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The $P(\phi)_2$ Green's Functions: Asymptotic Perturbation Expansion

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Abstract. The real time Green's functions in the $P(\phi)_2$ quantum field theory are infinitely differentiable functions of the coupling constant λ up to and including $\lambda = 0$. It follows that the perturbation series are asymptotic as $\lambda \to 0^+$.

1. Introduction

1.1. This paper is one of a series on perturbation theory and the general question of smooth dependence on coupling constants for $P(\phi)_2$ quantum field theory models. In Ref. [2] it was shown that the Schwinger functions or imaginary time Green's functions are infinitely differentiable in the coupling constant λ, for λ in an interval of the form $[0, \lambda_0)$. As a consequence the perturbation series for the Schwinger functions are asymptotic. In Ref. [3] it was shown that the real time Green's functions are infinitely differentiable on $(0, \lambda_0)$. In the present paper we show that the differentiability extends to the endpoint $\lambda = 0$, and hence that the standard perturbation series for these functions are asymptotic. We recall that the series definitely do not converge, as was shown by Jaffe [9].

The result tells us something about the structure of the Green's functions for small λ. For example in $(\phi^4)_2$, since the truncated four point function has a non-trivial series we can conclude that it is non-zero. This is a property not shared by free field theories and establishes a sense in which the theory is non-trivial.[3])

1.2. Our basic structure and notation are the same as in Ref. [3], where further references are given. The Hamiltonian has the form $H = H_0 + \lambda \int :\mathscr{P}(\varphi(x)): dx - E$ where H_0 is the free Hamiltonian for mass m_0, $\mathscr{P}$ is a lower semi-bounded polynomial, $\varphi(x)$ is the time zero field operator, and $E = E(m_0, \lambda, \mathscr{P})$ is chosen so $H \geqslant 0$. Certain aspects of our proofs are simpler if we take $H = H_0 + \lambda \int :\varphi(x)^d: dx - E$, d an even integer, and we shall work with this case. The results however are generally valid. We

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[1]) Research supported by Joint Awards Council/University Awards Committee of State University of New York/Research Foundation.

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[3]) Osterwalder and Seneor (to appear) and Eckmann, Epstein, and Fröhlich (to appear) study the perturbation series for the S-matrix, and are able to conclude that scattering is non-trivial for the model.

always take $\lambda \in [0, \lambda_0)$ with λ_0 sufficiently small so that the construction of Glimm, Jaffe, and Spencer [5, 6] is applicable. In particular then the physical mass $m(\lambda)$ satisfies $m(\lambda) \geqslant m_* > 0$ for $\lambda \in [0, \lambda_0)$, i.e. there is a uniform mass gap.

Time ordered products are defined by Nelson's technique [3, 10]. In the physical Hilbert space $\mathscr{H}$ let $\mathscr{H}^k$ be the scale associated with H. Then for $a_k \in C_0^\infty(R^1, \mathscr{L}(\mathscr{H}^1, \mathscr{H}^{-1}))$, supp $a_k \subset (t_-, \infty)$, and $\operatorname{Im}\theta \leqslant 0$, $\theta \neq 0$, we define inductively in $\mathscr{L}(\mathscr{H}^{2n+1}, \mathscr{H}^1)$:

$$E_0^\theta(t_+, t_-) = \exp(-i\theta(t_+ - t_-)H)$$

$$E_n^\theta(t_+, a_n, \ldots, a_1 t_-) = \int_{t_-}^{t_+} \exp(-i\theta(t_+ - s)H)a_n(s)E_{n-1}^\theta(s, a_{n-1}, \ldots, a_1, t_-)\, ds. \tag{1.1}$$

If $E_{n-1}^\theta \in \mathscr{L}(\mathscr{H}^{2n-1}, \mathscr{H}^1)$, then a priori we have only $E_n^\theta \in \mathscr{L}(\mathscr{H}^{2n-1}, \mathscr{H}^{-1})$. However, one shows that $[H, E_n^\theta] \in \mathscr{L}(\mathscr{H}^{2n-1}, \mathscr{H}^{-1})$ which implies $E_n^\theta \in \mathscr{L}(\mathscr{H}^{2n+1}, \mathscr{H}^1)$. The E_n^θ are analytic in $\operatorname{Im}\theta < 0$ and continuous in $\operatorname{Im}\theta \leqslant 0$, $\theta \neq 0$. If Ω is the vacuum in $\mathscr{H}(H\Omega = 0)$ then $(\Omega, E_n^\theta\Omega)$ is independent of $t_\pm$ provided supp $a_k \subset (t_-, t_+)$, and so $t_\pm$ are not specified. We also define

$$G_\theta(a_1, \ldots, a_n) = \sum_\pi (\Omega, E_n^\theta(a_{\pi(1)}, \ldots, a_{\pi(n)})\Omega)$$

where the sum is over all permutations π of $(1, \ldots, n)$.

The time zero fields $\varphi(g) = \int \varphi(x)g(x)\, dx$ define operators in $\mathscr{L}(\mathscr{H}^1, \mathscr{H}^{-1})$ with $\|\varphi(g)\|_{1,-1} \leqslant C\|g\|_1$ for some C independent of λ. Then for $f, g \in C_0^\infty(R^1)$, $f\varphi(g)$ is in $C_0^\infty(R^1, \mathscr{L}(\mathscr{H}^1, \mathscr{H}^{-1}))$. Now with $h = (h_1, \ldots, h_n)$, $h_n = f_n \otimes g_n$, we define Green's functions as distributions by

$$G_\theta(h) = G_\theta(f_1\varphi(g_1), \ldots, f_n\varphi(g_n)).$$

Wick ordered products $:\varphi^r:(g) = \int :\varphi(x)^r: g(x)\, dx$ may also be defined on $\mathscr{H}$ and satisfy for $r \leqslant d$ and $\lambda \in (0, \lambda_0)$

$$\| :\varphi^r:(g)\|_{1,-1} \leqslant C\lambda^{-1}D(g)\|g\|_\infty \tag{1.2}$$

where $D(g) = 1 + \text{diameter}(\operatorname{supp} g)$. Then $f :\varphi^r:(g)$ is in $C_0^\infty(R^1, \mathscr{L}(\mathscr{H}^1, \mathscr{H}^{-1}))$ and with $r = (r_1, \ldots, r_n)$ we define generalized Green's functions by

$$G_\theta(r, h) = G_\theta(f_1 :\varphi^{r_1}:(g_1), \ldots, f_n :\varphi^{r_n}:(g_n)). \tag{1.3}$$

We also define truncated Green's functions inductively by $G_\theta^T(r_1, h_1) = G_\theta(r_1, h_1)$ and

$$G_\theta^T(r, h) = G_\theta(r, h) - \sum_P \prod_{p \in P} G_\theta^T(r_p, h_p) \tag{1.4}$$

where the sum is over all proper partitions P of $(1, \ldots, n)$, $h_p = \{h_k\}_{k\in p}$, and $r_p = \{r_k\}_{k\in p}$. Both $G_\theta(r, h)$ and $G_\theta^T(r, h)$ are analytic in $\operatorname{Im}\theta < 0$ and continuous in $\operatorname{Im}\theta \leqslant 0$, $\theta \neq 0$.

Special cases of the $G_\theta(r, h)$ are the real time Green's functions $G(r, h) = G_\theta(r, h)|_{\theta=1}$ and the Schwinger functions $\hat{G}(r, h) = G_\theta(r, h)|_{\theta=-i}$. The $G(r, h)$ are the main items of interest, while the $\hat{G}(r, h)$ are special in that they have a representation as path space integrals (which indeed is crucial for the very construction of the model). Let dq be the Gaussian measure on real-valued $\mathscr{S}'(R^2)$ with mean zero and covariance

$(-\Delta + m_0^2)^{-1}$. We define in $L_p(\mathscr{S}', dq)$, $p < \infty$, $:q^r:(h) = \int :q(x)^r: h(x)\,dx$. For each bounded region $\Lambda \subset R^2$ let $d\nu_\Lambda$ be the measure

$$d\nu_\Lambda = \exp(-\lambda :q^d:(\chi_\Lambda))\,dq \Big/ \int \exp(-\lambda :q^d:(\chi_\Lambda))\,dq$$

and define

$$\hat{G}_\Lambda(r, h) = \int \prod_{k=1}^{n} :q^{r_k}:(h_k)\,d\nu_\Lambda.$$

Then $\hat{G}_\Lambda(r, h)$ converges as $|\Lambda| \to \infty$ [5, 6] and one version of the Feynman–Kac–Nelson formula is

$$\hat{G}(r, h) = \lim_{|\Lambda| \to \infty} \hat{G}_\Lambda(r, h). \tag{1.5}$$

1.3. The known smoothness results are the following [2, 3]. It is shown first for $\theta = -i$ and $\lambda \in [0, \lambda_0)$ and then for general θ and $\lambda \in (0, \lambda_0)$ that $G_\theta^T(r, h)$ is an infinitely differentiable function of λ and that

$$D^m G_\theta^T(r, h) = (-i\theta)^m \sum_{i,j} G_\theta^T(r, d; h, \chi(i, j)). \tag{1.6}$$

Here the sum is over $i, j \in Z^m$ and $\chi(i, j) = (\rho_{i_1} \otimes \chi_{j_1}, \ldots, \rho_{i_m} \otimes \chi_{j_m})$ where ρ_i and χ_j are partitions of unity indexed by Z^1. The χ_j are taken to have the form $\chi_j(x) = \chi(x - j)$ with $\chi \in C_0^\infty(R^1)$ and supp $\chi \subset (-1, 1)$. The ρ_i are special and are required to satisfy for some $\tau \geqslant n$,

(a) $\sum_i \rho_i = 1$ (1.7)

(b) supp $\rho_i \subset \{t : |t - i| \leqslant \max(1, |i|/3\tau)\}$.

(c) For $\alpha \geqslant 1$, there is a constant K_α such that

$$\|\rho_i^{(\alpha)}\|_1 \leqslant K_\alpha(|i| + 1)^{-\alpha+1}. \quad \text{Also} \quad \|\rho_i\|_1 \leqslant K_0.$$

The existence of such a partition for all τ is established in [3]. The crucial property is (1.7c) which says the ρ_i become smoother as $|i| \to \infty$, a property which leads to time-like clustering. In fact it follows from the mass gap that for fixed r, h, θ, N there exists a constant K such that

$$|G_\theta^T(r, d; h, \chi(i, j))| \leqslant K(|i| + 1)^{-N}(|j| + 1)^{-N}\lambda^{-(n+m)} \tag{1.8}$$

for all $\lambda \in (0, \lambda_0)$. This establishes the convergence of the sum in (1.6) and is the main ingredient in the proof of (1.6).

1.4. To extend control to $\lambda = 0$ at real time we must prove a bound like (1.8) uniform in $\lambda \in (0, \lambda_0)$. The strategy for accomplishing this is to make a finite expansion of $G^T(r, d; h, \chi(i, j))$ in powers of the interaction. This general technique has consistently been useful in constructive quantum field theory and has been called the pull-through formula. The point here is that by breaking up the operators $:\varphi^r:$ we avoid the bounds $\| :\varphi^r: \|_{1,-1} = \mathcal{O}(\lambda^{-1})$ which cause the trouble in (1.8). The new operators introduced all have the form $\lambda :\varphi^r:$ and the bound $\|\lambda :\varphi^r:\|_{1,-1} = \mathcal{O}(1)$ is quite satisfactory. Our main task is then to show that the delicate real-time clustering factors $(|i| + 1)^{-N}(|j| + 1)^{-N}$ still are present for each term in the expansion (Section

4). Once the bounds are proved the perturbation theory results follow directly (Section 5).

The expansion itself has the nicest form at imaginary time. The basic expansion step is then just integration by parts on path space. The relevant techniques have been established by Glimm and Jaffe [7]. In fact they give the whole argument of expansion plus clustering in this case (for different reasons). Our approach will be to develop a particular form of the imaginary time expansion (Section 2), and then generate the real time expansion by analytic continuation (Section 3).

2. Imaginary Time Expansion

We consider functions of the form $\int \prod_{k=1}^{n} :q(x_k)^{r_k}: w(x_1, \ldots, x_n)\, dx$. The kernels w will always be L_2 functions of compact support and so this is well defined in $L_p(\mathscr{S}', dq)$ for $p < \infty$ [1]. If $R(x) = \prod_{k=1}^{n} :q(x_k)^{r_k}:$ then the integration by parts formula is [7]:

$$\begin{aligned}\int &:q(y)^r: R(x)w(x, y)\, dx\, dy\, d\nu_\Lambda \\ &= \int :q(y)^{r-1}: C(y - z)(\delta R(x)/\delta q(z) - \lambda\chi_\Lambda(z)\, d:\varphi(z)^{d-1}: R(x)) \\ &\quad \times w(x, y)\, dx\, dy\, dz\, d\nu_\Lambda \end{aligned} \tag{2.1}$$

where the formal derivatives are to be evaluated by

$$\delta :q(x_k)^{r_k}: /\delta q(z) = r_k :q(x_k)^{r_k-1}: \delta(x_k - z),$$

and $C(y - z)$ is the kernel $(-\Delta + m_0^2)^{-1}(y, z)$ and is in $L_p(R^2)$ for $p < \infty$. This identity would be proved by making an explicit approximation by finite dimensional integrals, e.g. as in [1].

We want to iterate (2.1) and for this it is essential to use the language of graphs. The integrand on the left is labeled by a graph with $n + 1$ vertices, a vertex with r legs for each factor $:q(x)^r:$. The integrand on the right is a sum of terms labeled by graphs. The $\delta R/\delta q$ terms are labeled by graphs with a line joining a leg of the first vertex to each of the legs in the other vertices. The interaction term is labeled by d graphs joining this leg to each of d legs in a new vertex. One says that the identity consists of contracting the first leg in all possible ways.

Now beginning with $\hat{G}_\Lambda(r, h) = \int \prod_{k=1}^{n} :q^{r_k}: (h_k)\, d\nu_\Lambda$ contract the first leg in all possible ways, contract the second leg in all possible ways, and so on until all initial legs are contracted. The result is a sum over all possible contraction schemes, or equivalently a sum over all graphs $\Gamma = \mathscr{V}, \mathscr{L})$ which satisfy the following conditions. The vertices $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2$ consist of initial vertices $\mathscr{V}_1$ (depending on r), and a finite number of new vertices $\mathscr{V}_2$. The lines $\mathscr{L}$ are such that: (a) $\mathscr{V}_1$ legs are all contracted, (b) $\mathscr{V}_2$ vertices are joined to at least one $\mathscr{V}_1$ vertex, and (c) $\mathscr{V}_2$ vertices are not joined to each other. We call such a graph an *expansion graph*. Now for each Γ define r_v = number of uncontracted legs at vertex v, relabel h by $\mathscr{V}_1$, set $x = \{x_v\}_{v\in\mathscr{V}}$, and for a line l joining legs $l_1 \in v_1$ and $l_2 \in v_2$ define $x_{l_1} = x_{v_1}$, $x_{l_2} = x_{v_2}$. Then we have:

Theorem 2.1 [7]

$$\hat{G}_\Lambda(r, h) = \sum_{\Gamma} \int \prod_{v\in\mathscr{V}_1} h_v(x_v) \prod_{l\in\mathscr{L}} C(x_{l_1} - x_{l_2}) \prod_{v\in\mathscr{V}_2} - \lambda\chi_\Lambda(x_v) :q(x_v)^{r_v}: dx\, d\nu_\Lambda. \tag{2.2}$$

For the remainder of this chapter we rewrite this identity in a more suitable form.

Step I. For any expansion graph Γ, let $h = \{h_v\}_{v\in\mathscr{V}}$ be a set of functions indexed by $\mathscr{V}$, and define

$$\hat{I}_\Lambda(\Gamma, h) = \int \prod_{l\in\mathscr{L}} C(x_{l_1} - x_{l_2}) \prod_{v\in\mathscr{V}} h_v(x_v) :q(x_v)^{r_v}: dx\, dv_\Lambda. \tag{2.3}$$

Then with $h = \{h_v\}_{v\in\mathscr{V}_1}$ and $-\lambda\chi_\Lambda = \{-\lambda\chi_\Lambda\}_{v\in\mathscr{V}_2}$ the expansion becomes $\hat{G}_\Lambda(r, h) = \sum_\Gamma \hat{I}(\Gamma, h, -\lambda\chi_\Lambda)$. Furthermore we define $i = \{i_v\}_{v\in\mathscr{V}_2}$, $j = \{j_v\}_{v\in\mathscr{V}_2}$ and let $\chi(i, j) = \{\rho_{i_v} \otimes \chi_{j_v}\}_{v\in\mathscr{V}_2}$ where ρ satisfies (1.7) for some $\tau \geqslant |\mathscr{V}_2|$. Then we have

$$\hat{G}_\Lambda(r, h) = \sum_\Gamma \sum_{i,j} \hat{I}_\Lambda(\Gamma, h, -\lambda\chi_\Lambda\chi(i, j)). \tag{2.4}$$

The sums are finite, but since we eventually take $|\Lambda| \to \infty$ we prove the following lemma.

Lemma 2.2

$$|\hat{I}_\Lambda(\Gamma, h, \chi_\Lambda\chi(i, j))| = \mathscr{O}\left(\prod_{v\in\mathscr{V}_2} \exp(-m_0\,\epsilon\,(|i_v| + |j_v|))\right)$$

for some $\epsilon > 0$, uniformly in Λ.

Proof. It suffices to consider h_v, $v \in \mathscr{V}_1$, supported in lattice squares Δ_v. We first prove the bound with $\chi(i, j)$ replaced by $\chi(i', i, j) = \{\chi_{i'_v}\rho_{i_v} \otimes \chi_{j_v}\}_{v\in\mathscr{V}_2}$. We use the general bound

$$\left|\int \prod_{k=1}^{n} :q^{r_k}(x_k): w(x)\, dx\, dv_\Lambda\right| \leqslant K_{n,r}\|w\|_2 \tag{2.5}$$

which holds uniformly in Λ for w supported in lattice squares [5, 6]. Thus with $\Delta_v = \Delta_{i'_v, j_v}$ for $v \in \mathscr{V}_2$ we have for $p = |\mathscr{L}|/2$

$$\begin{aligned}|\hat{I}_\Lambda(\ldots, \chi(i', i, j))| &= \mathscr{O}\left\|\prod_l C_l\right\|_{L_2(\{\Delta_v\}_{v\in\mathscr{V}})}\\ &= \mathscr{O}\left(\prod_l \|C\|_{L_p(\Delta_{l_1}\times\Delta_{l_2})}\right)\\ &= \mathscr{O}\left(\prod_{v\in\mathscr{V}_2} \exp(-\tfrac{1}{4}m_0(|i'_v| + |j_v|)\right)\end{aligned}$$

since for each $v \in \mathscr{V}_2$, there is a line to $v' \in \mathscr{V}_1$ and

$$\|C\|_{L_p(\Delta_v\times\Delta_{v'})} = \mathscr{O}(\exp(-\tfrac{1}{2}m_0 d(\Delta_v, \Delta_{v'})) = \mathscr{O}(\exp(-\tfrac{1}{4}m_0(|i'_v| + |j_v|)).$$

Now we use

$$|\hat{I}_\Lambda(\ldots, \chi(i, j))| \leqslant \sum_{i':|i'_v - i_v| \leqslant 2 + |i_v|/3|\mathscr{V}_2|} |\hat{I}_\Lambda(\ldots, \chi(i', i, j))|$$

and obtain the required bound.

Step II. Let $\gamma = (\Gamma, \pi)$ be a pair consisting of a graph $\Gamma = (\mathscr{V}, \mathscr{L})$ and an ordering π of $\mathscr{V}$. We also write $\gamma = (V, \mathscr{L})$ with $V = (\mathscr{V}, \pi)$ a set of ordered vertices. In Hepp's terminology Γ is a Feynman graph and γ is a Dirac graph [8]. Given γ we define

$$\hat{I}_\Lambda(\gamma, h) = \int \prod_{l\in\mathscr{L}} C(x_{l_1} - x_{l_2}) \prod_{v\in V} h_v(x_v) :q(x_v)^{r_v}: dx^+\, dv_\Lambda \tag{2.6}$$

where dx^+ means we integrate $x = \{x_v\} = \{t_v, \vec{x}_v\}$ only over the region $t_{v_n} > \cdots > t_{v_1}$. Then $\hat{I}_\Lambda(\Gamma, h) = \sum_\pi \hat{I}_\Lambda((\Gamma, \pi), h)$ and the expansion becomes

$$\hat{G}_\Lambda(r, h) = \sum_\gamma \sum_{i,j} \hat{I}_\Lambda(\gamma, h, -\lambda\chi_\Lambda\chi(i, j)). \tag{2.7}$$

Step III. We now rewrite (2.6). First go to separate space-time variables replacing $x_v \in R^2$ by (t_v, x_v). Given γ, l joins a lower leg l_- to a higher leg l_+, and for $t_{l_+} > t_{l_-}$ we have

$$C(t_{l_+} - t_{l_-}, x_{l_+} - x_{l_-}) = \int (4\pi\omega_l)^{-1} \exp(-\omega_l(t_{l_+} - t_{l_-}) - ip_l(x_{l_+} - x_{l_-}))\, dp_l$$

where $\omega_l = \omega(p_l) = \sqrt{p_l^2 + m_0^2}$. We identify

$$\prod_l \exp(-ip_l(x_{l_+} - x_{l_-})) = \prod_v \exp(-ip_v x_v)$$
$$\prod_l \exp(-\omega_l(t_{l_+} - t_{l_-})) = \prod_v \exp(-\omega_v t_v) \tag{2.8}$$

where for any v we let $\mathscr{L}_v^+ (\mathscr{L}_v^-)$ be those lines joining v to a higher (lower) vertex and define

$$p_v = \sum_{l\in\mathscr{L}_v^-} p_l - \sum_{l\in\mathscr{L}_v^+} p_l$$
$$\omega_v = \sum_{l\in\mathscr{L}_v^-} p_l - \sum_{l\in\mathscr{L}_v^+} p_l. \tag{2.9}$$

Then we have, formally at least, with $p = \{p_l\}_{l\in\mathscr{L}}$, $h = \{f_v \otimes g_v\}_{v\in V}$.

$$\hat{I}_\Lambda(\gamma, h) = \int \prod_{l\in\mathscr{L}} (4\pi\omega_l)^{-1}\, dp_l \hat{I}_\Lambda(\gamma, h, p). \tag{2.10}$$

$$\hat{I}_\Lambda(\gamma, h, p) = \int \prod_{v\in V} :q(t_v, x_v)^{r_v}:\, e^{-\omega_v t} f_v(t_v) e^{-ip_v x_v} g_v(x_v)\, dt^+\, dx\, d\nu_\Lambda. \tag{2.11}$$

Since for $v \in V_1$ we have $r_v = 0$, $\hat{I}_\Lambda(\gamma, h, p)$ is proportional to $\prod_{v\in V_1} \tilde{g}_v(p_v)$. Furthermore one can show using (2.5), (2.8) that for fixed h, $|\hat{I}_\Lambda(\gamma, h, p)| \leqslant \text{const} \prod_{v\in V_1} |\tilde{g}_v(p_v)|$ uniformly in Λ. Combined with the following lemma this shows that the integral in (2.10) converges. The rigorous verification of (2.10) follows by introducing momentum cutoffs so that the interchange of integrations can be justified.

Lemma 2.3. For some constant K

$$\int \prod_{v\in V_1} |\tilde{g}_v(p_v)| \prod_{l\in\mathscr{L}} \omega_l^{-1}\, dp_l \leqslant K \prod_{v\in V_1} \|g_v\|_2.$$

Proof. For each $v \in V_1$, let N_v be those lines joining v to a V_2 vertex, let L_v be those lines joining v to a V_1 vertex, and let $\mathscr{L}_v = N_v \cup L_v$ be all lines at v. The basic estimate at the vertex v is

$$\int \left(\int |\tilde{g}_v(p_v)|^2 \prod_{l\in L_v} \omega_l^{-1}\, dp_l\right)^{1/2} \prod_{l\in N_v} \omega_l^{-1}\, dp_l$$
$$\leqslant \left(\int |\tilde{g}_v(p_v)|^2 \prod_{l\in\mathscr{L}_v} \omega_l^{-1+\epsilon}\, dp_l\right)^{1/2}$$
$$\leqslant K_v \|g_v\|_2. \tag{2.12}$$

For each $v \in V_1$, let L_v^+ be those lines joining v to a higher V_1 vertex. To set up the estimate (2.12) we successively apply the Schwarz in the variables $\{p_l\}_{l \in L_v^+}$ beginning with the first vertex and working up. Each time we split the two $\tilde{g}_v(p_v)$ factors which depend on p_l and split $\omega_l^{-1} = \omega_l^{-1/2}\omega_l^{-1/2}$. After doing this for all vertices less than v we have

$$\int \prod_{l \in M_v} \omega_l^{-1} \, dp_l \prod_{\substack{v' \in V_1 \\ v' \geqslant v}} \left(\int |\tilde{g}_{v'}(p_{v'})|^2 \prod_{l \in L_{v', <v}} \omega_l^{-1} \, dp_l \right)^{1/2} \prod_{u < v} K_u \|g_u\|_2$$

where M_v is all lines joining two V_1 vertices $\geqslant v$ or a V_2 vertex to a V_1 vertex $\geqslant v$, and $L_{v', <v}$ is all lines joining $v' \in V_1$ to a V_1 vertex less than v. Continuing to the last vertex gives the estimate.

Step IV. We now let $|\Lambda| \to \infty$. By a modification of (1.5), $\hat{I}(\gamma, h, p) = \lim \hat{I}_\Lambda(\gamma, h, p)$ exists and with $\hat{E} = E^{\theta = -i}$

$$\hat{I}(\gamma, h, p) = (\Omega, \hat{E}(\{e^{-\omega_v t} f_v : \varphi^{r_v} : (e^{-ip_v x} g_v)\}_{v \in V})\Omega), \tag{2.13}$$

where for $r_v = 0$ we define $:\varphi^{r_v}: (e^{-ip_v x} g_v) = \sqrt{2\pi}\tilde{g}_v(p_v) I$.

Since we have a suitable Λ-uniform bound on the p behavior using Lemma 2.3, $\hat{I}(\gamma, h) = \lim \hat{I}_\Lambda(\gamma, h)$ exists and

$$\hat{I}(\gamma, h) = \int \prod_{l \in \mathscr{L}} (4\pi\omega_l)^{-1} \, \mathrm{d}p_l \hat{I}(\gamma, h, p). \tag{2.14}$$

Finally by Lemma 2.2 we may take $|\Lambda| \to \infty$ in (2.7) and obtain

$$\hat{G}(r, h) = \sum_\gamma \sum_{i,j} \hat{I}(\gamma, h, -\lambda\chi(i, j)). \tag{2.15}$$

3. Real Time Expansion

3.1. We want to analytically continue the expansion (2.13), (2.14), (2.15) away from $\theta = -i$. We restrict ourselves to the region

$$\mathscr{D} = \{\theta \in C : \operatorname{Im} \theta < 0, \tfrac{1}{2} < |\theta| < 2\}$$

or its closure $\bar{\mathscr{D}}$ which includes $\theta = 1$. For any Dirac graph $\gamma = (V, \mathscr{L})$, $h = \{f_v \otimes g_v\}_{v \in V}$, $p = \{p_l\}_{l \in \mathscr{L}}$ and $\theta \in \bar{\mathscr{D}}$ we define

$$I_\theta(\gamma, h, p) = (\Omega, E^\theta(\{e^{-i\theta\omega_v t} f_v : \varphi^{r_v} : (e^{-ip_v x} g_v)\}_{v \in V})\Omega). \tag{3.1}$$

$$I_\theta(\gamma, h) = \int \prod_{l \in \mathscr{L}} (4\pi\omega_l)^{-1} \, dp_l I_\theta(\gamma, h, p). \tag{3.2}$$

Our goal is then to show

$$G_\theta(r, h) = \sum_\gamma \sum_{i,j} I_\theta(\gamma, h, -i\theta\lambda\chi(i, j)) \tag{3.3}$$

where the sum is over all expansion graphs γ with vertices $V = V_1 \cup V_2$ consisting of initial vertices V_1 (depending on r) and new vertices V_2, and where the vertex functions are $h = \{f_v \otimes g_v\}_{v \in V_1}$ (after relabeling) and $-i\theta\lambda\chi(i, j) = \{-i\theta\lambda\rho_i \otimes \chi_j\}_{v \in V_2}$.

3.2. The first difficulty is to show that the integral (3.2) is well-defined. Bounds on (3.1) using the estimates of [3] are not sufficient since they involve norms of the form $\|\partial^\alpha/\partial t^\alpha(e^{-i\theta\omega_v t}f_v)\|_1$ which grow in p. To avoid this we digress and define a time ordered product with shifted Hamiltonians.

Let $a_k \in C_0^\infty(R^1, \mathscr{L}(\mathscr{H}^1, \mathscr{H}^{-1}))$, supp $a_k \subset (t_-, \infty)$, and take $\Omega_k \geqslant 0$. Then for $\theta \in \bar{\mathscr{D}}$ we define inductively

$$E_0^\theta(t_+, \Omega_0, t_-) = \exp(-i\theta(H + \Omega_0)(t_+ - t_-))$$
$$E_n^\theta(t_+, \Omega_n, a_n, \ldots, \Omega_1, a_1, \Omega_0, t_-)$$
$$= \int_{t_-}^{t_+} \exp(-i\theta(H + \Omega_n)(t_+ - s))a_n(s)E_{n-1}^\theta(s, \Omega_{n-1}, a_{n-1}, \ldots, \Omega_0, t_-)\, ds.$$

Just as in the treatment of [3, 10] one establishes inductively the following points

$$i\theta((H + \Omega_n)E_n^\theta - E_n^\theta(H + \Omega_0))$$
$$= -\sum_{k=1}^{n} E_n^\theta(t_+, \ldots, \Omega_k, a_k', \ldots, t_-) + a_n(t_+)E_{n-1}^\theta. \tag{3.4a}$$

$$\|E_n^\theta(t_+, \Omega_n, a_n, \ldots, \Omega_0, t_-)\|_{2n+1,1} \leqslant K_n(1 + \Omega_0^n)\prod_k |a_k|_n$$
$$|a_k|_n = \sum_{\alpha=0}^{n} \int \|a_k^{(\alpha)}(t)\|_{1,-1}\, dt. \tag{3.4b}$$

(The point here is to use

$$\|E_n^\theta u\|_1 \leqslant \|(H + \Omega_n + 1)E_n^\theta u\|_{-1} \leqslant (1 + \Omega_0)\|E_n^\theta\|_{2n-1,-1}\|u\|_{2n+1}$$
$$+ \|(H + \Omega_n)E_n^\theta - E_n^\theta(H + \Omega_0)\|_{2n-1,-1}\|u\|_{2n+1}$$

together with part (a))

E_n^θ is continuous in the a_k. (3.4c)

$E_n^\theta \in \mathscr{L}(\mathscr{H}^{2n+1}, \mathscr{H}^1)$ is analytic in $\mathscr{D}$ and continuous in $\bar{\mathscr{D}}$. (3.4d)

Let $\hat{\theta} = -\operatorname{Re}\theta + i \operatorname{Im}\theta$ and $\check{a}(t) = a(-t)$. Then if supp $a_k \subset (t_-, t_+)$ we have the identity on $\mathscr{H}^{2n+1} \times \mathscr{H}^{2n+1}$

$$E_n^\theta(t_+, \Omega_n, a_n, \ldots, a_1, \Omega_0, t_-)^* = E_n^{\hat{\theta}}(-t_-, \Omega_0, \check{a}_1, \ldots, \check{a}_n, \Omega_n, -t_+) \tag{3.4e}$$

$$E_n^\theta(t_+, \Omega_n, a_n, \ldots, a_1, \Omega_0, t_-)$$
$$= e^{-i\theta\Omega_n t_+}E_n^\theta(t_+, e^{i\theta(\Omega_n - \Omega_{n-1})t}a_n, \ldots, e^{i\theta(\Omega_1 - \Omega_0)t}a_1, t_-)e^{i\theta\Omega_0 t_-}. \tag{3.4f}$$

3.3. We now develop general estimates on $I_\theta(\gamma, h, p)$ and $I_\theta(\gamma, h)$ for fixed $\gamma = (V, \mathscr{L})$. We use the notation

$$|f_v|_n = \sum_{\alpha=0}^{n} \|f_v^{(\alpha)}\|_1$$
$$|f| = \prod_{v\in V} |f_v|_{|V|}.$$

Lemma 3.1. $I_\theta(\gamma, h, p)$ is analytic in $\mathscr{D}$, continuous in $\bar{\mathscr{D}}$, and there is a constant K such that

$$|I_\theta(\gamma, h, p)| \leqslant K\prod_{v\in V_1} |\tilde{g}_v(p_v)| \prod_{v\in V_2} \lambda^{-1}D(g_v)\|g_v\|_\infty |f|$$

for all h, $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$.

Proof. For each vertex v define $\Omega_v = \sum_l \omega_l \geqslant 0$ where the sum is over all lines l joining a vertex preceding or equal to v to a vertex subsequent to v. Then $\Omega_v - \Omega_{v'} = -\omega_v$ (v' = predecessor of v) and so by (3.4f)

$$I_\theta(\gamma, h, p) = (\Omega, E_n^\theta(\{\Omega_v, f_v :\varphi^{r_v}: (e^{-ip_v x} g_v)\}_{v\in V})\Omega). \tag{3.5}$$

Now since

$$|f_v :\varphi^{r_v}: (e^{-ip_v x} g_v)|_{|V|} \leqslant \begin{cases} |f_v|_{|V|}\sqrt{2\pi}|\tilde{g}(p_v)| & v \in V_1 \\ |f_v|_{|V|}\lambda^{-1}D(g_v)\|g_v\|_\infty & v \in V_2 \end{cases} \tag{3.6}$$

the bound follows from (3.4b) and the θ dependence by (3.4d).

Lemma 3.2. $I_\theta(\gamma, h)$ is well defined, analytic in $\mathscr{D}$, continuous to $\bar{\mathscr{D}}$, and there is a constant K such that

$$|I_\theta(\gamma, h)| \leqslant K \prod_{v\in V_1} \|g_v\|_2 \prod_{v\in V_2} \lambda^{-1}D(g_v)\|g_v\|_\infty |f|$$

for all h, $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$.

Proof. Immediate from Lemma 3.1 and Lemma 2.3.

Definition. For any subset $\sigma \subset V$ let $f_\sigma = \{f_v\}_{v\in\sigma}$ and define for integer N.

$$|f_\sigma|_N = \sup_{\alpha_v:\Sigma\alpha_v = N} \prod_{v\in\sigma} |f_v^{(\alpha_v)}|_{|\sigma|}. \tag{3.7}$$

Lemma 3.3. For any N, N' there is a constant K such that if

(a) There is a partition (σ, σ') of V and $T \in (-\infty, \infty)$ such that $\operatorname{supp} f_v \subset (-\infty, T)$ for $v \in \sigma$ and $\operatorname{supp} f_v \subset (t, \infty)$ for $v \in \sigma'$.

(b) γ connects σ, σ'

then

$$|I_\theta(\gamma, h)| \leqslant K|f_\sigma|_N|f_{\sigma'}|_{N'} \prod_{v\in V_1} \|g_v\|_2 \prod_{v\in V_2} \lambda^{-1}D(g_v)\|g_v\|_\infty$$

for all $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$.

Proof. Let $\bar{v}$ be the last vertex in σ. If $v < \bar{v}$ for some $v \in \sigma'$ then $I_\theta(\gamma, h) = 0$. Thus we assume $v > \bar{v}$ for all $v \in \sigma'$ and have from (3.5).

$$\begin{aligned} I_\theta(\gamma, h, p) &= (\Omega, E^\theta(\{\Omega_v, a_v\}_{v\in\sigma'}, \Omega_{\bar{v}}, T)E^\theta(T, \{\Omega_v, a_v\}_{v\in\sigma})\Omega) \\ a_v(t) &= f_v(t) :\varphi^{r_v}: (e^{-ip_v x} g_v). \end{aligned} \tag{3.8}$$

This is formally true and can be proved by approximating the a_v by more regular operators. Since γ connects σ, σ' we have $\Omega_{\bar{v}} \geqslant m_0$ and thus

$$\begin{aligned} \|E^\theta(T, \{\Omega_v, a_v\}_{v\in\sigma})\Omega\| &\leqslant m_0^{-N}\|(H + \Omega_{\bar{v}})^N E^\theta(T, \{\Omega_v, a_v\}_{v\in\sigma})\Omega\| \\ &\leqslant K_1 \sum_{\alpha_v:\Sigma\alpha_v = N} \|E^\theta(T, \{\Omega_v, a_v^{(\alpha_v)}\}_{v\in\sigma})\Omega\| \\ &\leqslant K_2|f_\sigma|_N \prod_{v\in\sigma\cap V_1} |\tilde{g}_v(p_v)| \prod_{v\in\sigma\cap V_2} \lambda^{-1}D(g_v)\|g_v\|_\infty. \end{aligned}$$

Here the first step follows by the spectral theorem, the second by (3.4a) (in the case at hand the last term in (3.4a) is absent and $\Omega_0 = 0$), and the third step follows by (3.4b). Similarly using (3.4e) we obtain

$$\|E^{\theta}(\{\Omega_v, a_v\}_{v\in\sigma'}, \Omega_{\bar{v}}, T)^*\Omega\| \leqslant K_3|f_{\sigma'}|_{N'} \prod_{v\in\sigma'\cap V_1} |\tilde{g}_v(p_v)| \prod_{v\in\sigma'\cap V_2} \lambda^{-1}D(g_v)\|g_v\|_\infty$$

and combining these two bounds with Lemma 2.3 gives the result.

Lemma 3.4. For every N, there is a constant K such that if

(a) $\operatorname{Supp} f_v \subset (-T, T)$, $T \geqslant 1$
(b) There is a partition (σ, σ') of V such that

$$d\left(\bigcup_{v\in\sigma} \operatorname{supp} g_v, \bigcup_{v\in\sigma'} \operatorname{supp} g_v\right) \geqslant L \geqslant 1$$

(c) γ connects σ, σ'

then

$$|I_\theta(\gamma, h)| \leqslant K(T/L)^N \prod_{v\in V_1} D(g_v)\|g_v\|_2 \prod_{v\in V_2} \lambda^{-1}D(g_v)^2\|g_v\|_\infty|f|$$

for all $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$.

Proof. By (c) there exists a line l^* joining $l_1 \in v_1 \in \sigma$ and $l_2 \in v_2 \in \sigma'$ and we suppose for definiteness that $v_1 < v_2$. Now $I_\theta(\gamma, h, p)$ dpends on $p^* \equiv p_{l^*}$ only in the functions $e^{-i\theta\omega_v t}f_v : \varphi^{r_v}: (e^{-ip_v x}g_v)$, $v = v_1$ or v_2. Define $I_\theta(\gamma, h, p, q_1, q_2)$ by replacing p^* by q_1 in $p_{v_1} \equiv Q_{v_1} - p^*$ and p^* by q_2 in $p_{v_2} \equiv Q_{v_2} + p^*$. Let $\alpha_v \in C_0^\infty(R^1)$ satisfy $\alpha_v(x) = 1$ for $x \in \operatorname{supp} g_v$ and $\alpha_v(x) = 0$ for $d(x, \operatorname{supp} g_v) \geqslant \frac{1}{4}$. Now we claim that

$$I_\theta(\gamma, h, p) = \frac{1}{2\pi}\int \tilde{\alpha}_{v_1}(q_1 - p_*)\tilde{\alpha}_{v_2}(-q_2 + p^*)I_\theta(\gamma, h, p, q_1, q_2)\, dq_1\, dq_2. \tag{3.9}$$

The point is that $I^\theta(\gamma, h, p)$ can be regarded as a distribution evaluated at $e^{\pm ip^*x}g_v$, $v = v_1$ or v_2, and for any distribution ψ

$$\langle\psi, e^{\pm ip^*x}g_v\rangle = \frac{1}{\sqrt{2\pi}}\int \tilde{\alpha}_v(\pm q \mp p^*)\langle\psi, e^{\pm iqx}g_v\rangle\, dq$$

since $\alpha_v g_v = g_v$. We further define $\beta_{v_j} = (-\Delta + m_0^2)\alpha_{v_j}$ so that $\tilde{\alpha}_{v_j}(p) = \omega(p)^{-2}\tilde{\beta}_{v_j}(p)$ and write $p = (p', p^*)$, $p' = \{p_l\}_{l\neq l^*}$. Then interchanging integration variables we have

$$I_\theta(\gamma, h) = \int \tilde{\beta}_{v_1}(q_1 - p^*)\tilde{\beta}_{v_2}(-q_2 + p^*)W(p^*, p', q_1, q_2)\, dp^*\, dp'\, dq_1\, dq_2$$

$$W(p^*, p', q_1, q_2) = (2\pi)^{-1}\prod_l (4\pi\omega_l)^{-1}\omega(p^* - q_1)^{-2}\omega(p^* - q_2)^{-2} \times I_\theta(\gamma, h, p, q_1, q_2). \tag{3.10}$$

We will show that W is C^∞ in p^* and that for any N there is a constant K_1 such that

$$\|(\partial/\partial p_*)^N W\|_1 \leqslant K_1 T^N \prod_{v\in V_1} \|g_v\|_2 \prod_{v\in V_2} \lambda^{-1}D(g_v)\|g_v\|_\infty|f|. \tag{3.11}$$

This gives the result since the p^* integral in (3.10) can be estimated by

$$\begin{aligned}&\left|(2\pi)^{-1/2}\int \tilde{\beta}_{v_1}(q_1 - p^*)\tilde{\beta}_{v_2}(-q_2 + p^*)W(p^*, \ldots)\, dp^*\right| \\ &\quad = \left|\iint e^{-iq_1x}\beta_{v_1}(x)e^{iq_2y}\beta_{v_2}(y)\hat{W}(x - y, \ldots)\, dx\, dy\right| \\ &\quad \leqslant \|\beta_{v_1}\|_1\|\beta_{v_2}\|_1 \sup_{\substack{x\in \operatorname{supp}\beta_{v_1}\\ y\in \operatorname{supp}\beta_{v_2}}} |\hat{W}(x - y, \ldots)| \\ &\quad \leqslant K_2D(g_{v_1})D(g_{v_2})L^{-N}\int |(\partial/\partial p^*)^N W(p^*, \ldots)|\, dp^*.\end{aligned}$$

We now prove (3.11). We note that $I(\gamma, h, p, q_1, q_2)$ can be regarded as a distribution evaluated at $e^{\pm i\theta\omega^* t}f_v$, $v = v_1$ or v_2, and this is the only dependence on p^*. For any distribution ψ

$$\partial/\partial p^*\langle\psi, e^{\pm i\theta\omega^* t}f_v\rangle = \pm i\theta\frac{p^*}{\omega^*}\langle\psi, te^{\pm i\theta\omega^* t}f_v\rangle$$

Thus we have

$$\begin{aligned}&\partial/\partial p^* I_\theta(\gamma, h, p, q_1, q_2) \\ &\quad = i\theta\frac{p^*}{\omega^*}(I_\theta(\gamma, t_{v_1}h, p, q_1, q_2) - I_\theta(\gamma, t_{v_2}h, p, q_1, q_2))\end{aligned}$$

where $(t_{v_1}h)_v = t_{v_1}f_{v_1} \otimes g_{v_1}$ for $v = v_1$ and $(t_{v_1}h)_v = f_v \otimes g_v$ otherwise. Thus using the argument of Lemma 3.1, and the fact that $|t^\alpha f_v|_{|V|} \leqslant K_3T^\alpha|f_v|_{|V|}$ we have for $\alpha \leqslant N$

$$|(\partial/\partial p^*)^\alpha I_\theta(\gamma, h, p, q_1, q_2)| \leqslant K_4T^\alpha \prod_{v\in V_1}|\tilde{g}_v(p'_v)| \prod_{v\in V_2}\lambda^{-1}D(g_v)\|g_v\|_\infty|f|$$

where p'_v has p^* replaced by q_1 or q_2 if $v = v_1$ or v_2. Combining this with $(\partial/\partial p)^\alpha\omega(p)^{-2} = \mathcal{O}(\omega(p)^{-2})$ and $(\partial/\partial p)^\alpha\omega(p + q)^{-2} = \mathcal{O}(\omega(p + q)^{-2})$ we have

$$\begin{aligned}|(\partial/\partial p^*)^N W(p^*, p', q_1, q_2)| &\leqslant K_5T^N\prod_l \omega_l^{-1}\omega(p^* - q_1)^{-2}\omega(p^* - q_2)^{-2} \\ &\quad \times \prod_{v\in V_1}|\tilde{g}_v(p'_v)| \prod_{v\in V_2}\lambda^{-1}D(g_v)\|g_v\|_\infty|f|.\end{aligned}$$

Next we do the integral over q_j. This gives either $(\omega^{-2} * |\tilde{g}_v|)(p_v)$ if $v_j \in V_1$ or a constant if $v_j \in V_2$. Estimating the integral over p by Lemma 2.3 and using $\|\omega^{-2} * |\tilde{g}_v|\|_2 \leqslant \|\omega^{-2}\|_1\|g\|_2$ we obtain (3.11).

3.4. We now show that the sum in (3.3) converges and that the identity holds. For $h = \{f_k \otimes g_k\}$ we define $R(h) = 1 + \max_k R(f_k \otimes g_k)$ where $R(f_k \otimes g_k) = \sup\{|x| : x \in \operatorname{supp}(f_k \otimes g_k)\}$. For $i = \{i_v\}_{v\in V_2}$ we define $|i| = \max_v|i_v|$, etc.

Lemma 3.5. For every N there are constants K, ν such that

$$|I_\theta(\gamma, h, \lambda\chi(i, j))| \leqslant K(|i| + 1)^{-N}(|j| + 1)^{-N}R(h)^\nu\prod_{v\in V_1}\|g_v\|_2|f_v|_{|V|}$$

for all h, i, j, $\theta \in \bar{\mathscr{D}}$, and $\lambda \in (0, \lambda_0)$.

Proof. The case $V_2 = \varnothing$ follows from Lemma 3.2. Assuming $V_2 \neq \varnothing$, we set $C = 3|V_2|R(h)$ and consider three regions:

(I) $C^2 \geqslant |j|, |i|^2$. The bound in this region follows from Lemma 3.2 since $D(\chi_{j_v})\|\chi_{j_v}\|_\infty = \mathcal{O}(1)$ and $|\rho_{i_v}|_{|V|} = \mathcal{O}(1)$.

(II) $|i|^2 \geqslant |j|, C^2$. Let σ, σ' be a partition of V_2. We prove the bound on the subregion satisfying $d(\{0\} \cup i_\sigma, i_{\sigma'}) \geqslant |i|/|V_2|$ where $i_\sigma = \{i_v\}_{v\in\sigma}$. This is sufficient since as σ varies we cover all points. Since $\operatorname{supp} f_v \subset \{t: |t| \leqslant |i|/3|V_2|\}$ and $\operatorname{supp} \rho_{i_v} \subset \{t: |t - i_v| \leqslant |i|/3|V_2|\}$ we have that $\bigcup_{v\in V_1} \operatorname{supp} f_v$ and $\bigcup_{v\in\sigma} \operatorname{supp} \rho_{i_v}$ are separated from $\bigcup_{v\in\sigma'} \operatorname{supp} \rho_{i_v}$ by at least $|i|/3|V_2|$. Thus since γ connects σ' to $\sigma \cup V_1$ we have by Lemma 3.3

$$|I_\theta(\gamma, h, \lambda\chi(i,j))| \leqslant K_1 |\{\rho_{i_v}\}_{v\in\sigma'}|_{N+|\sigma'|} \prod_{v\in V_1} \|g_v\|_2 |f_v|_{|V|}.$$

By (1.7c) we have the bound

$$\begin{aligned} |\{\rho_{i_v}\}_{v\in\sigma'}|_{N+|\sigma'|} &\leqslant K_2 \sup_{\Sigma\alpha_v = N+|\sigma'|} \prod_{v\in\sigma'} (|i_v| + 1)^{-\alpha_v+1} \\ &\leqslant K_2 \Big(\inf_{v\in\sigma'} |i_v| + 1\Big)^{-N}. \end{aligned} \tag{3.12}$$

Since $\inf_{v\in\sigma'}|i_v| \geqslant |i|/|V_2|$ this gives a bound $\mathcal{O}(|i| + 1)^{-N}$ for any N which is sufficient in this region.

(III) $|j| \geqslant C^2, |i|^2$. Again we restrict attention to the subregion where $d(\{0\} \cup j_\sigma, j_{\sigma'}) \geqslant |j|/|V_2|$. Then $\bigcup_{v\in V_1} \operatorname{supp} g_v$ and $\bigcup_{v\in\sigma} \operatorname{supp} \chi_{j_v}$ are separated from $\bigcup_{v\in\sigma'} \operatorname{supp} \chi_{j_v}$ by $|j|/3|V_2|$. Since also $\operatorname{supp} f_v$ and $\operatorname{supp} \rho_{i_v}$ are contained the interval $(-\frac{3}{2}|j|^{1/2}, \frac{3}{2}|j|^{1/2})$ we have by Lemma 3.4 with $L = |j|/3|V_2|$ and $T = \frac{3}{2}|j|^{1/2}$

$$|I^\theta(\gamma, h, \lambda\chi(i,j))| \leqslant K_3(|j| + 1)^{-N} \prod_{v\in V_1} \|g_v\|_2 |f_v|_{|V|}$$

which suffices for this region. (We have used $D(g_v) \leqslant 2R(h) \leqslant 2|j|^{1/2}/3|V_2|$.)

Theorem 3.6. $G_\theta(r, h) = \sum_\gamma \sum_{i,j} I_\theta(\gamma, h, -i\theta\lambda\chi(i,j))$ for all $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$.

Proof. For $\frac{1}{2} < \alpha < 2$ define $f^\alpha(t) = \alpha^{-1} f(t/\alpha)$ and $h^\alpha = \{f_k^\alpha \otimes g_k\}$. Now the analysis of Section 2 holds with the partition of unity $\alpha\rho_{i_v}^\alpha$ replacing ρ_{i_v} and so with $\chi^\alpha(i,j) = \{\rho_{i_v}^\alpha \otimes \chi_{j_v}\}_{v\in V_2}$

$$\begin{aligned} G_{-i\alpha}(r, h) &= \hat{G}(r, h^\alpha) \\ &= \sum_\gamma \sum_{i,j} \hat{I}(\gamma, h^\alpha, -\lambda\alpha\chi^\alpha(i,j)) \\ &= \sum_\gamma \sum_{i,j} I_{-i\alpha}(\gamma, h, -\lambda\alpha\chi(i,j)). \end{aligned}$$

Thus the identity holds for $\theta = -i\alpha$. This is a determining set in $\mathscr{D}$ and since both sides are analytic in $\mathscr{D}$ by Lemma 3.2 and Lemma 3.5, the identity extends to $\mathscr{D}$. Similarly both sides are continuous and we extend to $\bar{\mathscr{D}}$.

Corollary 3.7. There exists K, ν such that for all h, $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$

$$|G_\theta(r, h)| \leqslant K D(h)^\nu \prod_{k=1}^n \|g_k\|_2 |f_k|_{2nd}.$$

Proof. Follows from Lemma 3.5 and Theorem 3.6. Note that for any expansion graph $|V| \leqslant 2nd$. We are allowed to replace $R(h)$ by

$$D(h) \equiv 1 + \operatorname{diam}\Big(\bigcup_k \operatorname{supp}(f_k \otimes g_k)\Big)$$

by translation invariance.

3.5. If we define $I_\theta(\Gamma, h) = \sum_\pi I_\theta((\Gamma, \pi), h)$ then our expansion can also be written as a sum over Feynman graphs.

$$G_\theta(r, h) = \sum_{\Gamma} \sum_{i,j} (I_\theta(\Gamma, h, -i\theta\lambda\chi(i, j))). \tag{3.13}$$

We use this form to write an expansion for truncated Green's functions.

Given $\Gamma = (\mathscr{V}, \mathscr{L})$ and $h = \{h_v\}_{v\in\mathscr{V}}$, let $\tilde{\Gamma}$ be the set of connected components of Γ. If $|\tilde{\Gamma}| = 1$ (i.e. if Γ is connected) we define $I_\theta^T(\Gamma, h) = I_\theta(\Gamma, h)$. For general Γ we define by induction on $|\tilde{\Gamma}|$

$$I_\theta^T(\Gamma, h) = I_\theta(\Gamma, h) - \sum_{Q|\tilde{\Gamma}} \prod_{q\in Q} I_\theta^T(\Gamma_q, h_q) \tag{3.14}$$

where the sum is over the proper partitions Q of $\tilde{\Gamma}$ and for $q \subset \tilde{\Gamma}$, Γ_q and h_q have the obvious meaning.

The expansion for $G_\theta^T(r, h)$ can now be written.

$$G_\theta^T(r, h) = \sum_{\Gamma} \sum_{i,j} I_\theta^T(\Gamma, h, -i\theta\lambda\chi(i, j)). \tag{3.15}$$

The path space version of this ($\theta = -i$) is due to Glimm and Jaffe [7], and (3.15) follows by analytic continuation. Or one can verify (3.15) directly from the definitions. The sum converges by an inductive argument based on Lemma 3.5 (c.f. Section 4).

4. Bounds on Derivatives

We now prove the uniform version of (1.8) at real time. We fix n, m; let $h = (f_1 \otimes g_1, \ldots, f_n \otimes g_n)$, f_k, $g_k \in C_0^\infty(R^1)$; and let $\chi(i, j) = (\rho_{i_1} \otimes \chi_{j_1}, \ldots, \rho_{i_m} \otimes \chi_{j_m})$ where ρ_i satisfies (1.7) with $\tau \geqslant 2(n + m)d$. Also define

$$|h| = \prod_{k=1}^{n} \|g_k\|_2 |f_k|_{2(n+m)d}. \tag{4.1}$$

Theorem 4.1. (a) For every N, there exist K, ν such that for all h, $\lambda \in (0, \lambda_0)$, and $i, j \in Z^m$:

$$|G^T(r, d; h, \chi(i, j))| \leqslant K(|i| + 1)^{-N}(|j| + 1)^{-N} R(h)^\nu |h|.$$

(b) There exist K, ν such that for all h, $\lambda \in (0, \lambda_0)$:

$$|D^m G^T(r, h)| \leqslant K D(h)^\nu |h|.$$

Proof. Part (b) follows from (a) and (1.6). We may take the diameter $D(h)$ instead of the distance to the origin $R(h)$ by the translation invariance of $D^m G^T(r, h)$.

To prove (a) we expand according to (3.15) with $I^T = I_{\theta=1}^T$.

$$\begin{aligned} G^T(r, d; h, \chi(i, j)) &= \sum_{\Gamma} \sum_{i',j'} I^T(\Gamma, h, \chi(i, j), -i\lambda\chi(i', j')) \\ &\equiv \sum_{\Gamma} \sum_{i',j'} I^T(\Gamma, \psi). \end{aligned} \tag{4.2}$$

Here the sum is over all expansion graphs $\Gamma = (\mathscr{V}, \mathscr{L})$ with initial vertices $\mathscr{V}_1 = \mathscr{V}_{1,1} \cup \mathscr{V}_{1,2}$, $\mathscr{V}_{1,1}$ depending on r and $\mathscr{V}_{1,2}$ consisting of m vertices with d legs. The vertex functions are denoted $\psi = \{\psi_v\}_{v\in\mathscr{V}}$ and are explicitly (after relabeling h, $\chi(i,j)$).

$$\begin{aligned} h &= \{f_v \otimes g_v\}_{v\in\mathscr{V}_{1,1}} \equiv \{\psi_v\}_{v\in\mathscr{V}_{1,1}} \\ \chi(i,j) &= \{\rho_{i_v} \otimes \chi_{j_v}\}_{v\in\mathscr{V}_{1,2}} \equiv \{\psi_v\}_{v\in\mathscr{V}_{1,2}} \\ -i\lambda\chi(i',j') &= \{-i\lambda\rho_{i_v} \otimes \chi_{j_v}\}_{v\in\mathscr{V}_2} \equiv \{\psi_v\}_{v\in\mathscr{V}_2}. \end{aligned} \tag{4.3}$$

To establish our bound we have only to show that for fixed Γ, N there exist K, ν such that

$$|I^T(\Gamma,\psi)| \leqslant K(|I|+1)^{-N}(|J|+1)^{-N}R(h)^{\nu}|h| \tag{4.4}$$

where with $\mathscr{V}^* = \mathscr{V}_{1,2} \cup \mathscr{V}_2$ we define

$$\begin{aligned} I &= (i,i') = \{i_v\}_{v\in\mathscr{V}^*} \\ J &= (j,j') = \{j_v\}_{v\in\mathscr{V}^*}. \end{aligned}$$

Now let $C = 9|\mathscr{V}^*|(R(h)+1)$ and distinguish three cases.

(I) $C^2 \geqslant |I|^2|J|$. By Lemma 3.2 we have $|I(\Gamma,\psi)| \leqslant K|h|$ for some constant K, since $|\mathscr{V}| \leqslant 2(n+m)d$. An inductive argument based on (3.14) then gives $|I^T(\Gamma,\psi)| \leqslant K|h|$ for some K, which suffices in this region.

(II) $|I|^2 \geqslant |J|, C^2$. In this region it suffices to show that for any N there is a K such that

$$|I^T(\Gamma,\psi)| \leqslant K(|I|+1)^{-N}|h|.| \tag{4.6}$$

Moreover we can let (σ,σ') be a fixed partition of $\mathscr{V}^*$ and consider only the subregion satisfying $d(\{0\}\cup I_\sigma, I_{\sigma'}) \geqslant |I|/|\mathscr{V}^*|$. In this region we have $\operatorname{supp} f_v \subset \{t: |t| \leqslant |I|/3|\mathscr{V}^*|\}$ and $\operatorname{supp}\rho_{i_v} \subset \{t: |t-i_v| \leqslant |I|/3|\mathscr{V}^*|\}$ (since $\tau \geqslant 2(n+m)d \geqslant |\mathscr{V}_*|$) and thus $\bigcup_{v\in\mathscr{V}_{1,1}} \operatorname{supp} f_v$ and $\bigcup_{v\in\sigma}\operatorname{supp}\rho_{i_v}$ are separated from $\bigcup_{v\in\sigma'}\operatorname{supp}\rho_{i_v}$ by at least $|I|/3|\mathscr{V}^*|$. We now distinguish two subcases. Let $\bar\sigma = \sigma \cup \mathscr{V}_{1,1}$ so that $\bar\sigma\cup\sigma' = \mathscr{V}$ and $\bar\sigma\cap\sigma' = \varnothing$.

(IIa) Γ connects $\bar\sigma$, σ'. By (3.12) we have

$$|\{\rho_{i_v}\}_{v\in\sigma'}|_{N+|\sigma'|} = \mathscr{O}((|I|+1)^{-N}) \tag{4.7}$$

and thus by Lemma 3.3 we have for any N, some K_1.

$$|I(\Gamma,\psi)| \leqslant K_1(|I|+1)^{-N}|h|. \tag{4.8}$$

The bound (4.6) now follows from (3.14) and an inductive argument since any partition Q of $\tilde\Gamma$ must contain a q such that Γ_q connects $\bar\sigma$, σ'. (More precisely we claim that for $q \subseteq \tilde\Gamma$, if Γ_q connects $\bar\sigma$, σ', then $|I^T(\Gamma_q,\psi_q)| = \mathscr{O}(|I|+1)^{-N}|h_q|$ where $h_q = \{h_v\}_{v\in\mathscr{V}_{1,1}\cap\mathscr{V}_q}$, $\mathscr{V}_q$ = vertices of Γ_q. We know that $|I(\Gamma_q,\psi_q)| = \mathscr{O}(|I|+1)^{-N}|h_q|$ by (4.8). Thus the claim holds for $|q| = 1$, and we proceed by finite induction on $|q|$ using (3.14) for $I^T(\Gamma_q,\psi_q)$ and isolating a connecting subgraph for each partition.)

(IIb) Γ does not connect $\bar\sigma$, σ'. In this case Γ splits into two graphs $\Gamma_{\bar\sigma}$, $\Gamma_{\sigma'}$ and we have

$$I^T(\Gamma,\psi) = I(\Gamma,\psi) - I(\Gamma_{\bar\sigma},\psi_{\bar\sigma})I(\Gamma_{\sigma'},\psi_{\sigma'}) - \sum_{Q|\tilde\Gamma}{}' \prod_{q\in Q} I^T(\Gamma_q,\psi_q) \tag{4.9}$$

where the sum is over all partitions of $\tilde\Gamma$ which do not refine $\Gamma_{\bar\sigma}$, $\Gamma_{\sigma'}$. Next $I(\Gamma,\psi)$ is written as $\int\prod_l(4\pi\omega_l)^{-1}\,dp_l\ (\Omega, E(\Gamma,\psi,p)\Omega)$ where $E(\Gamma,\psi,p) = \sum_\pi E((\Gamma,\pi),\psi,p)$ and where for $\gamma = (\Gamma,\pi)$ and $\psi_v = \psi_{v,1}\otimes\psi_{v,2}$ we define $E(\gamma,\psi,p)$ to be

$$E(\{\Omega_v, \psi_{v,1} : \varphi^{r_v} : (e^{-ip_v x}\psi_{v,2})\}_{v\in V}).$$

Then due to the time separation we have for some T (assuming σ' vertices precede $\bar{\sigma}$ vertices)

$$I(\Gamma, \psi) - I(\Gamma_{\bar{\sigma}}, \psi_{\bar{\sigma}})I(\Gamma_{\sigma'}, \psi_{\sigma'}) \\ = \int \prod_l (4\pi\omega_l)^{-1}\, dp_l(\Omega, E(\Gamma_{\bar{\sigma}}, \psi_{\bar{\sigma}}, p_{\bar{\sigma}}, T)E_0^{\perp}E(T, \Gamma_{\sigma'}, \psi_{\sigma'}, p_{\sigma'})\Omega). \tag{4.10}$$

Now by the spectral theorem and the uniform mass gap we have as in Lemma 3.3 with $N_1 = N + |\sigma'|$:

$$\begin{aligned}\|E_0^{\perp}E(T, \gamma_{\sigma'}, \psi_{\sigma'}, p_{\sigma'})\Omega\| &\leqslant m_*^{-N_1}\|H^{N_1}E(T, \gamma_{\sigma'}, \psi_{\sigma'}, p_{\sigma'})\Omega\| \\ &\leqslant K_1 \sum_{\alpha_v:\Sigma\alpha_v = N_1} \|E(T, \{\Omega_v, \psi_{v,1}^{(\alpha_v)} : \varphi^{r_v}: (e^{-ip_v x}\psi_{v,2})\}_{v\in\sigma'})\Omega\| \\ &\leqslant K_2|\{\rho_{i_v}\}_{v\in\sigma'}|_{N_1} \prod_{v\in V_{1,2}\cap\sigma'} |\tilde{\chi}(p_v)|.\end{aligned}$$

We also have

$$\|E(\gamma_{\bar{\sigma}}, \psi_{\bar{\sigma}}, p_{\bar{\sigma}}, T)^*\Omega\| \leqslant K_3 \prod_{v\in V_{1,1}} |\tilde{g}_v(p_v)| \prod_{v\in V_{1,2}\cap\sigma} |\tilde{\chi}_v(p_v)| \prod_{v\in V_{1,1}} |f_v|_{2(n+m)d}$$

and combining these bounds with (4.7) and Lemma 2.3 we obtain

$$|I(\Gamma, \psi) - (\Gamma_{\bar{\sigma}}, \psi_{\bar{\sigma}})I(\Gamma_{\sigma'}, \psi_{\sigma'})| \leqslant K_4(|I| + 1)^{-N}|h|. \tag{4.11}$$

The remaining terms in (4.9) can be bounded by an inductive argument since each partition Q in $\sum_Q'$ must have one q with $\mathscr{V}_q \cap \bar{\sigma} \neq \varnothing$ and $\mathscr{V}_q \cap \sigma' \neq \varnothing$. (In detail we claim that $|I^T(\Gamma_q, \psi_q)| = \mathcal{O}(|I| + 1)^{-N}|h_q|$ for all $q \subseteq \tilde{\Gamma}$ such that $\mathscr{V}_q \cap \bar{\sigma} \neq 0$ and $\mathscr{V}_q \cap \sigma' \neq 0$. By the argument leading to (4.11) we know that

$$(I(\Gamma_q, \psi_q) - I(\Gamma_{\mathscr{V}_q\cap\bar{\sigma}}, \psi_{\mathscr{V}_q\cap\bar{\sigma}})I(\Gamma_{\mathscr{V}_q\cap\sigma'}, \psi_{\mathscr{V}_q\cap\sigma'})| = \mathcal{O}(|I| + 1)^{-N}|h_q|.$$

Thus the claim holds for $|q| = 2$ and we proceed by induction on $|q|$ using (4.9) for $I^T(\Gamma_q, \psi_q)$ and isolating a subgraph with vertices in $\bar{\sigma}$ and σ'.)

(III) $|J| \geqslant C^2, |I|^2$. In this region it suffices to show for any N there is a constant K such that

$$|I^T(\Gamma, \psi)| \leqslant K(|J| + 1)^{-N}|h|. \tag{4.12}$$

As before we consider the subregion $d(\{0\} \cup J_o, J_{\sigma'}) \geqslant |J|/|\mathscr{V}^*|$ in which we have $\bigcup_{v\in\mathscr{V}_{1,1}} \operatorname{supp} g_v$ and $\bigcup_{v\in\sigma} \operatorname{supp} \chi_{j_v}$ separated from $\bigcup_{v\in\sigma'} \operatorname{supp} \chi_{j_v}$ by at least $|J|/3|\mathscr{V}^*|$. We also have from $|J| \geqslant C^2, |I|^2$ that $\bigcup_{v\in\mathscr{V}_{1,1}} \operatorname{supp} f_v$ and $\bigcup_{v\in\mathscr{V}^*} \operatorname{supp} \rho_{i_v}$ are contained in $(-\frac{3}{2}|J|^{1/2}, \frac{3}{2}|J|^{1/2})$.

(IIIa) Γ connects $\bar{\sigma}, \sigma'$. We apply Lemma 3.4 with $L = |J|/3|\mathscr{V}^*|$ and $T = \frac{3}{2}|J|^{1/2}$ and obtain for any N, some K_1.

$$|I(\Gamma, \psi)| \leqslant K_1(|J| + 1)^{-N}|h|. \tag{4.13}$$

An inductive argument now gives (4.12).

(IIIb) Γ does not connect $\bar{\sigma}, \sigma'$. Again we use the representation (4.9). Since $\bigcup_{v\in\bar{\sigma}} \operatorname{supp} \psi_v$ and $\bigcup_{v\in\sigma'} \operatorname{supp} \psi_v$ are space-like separated, the representation (4.10) holds by the locality of the theory. We now appeal to a particular form of some well known space-like cluster properties which follow from the mass gap. In the notation of [3], there are two rectangles $\mathcal{O}_{\bar\sigma}$, $\mathcal{O}_{\sigma'}$ such that supp $\psi_v \subset \mathcal{O}_{\bar\sigma}$ for $v \in \bar\sigma$, etc.; and such that

$$\Delta(\mathcal{O}_{\bar\sigma}, \mathcal{O}_{\sigma'}) = |J|/3|\mathscr{V}^*| - 3|J|^{1/2} \geqslant 3|J|^{1/2}.$$

Thus by equation (4.4) of [3]

$$\begin{aligned}|(\Omega, E(\gamma_{\bar\sigma}, \psi_{\bar\sigma}, p_{\bar\sigma}, T)E_0^{\perp}E(T, \gamma_{\sigma'}, \psi_{\sigma'}, p_{\sigma'})\Omega)| \\ \leqslant K_1 R(h)\exp(-m_* 3|J|^{1/2})(\|HE_{\sigma'}\Omega\| \, \|E_{\bar\sigma}{}^*\Omega\| + \|HE_{\bar\sigma}\Omega\| \, \|HE_{\sigma'}^*\Omega\|) \\ \leqslant K_2(|J| + 1)^{-N} \prod_{v\in V_{1,1}} |\tilde g_v(p_v)| \prod_{v\in V_{1,2}} |\tilde\chi(p_v)| \prod_{v\in V_{1,1}} |f_v|_{2(n+m)d}.\end{aligned}$$

Combining this with Lemma 2.3 we have

$$|I(\Gamma, \psi) - I(\Gamma_{\bar\sigma}, \psi_{\bar\sigma})I(\Gamma_{\sigma'}, \psi_{\sigma'})| \leqslant K_3(|J| + 1)^{-N}|h|. \tag{4.14}$$

The remaining terms in (4.9) are bound by an inductive argument as before.

Corollary 4.2. Given h, m there is a constant M such that $|D^m G_\theta^T(r, h)| \leqslant M$ for all $\theta \in \bar{\mathscr{D}}$, $\lambda \in (0, \lambda_0)$.

Proof. By scaling Theorem 4.1b we have $|D^m G_\theta^T(r, h)| \leqslant M_1 \exp(|\theta| + |\theta|^{-1})$ for θ real and non-zero, $\lambda \in (0, \lambda_0)$. The same bound holds for θ negative imaginary by the path space treatment [2]. Since $D^m G_\theta^T(r, h)$ is analytic in Im $\theta < 0$ and is not too singular at 0 or ∞, one can apply the Phragmen–Lindelhof technique to obtain $|D^m G_\theta^T(r, h)| \leqslant M_1 \exp(2|\theta| + 2|\theta|^{-1})$ for all Im $\theta < 0$, $\lambda \in (0, \lambda_0)$ (see [3] for details).

5. Perturbation Theory

Theorem 5.1. The truncated Green's functions $G^T(h) = G^T(h, \lambda)$, defined and C^∞ on $(0, \lambda_0)$, have C^∞ extensions to $[0, \lambda_0)$. As a consequence they have asymptotic expansions near zero, i.e. for any integer N.

$$|G^T(h, \lambda) - \sum_{k=0}^{N} D^k G^T(h, 0)\lambda^k/k!| = \mathcal{O}(\lambda^{N+1}).$$

Proof. We know that for any k and $\lambda_1, \lambda_2 \in (0, \lambda_0)$, $D^k G^T(\lambda_2) - D^k G^T(\lambda_1) = \int_{\lambda_1}^{\lambda_2} D^{k+1} G^T(\lambda')\, d\lambda'$. Thus by Theorem 4.1 $D^k G^T(\lambda)$ has a continuous extension to $[0, \lambda_0)$, denoted $g_k(\lambda)$, and $g_k(\lambda) = g_k(0) + \int_0^\lambda g_{k+1}(\lambda')\, d\lambda'$. It follows that for all k and all $\lambda \in [0, \lambda_0)$ g_k is differentiable and $Dg_k = g_{k+1}$, hence g_0 is C^∞ and $D^k g_0 = g_k$. Q.E.D.

The point of the theorem is of course that the coefficients $D^k G^T(h, 0)$ can be calculated exactly as we now demonstrate. We define a vacuum graph to be one with $\mathscr{V}_2 = \varnothing$. The Feynman propagator is the usual.

$$\begin{aligned}\Delta_F(t, x) &= \theta(t)\Delta_+(t, x) + \theta(-t)\Delta_+(-t, x) \\ \Delta_+(t, x) &= \int (4\pi\omega)^{-1} \exp(ipx - i\omega t)\, dp.\end{aligned} \tag{5.1}$$

Theorem 5.2

$$D^m G^T(h, 0) = (-i)^m {\sum_{\Gamma}}'' \sum_{i,j} \left(\int dx \prod_{l\in\mathscr{L}} \Delta_F(x_{l_1} - x_{l_2}) \prod_{v\in\mathscr{V}_{1,1}} h_v(x_v) \prod_{v\in\mathscr{V}_{1,2}} (\rho_{i_v} \otimes \chi_{j_v})(x_v) \right) \tag{5.2}$$

where the sum is over all connected vacuum Feynman graphs on n vertices with one leg ($\mathscr{V}_{1,1}$) and m vertices with d legs ($\mathscr{V}_{1,2}$).

Proof. For a vacuum graph the Hilbert space structure in $I(\gamma, h, p)$ disappears and we have for any λ

$$I(\gamma, h, p) = \int \prod_{v \in V_1} e^{-i\omega_v t_v} f_v(t_v) e^{-ip_v x_v} g_v(x_v)\, dt^+\, dx$$

Then by reversing steps in § II we have for $I(\Gamma, h)$.

$$I(\Gamma, h) = \int dx \prod_{l \in \mathscr{L}} \Delta_F(x_{l_1} - x_{l_2}) \prod_{v \in \mathscr{V}_1} h_v(x_v). \tag{5.3}$$

Thus the theorem states that

$$D^m G^T(h, 0) = (-i)^m \sum_{\Gamma}{}'' \sum_{i,j} I(\Gamma, h, \chi(i, j)). \tag{5.4}$$

In this form we see that the sum over i, j is absolutely convergent by (4.4) since $I = I^T$ for a connected graph.

To prove (5.4) we first claim that for $\theta \in \bar{\mathscr{D}}$, $G_\theta(r, h; 0) = \lim_{\lambda \to 0^+} G_\theta(r, h; \lambda)$ exists and is given by

$$G_\theta(r, h; 0) = \sum_{\Gamma}{}' I_\theta(\Gamma, h) \tag{5.5}$$

where the sum is over vacuum graphs. This holds for $\theta = -i$ since from (2.4) we have $\hat{G}(r, h; 0) = \sum_\Gamma' \hat{I}(\Gamma, h)$. We extend to θ negative imaginary by scaling. By Corollary 3.7, $G_\theta(r, h, \lambda)$ is uniformly bounded on $\mathscr{D}$ and so by Vitali's theorem the $\lambda \to 0^+$ limit exists for $\theta \in \mathscr{D}$ and is analytic. Since the limit agrees with the analytic function $\sum_\Gamma' I_\theta(\Gamma, h)$ on a determining set they must be equal, i.e. (5.5) holds for $\theta \in \mathscr{D}$. Now by Corollary 4.2 the functions $G_\theta(r, h; \cdot)$, $\theta \in \mathscr{D}$, are a uniformly equicontinuous family on $(0, \lambda_0)$. It follows that the convergence $G_{\theta_j}(r, h; \cdot) \to G_\theta(r, h; \cdot)$ as $\theta_j \to \theta \in \bar{\mathscr{D}}$ is uniform in $\lambda \in (0, \lambda_0)$ and thus we obtain (5.5) for all $\theta \in \bar{\mathscr{D}}$.

From the representation (5.3) one can now argue inductively that $G^T(r, h; 0) = \sum_\Gamma'' I(\Gamma, h)$ where the sum is over connected vacuum graphs.

Finally by Theorem 4.1 we may take the $\lambda \to 0^+$ limit in (1.6) to obtain $D^m G^T(h, 0) = (-i)^m \sum_{i,j} G^T(1, d; h, \chi(i, j); 0)$. Inserting $G^T(1, d; h, \chi(i, j); 0) = \sum_\Gamma'' I(\Gamma, h, \chi(i, j))$ and interchanging summations gives (5.4).

Remark. If one formally takes $\sum_{i,j} \int dx = \int dx \sum_{i,j}$ and uses $\Sigma_{i_v, j_v} \rho_{i_v} \otimes \chi_{j_v} = 1$, then we have the standard

$$D^m G^T(h, 0) = (-i)^m \sum_{\Gamma}{}'' \left(\int dx \prod_{l \in \mathscr{L}} \Delta_F(x_{l_1} - x_{l_2}) \prod_{v \in \mathscr{V}_{1,1}} h_v(x_v) \right).$$

Our expression (5.2) is one way of making this precise.

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