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# Angular Momentum Projection in the Two-Center Model by Projection of Hilbert Spaces

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*Abstract.* The theory of radial canonical transforms is used as a tool for angular momentum projection in quantum-mechanical problems with central interactions. The method is developed for the case where an $n$-body system is described by two fragments of internal angular momentum zero with oscillator wave functions of arbitrary width, with particular attention to the two-center model. By way of example the applicability of the effective Brink–Boeker nucleon–nucleon interaction for functions with independently varied width parameters is tested. A negative answer results.

## 1. Introduction

The theory of canonical transforms on Hilbert spaces has received increasing attention in recent years. First representations of the symplectic group $\mathcal{Sp}(2n, \mathbb{R})$ on Hilbert spaces were discussed [1, 2]. A representation of this group on $\mathcal{L}^2(\mathbb{R}^n)$ was shown [3] to be a quantum-mechanical representation of linear canonical transformations in $2n$-dimensional phase space.

More general transforms [4–11] were studied and two lines of investigations developed which shall be of interest to us. On one hand complex extensions of the above mentioned representations of $\mathcal{Sp}(2n, \mathbb{R})$ were studied [6, 7], and the relevance of such transforms to nuclear physics emerged [5, 7, 10, 11]. On the other hand radial canonical transforms were analyzed [4, 7, 8] but these have not been applied to many-body problems up to now.

The purpose of the present paper is to show that such applications are indeed possible. The most natural field of application for radial canonical transforms is that of angular momentum projection or decomposition for complicated systems, and we can hope for success whenever the non-projected system is described essentially by ordinary canonical transforms.

Particularly we shall discuss systems of $n$ particles that are approximated by an ansatz of two fragments. We shall find that these problems are amenable to treatment by radial canonical transforms if the fragments are described by superpositions of oscillator functions of arbitrary width and angular momentum zero. Further the two-particle interaction is taken to be central. In this framework we shall solve the problem of angular momentum projection for the two-center model that has received increasing attention recently [12].

By way of application we perform a two-center calculation of $^6$Li with the effective

Brink–Boeker interaction [13]. This calculation can be compared with previous non-projected [14] or approximately projected calculations [5]. As we are able to handle oscillator functions of different width we can test the usefulness of the Brink–Boeker interaction for functions of different width in the two fragments, i.e. upon departure from the oscillator cluster model. We find that this interaction is indeed not appropriate for such more general wave functions.

## 2. Canonical Transforms

In this section we shall discuss some results about canonical transforms that will be useful in what follows. Much of the presented material is well-known.

We shall start with the Moshinsky–Quesne representation [3] of $\mathscr{Sp}(2n, \mathbb{R})$. Consider the $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

where $A$, $B$, $C$, $D$ are in turn real $n \times n$ matrices fulfilling the conditions

$$\begin{aligned} B\tilde{A} &= A\tilde{B} \\ C\tilde{D} &= D\tilde{C} \\ D\tilde{A} - C\tilde{B} &= I \end{aligned} \tag{2.2}$$

where $\sim$ denotes transposition of matrices and later of column vectors, and $I$ is a $n \times n$ unit matrix. Then the integral kernels

$$K(\bar{\mathbf{x}}, \mathbf{x}) = [(2\pi)^n |\det B|]^{-1/2} \exp\left\{\frac{i}{2}(\bar{\mathbf{x}}\tilde{B}^{-1}\tilde{D}\bar{\mathbf{x}} - 2\bar{\tilde{\mathbf{x}}}\tilde{B}^{-1}\mathbf{x} + \tilde{\mathbf{x}}\tilde{A}\tilde{B}^{-1}\mathbf{x})\right\} \tag{2.3}$$

with $\bar{\mathbf{x}}, \mathbf{x} \in \mathbb{R}^n$ form a unitary projective representation [3] on $\mathscr{L}^2(\mathbb{R}^n)$ of the matrix group $\mathscr{Sp}(2n, \mathbb{R})$ given in (2.1). Clearly this expression is only meaningful if $\det B \neq 0$. If $\det B = 0$, we may obtain the kernel $K(\bar{\mathbf{x}}, \mathbf{x})$ by decomposing the matrix into a product of two matrices with regular $B$ or by taking an appropriate limit. Often Dirac $\delta$-functions will appear. This will always happen for 'point transformations' where $B$ is a null matrix. The kernels $K(\bar{\mathbf{x}}, \mathbf{x})$ may be considered as the quantum-mechanical representation of linear canonical transformations [3], but we shall not deal with this aspect. Note that the above representation is equivalent to that given by Bargmann [1] in a Hilbert space of analytic functions.

We wish now to consider the extension of the kernel (2.3) to $\mathscr{Sp}(2n, \mathbb{C})$, i.e. to complex matrices $A$, $B$, $C$, $D$. It is readily seen that the resulting kernels do not represent this group as their folding yields on occasion integrals that grow to infinity. Nevertheless we are interested in these extensions and we shall call them canonical transforms [6]. These transforms were analyzed from two points of view. On one hand Wolf [6] discusses the possibility of retaining unitarity of the integral transformation wherever this is possible by a change of measure, i.e. by considering transformations from one Hilbert space to another, rather than within one Hilbert space. As he treated this subject extensively we shall not go into details.

We shall be more interested in an approach that is a generalization of the one given in [7]. We consider the transforms resulting from the complex extensions, but

retaining the measure in the function space. The corresponding integral operators are not unitary and in general also unbounded. As mentioned above they do not form a representation of $\mathcal{Sp}(2n, \mathbb{C})$ but we can formulate what we shall call the weak representation theorem.

*Theorem.* If we fold the kernel of two canonical transforms and the corresponding integral exists (eventually in the sense of a Dirac $\delta$) then the resulting kernel is equal, up to a phase, to the one we obtain if we construct a kernel from the product of the corresponding symplectic transformations in $\mathcal{Sp}(2n, \mathbb{C})$.

The proof of this theorem may be obtained by carrying out explicitly the integration over gaussians. But actually we can simply argue by analytic continuation from the $\mathcal{Sp}(2n, \mathbb{R})$ case. This argument must hold as long as the integral converges. Some attention has to be given to the cases where $\delta$ functions appear, but these can either be handled by appropriate limits or by discussing the problem in Bargmann–Segal space where no distributions occur [15].

From the weak representation theorem we see immediately that the subsemigroup of $\mathcal{Sp}(2n, \mathbb{C})$ corresponding to bounded operators is represented by canonical transforms. The existence of all integrals follows immediately from the boundedness. This semigroup was discussed extensively [7, 16] but recently it was pointed out [10, 11] that the restriction to the semigroup is not useful for those applications in many-body theory which we wish to consider. Indeed singular integral transforms corresponding to unbounded operators are often used. It was shown [17] that for applications to variational or projection calculations in quantum mechanics it is sufficient that the transformation maps a dense subspace of $\mathcal{L}^2(\mathbb{R}^n)$ into a dense subspace of $\mathcal{L}^2(\mathbb{R}^n)$. Table I shows three singular canonical transforms [11] often used in nuclear physics.

## 3. Radial Canonical Transforms and Angular Momentum Projection

In this section we shall discuss some results on radial canonical transforms [4] from the point of view of angular momentum projection. We consider a $\mathcal{Sp}(2, \mathbb{C})$ subgroup of $\mathcal{Sp}(2n, \mathbb{C})$ given by matrices

$$\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix} \qquad ad - bc = 1 \tag{3.1}$$

where $I$ is a $n \times n$ unit matrix. The corresponding canonical transform (2.3) acts in the same way on all variables. Such a transform clearly commutes with the group $\mathcal{O}(n)$ of orthogonal transformations in $\mathbb{R}^n$. We may therefore introduce a radial variable $r = (\sum_{i=1}^{n} x_i^2)^{-1/2}$ and angular variables. Then we can project one irreducible representation $\lambda$ of $\mathcal{O}(n)$ by integrating with a basis function $\Gamma_\lambda$ of this representation and we obtain the kernel of a radial canonical transform as

$$k'(\bar{r}, r) = \int \Gamma_\lambda^* K(\bar{\mathbf{x}}, \mathbf{x}) \Gamma_\lambda. \tag{3.2}$$

Here the integration has to run over all angular variables. In [4] this method was used to pass from a two-dimensional system to the radial system. It yields with $\Gamma_\mu = e^{i\mu\varphi}$; $\mu$ integer

$$k^\mu(\bar{r}, r) = (-i)^\mu |b^{-1}| (\bar{r}r)^{1/2} \mathfrak{J}_\mu(b^{-1}\bar{r}r) \exp\{(i/2b)(d\bar{r}^2 + ar^2)\} \tag{3.3}$$

Table I
The complex canonical transformations corresponding to three integral transforms as well as the operator properties of the transforms are listed.

| | Integral transform | Complex canonical transformation | Type of operator | |
|---|---|---|---|---|
| | | | Transform | Inverse |
| Hill–Wheeler [18] | $\left(\frac{\alpha}{2\pi}\right)^{1/2} \exp\left\{-\frac{\alpha}{2}(x - x')^2\right\}$ | $\begin{pmatrix} 1 & 1/i\alpha \\ 0 & 1 \end{pmatrix}$ | bounded | unbounded |
| Hackenbroich, Wiedmann [19, 20] | $\pi^{-1/2} \exp\left\{-\frac{\alpha}{2}x^2 + 2ixx' + \frac{1}{\alpha}x'^2\right\}$ | $\begin{pmatrix} 1/\alpha & -\frac{1}{2} \\ 1 & -i/2\alpha \end{pmatrix}$ | unbounded | unbounded |
| Sünkel–Wildermuth [21] | $\left(\frac{\alpha}{2\pi}\right)^{1/2} \exp\left\{-\frac{\alpha}{2}(x - ix')^2\right\}$ | $\begin{pmatrix} i & -1/\alpha \\ 0 & -i \end{pmatrix}$ | unbounded | unbounded |

Here we introduced the factor $(\bar{r}r)^{1/2}$ such that this kernel provides a projective unitary representation of $\mathcal{Sp}(2, \mathbb{R})$ under the measure $dr$ with integration range from 0 to $\infty$ [4]. Actually the last statement was shown to be true for any real $\mu$ [4]. On the other hand it is easy to see that we will obtain $\mu$ integer from any even-dimensional space and $\mu$ semi-integer for any odd-dimensional space. Particularly for three dimensions we have

$$\mu = (l(l+1) + \tfrac{1}{4})^{1/2} \tag{3.4}$$

where $l$ is the angular momentum. Based on our previous arguments it is trivial to see that the weak representation theorem must hold for the continuation to $\mathcal{Sp}(2, \mathbb{C})$ of (3.3).

If we wish to apply these results practically we will find gaussian kernels (say in three dimensions) but the preceding constant factors will in general not agree with (2.3). This is of no consequence as the projection procedure only implies the gaussians. Yet in order to obtain the desired result we have to divide by $(2\pi|b|)^{-3/2}$ and multiply by the factor actually preceding the exponential in the unprojected kernel.

We can proceed similarly for canonical transforms in Wolf's sense [6, 8]. This allows to pass from the Bargmann–Segal transform [1, 6, 7], that is a generating function for Hermite polynomials to the Barut–Girardello transform [7, 8, 22], that is a generating function for Laguerre polynomials. The procedure is quite similar as above and is discussed in [7, 8].

## 4. Angular Momentum Projection in the Two-Center Model

The two-center model describes an $N$-nucleon system in terms of two fragments with $N_1$ and $N_2$, $(N = N_1 + N_2)$ nucleons respectively. The orbital wave functions of the fragments $\varphi_i$, $i = 1, 2$ are given as superpositions of products of single particle functions located at center $i$. The $N$-nucleon function reads as

$$\Phi = \mathcal{A}(\varphi_1\varphi_2\Theta) \tag{4.1}$$

where $\mathcal{A}$ is an antisymmetrizer and $\Theta$ an appropriate spin–isospin function. Clearly this ansatz is neither translationally invariant nor does it have good orbital angular momentum. If the single particle functions are oscillator functions, the problem of translational invariance can be handled. It is trivial if the same oscillator frequencies are used at both centers but recently methods became available for the case of different frequencies [10, 20]. For this purpose as well as for the purpose of angular momentum projection we have to realize that the above ansatz is related by a Hill–Wheeler transform [18] to a resonating group ansatz [24]. This ansatz uses so-called cluster functions that read for two fragments as

$$\Psi = \mathcal{A}(\psi_1\psi_2\chi\Theta) \tag{4.2}$$

where $\psi_1$ and $\psi_2$ are translationally invariant orbital functions of the Jacobi vectors within each fragment while $\chi$ is a relative motion function for the fragments. Again we shall assume $\psi_1$ and $\psi_2$ to be oscillator functions of possibly different width. For given $\psi_1$ and $\psi_2$ we can then derive the resonating group equation, which is an integro-differential equation for $\chi$. If we allow different functions we obtain a set of coupled integro-differential equations. The kernel for such equations results from integrating a matrix element of the Hamiltonian over the internal coordinates as far as possible. This

kernel must be a sum of terms of the type

$$K(\bar{\mathbf{x}}, \mathbf{x}) = \int \bar{\psi}_1 \bar{\psi}_2 \, \mathrm{Op} \, \psi_1 \psi_2. \tag{4.3}$$

Here $\bar{\psi}_1$, $\bar{\psi}_2$ are functions of the same type as $\psi_1$ and $\psi_2$. Op. is an operator consisting of one term from the Hamiltonian and a permutation stemming from the anti-symmetrizer and $\bar{\mathbf{x}}$ and $\mathbf{x}$ are the relative motion vectors of the fragments in bra and ket. If the Hamiltonian is decomposed into derivatives, powers and gaussians, this integral may be performed analytically and the result is a polynomial in $\bar{\mathbf{x}}$, $\mathbf{x}$ multiplied by a gaussian kernel or eventually a $\delta$ function [25]. In these variables we can write

$$K(\bar{\mathbf{x}}, \mathbf{x}) = \eta(\bar{\mathbf{x}}, \mathbf{x}) H(\bar{\mathbf{x}}, \mathbf{x}) \tag{4.4}$$

where $\eta(\bar{\mathbf{x}}, \mathbf{x})$ is the polynomial and $H(\bar{\mathbf{x}}, \mathbf{x})$ is a Moshinsky–Quesne transform for $\mathscr{Sp}(6, \mathbb{C})$ as discussed in Section 2. There exists another Moshinsky–Quesne transform $H_{\alpha\beta\gamma}(\bar{\mathbf{x}}, \mathbf{x})$ such that

$$K(\bar{\mathbf{x}}, \mathbf{x}) = \lim_{\alpha,\beta,\gamma \to 0} \vartheta\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}\right) H_{\alpha\beta\gamma}(\bar{\mathbf{x}}, \mathbf{x}) \tag{4.5}$$

holds, where $\vartheta$ is a polynomial in the derivatives. This implies that $H_{\alpha\beta\gamma}(\bar{\mathbf{x}}, \mathbf{x})$ generates $K(\bar{\mathbf{x}}, \mathbf{x})$. If we restrict ourselves to central interactions and fragments of angular momentum zero, particularly fragments consisting of $^1S$ states or closed shells, the transform $H_{\alpha\beta\gamma}(\bar{\mathbf{x}}, \mathbf{x})$ factorizes and acts in the same way on all components. With other words we may say that it corresponds to an element of $\mathscr{Sp}(6, \mathbb{C})$ of the type given in (3.1). Using (3.3) we can then write the corresponding kernel in radial space for a given $\mu = (l(l+1) + \frac{1}{4})^{1/2}$ and we obtain the radial kernel $h^{\mu}_{\alpha\beta\gamma}(\bar{r}, r)$ from which we can get the angular momentum projection of $K(\bar{\mathbf{x}}, \mathbf{x})$ as

$$k^{\mu}(\bar{r}, r) = \lim_{\alpha,\beta,\gamma \to 0} \vartheta\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}\right) h^{\mu}_{\alpha\beta\gamma}(\bar{r}, r). \tag{4.6}$$

Actually we have interchanged limits to obtain this result, but practically this proves possible in all cases of interest.

We were thus able to use the radial transform developed in [4] as given in (3.3) to perform angular momentum projection for the terms of a resonating group kernel if the mentioned restrictions are made for functions and interactions. We could then set up the integro-differential equation for the radial part of the relative motion function $\chi$. Note, that it is not so much the result which is of interest, as rather the method by which it was obtained. The radial equation could have been constructed by separation of variables in a standard way.

We shall now go one step further and consider the Hill–Wheeler transform for a three dimensional problem. As it is to affect all components in the same way as in one dimension, it must be a canonical transform characterized by the element

$$\left(\frac{\zeta}{2\pi}\right)^{3/2} \exp\left\{-\frac{\zeta}{2}(\mathbf{x} - \mathbf{y})^2\right\} \Leftrightarrow \begin{pmatrix} I & (1/i\zeta)I \\ 0 & I \end{pmatrix} \tag{4.7}$$

of $\mathscr{Sp}(6, \mathbb{C})$. Comparing with (3.1) we see immediately that we can also construct a radial Hill–Wheeler transform according to (3.3).

Note that the weak representation theorem of Section 2 holds both for the

transforms associated with $\mathcal{SU}(6, \mathbb{C})$ and the corresponding radial transforms. We may conclude that the angular momentum projection of the Hill–Wheeler transformed

$$W_{\alpha\beta\gamma}(\bar{\mathbf{y}}, \mathbf{y}) = \left(\frac{\zeta}{2\pi}\right)^3 \int \exp\left\{-\frac{\zeta}{2}(\bar{\mathbf{x}} - \bar{\mathbf{y}})^2\right\} \times H_{\alpha\beta\gamma}(\bar{\mathbf{x}}, \mathbf{x}) \exp\left\{-\frac{\zeta}{2}(\mathbf{x} - \mathbf{y})^2\right\} d\bar{\mathbf{x}}\, d\mathbf{x} \tag{4.8}$$

of $H_{\alpha\beta\gamma}(\bar{\mathbf{x}}, \mathbf{x})$ must be equal to the radial Hill–Wheeler transformed of $h^{\mu}_{\alpha\beta\gamma}(\bar{r}, r)$, and we denote it by $w^{\mu}_{\alpha\beta\gamma}(\bar{s}, s)$. It is obtained by simply inserting the coefficients for $W_{\alpha\beta\gamma}(\bar{\mathbf{y}}, \mathbf{y})$ in equation (3.4).

For the Hill–Wheeler transformed of $k^{\mu}(\bar{r}, r)$ we then get

$$v^{\mu}(\bar{s}, s) = \lim_{\alpha,\beta,\gamma \to 0} \vartheta\left(\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}\right) w^{\mu}_{\alpha\beta\gamma}(\bar{s}, s) \tag{4.9}$$

which must be the angular momentum projection of the Hill–Wheeler transformed $V(\bar{\mathbf{y}}, \mathbf{y})$ of $K(\bar{\mathbf{x}}, \mathbf{x})$. This result allows us to set up a radial Hill–Wheeler equation. Further we wish to recall that $V(\bar{\mathbf{y}}, \mathbf{y})$ is up to a constant equal to the matrix element of the operator Op. (4.3) with two-center functions (4.1) in bra and ket. For this to hold $\varphi_1$ and $\varphi_2$ have to be chosen as oscillator functions with equal width parameter $\zeta$. This condition is necessary to allow trivial elimination of the center of mass motion. The distance of separation of the centers is $D = [N/(N_1N_2)]^{1/2}|\mathbf{y}|$ for the ket and $\bar{D} = [N/(\bar{N}_1\bar{N}_2)]^{1/2}|\bar{\mathbf{y}}|$ for the bra. But then we can conclude that

$$v^{\mu}(\bar{D}[N/(\bar{N}_1\bar{N}_2)]^{-1/2}, D[N/(N_1N_2)]^{-1/2}) \tag{4.10}$$

is the angular momentum projected matrix element in the two-center model with $\mu = (l(l+1) + \frac{1}{4})^{1/2}$. The above mentioned constants as well as the factor $(\bar{s}s)^{1/2}$ stemming from the change of measure will cancel with the corresponding factors in the norm.

The generalization to different width parameters for bra and ket is trivial while the case of different oscillator width for $\varphi_1$ and $\varphi_2$ is somewhat more complicated. We shall not go into details here and restrict ourselves to say that already the separation of the center of mass motion requires interpretation in terms of the Hill–Wheeler method. If this separation is performed the remaining matrix element will always be of the form $V(\bar{\mathbf{y}}, \mathbf{y})$ and angular momentum projection proceeds as before. We shall return to the somewhat more complicated interpretation of this case in the example. A more detailed discussion is given elsewhere (26).

## 5. A Two-Center Calculation of $^6$Li.

The calculation presented in this section serves two purposes. First it is to illustrate the method discussed in the previous section. The second and more physical purpose is to investigate the range of applicability of the Brink–Boeker potential [13]. It is given by

$$V(r_{ij}) = S_1(1 - m_1 + m_1P_M)e^{-r_{ij}^2/\mu_1^2} + S_2(1 - m_2 + m_2P_M)e^{-r_{ij}^2/\mu_2^2} \tag{5.1}$$

and we shall use the parameter set $S_1 = -140.6$ MeV, $m_1 = 0.4864$, $\mu_1 = 1.4$ fm, $S_2 = 389.5$ MeV, $m_2 = -0.529$, $\mu_2 = 0.7$ fm, and $P_M$ is the Majorana operator.

The Brink–Boeker potential is an effective potential derived for the oscillator shell model, but recently it was shown to be useful in the oscillator cluster model [5] and in the two-center model [14]. In both cases the width parameters for the fragments were chosen equal.

To deal with reaction problems in somewhat heavier systems methods using integral transforms were developed [5, 7, 18, 19, 20, 27] and successfully applied for oscillator functions of equal frequencies in the fragments. Recently such methods have been generalized to include different frequencies [10, 20, 23, 28]. A realistic nucleon–nucleon potential is too complicated for such extended calculations and it therefore is of interest to know whether the Brink–Boeker potential is acceptable; the main advantage of this potential is that, due to the core, no collapse for shell model functions from $^4$He to $^{40}$Ca occurs.

We use the $^6$Li system to be able to compare with the results for the approximate projection [5] and to some extent with the results of [14]. Also this system is simple for calculations yet complicated enough to give a significant result. Calculations were carried out for an $\alpha$ and a $d$ structure at the two centers. Figure 1 shows the result for the binding energy as a function of the distance $D$ of the centers. First a common width parameter was chosen and optimized for any value of $D$. The corresponding results are given in the dashed curves labelled by $P$ and $NP$ respectively for the angular momentum projected and the non-projected case. The full lines show the corresponding results with independent variation of the width parameter, $PD$ denoting

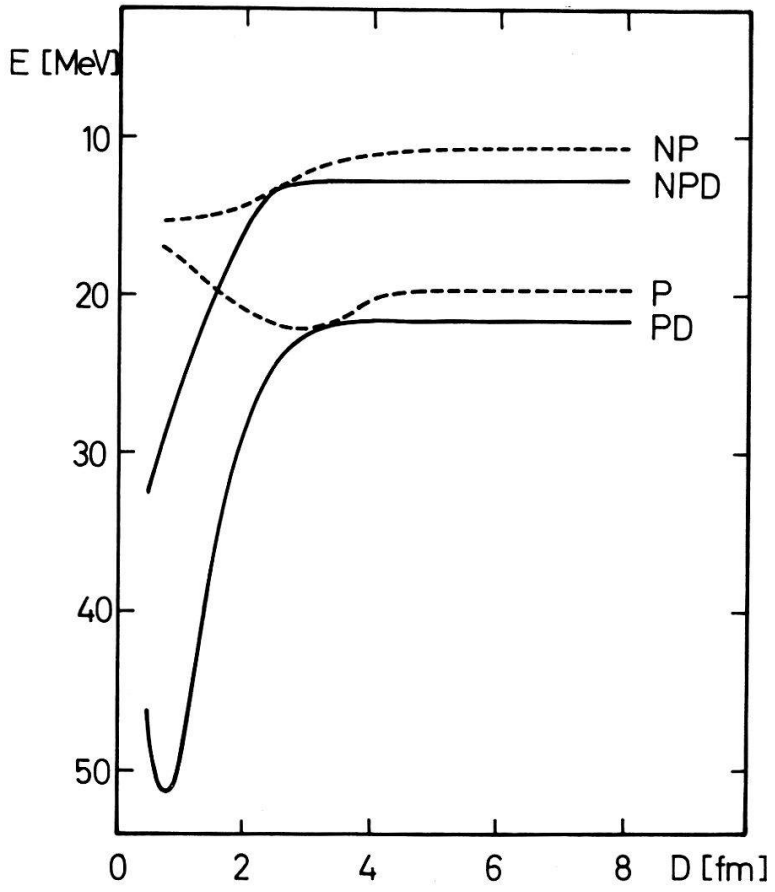


Figure 1
$^6$Li binding energy versus cluster distance. Angular momentum projected results with equal ($P$) and independent ($PD$) width parameters as well as non-projected results for these two cases ($NP$, $NPD$) are shown.

the projected and *NPD* the non-projected calculation. The effect of the angular momentum projection is evident.

On the other hand the calculations *PD* and *NPD* show that the Brink–Boeker potential may not be used beyond the so-called 'oscillator limit', i.e. equal frequencies in all clusters. This is stressed by the fact that asymptotically the deuteron is much larger than the $\alpha$ particle, but near the unphysically deep minimum the variation yields a deuteron that is about 10% smaller than the $\alpha$ particle. This negative result is in agreement with findings of Hackenbroich [29].

For comparison we performed a calculation with a potential given in [30]. This potential is fitted to low-energy nucleon–nucleon data but has no core. Consequently the $\alpha$ width has to be kept fixed. The result yields a binding energy of 24.6 MeV and a deuteron about 7% larger than the $\alpha$.

Next we compared our result for equal width with the approximative result of [5] denoted by *A* in Figure 2. We see that the curves are similar for small distances and deviate for larger ones as we expect for an expansion in the vicinity of $D = 0$. Note also that the exact calculation is less time-consuming than the approximative one.

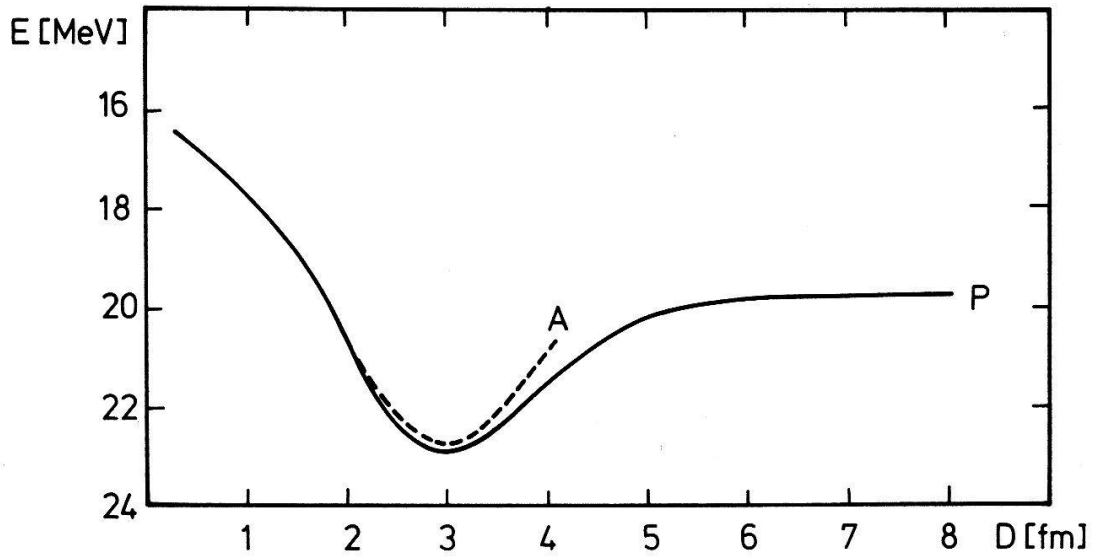


Figure 2
A comparison between the approximatively angular momentum projected result (*A*) of Ref. [5] and the corresponding result for exact projection (*P*).

Finally we wish to make a remark on the calculations with different width: The elimination of the center of mass motion requires that we have to adapt the frequency for both center of mass vectors of the clusters such that they are related by an orthogonal transformation. Details for this procedure are given in [20, 28]. For our purpose it is significant that we have some freedom to choose this frequency, and we chose it to be equal to the internal frequency of the $\alpha$. Further consequences of this freedom for the two-center model will be discussed in [26].

## 6. Conclusion

We have seen that it is possible to use radial Hilbert spaces and radial canonical transforms to perform angular momentum projection in the two-cluster and the two-center model if we restrict ourselves to central interactions and fragments of orbital angular momentum zero.

The method resulting for the two-center model makes the calculation, including projection only slightly more involved than the unprojected one. Further this method is simpler than the approximate one, given in [5]. Despite the usefulness of this result we wish to mention some further developments for which the general idea proposed might be useful. First one would certainly like to abandon both restrictions mentioned above, namely orbital angular momentum zero for the fragments and central operators. This will require transforms between radial Hilbert spaces corresponding to different angular momentum, and their theory still has to be developed.

If such an extension of the theory proves possible it can also be used to discuss radial matrix elements in hyperspherical coordinates. This is presently not possible as the operators of interest do not depend on the radial variable only. One could further hope to perform angular momentum projection in a three-center model, but this problem presents considerable extra difficulties.

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