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On the Derivation of Bounds for the Ladder Graphs of a Scattering Amplitude[1])

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Abstract. The derivation of exact upper and lower bounds for the sum of scalar ladder graphs is described in detail. The bounds are consistent with Regge behaviour. The sum of ladder graphs diverges if the coupling constant is large. This divergence occurs as well in the $g\phi^3$ theory in four space-time dimensions as in the $g\phi^4$ theory in two dimensions.

1. Introduction

As announced in a recent letter [1], we have revisited the ladder diagrams of a scalar $g\phi\psi^2$ theory. In this article we display the techniques which we used in the derivation of our results. For completeness we repeat the discussion of these results, adding one point which has been established after the drawing up of our letter.

The investigations reported here started with the recognition that the derivation of rigorous and nontrivial bounds for the sum of ladder graphs is a practicable undertaking. We found it worthwhile to devote some effort to the construction of such bounds because the information they provide is complementary to that provided by more usual methods. In particular, the extraction of the asymptotic behaviour of the sum of ladder graphs from the asymptotic behaviour of the individual graphs is quite an art because it is not sufficient to take only the leading term of each graph [2]. Exact bounds test the reliability of formal manipulations with asymptotic forms. Furthermore the bounds we are able to derive are valid at all energies and are not only constraints on the high-energy behaviour of the ladder graphs.

Our work has some resemblance with the investigations initiated by Tiktopoulos and Treiman [3]. These authors establish rigorous bounds for the absorptive part of the forward multiperipheral scattering amplitude. Our bounds are bounds of the modulus of the full amplitude and we are not limited to forward scattering. Therefore our bounds may reveal singularities which are only present in the real part of the scattering amplitude.

We study the ladder graphs shown in Figure 1; the details of our notations will be explained in Section 2. We consider these graphs at fixed momentum transfer t and variable energy s. The techniques we shall describe allow the derivation of upper bounds for $|T_n(s, t)|$ for any s in the complex plane cut along $((n\mu)^2, \infty)$, including

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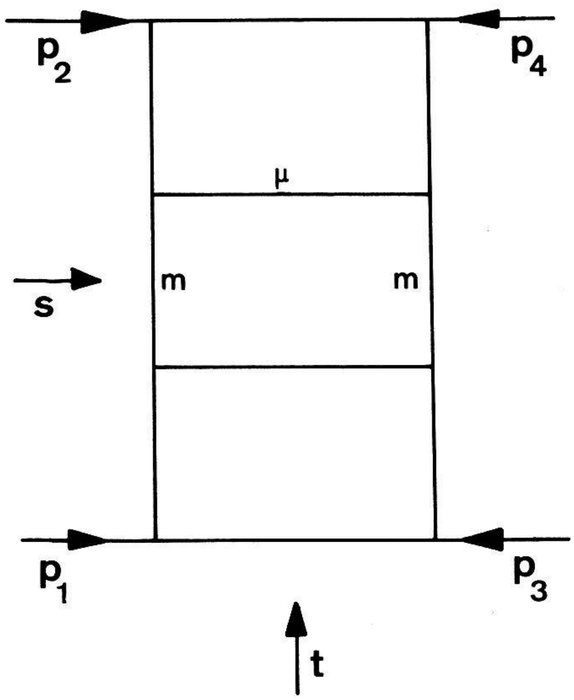

Figure 1
The ladder graph $T_4(s, t)$. The mass of the external particles and of the particles represented by vertical lines is m. The particles corresponding to the horizontal rungs is μ. The variables s and t are defined in the standard way: $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$.

the cut. In this article, we shall restrict ourself to the u-channel where s and t are negative, with one exception in Section 5. We hope to explore the s-channel in a forthcoming publication. The advantage of the u-channel is that in this channel we can derive not only upper bounds for $T_n(s, t)$ but also lower bounds. This allows a control of the convergence of the series of ladder graphs.

Our results are consistent with a high-energy behaviour dominated by a Regge pole. They give bounds for the position $\alpha(t)$ of this pole which are consistent with the most refined estimations [4, 5]. Furthermore, our results show that the series of ladder graphs fails to converge for all t's if the coupling constant is large. They indicate that, once the coupling constant is such that $\alpha(0) > 0$, the sum of ladder graphs has a pole at a negative, physical value of t at which $\alpha(t) = 0$. The appearance of this ghost has apparently not been noticed explicitly until now. What has been noticed is that $\alpha(0)$ may exceed 1. Then, the ladder graphs violate the Froissart bound and other graphs have to correct this defect [6]. We prove that compensations between ladder and other graphs must be effective earlier, as soon as $\alpha(0) > 0$.

Our basic formulas are given in Section 2. The derivation of an upper bound is presented in Section 3 whereas a lower bound is constructed in Section 4. The matter of these two Sections is of a highly technical nature. Our results are discussed and extended in Section 5. The details of our computations are collected in four Appendices.

2. Basic Formulas

Let $T_n(s, t)$ designate the contribution of the ladder graph with n rungs (Fig. 1) to the scattering amplitude of two mesons of mass m. The diagram describes the exchange of n mesons of mass μ. The usual invariants s, t and u are defined by:

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2, \quad s + t + u = 2m^2 + 2\mu^2. \tag{2.1}$$

Our goal is to find properties of the sum $T(s, t)$ of all ladder graphs with $n \geqslant 2$:

$$T(s, t) = \sum_{n=2}^{\infty} T_n(s, t). \tag{2.2}$$

Notice that the Born term $T_1(s, t) = -g^2/(s - \mu^2)$ has been excluded from this sum. g is our coupling constant.

We start with the Okubo–Feldman representation [7] of the Bethe–Salpeter amplitude. It has been shown in [8] that this representation leads to the Fourier representation of $T_n(s, t)$ with respect to s:

$$T_n(s, t) = i16\pi^2\lambda^n \int_0^{\infty} dz\bar{f}_n(z, t)e^{izs}, \tag{2.3}$$

with $\lambda = (g^2/16\pi^2)$. The Fourier transform $\bar{f}_n(z, t)$ is an analytic function of z which is regular in the lower half plane $\operatorname{Im} z \leqslant 0$ if $t < 4m^2$. Furthermore

$$|\bar{f}_n(z, t)e^{izn^2\mu^2}| < \text{Cst.}\ |z|^{-1/2} \tag{2.4}$$

in this half-plane. This allows one to shift the integration path in (2.3) onto the negative imaginary axis if $s < n^2\mu^2$. The result is the much more convenient Laplace representation:

$$T_n(s, t) = 16\pi^2\lambda^n \int_0^{\infty} dzf_n(z)e^{zs} \tag{2.5}$$

valid for $t < 4m^2$ and $s < n^2\mu^2$; $f_n(z, t) = \bar{f}_n(-iz, t)$. All what follows reduces to the derivation and application of specific properties of the Laplace transform $f_n(z, t)$. This forces us to write down explicitly its cumbersome expression. If $n \geqslant 2$, $f_n(z, t)$ is given by the following multiple integral:

$$\begin{aligned} f_n(z, t) = \left(\frac{z}{2}\right)^{n-1} \int_0^{\infty} dq_1 \cdots \int_0^{\infty} dq_{n-1} \int_0^{\infty} dx_1 \int_{-x_1}^{+x_1} dy_1 \cdots \int_{x_{k-1}}^{\infty} dx_k \\ \times \int_{y_{k-1}-x_k+x_{k-1}}^{y_{k-1}+x_k-x_{k-1}} dy_k \cdots \int_{x_{n-2}}^{\infty} dx_{n-1} \int_{y_{n-2}-x_{n-1}+x_{n-2}}^{y_{n-2}+x_{n-1}-x_{n-2}} dy_{n-1} \\ \times \exp[-z\Delta_n(t, q, x, y)] \end{aligned} \tag{2.6}$$

where $\Delta_n(t, q, x, y)$ is the function:

$$\begin{aligned} \Delta_n(t, q, x, y) = n\mu^2 + \sum_{k=1}^{n-1} \Bigg\{\Bigg[-\tfrac{1}{4}t(x_k^2 - y_k^2) \\ + m^2x_k^2 + \mu^2(1 + x_k)\left(1 + \sum_{j=1}^{k-1} (1 + x_j)\right)\Bigg]q_k \\ + \mu^2 \frac{1}{q_k}\left[1 + \sum_{j=k+1}^{n-1} (1 + x_j)q_j\right]^2 + \mu^2 x_k\Bigg\}. \end{aligned} \tag{2.7}$$

The preceding formulas are obtained from equation (2.3) in [8] by reversing the numbering of the variables [3]). The most important property of $f_n(z, t)$ is that it is positive. This implies that $T_n(s, t)$ is positive for $s < n^2\mu^2$ and $t < 4m^2$. What is more important is that upper and lower bounds for $f_n(z, t)$ produce, via (2.5), upper and lower bounds for $T_n(s, t)$.

Matters simplify considerably in the case of forward scattering, $t = 0$. Then Δ_n does not depend on the y_k's and the y_k-integrations in (2.6) become trivial. In fact we shall restrict ourself to this case, with one exception in Section 5. It is readily seen that the results we shall obtain for $t = 0$ give automatically some crude information concerning the general case $t \neq 0$. We write, making the m^2-dependence of Δ_n explicit:

$$\Delta_n(t, q, x, y) = F_n(t, q, x, y, m^2),$$
$$\Delta_n(0, q, x, y) = G_n(q, x, m^2).$$

We notice that the limits of integration in (2.6) are such that $|y_k| < x_k$. Under these circumstances:

$$G_n(q, x, m^2 - \tfrac{1}{4}t) \leqslant F_n(t, q, x, y, m^2) \leqslant G_n(q, x, m^2) \tag{2.8}$$

if $0 \leqslant t \leqslant 4m^2$ and

$$G_n(q, x, m^2) \leqslant F_n(t, q, x, y, m^2) \leqslant G_n(q, x, m^2 - \tfrac{1}{4}t) \tag{2.9}$$

if $t \leqslant 0$. These inequalities imply that if $U(s, m^2)$ and $L(s, m^2)$ are upper and lower bounds of $T(s, 0)$, these functions give bounds for $T(s, t)$ provided m^2 is replaced by $m^2 - \frac{1}{4}t$ at some places:

$$L(s, m^2) < T(s, t) < U(s, m^2 - \tfrac{1}{4}t) \tag{2.10}$$

if $0 \leqslant t \leqslant 4m^2$ and

$$L(s, m^2 - \tfrac{1}{4}t) < T(s, t) < U(s, m^2) \tag{2.11}$$

if $t \leqslant 0$.

3. An Upper Bound

In this Section we shall derive an upper bound for the sum $T(s, 0)$ of the ladder graphs for forward scattering. The bound is valid if $s < \mu^2$. Our method is similar to that used by Tiktopoulos and Treiman in [3]. The required effort is modest and the result is optimal in the sense that it gives the exact behaviour of $T(s, 0)$ in the limit $\mu \to 0$.

If $t = 0$, (2.6) becomes

$$f_n(z) = z^{n-1} \int_0^\infty dq_{n-1} \cdots \int_0^\infty dq_1 \int_0^\infty dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \cdots \int_0^{x_2}$$
$$\times\, dx_1 (x_{n-1} - x_{n-2}) \cdots (x_2 - x_1) x_1 \exp[-z\Delta_n(0, q, x, 0)] \tag{3.1}$$

[3]) The careful reader has noticed that the T_n's have different signs in [1] and [8] and in the present work. In [1] and [8] we adopted the definition of the transition matrix T commonly used in perturbation theory. This T is related to the S-matrix by $S = 1 - iT$. In this work we switch to the definition $S = 1 + iT$; the T_n's have positive absorptive parts in the s-channel and are real and positive in the u-channel.

with $f_n(z) = f_n(z, 0)$. As $\Delta_n(0, q_k, x_k, 0)$, defined in (2.7), is a sum of positive terms, one gets an upper bound for $f_n(z)$ by dropping some of these terms. The upper bound we shall use is obtained by keeping only the term $\mu^2(1 + x_1)$ and a part of the terms which are linear in the q_k's:

$$\Delta_n(0, q, x, 0) \to \mu^2(1 + x_1) + \sum_{k=1}^{n-1} x_k(m^2 x_k + \mu^2) q_k. \tag{3.2}$$

It is somewhat surprising that we shall get sensible results after such a drastic mutilation. Once the substitution (3.2) is performed, the q_k-integrations in (3.1) are trivial. One gets an upper bound for $f_n(z)$ which has the form:

$$f_n(z) < \int_0^\infty dx \frac{e^{-z(1+x)\mu^2}}{x(m^2x + \mu^2)} g_n(x). \tag{3.3}$$

The functions $g_n(x)$ can be defined by the recurrence relation:

$$g_n(x) = \int_0^x dx' \frac{x - x'}{x'(m^2x' + \mu^2)} g_{n-1}(x') \tag{3.4}$$

for $n \geqslant 3$ and $g_2(x) = x$.

If the sum $T(s, 0)$ of ladder graphs converges, it has the Laplace representation

$$T(s, 0) = 16\pi^2 \int_0^\infty dz f(z) e^{zs} \tag{3.5}$$

for $s < (4\mu^2)$. The Laplace transform $f(z)$ is given by:

$$f(z) = \sum_{n=2}^\infty \lambda^n f_n(z) < \int_0^\infty dx \frac{e^{-z(1+x)\mu^2}}{x(m^2x + \mu^2)} g(x). \tag{3.6}$$

The function $g(x)$ is the sum of the $\lambda^n g_n$'s:

$$g(x) = \sum_{n=2}^\infty \lambda^n g_n(x). \tag{3.7}$$

The recurrence relation (3.4) implies that $g(x)$ is the solution of the following Volterra integral equation:

$$g(x) = \lambda^2 x + \lambda \int_0^x dx' \frac{x - x'}{x'(m^2x' + \mu^2)} g(x'). \tag{3.8}$$

This equation has a unique solution for all values of λ which coincides with the sum of the series (3.7). This means that *the sum $f(z)$ of the Laplace transforms of the ladder diagrams converges for all values of the coupling constant;* $f(z)$ is an entire function of λ for $z > 0$.

Equation (3.8) is discussed in Appendix A. The results given there imply that $g(x)$ has the following upper bound:

$$g(x) < \begin{cases} \text{Cst.}\ x & \text{for } x < \mu^2/m^2, \\ \text{Cst.}\ x^{\alpha_1+2} & \text{for } x > \mu^2/m^2, \end{cases} \tag{3.9}$$

where the exponent α_1 is

$$\alpha_1 = \frac{1}{2}\left(-3 + \sqrt{1 + \frac{4\lambda}{m^2}}\right). \tag{3.10}$$

Inserting (3.9) into (3.6), one gets:

$$f(z) < \left[\bar{C}_1 + \bar{C}_2 \frac{1}{z^{\alpha_1+1}}\right] e^{-z\mu^2}. \tag{3.11}$$

This upper bound is singular at $z = 0$ and is nonintegrable if α_1 is larger than 0. This means that the upper bound (3.11) of the Laplace transform $f(z)$ produces a convergent upper bound for $T(s, 0)$ only if the coupling constant is small enough: $\lambda < 2m^2$.

Our bound for $T(s, 0)$ is obtained by combining (3.11) and (3.5). It has the form:

$$T(s, 0) < \left[\frac{C_1}{|s - \mu^2|} + C_2 \frac{|s - \mu^2|^{\alpha_1}}{\sin|\pi\alpha_1|}\right]. \tag{3.12}$$

To convince our reader that the constants C_1 and C_2 are known finite functions of the masses and coupling constant, we give their precise expressions:

$$C_1 = \frac{\lambda^{3/2}}{m} sh\left(\frac{2\lambda^{1/2}}{m}\right),$$

$$C_2 = \pi\left(\frac{m}{\mu^2}\right)^{2(\alpha_1+1)} \frac{\lambda}{m^2}\left[1 + \frac{\lambda^{1/2}}{m} sh\left(\frac{2\lambda^{1/2}}{m}\right)\right].$$

The bound (3.12) is valid if $s < \mu^2$ and $\lambda < 2m^2$; the corresponding range of α_1 is $(-1, 0)$. The second term dominates the first one in the limit $s \to \infty$. This term behaves as the asymptotic form of an amplitude dominated by a Regge pole α_1. It is common belief that the sum of ladder graphs has in fact an asymptotic Regge behaviour. Our result shows that the position α of its Regge pole has to be below our α_1. As a matter of fact, inspection of (3.12) and (3.10) shows that α_1 is exactly the known position of the Regge pole of $T(s, 0)$ in the case of massless rungs, $\mu = 0$ [9]. It is in this precise sense that our result is optimal. Our μ-independent upper bound α_1 of α is the value taken by α in the limit $\mu = 0$.

We cannot infer a divergence of the sum of ladder graphs from the divergence of the upper bound (3.11) at $\lambda = 2m^2$. The question of the possible divergence of the series $\sum_n T_n$ will be settled in the next Section by the construction of a lower bound. In any case the method of the present Section gives information beyond $\lambda = 2m^2$. If we consider the derivative $(\partial^n/\partial s^n)T(s, 0)$ instead of $T(s, 0)$, we get:

$$\left|\frac{\partial^n}{\partial s^n} T(s, 0)\right| < \frac{1}{|s - \mu^2|^n} \times \left[\frac{n!\, C_1}{|s - \mu^2|} + C_2 \frac{|\alpha_1(\alpha_1 - 1)\cdots(\alpha_1 - n + 1)|}{\sin|\pi\alpha_1|} |s - \mu^2|^{\alpha_1}\right]. \tag{3.13}$$

This bound is finite and valid for $-1 < \alpha_i < n$, i.e. $\lambda < (2 + n^2 - 3n)m^2$.

The left hand side of (3.12) is a function $U(s, m^2)$ in the sense of Section 2. The rules (2.10) and (2.11) give upper bounds for $T(s, t)$ at $t \neq 0$. In particular, the

t-dependence of the position $\alpha(t)$ of the Regge pole must be such that:

$$\alpha(t) < \begin{cases} \frac{1}{2}\left(\sqrt{1 + \frac{4\lambda}{m^2 - \frac{1}{4}t}} - 3\right) & \text{for } 0 \leqslant t \leqslant 4m^2, \\ \frac{1}{2}\left(\sqrt{1 + \frac{4\lambda}{m^2}} - 3\right) & \text{for } t \leqslant 0. \end{cases} \tag{3.14}$$

4. A Lower Bound

The results of the preceding Section show that $T(s, t)$ is an analytic function of λ, regular in $|\lambda| < 2m^2$ if $t \leqslant 0$ and $|\lambda| < 2(m^2 - \frac{1}{4}t)$ $0 \leqslant t < 4m^2$. They create a kind of suspense by suggesting a divergence without proving it. In this Section we shall prove that the divergence is really there. We can do this mainly because all the T_n's have the same positive sign. In these fortunate circumstances a divergent series of lower bounds of the T_n's is a sure indication that $\sum_n T_n$ itself diverges. The circumstances are fortunate but they do not save our pain, as the reader will recognize soon. Paradoxically, given a divergent series it might be hard to find a series of lower bounds which is divergent too.

In the first part of the present Section, we shall derive a lower bound for $f_n(z)$, the Laplace transform of $T(s, 0)$. This is much more involved than the crude majorization performed in Section 3. We shall sketch here the main steps of the derivation, indications concerning the details of the calculations being collected in Appendix B.

Looking at definition (3.1), we see that one gets a lower bound of $f_n(z)$ by majorizing $\Delta_n(0, q_k, x_k, 0)$. According to (2.7) such a majorization is obtained by replacing m and μ by M, the largest of these masses:

$$M = \max(m, \mu). \tag{4.1}$$

We shall do this in order to simplify the expression of our results.

The results of Section 3 show that a divergence of $\sum_n T_n(s, 0)$ can only result from the singularity of $f(z)$ at $z = 0$. Therefore, the lower bound we have in mind has to be as good as possible near $z = 0$. On the other hand, it will do no harm if the validity of the bound is restricted to a finite interval, $0 \leqslant z < M^{-2}$ for instance.

Unfortunately, it is impossible to find a majorization of $\Delta_n(0, q_k, x_k, 0)$ which eliminates the $1/q_k$-terms. We have to keep these terms in the treatment of the q_k-integrals. We show in the first part of Appendix B how this can be done. The result is:

$$f_n(z) > \int_0^\infty dx_{n-1} \int_0^{x_{n-1}} dx_{n-2} \cdots \int_0^{x_2} dx_1 (x_{n-1} - x_{n-2}) \cdots (x_2 - x_1) x_1 \times \prod_{k=1}^{n-1} \frac{1}{A_k} \exp\{-zM[M(1 + x_k) + 2\sqrt{A_k}]\}, \tag{4.2}$$

the A_k's being defined in (B.6).

As is readily seen, it follows from recurrence relation (B.6) that $A_k(x_1, \ldots, x_k)$ is an increasing function of its arguments. This implies that the exponential in the integral of (4.2) damps the large x contributions. Therefore, we do not lose too much by restricting the integral to values of x_k which are smaller than $(M^2 z)^{-1}$. As a matter of fact, we shall replace the domain of integration $0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_{n-1} < \infty$ of

(4.2) by a smaller one, in which the limits for x_k are no longer variable but fixed. This new domain depends on a constant a ($a > 1$) and is defined by:

$$0 \leqslant x_1 \leqslant a \leqslant \cdots \leqslant a^{k-1} \leqslant x_k \leqslant a^k \leqslant \cdots \leqslant x_{n-1} \leqslant a^{n-1}. \tag{4.3}$$

According to our preceding remark, we choose a such that:

$$a^{n-1} \leqslant \frac{1}{M^2 z}, \tag{4.4}$$

(remember that we take $z < M^{-2}$). This change in the domain of integration having been done, we can replace, in (4.2), the A_k's by their maximum value on the domain (4.3). These quantities being increasing functions:

$$A_k(x_1, \ldots, x_k) \leqslant A_k(a, \ldots, a^k) \tag{4.5}$$

and we show in the second part of Appendix B that:

$$A_k(a, \ldots, a^k) < M^2 \frac{a^{2k+1}}{(\sqrt{a} - 1)^2}. \tag{4.6}$$

If we insert this upper bound into (4.2), we get:

$$f_n(z) > \left(\frac{\sqrt{a} - 1}{M}\right)^{2(n-1)} a^{-[n^2-1]} \exp\left\{-zM^2\left[n - 1 + \frac{3\sqrt{a} - 1}{\sqrt{a} - 1}\frac{a^n}{a - 1}\right]\right\}$$
$$\times \int_{a^{n-2}}^{a^{n-1}} dx_{n-1} \cdots \int_0^a dx_1 (x_{n-1} - x_{n-2}) \cdots (x_2 - x_1) x_1. \tag{4.7}$$

The x-integral is evaluated in the last part of Appendix B. The result gives the final version of our lower bound for $f_n(z)$:

$$f_n(z) > M^2 C(a) \left(\frac{D(a)}{M^2}\right)^n e^{-z(n-1)M^2}. \tag{4.8}$$

$C(a)$ and $D(a)$ are known functions:

$$C(a) = \frac{a^8(3a - 1)}{12(\sqrt{a} - 1)^2(a - 1)^5 P_+^3} \exp\left[-\frac{(3\sqrt{a} - 1)a}{(\sqrt{a} - 1)(a - 1)}\right],$$
$$D(a) = \frac{1}{a^4}(\sqrt{a} - 1)^2(a - 1)^2 P_+, \tag{4.9}$$

with P_+ defined in (B.24). The validity of the bound (4.8) is restricted to:

$$n \geqslant 3 \quad \text{and} \quad 1 < a^{n-1} \leqslant \frac{1}{M^2 z}. \tag{4.10}$$

The last restriction has already been used in the minorization performed on the exponential appearing in (4.7) in order to obtain (4.8).

Up to now we considered $f_n(z)$ for fixed values of z ($z < M^{-2}$) and n and had to choose an a fulfilling (4.10). We can as well consider z and a fixed ($a > 1$); then, the bound (4.8) is true as long as:

$$3 \leqslant n \leqslant N(z, a) = 1 + \left[\frac{\log(1/M^2 z)}{\log a}\right], \qquad a > 1, \qquad z < (Ma)^{-2}. \tag{4.11}$$

Here $[x]$ means the largest integer which is smaller than x. Therefore we can write for the Laplace transform $f(z)$ of the sum of ladder graphs:

$$f(z) = \sum_{n=2}^{\infty} \lambda^n f_n(z) > \lambda^{N(z,a)} f_{N(z,a)}(z) > C(a) e^{-z(N(z,a)-1)M^2} \left(\frac{\lambda}{M^2} D(a) \right)^{N(z,a)}. \quad (4.12)$$

It may be worthwhile to spell out our main trick explicitly: we minorize $f(z)$ by replacing the whole sum $\sum_n \lambda^n f_n(z)$ by a lower bound of one single term, the term corresponding to $n = N(z, a)$. Taking the explicit form (4.11) of $N(z, a)$ into account, (4.12) implies:

$$f(z) > \bar{C}_3 \frac{1}{z^{\bar{\alpha}_2 + 1}} e^{-z\mu^2}. \quad (4.13)$$

This inequality is valid if $z < (Ma)^{-2}$; it is the counterpart of the inequality (3.11) of the preceding Section. The exponent $\bar{\alpha}_2$ is defined by

$$\bar{\alpha}_2 = -1 + \frac{\log((\lambda/M^2)D(a))}{\log a}, \quad (4.14)$$

and $\bar{C}_3$ is a known function of λ, a and M. It is at this stage that it becomes evident that the sum of ladder graphs diverges for large coupling constants. The exponent $\bar{\alpha}_2$ is an increasing function of λ which is positive if $\lambda > M^2 a/D(a)$. For such values of λ, $f(z)$ is non integrable at $z = 0$ and $\sum_n T_n(s, 0)$ diverges.

Our results contain the constant a whose value is still free. For our purposes, it is clearly convenient to choose the value of a as a function $a(\lambda)$ of the coupling constant λ in such a way that $\bar{\alpha}_2$ becomes maximal. Then we may replace $\bar{\alpha}_2$ in (4.10) by:

$$\alpha_2\left(\frac{\lambda}{M^2}\right) = -1 + \max_{a>1} \frac{\log((\lambda/M^2)D(a))}{\log a}, \quad (4.15)$$

and $\bar{C}_3$ becomes a function of M and λ.

To get our lower bound for $T(s, 0)$, we notice that (3.5) implies the inequality:

$$T(s, 0) > \int_0^{(1/(Ma)^2)} dz [f(z) e^{\kappa^2 z}] e^{z(s-\kappa^2)}. \quad (4.16)$$

κ is an arbitrary mass larger than 2μ. As (3.5) is valid for $s < 4\mu^2$, the same is true for (4.13). If we insert the lower bound (4.10) into (4.13), we find:

$$T(s, 0) > C_3 \frac{|s - \kappa^2|^{\alpha_2}}{|\alpha_2|} \quad (4.17)$$

for $s < 4\mu^2$. As a curiosity we give a value of C_3 obtained from a crude lower bound of $\bar{C}_3$:

$$C_3 = \frac{2}{3(a-1)^7} \exp\left[-\frac{6a^2}{(a-1)^2} - \frac{1}{e \log a} \right] \left(\frac{\kappa^2 - \mu^2}{M^2} \right)^{\alpha_2} \frac{1}{M^{2(\alpha_2+1)}} \gamma, \quad (4.18)$$

$$\gamma = \begin{cases} \dfrac{\lambda}{M^2} D(a) & \text{if } \dfrac{\lambda}{M^2} D(a) \leqslant 1, \\ 1 & \text{if } \dfrac{\lambda}{M^2} D(a) > 1. \end{cases}$$

In these relations, a stands for the function $a(\lambda)$ defined above. It turns out that this function is such that C_3 is nonvanishing for every finite value of λ.

It is evident that the bound (4.10) for $f(z)$ gives a lower bound for $|(\partial^n/\partial s^n)T(s, 0)|$ which is similar to the upper bound (3.13) and is valid if $-1 < \alpha_2 < n$. Furthermore, α_2 provides a lower bound of $\alpha(t)$ by applying the rules (2.10) and (2.11).

5. Discussion of the Results

We discuss the implications of our results on the sum $T(s, t)$ of ladder graphs for $t < 0$ and $s < \mu^2$. For the sake of simplicity, we shall assume $\mu \leqslant m^2$, but our main conclusions remain true if $\mu > m$. With our assumption $M = m$ and the exponents α_1 and α_2 become functions of λ/m^2 and $4\lambda/(4m^2 - t)$ respectively.

First we would like to emphasize that our upper bounds provide an explicit proof that *the series of ladder graphs converges as long as the coupling constant is small enough.* This completes the results of Mattioli [10] who studied the convergence in the t-channel.

According to (3.12) and (2.11), our bounds have the form:

$$T(s, t) < \left[\frac{C_1}{|s - \mu^2|} + C_2 \frac{|s - \mu^2|^{\alpha_1(\lambda/m^2)}}{\sin|\pi\alpha_1(\lambda/m^2)|}\right], \tag{5.1}$$

$$T(s, t) > C_3 \frac{|s - \kappa^2|^{\alpha_2(4\lambda/(4m^2 - t))}}{|\alpha_2(4\lambda/(4m^2 - t))|}. \tag{5.2}$$

These inequalities hold for $t < 0$, $s < \mu^2$ if $\alpha_1(\lambda/m^2) < 0$, resp. $\alpha_2[4\lambda/(4m^2 - t)] < 0$. *They are compatible with the Regge behaviour obtained from a study of the asymptotics of $T_n(s, t)$ for $s \to -\infty$.* Our $\alpha_1(\lambda/m^2)$ and $\alpha_2[4\lambda/(4m^2 - t)]$ are indeed upper and lower bounds of the position $\alpha(t)$ of the Regge pole as it has been estimated by various authors [2, 4, 5]. The bounds (5.1) and (5.2) have the virtue to be exact and to hold for all s $(s < \mu^2)$. They are more than a statement concerning the asymptotic behaviour of $T(s, t)$.

Our next point concerns *the divergence of the series of ladder graphs for large values of the coupling constant.* We define two values λ_1 and λ_2:

$$\alpha_1(\lambda_1) = 0 \qquad \alpha_2(4\lambda_2/4m^2 - t) = 0.$$

The inequalities (5.1) prove the existence of a critical value $\lambda_{crit}(t)$ of the coupling constant $(\lambda_1 < \lambda_{crit} < \lambda_2)$ such that the series of ladder graphs converges if $\lambda < \lambda_{crit}$ and becomes unbounded as λ approaches λ_{crit} (Fig. 2). Unfortunately we cannot infer from (5.1) the nature of the singularity at λ_{crit}. However, the form of the bounds suggests that the singularity is a first order pole. Such a pole would be no surprise at positive t. There, it would correspond to the occurrence of an S-wave bound state once the coupling constant is large enough. On the other hand, our singularity at negative t is physically inacceptable. The fact that the series for $(\partial/\partial s)T(s, t)$ converges for values of λ at which the upper bound of $T(s, t)$ is already divergent (cf. (3.13)) is an indication that the singular term of $T(s, t)$ is s-independent. This leads us to guess that we have to do with the occurrence of an S-wave ghost.

Until now we have presented the situation at fixed momentum transfer t and variable coupling constant λ. Physically, it is more interesting to keep λ fixed and to vary t. We can do this if we are able to replace the t-independent upper bound (5.1)

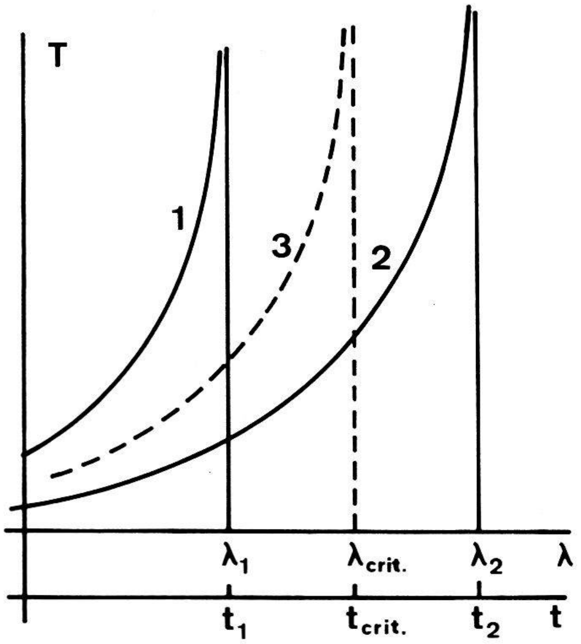

Figure 2
A qualitative picture of the situation at fixed t (upper horizontal scale) and at fixed λ (lower horizontal scale). The curves (1) and (2) represent an upper and a lower bound of $T(s, t)$ as a function λ (or t). $T(s, t)$ is given by curve (3): it is forced to go to infinity at $\lambda = \lambda_{\text{crit}}$ (or $t = t_{\text{crit}}$).

by a t-dependent one. A rather crude estimate (see Appendix C) shows indeed that we can replace α_1 in (5.1) by:

$$\alpha_3(t) = -1 + \frac{4\lambda}{4m^2 - t}, \tag{5.3}$$

or, more adroitly by:

$$\alpha_4(t) = \min\{\alpha_1, \alpha_3(t)\}. \tag{5.4}$$

We notice that $\lim_{t\to-\infty}\alpha_4(t) = -1$. Therefore the *sum of ladders converges for every given value of the coupling constant if t is sufficiently negative.* If $\alpha_4(0) < 0$, this convergence holds for all negative t's. On the contrary, if $\lambda > \lambda_1$, $\alpha_4(0) > 0$ and this convergence is certain only for $t < t_1$ ($\alpha_4(t_1) = 0$). If $\lambda > \lambda_2(0)$, the series diverges certainly from $t = 0$ down to some t defined by $\alpha_2[4\lambda/(4m^2 - t_2)] = 0$. Therefore, there exists a critical value t_{crit} of t ($t_1 < t_{\text{crit}} < t_2$) such that the sum of ladders graphs converges for $t < t_{\text{crit}}$ and becomes unbounded as $t \to t_{\text{crit}}$ from below (Fig. 2).

As far as we know, this divergence has not been noticed explicitly until now. The work concerned with the absorptive part of $T(s, 0)$ [3] bypasses this divergence because it is the real part of $T(s, 0)$ which diverges. On the other hand, most investigations have been focused on refined estimates of the Regge pole rather than the residue function. In the study of the homogeneous Bethe–Salpeter equation, the existence of negative t solutions has been recognized [11] and their connection with the Regge pole established [12]. In the context of these results our divergence has a natural explanation. One expects that, like any bound state, the negative t bound states induce poles in the scattering amplitude and force its perturbation expansion to diverge. We prove irrefutably that there is no mechanism built into the ladder graphs which prevents this divergence.

Clearly, there is little doubt that the defect we have discovered is due to a partial summation of the perturbation series. We must confess that we are unable to tell what sequence of diagrams succeeds in killing our ghost. All we can say is that this is not achieved by simply restoring $s - u$ symmetry through addition of $T(u, t)$ to $T(s, t)$.

To establish this last point, we notice that if a ghost is present in the full series $\sum_n T_n$ it is also present in its remainder:

$$T^{(N)}(s, t) = \sum_{n \geqslant N} T_n(s, t). \tag{5.5}$$

As $T_n(s, t)$ has its s-plane cut on $((n\mu)^2, \infty)$, $T^{(N)}(s, t)$ is holomorphic in the s-plane cut along $((N\mu)^2, \infty)$. This means that $T^{(N)}(s, t)$ is real for s below $(N\mu)^2$ and bounds of the type (5.1) and (5.2) should hold there. It is readily seen that $T^{(N)}(s, t)$ has such bounds with the same α_1 and α_2 if $s < N\mu^2$. This result is obtained just by keeping the term $n\mu^2$ in expression (2.7) of Δ_n. The domain $s < N\mu^2$ is not optimal but is sufficient for our purpose. The advantage of dealing with $T^{(N)}$ rather than T is that the bounds for $T^{(N)}(s, t)$ and $T^{(N)}(u, t)$ have a large common domain of validity if N is large. In this domain we have simultaneously $s < N\mu^2$ and $u < N\mu^2$. In terms of s and t it is defined by:

$$-(N - 2)\mu^2 + 2m^2 - t < s < N\mu^2, \quad t > -(N - 2)\mu^2 + 2m^2. \tag{5.6}$$

Suppose now that N is so large that it contains a segment of the line $t = t_{\text{crit}}$. Then we know that $T^{(N)}(s, t)$ and $T^{(N)}(s, u)$ both exist and are both positive below this line and that they are unbounded at t_{crit}. Therefore their sum is also unbounded (Fig. 3).

Some doubts concerning the relevance of our results may rise due to the fact that we are dealing with a ϕ^3-theory. As the hamiltonian of such a theory has no finite lower bound, it is not clear whether the theory exists at all. If this is the case, there is no sense in being worried by the divergence of the sum of a particular set of diagrams. However, this objection does not hold really. We have discussed ϕ^3 ladder graphs for

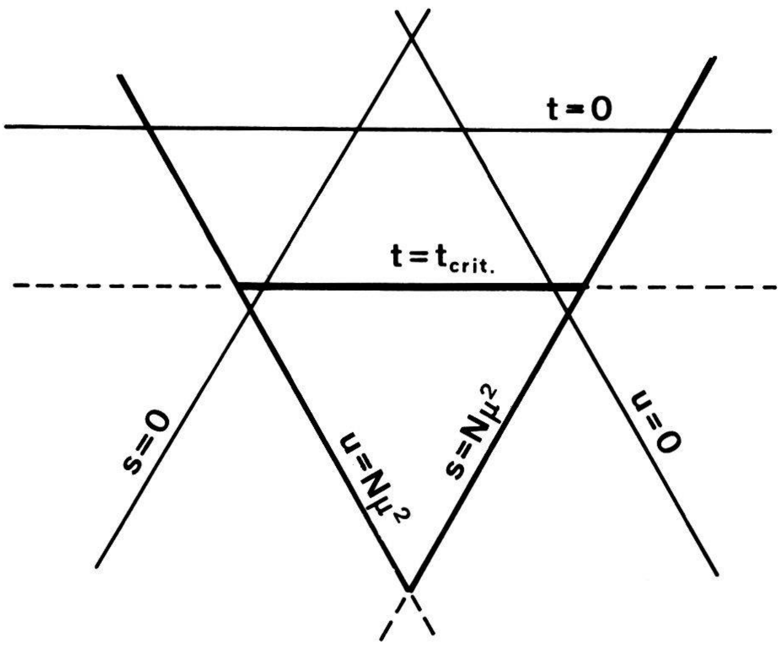

Figure 3
$T(s, t)$ is known to be positive and bounded on the left of the line $s = N\mu^2$ and below the line $t = t_{\text{crit}}$ in the s-t-u-plane. $T(s, u)$ is positive and bounded on the right of $u = N\mu^2$ and below $t = t_{\text{crit}}$. As $T(s, t)$ and $T(s, u)$ are both infinite on the heavy segment of $t = t_{\text{crit}}$, the same is true for their sum.

the sake of simplicity. We are able to treat ladders resulting from the exchange of objects which are more complicated than bare particles. For instance, we can take the exchange of bubbles in a respectable ϕ^4-theory (Fig. 4). In order to avoid any renormalization problem, we can even abandon the four dimensional world and consider $(\phi^4)_2$, ϕ^4 in two space-time dimensions. This theory is known to exist if the coupling is weak enough [13]. Using the same techniques as in the preceding Sections, one proves that the sum of diagrams shown in Figure 3 diverges once the coupling constant exceeds a critical value. If $(\phi^4)_2$ turns out to exist at this value, this divergence must be compensated by some mechanism.

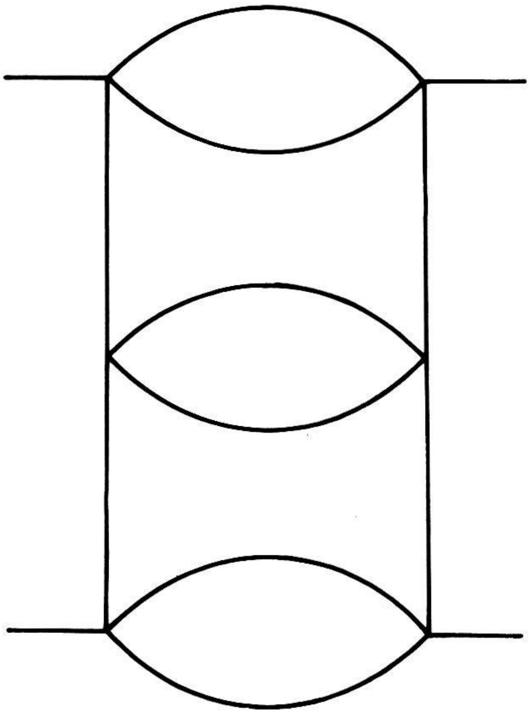

Figure 4
A ladder graph of a ϕ^4 theory generated by repeated exchange of bubbles.

Finally, we present the quantitative aspects of our results. As the Laplace transform $f(z, t)$ exists for all values of λ, it gives informations on $T(s, t)$ beyond the critical value λ_{crit}, for instance through the inequalities (3.13). This implies that α_1 and α_2 are upper and lower bounds of the Regge pole α for all values of λ. Figure 5 gives curves for $\alpha_1(\lambda, t)$ and $\alpha_2(\lambda, t)$ for $t = 0$ and $\mu = m$. $\alpha_1(\lambda, 0)$ is defined in (3.10) and the lower curve for $\alpha_2(\lambda, 0)$ is obtained from (4.12). There is a second curve for $\alpha_2(\lambda, 0)$ which results from an improved lower bound. The improvement consists in the use of an upper bound for $A_k(x_1, \ldots, x_k)$ which depends on x_k instead of the constant bound (4.6). Inspection of Figure 4 shows that

$$2m^2 < \lambda_{\text{crit}} < 11.03 \text{ m}^2. \tag{5.7}$$

This means that a ghost is present at $t < 0$ if $\lambda > 11.03 \text{ m}^2$.

Figure 4 contains also a curve representing an estimate by Chang and Rosner [5]:

$$\alpha(\lambda) \simeq 1.4669\left(\frac{\lambda}{m^2}\right)^{1/4} - 0.37943 + 0\left(\frac{1}{\lambda^{1/4}}\right). \tag{5.8}$$

A striking feature of this result is the asymptotic behaviour $\alpha(\lambda, 0) \sim \lambda^{1/4}$ whereas

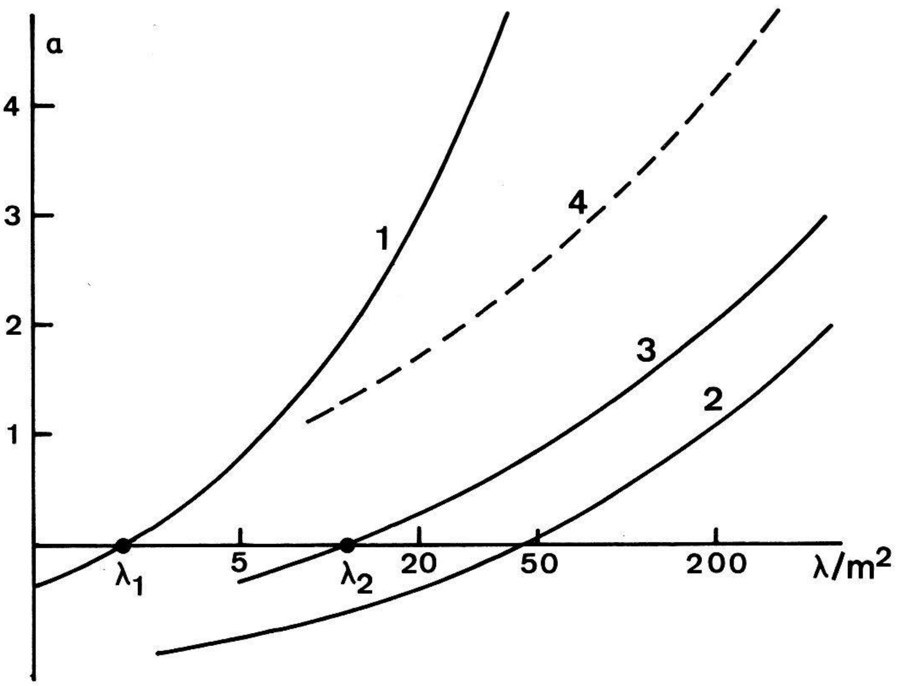

Figure 5
The bounds of the Regge-pole α in the case $t = 0$ and $m = \mu$. The curve (1) is an upper bound; curve (2) is the lower bound obtained in this work and (3) is an improved lower bound. The dashed curve (3) represents an asymptotic estimate derived by Chang and Rosner [5].

$\alpha(\lambda, 0) \sim \lambda^{1/2}$ if $\mu = 0$. As a matter of fact, our lower bound $\alpha_2(\lambda, 0)$ exhibits this behaviour. As shown in Appendix D,

$$\alpha_2(\lambda) \sim 1.016\left(\frac{\lambda}{m^2}\right)^{1/4}. \tag{5.9}$$

It is also possible to derive an improved upper bound which has a $\lambda^{1/4}$ behaviour:

$$\alpha_1(\lambda) \sim 4.97\left(\frac{\lambda}{\mu^2}\right)^{1/4}, \tag{5.10}$$

if $\mu \leqslant m$. The consistency of (5.8) with (5.9) and (5.10) is evident.

Appendix A. Discussion of a Volterra-type Equation

The integral equation (3.8) implies that $g(x)$ satisfies a second order homogeneous differential equation. (3.8) can be solved by finding the adequate solution of this differential equation. We prefer a more direct construction of simple upper bounds of $g(x)$. Such bounds are obtained by majorizing the kernel or (and) the inhomogeneous term of (3.8).

For instance $h(x)$, solution of

$$h(x) = \lambda^2 x + \frac{\lambda}{\mu^2}\int_0^x dx'\left(\frac{x}{x'} - 1\right)h(x') \tag{A.1}$$

is larger than $g(x)$; it is rather close to $g(x)$ for small values of x, One finds:

$$g(x) < h(x) = \lambda^2 x \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!}\left(\frac{\lambda x}{\mu^2}\right)^n < \mu\lambda^{3/2}x^{1/2} sh\left(\frac{2}{\mu}(\lambda x)^{1/2}\right). \tag{A.2}$$

We use this first bound in the construction of a second one which is better at large x. (3.8) is rewritten as follows for $x > (\mu^2/m^2)$:

$$g(x) = g_1(x) + \int_{\mu^2/m^2}^{x} dx' \frac{x - x'}{x'(m^2x' + \mu^2)} g(x') \tag{A.3}$$

$$g_1(x) = \lambda^2 x + \lambda \int_0^{\mu^2/m^2} dx' \frac{x - x'}{x'(m^2x' + \mu^2)} g(x'). \tag{A.4}$$

Now, $j(x)$, solution of

$$j(x) = \lambda^2\left(1 + \frac{\lambda^{1/2}}{m} sh\left(\frac{2\lambda^{1/2}}{m}\right)\right)x + \frac{\lambda}{m^2}\int_{\mu^2/m^2}^{x} dx' \frac{x - x'}{x'^2} j(x') \tag{A.5}$$

is an upper bound of $g(x)$ for $x > (\mu^2/m^2)$. This is so because the inhomogeneous term in (A.5) is an upper bound of $g_1(x)$ obtained from (A.2) and the kernel of (A.5) majorizes the kernel of (A.3). The solution of (A.5) is

$$j(x) = \text{const}\left[\nu\left(\frac{m^2x}{\mu^2}\right)^{\nu} - (1 - \nu)\left(\frac{m^2x}{\mu^2}\right)^{1-\nu}\right] \tag{A.6}$$

with $\nu = \frac{1}{2}(1 + \sqrt{1 + (4\lambda/m^2)})$. (A.2) and (A.6) give the upper bound (3.9).

Appendix B. A Lower Bound for $f_n(z)$

This Appendix contains the technicalities related to the discussion in Section 4.

1. Performing the q_k-integrations

As a function of the q_k's, Δ_n has the following structure (cf. (2.7)):

$$\Delta_n = \sum_{k=1}^{n-1}\left[a_k q_k + b_k^2 \frac{1}{q_k}\right] + c. \tag{B.1}$$

The a_k's are independent of the q_j's and b_k is a linear function of the q_j's with $j > k$, $b_k = b_k(q_{k+1}, \ldots, q_{n-1})$. Therefore q_1 appears only in the first term of the sum in the right-hand side of (B.1) and the q_1-integration requires the evaluation of the following integral:

$$J = \int_0^{\infty} dq_1 \exp\left\{-z\left[a_1 q_1 + b_1^2 \frac{1}{q_1}\right]\right\}. \tag{B.2}$$

We replace this integral by its lower bound

$$\bar{J} = \frac{1}{za_1}\exp[-2za_1^{1/2}b_1] \tag{B.3}$$

and we are left with the following integral over the $(n - 2)$ remaining q_k's:

$$\int_0^{\infty} dq_{n-1} \cdots \int_0^{\infty} dq_2 \exp\left\{-z\left[a_1^{1/2}b_1 + \sum_{k=2}^{n-1}\left(a_k q_k + b_k^2 \frac{1}{q_k}\right)\right]\right\}. \tag{B.4}$$

We see that the fact that $(1/q_1)$ is multiplied, in Δ_n, by the square of a linear form in the q's has a wonderful consequence. The structure of the exponent in (B.4) with respect to $q_2, \ldots, q_{n-1}$ is exactly the same as the structure of Δ_n with respect to

$q_1, \ldots, q_{n-1}$. Therefore we can minorize the q_2-integration in the same way we have minorized the q_1-integration. Repeating this procedure we get

$$\int_0^\infty dq_1 \cdots \int_0^\infty dq_{n-1} \exp[-z\Delta_n] > \frac{1}{z^{n-1}} e^{-zc} \prod_{k=1}^{n-1} \frac{1}{A_k} \exp[-2z\mu A_k^{1/2}]. \quad \text{(B.5)}$$

The A_k's are functions of the x's defined by the following recurrence relation:

$$A_k = a_k + 2M(1 + x_k) \sum_{j=1}^{k-1} A_j^{1/2}; \qquad A_1 = a_1. \quad \text{(B.6)}$$

The definitions of c and a_k are:

$$c = M^2 \sum_{k=1}^{n-1} (1 + x_k) \quad \text{(B.7)}$$

$$a_k = M^2\left[x_k^2 + (1 + x_k)\left(1 + \sum_{j=1}^{k-1} (1 + x_j)\right)\right].$$

2. *An upper bound for A_k*

The recurrence relation (B.6) can be transformed in a relation relating only A_k and A_{k-1}:

$$A_k = M^2\left[\frac{1}{1 + x_k} (x_k - x_{k-1})(x_k + x_{k-1} + x_k x_{k-1}) + (1 + x_k)(1 + x_{k-1})\right]$$

$$+ 2M(1 + x_k)A_{k-1}^{1/2} + \frac{1 + x_k}{1 + x_{k-1}} A_{k-1}, \qquad k \geqslant 2. \quad \text{(B.8)}$$

We notice that the factor of M^2 in the first term of the right-hand side is smaller than $(1 + x_k)^2$. Therefore:

$$A_k \leqslant (1 + x_k)\left[M^2(1 + x_k) + 2MA_{k-1}^{1/2} + \frac{1}{1 + x_{k-1}} A_{k-1}\right]. \quad \text{(B.9)}$$

This inequality is verified by $A_k(a, \ldots, a^k) \equiv A_k(a)$ and $A_{k-1}(a)$ with $x_k = a^k$ and $x_{k-1} = a^{k-1}$. It leads to:

$$(A_k(a))^{1/2} < M(1 + a^k) + (aA_{k-1}(a))^{1/2}. \quad \text{(B.10)}$$

This gives the bound (4.6) if one notices that $A_1(a) = M^2(a^2 + a + 1)$.

3. *Performing the x_k-integrations*

According to (4.7), we have to estimate the following integral:

$$J_n = \int_{a^{n-2}}^{a^{n-1}} dx_{n-1} \cdots \int_0^a dx_1 (x_{n-1} - x_{n-2}) \cdots (x_2 - x_1) x_1. \quad \text{(B.11)}$$

The variables x_k are replaced by variables u_k whose range is (0, 1):

$$x_k = a^{k-1}[1 + (a - 1)u_k] \qquad \text{for } k \geqslant 2 \quad \text{and} \quad x_1 = au_1.$$

One gets:

$$J_n = a^{[n^2-4n+7]}(a-1)^{2n-5}\bar{J}_n, \tag{B.12}$$

with

$$\bar{J}_n = \int_0^1 du_1 \cdots \int_0^1 du_{n-1} u_1[1+(a-1)u_2 - u_1] \prod_{k=3}^{n-1} [1 + au_k - u_{k-1}]. \tag{B.13}$$

We define a set of functions $\phi_n(u)$:

$$\phi_n(u_n) = \int_0^1 du_1 \cdots \int_0^1 du_{n-1} u_1[1+(a-1)u_2 - u_1] \prod_{k=3}^{n} [1 + au_k - u_{k-1}] \tag{B.14}$$

for $n \geqslant 3$ and:

$$\phi_2(u_2) = \int_0^1 du_1 u_1[1+(a-1)u_2 - u_1]. \tag{B.15}$$

These definitions are such that:

$$\bar{J}_n = \int_0^1 du \phi_{n-1}(u), \qquad n \geqslant 3. \tag{B.16}$$

$\phi_n(u)$ is a linear function of u:

$$\phi_n(u) = \rho_n^{(1)} + u\rho_n^{(2)}. \tag{B.17}$$

Let $\boldsymbol{\rho}_n$ be the column-vector whose components are $\rho_n^{(1)}$ and $\rho_n^{(2)}$. The definitions (B.14) and (B.15) imply the recurrence relation:

$$\boldsymbol{\rho}_n = P\boldsymbol{\rho}_{n-1}, \qquad n \geqslant 3, \tag{B.18}$$

where P is the matrix:

$$P = \begin{pmatrix} 1/2 & 1/6 \\ a & a/2 \end{pmatrix}. \tag{B.19}$$

Let P_+ and P_- be the two eigenvalues of P and $\boldsymbol{\rho}_+$ and $\boldsymbol{\rho}_-$ corresponding eigenvectors. $\boldsymbol{\rho}_2$ is a linear combination of $\boldsymbol{\rho}_+$ and $\boldsymbol{\rho}_-$:

$$\boldsymbol{\rho}_2 = r_+\boldsymbol{\rho}_+ + r_-\boldsymbol{\rho}_-. \tag{B.20}$$

The iteration of (B.18) gives:

$$\boldsymbol{\rho}_n = P^{n-2}\boldsymbol{\rho}_2 = r_+P_+^{n-2}\boldsymbol{\rho}_+ + r_-P_-^{n-2}\boldsymbol{\rho}_- \tag{B.21}$$

for $n \geqslant 2$. This result determines completely $\phi_n(u)$ and gives, via (B.16):

$$\bar{J}_n = r_+P_+^{n-3}(\rho_+^{(1)} + \tfrac{1}{2}\rho_+^{(2)}) + r_-P_-^{n-3}(\rho_-^{(1)} + \tfrac{1}{2}\rho_-^{(2)}), \qquad n \geqslant 3. \tag{B.22}$$

After evaluation of the various quantities appearing in this expression, one recognizes that it verifies the following inequality

$$\bar{J}_n > \tfrac{1}{12}(3a-1)P_+^{n-3}, \qquad n \geqslant 3, \tag{B.23}$$

with

$$P_+ = \tfrac{1}{2}[\tfrac{1}{2}(a+1) + (\tfrac{1}{4}(a-1)^2 + \tfrac{2}{3}a)^{1/2}]. \tag{B.24}$$

(B.12) and (B.23) give the desired result.

Appendix C. An Upper Bound at $t < 0$

This bound has been established by Ph. Caussignac. $f_n(z)$ is majorized by the substitution:

$$\Delta_n \to \sum_{k=1}^{n-1} \{[-\tfrac{1}{4}t(x_k^2 - y_k^2) + m^2 x_k^2 + \mu^2 x_k] q_k + \mu^2(1 + x_k)\}, \tag{C.1}$$

and by replacing the limits of integration on y_k in (2.6) by $-x_k$ and $+x_k$. Performing the q_k- and y_k-integrations, one gets:

$$f_n(z, t) < \frac{1}{(n-1)!} (g(z, t))^{n-1} e^{-\mu^2 z}, \tag{C.2}$$

where:

$$g(z, t) = 4 \int_0^\infty dx \frac{1}{x(4m^2 - t) + \mu^2} e^{-\mu^2 x z}. \tag{C.3}$$

(C.2) implies:

$$f(z, t) < \sum_{n=2}^{\infty} \lambda^n f_n(z, t) < \lambda \exp[\lambda g(z, t) - \mu^2 z]. \tag{C.4}$$

Insertion of the simple bounds:

$$g(z, t) < \begin{cases} \dfrac{4}{4m^2 - t} \log\left(\dfrac{2e(4m^2 - t)}{\mu^4 z}\right) & \text{if } z \leqslant \dfrac{4m^2 - t}{\mu^4} \\ \dfrac{1}{z\mu^4} & \text{if } z > \dfrac{4m^2 - t}{\mu^4} \end{cases} \tag{C.5}$$

into (C.4) gives an upper bound for $f(z, t)$ which has the same form as the bound (3.11) of $f(z)$, with $\alpha_1 \to \alpha_3 = -1 + 4\lambda/(4m^2 - t)$.

Appendix D. Strong Coupling Limit of $\alpha_2(\lambda)$

If λ is large, the maximum in (4.12) occurs at a value of a which is close to 1. For $a \simeq 1$, (4.9) tells us that:

$$D(a) \simeq C(a - 1)^4 \tag{D.1}$$

with $C = \frac{1}{8}(1 + \sqrt{\frac{2}{3}})$. The important point is that $D(a)$ vanishes as the fourth power of $(a - 1)$. With (D.1) one has, at the maximum of $(\log[(\lambda/M^2)D(a)]/\log a)$:

$$\log\left(\frac{\lambda}{M^2} D(a)\right) = a \log a \frac{D'(a)}{D(a)} \simeq 4. \tag{D.2}$$

Therefore:

$$(a - 1)^4 \simeq \frac{e^4}{(\lambda/M^2)C},$$

and

$$\alpha_2 \simeq 4/\log a \simeq \frac{4}{e} C^{1/4}\left(\frac{\lambda}{M^2}\right)^{1/4}. \tag{D.3}$$

This is the result (5.9).

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