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Note on the Spectrum of Boltzmann's Collision Operator

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Abstract. The discrete spectrum of the Boltzmann collision operator for a hard-sphere gas is studied. We prove that beyond a critical 'angular momentum' l_0 no eigenvalues exist. A 'proof by computer' gives $l_0 = 3$, which is in accordance with a conjecture made by Jenssen.

1. Introduction

In this paper we consider the hard-sphere collision operator I which appears in the linearized Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = If. \tag{1.1}$$

One of the small gaps in our knowledge of the spectrum of Boltzmann's collision operator concerns its spectrum in the invariant subspaces, which are present due to rotational invariance. Each invariant subspace may be labeled by a number l ($l = 0, 1, 2, \ldots$) and leads to a reduced collision operator I_l. We are interested in the point spectrum of I_l for various l. I_l can be written as

$$I_l = -\nu + K_l \tag{1.2}$$

where ν is the collision frequency (a multiplication operator) and K_l is a compact integral operator. A description of the spectrum of I_l was given by previous authors for the lowest orders in l. First of all, Kuščer and Williams [1] investigated the case $l = 0$. They proved the existence of an infinite set of eigenvalues (relaxation constants) in the interval $(-1, 0]$, with -1 as the only accumulation point. Besides that, the spectrum consists of an essential part (often called 'continuum') extending from -1 to $-\infty$. The positive real axis belongs to the resolvent set. The last two properties are general features of the operator I. The case $l = 1$ was treated by Yan [2]. He also found an infinite set of eigenvalues between -1 and 0. Jenssen [3] considered general values of l, and he showed that an infinite sequence of eigenvalues exists for $l < 3$. For $l \geqslant 3$ only a finite number can occur, but the numerical calculations of Jenssen even lead to the conjecture that no eigenvalues exist for $l \geqslant 3$. In this paper we show that the eigenvalues indeed disappear for large values of l, and a numerical consideration different from Jenssen's gives strong evidence that $l = 3$ is the first value for which eigenvalues are absent. While Jenssen in his calculations used a big matrix to approximate I_l, we treated the full collision operator. In our numerical calculations we had to evaluate certain integrals which estimate the norm of an operator.

2. The Spectrum of I_l for Large l

For a hard-sphere gas the collision operator (1.2) is known explicitly [6, 7],

$$\nu(v) = \frac{1}{2}\left[\exp(-\tfrac{1}{2}v^2) + \left(v + \frac{1}{v}\right)\int_0^v \exp(-\tfrac{1}{2}x^2)\,dx\right] \qquad \mathbf{v} \in \mathbb{R}^3,\ v = |\mathbf{v}| \tag{2.1}$$

$$K_l(v, v') = K_l^{(2)} - K_l^{(1)} \tag{2.2}$$

where

$$K_l^{(i)}(v, v') = 2\pi \int_{-1}^{1} K^{(i)}(\mathbf{v}, \mathbf{v}')P_l(z)\,dz \qquad i = 1, 2 \tag{2.3}$$

(z denotes the cosine of the angle between $\mathbf{v}$ and $\mathbf{v}'$)

$$K^{(1)}(\mathbf{v}, \mathbf{v}') = \frac{1}{8\pi}|\mathbf{v} - \mathbf{v}'| \exp[-\tfrac{1}{4}(v^2 + v'^2)] \tag{2.4}$$

$$K^{(2)}(\mathbf{v}, \mathbf{v}') = \frac{1}{2\pi|\mathbf{v} - \mathbf{v}'|} \exp\left[-\frac{1}{8}\left(|\mathbf{v} - \mathbf{v}'|^2 + \frac{(v^2 - v'^2)^2}{|\mathbf{v} - \mathbf{v}'|^2}\right)\right]. \tag{2.5}$$

With this definition the operator I_l acts in the Hilbert space $L_2(v; v^2\,dv)$. Jenssen used the space $L_2(v; \pi^{-3/2} v^2 e^{-v^2}\,dv)$. If J_l denotes his collision operator and U: $L_2(v; \pi^{-3/2} v^2 e^{-v^2}\,dv) \to L_2(v; v^2\,dv)$ the unitary transformation

$$(Uf)(v) = (2\pi)^{-3/4} e^{-(v^2/4)} f(v/\sqrt{2})$$

we have

$$I_l = \frac{\sqrt{\pi}}{2} U J_l U^{-1}.$$

The eigenvalue problem for I_l

$$I_l f = \lambda f \qquad (\lambda > -1) \tag{2.6}$$

can also be written as

$$g = C_\lambda K_l C_\lambda g \equiv B_\lambda(l) g \tag{2.7}$$

where $g = (\nu + \lambda)^{1/2} f$ and $C_\lambda = (\nu + \lambda)^{-(1/2)}$. For $\lambda > -1$, C_λ is a bounded operator. However, it is known [1] that the strong limit $\lambda \to -1$ of $B_\lambda(l)$ exists and is a bounded operator $B_{-1}(l)$. Furthermore $\|B_\lambda(l)\| \leqslant \|B_{-1}(l)\|$ uniformly in $\lambda \geqslant -1$. In the following we put $B_{-1}(l) = B(l)$ and $C_{-1} = C$.

Equation (2.6) has a solution if and only if (2.7) has one. But (2.7) has no solution if $\|B(l)\| < 1$. Then I_l has no eigenvalues $\lambda > -1$. We shall show that $\|B(l)\| \to 0$ as $l \to \infty$, or equivalently

Theorem. There exists an integer $l_0 > 0$ such that for $l \geqslant l_0$ there are no eigenvalues of the operator I_l in the gap $(-1, 0]$.

Proof. We present a short proof which shows the disappearance of the eigenvalues for large l, but which gives no reasonable value for l_0 (i.e. one close to $l_0 = 3$) if these estimates are used to calculate l_0.

Since the kernel $K_l^{(1)}(v, v')$ is much better behaved than $K_l^{(2)}(v, v')$ we only consider that contribution to $B(l)$ which comes from the latter. The kernel $K_l^{(1)}(v, v')$ could be treated in the same way. First we estimate (2.3) by the familiar Schwarz inequality

$$|K_l^{(2)}(v, v')| \leqslant \frac{2\pi\sqrt{2}}{\sqrt{2l+1}} \left(\int_{-1}^{1} |K^{(2)}(\mathbf{v}, \mathbf{v}')|^2 \, dz\right)^{1/2} \tag{2.8}$$

The integral on the right-hand side can be evaluated explicitly

$$\begin{aligned} |K_l^{(2)}(v, v')| &\leqslant \frac{1}{\sqrt{2l+1}} \frac{e^{-v^2/4} e^{v'^2/4}}{\sqrt{vv'}} \left(\int_0^{v'} \frac{e^{-u^2}\, du}{\sqrt{u^2 + v^2 - v'^2}}\right)^{1/2} \\ &\leqslant \frac{2}{\sqrt{2l+1}} \frac{e^{-v^2/4} e^{v'^2/4}}{\sqrt{vv'}(v^2 - v'^2)^{1/4}} \left(\int_0^{v'} e^{-u^2}\, du\right)^{1/2} \\ &\equiv \frac{2}{\sqrt{2l+1}} S(v, v') \qquad \text{for} \quad v' \leqslant v. \end{aligned} \tag{2.9}$$

For $v' \geqslant v$ one has to interchange v and v'.

Now we use a result from the theory of integral operators [4]. Given a symmetric kernel $t(x, x')$ of an integral operator T, defined for instance in the space $L_2(\mathbb{R}^+)$, one has an estimate for the norm of the operator, namely

$$\|T\| \leqslant \sup_{x \in \mathbb{R}^+} \int_0^\infty |t(x, x')| \, dx'$$

provided the integral exists. In a L_2-space with measure $x^2\, dx$ we have

$$\|T\| \leqslant \sup_{x \in \mathbb{R}^+} \int_0^\infty xx' |t(x, x')| \, dx'. \tag{2.10}$$

In this paper, it is always this estimate which is used to bound operator norms. Putting

$$B^{(2)}(l) = CK_l^{(2)}C \tag{2.11}$$

we get

$$\|B^{(2)}(l)\| \leqslant \frac{2}{\sqrt{2l+1}} \sup_{v \in \mathbb{R}^+} \int_0^\infty \frac{vv' S(v, v')}{(\nu(v) - 1)^{1/2}(\nu(v') - 1)^{1/2}} \, dv' \tag{2.12}$$

By investigating the integral for $v' \leqslant 1$ and $v' \geqslant 1$, and using the fact that

$$\nu(v) - 1 \sim v^2 \qquad v \to 0 \tag{2.13}$$

and

$$\nu(v) - 1 \sim v \qquad v \to \infty \tag{2.14}$$

one readily shows that the supremum in (2.12) is finite. Therefore (2.12) tells us that $\|B(l)\| < 1$ if $l \geqslant l_0$ for some l_0. This finishes our proof.

One could evaluate the integral (2.12) numerically and calculate l_0, but because of the slow decay in l of the right-hand side of (2.12), attempts in this direction were not successfull. To get better numerical results one has to improve the estimates. Indeed it is possible to show that the norm of $B(l)$ vanishes as l^{-1}. This will be sketched in the next section.

3. Calculation of l_0

To get a good value for l_0 one can refine the analysis. First, one can calculate $K_l^{(1)}(v, v')$ explicitly

$$\begin{aligned} K_l^{(1)}(v, v') &= \frac{e^{-v^2/4}e^{-v'^2/4}v'}{2l+1}\left[\frac{v'^{l+1}}{(2l+3)v^{l+1}} - \frac{v'^{l-1}}{(2l-1)v^{l-1}}\right] && v' \leqslant v \\ &= K_l^{(1)}(v', v) && v' \geqslant v. \end{aligned} \tag{3.1}$$

With this expression one forms $B^{(1)}(l) = CK_l^{(1)}C$ and using (2.10) one easily checks that its norm is damped like l^{-3}. The kernel $K_l^{(2)}(v, v')$, however, cannot be expressed in a simple form. But one can separate the singularity by defining two operators $Q_l(v, v')$ and $T_l(v, v')$ according to

$$K_l^{(2)}(v, v') = Q_l(v, v') + T_l(v, v') \tag{3.2}$$

where

$$\begin{aligned} Q_l(v, v') &= e^{-v^2/4}e^{v'^2/4}\int_{-1}^{+1}\frac{P_l(z)}{|\mathbf{v} - \mathbf{v}'|}\,dz = \frac{2}{(2l+1)}\frac{v'^l}{v^{l+1}}e^{-v^2/4}e^{v'^2/4} && v' \leqslant v \\ &= Q_l(v', v) && v' \geqslant v. \end{aligned} \tag{3.3}$$

$$\begin{aligned} T_l(v, v') &= \frac{e^{v'^2/4}e^{-v^2/4}}{v}\int_{-1}^{+1}\frac{P_l(z)\left(\exp\left[-\frac{v'^2}{2}\frac{(z-t)^2}{(1+t^2-2zt)}\right]-1\right)}{(1+t^2-2zt)^{1/2}}\,dz && v' \leqslant v \\ &= T_l(v', v) && v' \geqslant v,\ t = \frac{v'}{v}. \end{aligned} \tag{3.4}$$

It follows from (3.3) that $\|CQ_lC\| \to 0$ for $l \to \infty$.

The kernel $T_l(v, v')$ can be tracted further by a partial integration. One uses

$$\int P_l(z) = \frac{1}{2l+1}(P_{l+1}(z) - P_{l-1}(z)) \tag{3.5}$$

which brings in a power l^{-1}. The difference between the two Legendre polynomials can be estimated by

$$|P_{l+1}(z) - P_{l-1}(z)| \leqslant \frac{4\sqrt{2}}{\pi}(1 - |z|)^{1/2}. \tag{3.6}$$

This is easily shown by an integral representation for the Legendre polynomials [5]. Further estimates of the integrand (after the partial integration) can be carried out. Finally one gets an estimate of the norm of $T_l(v, v')$ which decays like l^{-1}. It was along this line we got numerically reasonable results. The numerical work consisted of the evaluation of the integrals which estimate the norm of $B(l)$. I am very thankful to Dr. W. Schnider from the Institut fuer Elektronik at the ETH who did the integrations on the computer. Using Simpson's rule and the method of Gaussian quadrature it was seen that for $l \geqslant 5$ the norm of $B(l)$ was significantly smaller than unity. The cases $l = 3, 4$ remained undecided by the method of this section. But for these two cases the integration (2.3) was done explicitly, and indeed, the norm of $B(l)$ was also found to be smaller than unity. Since nowhere in our calculations does any approximation of the collision operator occur and the integrations can be done with high accuracy, we are convinced that Jenssen's conjecture is right.

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