, m\}$, the functions

$$\int \prod_{\sigma \in \pi} \left\{ dy_\sigma \prod_{i \in \sigma} g_i^*(y_i, t_i) \right\} \left\langle T \prod_{\sigma \in \pi} v^{|\sigma|}(y_\sigma) \right\rangle \tag{18}$$

are in $\mathscr{S}(\mathbb{R}^m)$ in the variables $t_1, \ldots, t_m$ with bounds uniform in λ for $\lambda \in [0, \lambda_0]$. Here g^* means $\bar{g}$ (complex conjugate) or g; $y_i = y_{\sigma(i)}$ with $\sigma(i)$ determined by $i \in \sigma$.

Postponing the proof of this lemma we now state the main result of this paper.

Theorem. For non-overlapping $\{\tilde{f}_i\} \in \mathscr{S}(G)$, the connected part of the S-matrix element $\langle \hat{f}_1^{\text{out}} \cdots \hat{f}_n^{\text{out}} \mid \hat{f}_{n+1}^{\text{in}} \cdots \hat{f}_{n+m}^{\text{in}} \rangle$ is given by

$$\sum_{\pi} (-i\lambda)^{|\pi|} \int \prod_{i=1}^{n+m} dt_i \int \prod_{\sigma \in \pi} dy_\sigma \prod_{i=1}^{n} (-\bar{g}_i)(y_i, t_i) \prod_{i=n+1}^{n+m} g_i(y_i, t_i) \left\langle T \prod_{\sigma \in \pi} v^{|\sigma|}(y_\sigma) \right\rangle^T \tag{19}$$

where $\hat{f}_i$ and g_i are given by (13) and (16) respectively. The coefficients of $(-i\lambda)^{|\pi|}$ in (19) are bounded uniformly in λ for $\lambda \in [0, \lambda_0]$.

The proof of this theorem follows immediately from equation (17) and from Lemma 3.

Corollary. Suppose the interaction polynomial $P(\xi)$ is of the form ξ^{2n} + lower order terms. Then for non-overlapping $\{\tilde{f}_i\} \in \mathscr{S}(G)$

$$\langle \hat{f}_1^{\text{out}} \cdots \hat{f}_m^{\text{out}} \mid \hat{f}_{m+1}^{\text{in}} \cdots \hat{f}_{2n}^{\text{in}} \rangle_{\text{connected}}$$

$$= -i\lambda(2n)! \int \prod_{i=1}^{m} \frac{\bar{\hat{f}}_i(\vec{p}_i)}{2\omega_i}\, d\vec{p}_i \prod_{i=m+1}^{2n} \frac{\hat{f}_i(\vec{p}_i)}{2\omega_i}\, d\vec{p}_i$$

$$\times\, \delta\left(\sum_1^m \omega_i - \sum_{m+1}^{2n} \omega_i\right) \delta\left(\sum_1^m \vec{p}_i - \sum_{m+1}^{2n} \vec{p}_i\right) + O(\lambda^2).$$

Hence for λ sufficiently small, the S-matrix is non-trivial.

We now prove Lemma 3, following closely Hepp's arguments in the proof of Theorem 3.1 in [11]. In a crucial way we will use the support and decay properties of the test functions g_i and the fact that the finite mass renormalization has been done correctly (see Lemma 4). Our first goal is to partially remove the time ordering in (18).

The test functions $g_i(x, t)$ have their support 'essentially' in a neighborhood of

$$\{(x, t) \mid x^0 = t, \vec{x} = \vec{v}t \text{ for some } \vec{v} \in \mathscr{V}_i\}$$

where $\mathscr{V}_i = \{\vec{v} \mid \vec{v} = \vec{p}/\omega, (p^0, \vec{p}) \in \operatorname{supp} \tilde{g}_i \text{ for some } p^0\}$ is the set of velocities admitted by g_i. More precisely, $g_i(x, t)$ is in $\mathscr{S}(\mathbb{R}^1)$ in the variable $\tau = x^0 - t$ and a smooth solution of the Klein–Gordon equation in the variables $(t, \vec{x})$. From this one easily derives the following uniform estimates on g_i and all its derivatives

$$|g_i(x, t)| \leqslant \text{const} \begin{cases} (1 + |x^0 - t|)^{-M} & \text{for } \operatorname{dist}(t^{-1}\vec{x}, \mathscr{V}_i) < \eta \\ (1 + |x^0 - t|)^{-M}(1 + \vec{x}^2 + t^2)^{-N} & \text{otherwise.} \end{cases} \tag{20}$$

Inequalities (20) hold for all $M, N \in \mathbb{Z}_+$, $\eta > 0$; the constant depending on g_i, M, N and η but not on x and t, see Refs. [16] or [12].

Because $\{\tilde{g}_i\}$ are non-overlapping, there exists $\eta \in (0, \frac{1}{2})$ such that the sets

$$S_i(t_i) = \{x \mid |t_i^{-1}x^0 - 1| < \eta, \operatorname{dist}(t_i^{-1}\vec{x}, \mathscr{V}_i) < \eta\}$$

are mutually space-like separated, whenever

$$\left| \frac{t_i - t_j}{t_i + t_j} \right| < \eta. \tag{21}$$

Now we choose and fix $t_1, \ldots, t_m$ and set $t = \sup|t_i|$. We study (18) for various sectors in the space of the t_i's. All bounds will be uniform for $\lambda \in [0, \lambda_0]$.

(a) Suppose $|t_i - t_j| > \alpha t^\beta$, with $\alpha = \eta/n$, $0 < \beta \leqslant 1$ and $\sigma(i) = \sigma(j)$. Then either $|t_i - y_\sigma^0| > (\alpha/2)t^\beta$ or $|t_j - y_\sigma^0| > (\alpha/2)t^\beta$ and (18) with all its derivatives is $o(t^{-M})$ for all $M \in Z_+$, due to (20).

(b) Suppose $|t_i - t_j| \leqslant \alpha t^\beta$ whenever $\sigma(i) = \sigma(j)$. Let us consider the sector where $t > 1$ and

$$t = t_1 \geqslant t_2 \geqslant t_3 \geqslant \cdots \geqslant t_m, \tag{22'}$$

and for some $k \in \{1, \ldots, n\}$

$$\begin{aligned} &t_i - t_{i+1} \leqslant \alpha t^\beta \qquad \text{for } i = 1, \ldots, k-1 \\ &t_k - t_{k+1} > \alpha t^\beta. \end{aligned} \tag{22''}$$

(Other sectors are dealt with similarly.) In this sector, for $i, j < k$

$$|t_i - t_j| \leqslant \eta t^\beta \quad \text{and} \quad t \geqslant t_i \geqslant t - \eta t^\beta > t/2 \tag{23}$$

but for $i \leqslant k < j$

$$t_i - t_j > \alpha t^\beta \quad \text{and} \quad \sigma(i) \neq \sigma(j). \tag{24}$$

It follows that the partition π of $\{1, \ldots, n\}$ consists in a partition π_1 of $\{1, \ldots, k\}$ and a partition π_2 of $\{k+1, \ldots, m\}$. Again by (20) we make only an error of $o(t^{-N})$ if we restrict the y_σ-integrations of (18) to the region $\mathscr{R}_\eta$ defined by

$$|y_i^0 - t_i| < \frac{\alpha}{2} t^\beta, \qquad \text{for all } i \tag{25}$$

and

$$\operatorname{dist}\left(\frac{\vec{y}_i}{t_i}, \mathscr{V}_i\right) < \eta \qquad \text{for } i = 1, \ldots, k.$$

In this region $y_i^0 > y_j^0$ for $i \leqslant k < j$ (by (24)) and by (25) and (23)

$$|t_i^{-1} y_i^0 - 1| < \eta \quad \text{and} \quad \left|\frac{t_i - t_j}{t_i + t_j}\right| < \eta \qquad \text{for } i, j \in \{1, \ldots, k\}.$$

Therefore by (21), in $\mathscr{R}_\eta$ the points y_σ, $\sigma \in \pi_1$, are mutually space-like separated and

$$T \prod_{\sigma \in \pi} v^{|\sigma|}(y_\sigma) = \prod_{\sigma \in \pi_1} v^{|\sigma|}(y_\sigma) \left(T \prod_{\sigma \in \pi_2} v^{|\sigma|}(y_\sigma)\right).$$

We conclude that in the sector determined by (22), up to an error of $o(t^{-N})$, (18) is equal to

$$\int \prod_{\sigma \in \pi} \left\{dy_\sigma \prod_{i \in \sigma} g_i^*(y_i, t_i)\right\} \left\langle \prod_{\sigma \in \pi_1} v^{|\sigma|}(y_\sigma) \left(T \prod_{\sigma \in \pi_2} v^{|\sigma|}(y_\sigma)\right)\right\rangle. \tag{26}$$

By Schwarz' inequality the modulus of (26) is bounded by

$$\left\| \int \prod_{\sigma \in \pi_1} \left\{dy_\sigma \prod_{i \in \sigma} g_i^*(y_i, t_i) v^{|\sigma|}(y_\sigma)\right\} \right\rangle \right\| \left\| T \prod_{\sigma \in \pi_2} \left\{dy_\sigma \prod_{i \in \sigma} g_i^*(y_i, t_i) v^{|\sigma|}(y_\sigma)\right\} \right\rangle \right\|. \tag{27}$$

The second factor in (27) grows at most polynomially in the t_i's, while we claim that the first factor in (27), in the sector determined by (22), is $o(t^{-N})$ for any N. Proving this claim we end the proof of Lemma 3.

We expand the square of the first factor in (27) into a sum of products of truncated vacuum expectation values and show that each factor

$$\left\langle \int \prod_{\sigma}{}' \left\{ dy_\sigma \prod_{i\in\sigma} g_i^*(y_i, t_i) v^{|\sigma|}(y_\sigma) \right\} \right\rangle^T \tag{28}$$

in that expansion is $o(t^{-N})$. The product $\prod_0'$ now runs over σ contained in some subset of two copies of π_1. In case there are more than two factors g^* in (28) the proof goes exactly as in Ref. [11] (with an appropriate choice of β) and we will not repeat it here.

For the remaining cases we claim (28) is zero. Namely

$$1.\quad \left\langle \int dy g_i^*(y, t_i) v^1(y) \right\rangle^T = \int dy g_i^*(y, t_i) \cdot \langle v^1(0) \rangle = \tilde{g}_i^*(0) e^{\pm i t_i m} \langle v^1(0) \rangle = 0$$

because of the support properties of $\tilde{g}_i^*$.

$$2.\quad \left\langle \int dy g_i^*(y, t_i) g_j^*(y, t_j) v^2(y) \right\rangle = 0$$

as for $i \neq j$ and $\{\tilde{g}_i^*\}$ non-overlapping, $\operatorname{supp} \tilde{g}_i^* \cap \operatorname{supp} \tilde{g}_j^* = \varnothing$. Of course, terms of this form with $i = j$ do not occur in (28).

$$3.\quad \left\langle \int dy_1 dy_2 g_i^*(y_1, t_i) g_j^*(y_2, t_j) v^1(y_1) v^1(y_2) \right\rangle^T = 0$$

because of the following lemma.

Lemma 4. The Fourier transform $\langle \tilde{v}^1(p)\tilde{v}^1(q)\rangle^T$ of $\langle v^1(y_1)v^1(y_2)\rangle^T$ is of the form

$$\text{const } \delta(p+q) \int_{m'}^{\infty} d\rho(a^2)\theta(p^0)\delta(p^2 - a^2).$$

Proof. Using integration by part (2) we find

$$\langle \Phi_1 \Phi_2 \rangle^T = C(x_1 - x_2) - \lambda \int C(x_1 - y) C(x_2 - y) \langle V^2(y) \rangle \, dy + \lambda^2 \int C(x_1 - y_1) C(x_2 - y_2) \langle V^1(y_1) V^1(y_2) \rangle^T \, dy_1 \, dy_2.$$

Hence $\lambda^2 \langle V^1 V^1 \rangle^T = C^{-1} \langle \Phi_1 \Phi_2 \rangle^T C^{-1} - C^{-1} + \lambda \langle V^2 \rangle$.
Analytic continuation of this equation to real times shows that the Fourier transform of $\langle T v^1 v^1 \rangle^T$ has no singularity on the mass shell. Hence it must have a Källen–Lehmann representation of the form

$$\text{const } \delta(p+q) \int_{m'} d\rho(a^2)(p^2 - a^2 + i\epsilon)^{-1}.$$

The lemma follows immediately. This ends our proof of Lemma 3.

V. Concluding Remarks

Our expansion (19) of the S-matrix elements reproduces ordinary perturbation expansion only in lowest (non-trivial) order. To get more of the perturbation expansion one would have to continue using integration by parts in (3) and then repeat the rest of our arguments. This method is suitable to prove that the perturbation expansion of the S-matrix is asymptotic. The advantage of this procedure over the straightforward Taylor expansion with remainder term is that one never has to deal with derivatives of the mass $m(\lambda)$.

Our arguments should also be applicable to show that other models with weak coupling have a non-trivial S-matrix. The essential input into our proof is the existence of an isolated mass hyperboloid and the local integrability of the (generalized) Euclidean Green's functions.

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Note added in proof: After the publication of the preprint of this work, another preprint appeared [18], containing similar results.

Appendix I: Proof of Lemma 1

We use the calculus of formal power series, see Ref. [16]. Let F, G be functions from subsets of $\{1, 2, \ldots\}$ to $\mathbb{C}$ and define $F + G$, $F * G$, F^{-1} and I by

$$(F + G)(N) = F(N) + G(N)$$

$$(F * G)(N) = \sum_{N_1 \cup N_2 = N} F(N_1)G(N_2)$$

$$(F^{-1} * F)(N) = I(N) = \begin{cases} 1 & \text{if } N = \varnothing, \\ 0 & \text{otherwise.} \end{cases}$$

Now for $A = (A_1, A_2, \ldots)$ we define F_A, $F_{A,k}$ and F_A^T by

$$F_A(N) = \left\langle \prod_{i \in N} A_i \right\rangle, \; A_{A,k}(N) = F_A(N \cup \{k\}), \qquad \text{for } k \notin N,$$

$$F_{A,k}^T = F_{A,k} * F_A^{-1}.$$

The order of the factors in $\prod_{i \in N} A_i$ is corresponding to the indices. For $A_i = \Phi_i = \Phi(x_i)$, and $A \equiv \Phi = (\Phi_1, \Phi_2, \ldots)$ $F_\Phi^T(N)$ is a Euclidean Green's function and

$$F_\Phi^T(N) = (F_{\Phi,k} * F_\Phi^{-1})(N) \tag{A1}$$

is the truncation of it.

We want to use the integration by parts formula (2) on the factor $F_{\Phi,k}$ in (A1). We write Ψ_k for the operator $C_k(\delta_k - \lambda V_k)$, with C_k and δ_k as in Section II and $V_k = {:}P^{(1)}(\Phi(y_k)){:}$. By convention $\delta_k \Omega = 0$. Then for $k \notin N$,

$$\begin{aligned} F_{\Phi,k}(N) &= \left\langle \prod_{i \in N \cup \{k\}} \Phi_i \right\rangle \\ &= \left\langle \Phi_k \prod_{i \in N} \Phi_i \right\rangle \\ &= (\langle \Phi_k \rangle + \lambda C_k \langle V_k \rangle) \left\langle \prod_{i \in N} \Phi_i \right\rangle + \left\langle C_k(\delta_k - \lambda V_k) \prod_{i \in N} \Phi_i \right\rangle \\ &= (\langle \Phi_k \rangle + \lambda C_k \langle V_k \rangle) F_\Phi(N) + \sum_{\substack{l \in N \\ l < k}} C_{kl} F_\Phi(N \backslash \{l\}) + F_{(\Phi_1, \Phi_2, \ldots, \Psi_k, \ldots), k}(N). \end{aligned} \tag{A2}$$

Here $C_{kl} = C(x_k - x_l)$, and the sum $\sum_{l \in N, l<k}$ in (A2) comes from moving $\Psi_k = C_k(\delta_k - \lambda V_k)$ to its right place using $[\Psi_k, \Phi_l] = C_{kl}$.

Substituting (A2) in (A1) we find, again for $k \notin N$,

$$F^T_{\Phi,k}(N) = (\langle \Phi_k \rangle + \lambda C_k \langle V_k \rangle) I(N) + \sum_{\substack{l \in N \\ l<k}} C_{kl} I(N \backslash \{l\}) + F^T_{(\Phi_1, \Phi_2, \ldots, \Psi_k, \ldots), k}(N) \quad \text{(A3)}$$

If N contains more than one element then the first two terms on the r.h.s. of (A3) vanish and we find

$$F^T_{\Phi,k}(N) = F^T_{(\Phi_1, \Phi_2, \ldots, \Psi_k, \ldots), k}(N). \quad \text{(A4)}$$

By definition of $F_{A,k}$ we may write (A4) as

$$F^T_{(\Phi_1, \Phi_2, \ldots, \Psi_k, \ldots), k-1}(N \cup \{k\} \backslash \{k-1\})$$

(for those N which contain $k - 1$). Hence we can repeat our argument and integrate by parts with respect to Φ_{k-1}. Notice that we never have to move Φ_{k-1} across Ψ_k. Finally after sufficiently many steps we find with $\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_k, \ldots)$

$$F^T_{\Phi,k}(N) = F^T_{\Psi,k}(N)$$

for N containing more than one element. Now we choose $N = \{1, 2, \ldots, k - 1,\}$ $k > 2$ and find that

$$F^T_{\Phi,k}(N) = \left\langle \prod_{i=1}^{k} \Phi_i \right\rangle^T$$

$$= F^T_{\Psi,k}(N) = \left\langle \prod_{i=1}^{k} C_i(\delta_i - \lambda V_i) \right\rangle^T. \quad \text{(A5)}$$

Expanding the product in the last term of (A5) proves the lemma.

Appendix 2: Proof of Lemma 2

We only prove (7), the proof of (8) is similar. We use the methods of [15] to construct the analytic continuation $\mathfrak{S}_k(\mu \mathbf{x})$ of $\mathfrak{S}_k(\mathbf{x}) = \langle \prod_{i=1}^k \Phi_i \rangle$ and to find bounds on it.

Lemma A1. Let $\mathbf{x} \in \mathbb{R}^{2k}$ with $x_i^0 \neq x_j^0$ for all $i \neq j$ and $\mu = i - \delta$ with $0 < \delta < \frac{1}{2}$. Then there is a constant c, depending on k only (in particular not on λ for $\lambda \in [0, \lambda_0]$) such that

$$|\mathfrak{S}_k(\mu \mathbf{x})| \leqslant c(1 - \ln \epsilon)^{2k} \cdot \delta^{-(2k+1/2)} \quad \text{(A6)}$$

where

$$\epsilon = \min\{1, |x_i^0 - x_j^0| \text{ for } 1 \leqslant i < j \leqslant k\}.$$

Proof. The estimates of [6] and of [8] show that

$$|\mathfrak{S}_k(\mathbf{x})| < \alpha^k k! \prod (1 - \ln \epsilon)^{2k}$$

for some constant α not depending on λ for $\lambda \in [0, \lambda_0]$. (We remark that this bound is the source of the uniformity in λ of *all* the estimates in this paper.) Now we look at the sector in $\mathbb{R}^{2k}$ where

$$x_1^0 > x_2^0 > \cdots > x_k^0.$$

(Other sectors are dealt with similarly.) In this sector, using the notation of [15],

$$\mathfrak{S}_k(\mathbf{x}) = S_{k-1}(\boldsymbol{\xi})$$
$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_{k-1}), \quad \xi_i = (\xi_i^0, \vec{\xi}_i) \in \mathbb{R}^2, \quad \xi_i = x_{i+1} - x_i.$$

Then the methods of [15] show that for Re $\mu < 0$ and some constant $c = c(k)$

$$|\mathfrak{S}_k(\mu\boldsymbol{\xi})| = |S_{k-1}(\mu\boldsymbol{\xi})|$$
$$\leqslant c|\arg \mu - \pi/2|^{-2k}(1 - \ln \epsilon')^{2k}$$

where $\epsilon' = \min\{1, |\mathrm{Re}\, \mu\xi_i^0| \text{ for } i = 1, 2, \ldots, k-1\}$. Hence Lemma (A1), as $|\arg \mu - \pi/2| > \delta/2$ and $(1 - \ln \epsilon') \leqslant (1 - \ln \epsilon)(1 - \ln \delta)$.

From estimate (A6) it follows by standard methods that (6) defines a tempered distribution. Namely with $\mu = i - \delta, f \in \mathscr{S}(\mathbb{R}^{2k})$ and

$$\tau_\delta(f) = \int \mathfrak{S}_k(\mu\mathbf{x})f(x)\, d\mathbf{x}, \tag{A7}$$

$$\frac{d^\gamma}{d\delta^\gamma} \tau_\delta(f) = \int \mathfrak{S}_k(\mu\mathbf{x}) \sum_{i_1, i_2, \ldots, i_\gamma} \left(\prod_{k=1} \frac{\partial}{\partial x_{i_k}^0} x_{i_k}^0\right) f(\mathbf{x})\, d\mathbf{x}. \tag{A8}$$

Choosing $\gamma = 2k + 1$ and using (A6) we find by integrating (A8) γ times that (A7) is uniformly bounded as $\delta \to 0$ and converges (to a tempered distribution). The remaining assertions in Lemma 2 follow immediately from the results of [15].

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