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The Massive Thirring-Schwinger Model (QED_2): Convergence of Perturbation Theory and Particle Structure

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Abstract. The equivalence of the massive Thirring-Schwinger model in two space-time dimensions (QED_2) on the charge-0-sector and the 'sine-Gordon' theory with a bare mass proportional to the electric charge is reconsidered and rigorously derived from a more general equivalence theorem. It is then shown that for sufficiently large electric charge the Feynman perturbation expansion of the Euclidean Green's functions in the Fermion mass M about $M = 0$ converges. Existence of one particle states (Fermion-anti-Fermion bound states) and nontriviality of the S-matrix are proven. The dependence of the scattering matrix on the charges at infinity is analyzed.

Introduction

In this paper we study the massive Thirring-Schwinger model (QED_2) in two space-time dimensions on the charge 0-sector. This model was previously discussed in References [4], [15], [16], [17] and references given there. It is a model for massive, relativistic Fermions – *formally* described by a Dirac two-spinor 'field' ψ – with current-current and Coulomb self-interaction. The formal Lagrangian of this model is given by

$$\mathscr{L}(\psi) = \mathscr{L}_0(\psi) - \mathscr{L}_I(\psi),$$

with

$$\mathscr{L}_I(\psi) = \mathscr{L}_1(g, e; \psi) + M \cdot N(\bar{\psi}\psi), \tag{0.1}$$

and

$$\mathscr{L}_1(g, e; \psi) = \frac{g}{2} N(j^\mu j_\mu) + \frac{\pi e^2}{2} N(\tilde{j}^0 V_c * \tilde{j}^0)$$

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Here $\mathscr{L}_0(\psi)$ is the Lagrangian of a free Dirac two-spinor field ψ of mass 0; the symbols $N(\bar{\psi}\psi)$ etc. denote suitable normal products which will be given a precise meaning by the constructions of section 1; $j^\mu = N(\bar{\psi}\gamma^\mu\psi)$ is the conserved current. Furthermore $\tilde{j}^0 = j^0 + j_c^0$, where j_c^0 is a formal c-number current specifying charges at infinity; $V_c(x) = \frac{1}{2}|x|$ is the one-dimensional Coulomb potential.

Since the only manifestation of electromagnetism in one space dimension is the Coulomb interaction between charges, the model defined by the Lagrangian (0.1) may be viewed as a model of Quantum Electrodynamics (QED_2) in two space-time dimensions. (In the two-dimensional world the photon is a real particle only if it is *massive*. Massive QED_2 has been discussed in [17].)

The interaction $N(\tilde{j}^0 V_c * \tilde{j}^0)$ is extremely long range. As a consequence, electric charge is completely confined, i.e. there are no charged physical states, at least for $g > 0$ and M^2/e^2 sufficiently small. (This is shown in Section 2.) Under these circumstances the charged fields ψ and $\bar{\psi}$ may not be well-defined. In any event, they do *not* have a well-defined time-evolution. (This situation changes for $g > g_{\text{critical}} \geqslant 0$ and M^2/e^2 sufficiently large: then there exist Poincaré covariant charged sectors of charge $\pm q \neq 0$.)

It is a common experience that in constructive quantum field theory models for interacting Fermi fields are much more difficult to analyze than models of interacting (scalar) Bose fields. One reason for this is that there is no natural Euclidean *field* theory for Fermions, whereas much of the success in constructing relativistic, interacting Bose fields is due to the existence of a powerful Euclidean *field* theory formalism for Bosons; see [11, 37]. A second reason for these difficulties is that Fermi field theories tend to have more serious ultraviolet divergencies than theories of (scalar or neutral vector) Bose fields. The theory formally defined in (0.1) is *renormalizable* but not *superrenormalizable*, and so far there are no techniques allowing for a direct construction of non-superrenormalizable models.

The basic idea of how to avoid all these difficulties and the starting point of this paper is to map the Fermi field theory formally defined in (0.1) isomorphically onto a quantum field theory involving only scalar Bose fields; [3, 4, 37]. The fact that, in our case, it is possible to realize this idea and make it a powerful tool in the analysis of QED_2 is a two-dimensional 'miracle': In two space-time dimensions neutral, local bilinear composite fields formed out of ψ and $\bar{\psi}$ can be identified with local functions of a scalar, neutral Bose field (in this connection cf. [7, 8, 27]).

We propose to prove

Theorem 1: For $g > 0$ and on the charge-0-sector QED_2, formally defined by (0.1), is equivalent to the *sine-Gordon theory* with a *non-vanishing mass term* defined by the following formal Lagrangian:

$$\mathscr{L}(\varphi) = \mathscr{L}_0(\varphi) - \mathscr{L}_I(\varphi),$$

where $\mathscr{L}_0(\varphi)$ is the Lagrangian of a free, neutral, scalar field φ with mass m_0, and

$$\mathscr{L}_I(\varphi) = \lambda :\cos(\varepsilon\varphi(x, t) + \theta):_1. \tag{0.2}$$

(Here $:\text{——}:_1$ denotes normal ordering with respect to bare mass 1; (see e.g. [17, 37]). More precisely: If one identifies

$$j^\mu \quad \text{with} \quad -\frac{\varepsilon}{2\pi}\,\varepsilon^{\mu\nu}\,\partial_\nu\varphi$$

$$\begin{aligned} N(\bar{\psi}\psi) &\quad \text{with} \quad :\cos(\varepsilon\varphi + \theta):_1, \quad \theta \in [0, 2\pi) \\ iN(\bar{\psi}\gamma_5\psi) &\quad \text{with} \quad -:\sin(\varepsilon\varphi + \theta):_1, \end{aligned} \tag{0.3}$$

and

$$\frac{\pi}{2}:j^0 V_c * j^0: \quad \text{with} \quad \frac{\varepsilon^2}{4\pi}\frac{1}{2}:\varphi^2:$$

('Schwinger mechanism', [43]), and if one sets

$$m_0^2 = \frac{e^2}{1 + g/\pi}, \qquad \lambda = M, \qquad \frac{\varepsilon^2}{4\pi} = \frac{1}{1 + g/\pi} \tag{0.4}$$

then the Wightman functions of the two models are *identical*. (The interpretation of the so far unspecified angle $\theta \in [0, 2\pi)$ is described in Section 1.)

We give a rigorous proof of Theorem 1 in Section 1, anticipating the results of Section 2 (existence and Wightman axioms for the sine-Gordon theory; see also [17]).

Technically the main ingredients for our arguments in Section 1 are

(*a*) A lemma of Cauchy's asserting that

$$\frac{\prod_{1 \leqslant i < j \leqslant n} |z_i - z_j|^\alpha |w_i - w_j|^\alpha}{\prod_{i,j=1}^{n} |z_i - w_j|^\alpha} = \left|\operatorname{Det}\left[\frac{1}{z_i - w_j}\right]\right|^\alpha$$

where the z_i's and w_j's are complex numbers (and $0 \leqslant \alpha \leqslant 2$), and

(*b*) Bogoliubov transformations.

In the course of the proof we find it convenient to view QED$_2$ as the limit of massive QED$_2$ (discussed in [17]) as the mass of the photon tends to 0.

In Section 2 we prove

Theorem 2: The Feynman perturbation expansion of the Euclidean Green's functions (EGF's) of the sine-Gordon theory (0.2) in λ ($\equiv M$; for given $\varepsilon^2/4\pi \equiv (1 + g/\pi)^{-1} < 1$) about $\lambda = 0$ converges, provided m_0^2 ($\equiv e^2(1 + g/\pi)^{-1}$) is large enough.

The proof of Theorem 2 is based on the cluster expansion of Glimm, Jaffe and Spencer [21] and a result of Dimock [6]. Although, following the strategy of [21], one can quite easily derive many different versions of formal cluster expansions, some of which may possibly be more appropriate for the analysis of the sine-Gordon theory than the one of [21], we apply the expansion developed in detail in [21, 38]. This keeps our analysis reasonably short. Because of the *non-polynomial sine-Gordon action* $-\lambda:\cos(\varepsilon\varphi + \theta):_1$ – the application of the cluster expansion of [21] originally designed for the $P(\varphi)_2$-models to sine-Gordon is however slightly more than an exercise and requires some new estimates on Gaussian integrals and a more careful combinatorial analysis.

In Section 3 we prove

Theorem 3: For given $\varepsilon^2/4\pi < 1$ and for $|\lambda/m_0^2|$ sufficiently small (depending on ε) the energy-momentum spectrum of the sine-Gordon theory has an isolated one particle shell of mass $m > 0$.

The proof of this result is based on a powerful expansion of Spencer's [39] that yields decay estimates for one particle irreducible Green's functions. Again the non-polynomial character of $:\cos(\varepsilon\varphi + \theta):_1$ requires a number of somewhat involved combinatorial arguments that form the technical core of Section 3.

In Section 4 we prove the nontriviality of the scattering matrix of the sine-Gordon theory and investigate its dependence on the angle θ (or, in QED_2-language, the dependence of scattering amplitudes of QED_2 on the charges at infinity). The proof for $S \neq I$ is based on results of [13]; see also [34]. One proves that perturbation theory in λ about $\lambda = 0$ is asymptotic to the scattering amplitudes and then checks that lowest order perturbation theory is nontrivial. The result that $S \neq I$ can then be extended to larger values of $|\lambda|$ by combining Theorems 2, 3 with [13] and applying an edge of the wedge type argument.

Finally we suggest that certain kernels important for the analysis of elastic two body scattering have actually a convergent perturbation expansion in λ about $\lambda = 0$.

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Equivalence of QED_2 and Massive 'sine-Gordon' Theory; (Theorem 1)

In this section we prove Theorem 1 of the introduction, (assuming existence and Wightman axioms for the sine-Gordon theory). The strategy of our proof of Theorem 1 is as follows:

Step 1: Prove Theorem 1 for $\lambda = M = 0$. In this case both models can be solved explicitly, and the proof is quite straightforward.

Step 2: Prove that the perturbation expansions of the EGF's (Schwinger functions of the fields j^μ, $N(\bar{\psi}\psi), \ldots, -(\varepsilon/2\pi)\varepsilon^{\mu\nu}\,\partial_\nu\varphi, :\cos(\varepsilon\varphi + \theta):_1, \ldots$, respectively, in M, λ, respectively, about $M = \lambda = 0$, are identical to all orders in $M = \lambda$.

Step 3: For $e^2 > 0$ and small M the perturbation expansion of the EGF's in M converges. (This result was announced in [15]; in Section 2 we present a detailed proof; see Theorem 2 of the Introduction.)

(*a*) *Step 1*: Suppose first that $g = e = M = 0$. Then the theory with Lagrangian $\mathscr{L}_0(\psi) = \mathscr{L}(\psi)$ has two conserved currents

$$j^\mu \text{ and } j_5^\mu = N(\bar{\psi}\,\gamma^\mu\gamma_5\psi). \tag{1.1}$$

(When doing explicit calculations we may use the representation $\gamma_0 = \sigma_x$, $\gamma_1 = i\sigma_y$, $\gamma_5 = -\sigma_z$, where the σ's are the Pauli matrices.) In two space-time dimensions

$$j_5^\mu = \varepsilon^{\mu\nu} j_\nu \tag{1.2}$$

and this combined with current conservation yields

$$\Box j^\mu = 0. \tag{1.3}$$

From (1.1) to (1.3) one can conclude that

$$j^\mu = Z\varepsilon^{\mu\nu}\,\partial_\nu\varphi \tag{1.4a}$$

or

$$j^\mu = Z\,\partial^\mu\varphi \tag{1.4b}$$

where φ is a neutral scalar massless field. Direct computation shows

$$\langle j^\mu(x) j^\nu(y)\rangle_0 = \frac{1}{\pi}\,\varepsilon^{\mu\rho}\,\varepsilon^{\nu\sigma}\,\langle\partial_\rho\varphi(x)\,\partial_\sigma\varphi(y)\rangle_0$$

and

$$\langle\prod_{i=1}^{n} j^{\mu_i}(x_i)\rangle_0^T = 0 \quad \text{for all } n > 2$$

which rules out (1.4b) and shows that (1.4a) is correct with

$$Z = -\frac{1}{\sqrt{\pi}} \tag{1.5}$$

(the sign is purely conventional). We define the fields

$$\sigma_e(x) = \tfrac{1}{2}{:}\bar{\psi}(1 + e\gamma_5)\psi{:}, \quad e = \pm 1 \tag{1.6}$$

We denote the EGF's of the fields σ_e by $S(x_1, e_1, \ldots, x_n, e_n)$. Then a direct calculation shows that

$$S(x_1, e_1, \ldots, x_n, e_n) = 0 \qquad \text{unless} \qquad \sum_{i=1}^{n} e_i = 0 \tag{1.7}$$

(which follows also from chiral invariance), and

$$S(x_1, 1, \ldots, x_n, 1, x_1', -1, \ldots, x_n', -1) = \left|\operatorname{Det}\left(\frac{1}{z_i - z_j'}\right)\right|^2 \tag{1.8}$$

where we have set $x = (\mathbf{x}, t)$ and $z = \mathbf{x} + it \in \mathbb{C}$.

It is shown in [44] that ${:}e^{\pm i\alpha\varphi}{:}_1(x)$ is a quantum field satisfying all Wightman axioms [26, 42] if φ is a free, neutral, scalar field of mass 0. In [16, 17] we have calculated the EGF's of the fields: $e^{\pm i(2\sqrt{\pi}\varphi+\theta)}{:}_1$. These are Gaussian expectation values of Euclidean fields [32, 37] denoted by $\chi_{\pm 1}^{\theta}$ (or also ${:}e^{\pm i(2\sqrt{\pi}\varphi+\theta)}{:}_1$; in our notation we do not distinguish between the relativistic and the Euclidean scalar field, and $\langle\text{——}\rangle_0$ denotes both the free v.e.v. and the free, Gaussian expectation value). The result is:

$$\langle\prod_{i=1}^{n}\chi_{e_i}^{\theta}(x_i)\rangle_0 = 0, \qquad \text{unless} \qquad \sum_{i=1}^{n} e_i = 0,$$

and

$$\langle\prod_{i=1}^{n}\chi_1^{\theta}(x_i)\prod_{j=1}^{n}\chi_{-1}^{\theta}(x_j')\rangle_0 = \frac{\prod_{1\leqslant i<j\leqslant n}|x_i - x_j|^2|x_i' - x_j'|^2}{\prod_{i,j=1}^{n}|x_i - x_j'|^2}, \tag{1.9}$$

see [3, 16]. By a lemma of Cauchy's, see, e.g., [10], the r.h.s. of (1.9) is equal to

$$\left|\operatorname{Det}\left[\frac{1}{z_i - z_j'}\right]\right|^2.$$

Therefore

$$S(x_1, 1, \ldots, x_n', -1) = \langle\prod_{i=1}^{n}\chi_1^{\theta}(x_i)\chi_{-1}^{\theta}(x_i')\rangle_0 \tag{1.10}$$

This equation tells us that S is the expectation value of a product of Euclidean fields, denoted $\sigma_\pm$, and we may identify $\sigma_\pm$ with $\chi^\theta_{\pm 1}$. The choice of θ is irrelevant at this point, due to chiral invariance. We define

$$c = \tfrac{1}{2}(\chi^\theta_{+1} + \chi^\theta_{-1}) = \tfrac{1}{2}(\sigma_+ + \sigma_-),$$

and

$$s = \frac{1}{2i}(\chi^\theta_{+1} - \chi^\theta_{-1}) = \frac{1}{2i}(\sigma_+ - \sigma_-) \tag{1.11}$$

Then

$$[j^\mu(x), c(y)] = \varepsilon^{\mu\nu}\,\partial_\nu D(x-y)s(y), \tag{1.12}$$

and

$$[j^\mu(x), s(y)] = -\varepsilon^{\mu\nu}\,\partial_\nu D(x-y)c(y), \tag{1.13}$$

where D is the commutator function of the field φ. Equations (1.12) and (1.13) are a straightforward computation which we leave to the reader.

Next we consider the case where $e \neq 0$, but $g = M = 0$. This model is the Schwinger model [43]. It has been discussed extensively in the literature; see e.g. [29, 43, 45], and references given there. Rather than giving independent arguments at this point we just quote the main result about the Schwinger model on the charge 0-sector:

$$\langle j^\mu(x) j^\nu(y)\rangle_e = \frac{1}{\pi}\,\varepsilon^{\mu\rho}\,\varepsilon^{\nu\sigma}\langle\partial_\rho\varphi(x)\,\partial_\sigma\varphi(y)\rangle_e, \tag{1.14}$$

where now φ is a free, neutral, scalar field of mass $m_0 = |e|$, and $\langle\text{——}\rangle_e$ denotes v.e.v.'s in the Schwinger model. Moreover

$$\langle\prod_{j=1}^{n} j^{\mu_j}(x_j)\rangle_e^T = 0, \quad \text{for all } n > 2.$$

Therefore we can make the identification

$$j^\mu = -\frac{1}{\sqrt{\pi}}\,\varepsilon^{\mu\nu}\,\partial_\nu\varphi. \tag{1.15}$$

It is known that in the Schwinger model ($e \neq 0, g = M = 0$) the currents j^μ, c and s do not have anomalous dimensions and that the commutation relations (1.12) and (1.13) remain unchanged when passing from $e = 0$ to $e = \pm m_0$.

We now note that the operators $\{e^{ij^0(f)}, e^{ij^1(g)}\}$ with f and g in $\mathscr{S}_{\text{real}}(\mathbb{R}^1)$ generate an irreducible algebra of bounded operators on the Fock space of the free field φ. Therefore, in principle, the commutation relations (1.12) and (1.13) ought to suffice to determine the currents σ_+, σ_-, and c, s (up to normalization and an orthogonal transformation of the doublet (c, s)). The result of such reasoning is easily seen to be

$$\begin{aligned} c(x) &= Z{:}\cos(2\sqrt{\pi}\varphi(x) + \theta){:}_1 \\ s(x) &= Z{:}\sin(2\sqrt{\pi}\varphi(x) + \theta){:}_1, \end{aligned} \tag{1.16}$$

for some arbitrary $\theta \in [0, 2\pi)$. Conventional normalization yields $Z = 1$.

Although (1.16) is, formally, the most general solution of the commutation relations (1.12), (1.13) on the Fock space of φ, in practice one encounters domain problems. Such problems are one of the reasons why the Schwinger model is not entirely trivial. Fortunately the result (1.16) turns out to be correct.

From (1.16) we conclude:

$$\sigma_{\pm}(x) = :e^{\pm i(2\sqrt{\pi}\varphi(x)+\theta)}:_1. \tag{1.17}$$

It is no longer true that

$$\langle \prod_{i=1}^{n} \sigma_{e_i}(x_i)\rangle_e = 0, \qquad \text{unless} \qquad \sum_{i=1}^{n} e_i = 0,$$

since for $e \neq 0$ chiral invariance is broken.

(Formally, all these results follow from equations (1.2) and (1.8) by inserting $j^0 = (-1/\sqrt{\pi})\,\partial_x\varphi$ into the interaction term $\pi e^2/2{:}j^0 V_c * j^0{:}$ and then doing two integrations by part. Even though this argument is purely heuristic one can make it *precise*: In [17] it is indicated how to construct massive QED_2, i.e., a model with a $\sqrt{\pi}\, e j^\mu A_\mu$-coupling, where A_μ is a neutral vector boson of bare mass m and j^μ is the current of a Dirac two-spinor field. If the bare mass M of the Dirac field is 0 we know from (1.4), (1.10) and (1.11) that $j^\mu = (1/\sqrt{\pi})\varepsilon^{\mu\nu}\,\partial_\nu\varphi$, and $\sigma_\pm = :e^{\pm i(2\sqrt{\pi}\varphi+\theta)}:_1$. For $M = 0$ this model can be solved explicitly by integrating out the Euclidean vector field associated with A_μ in the EGF's of the currents j^μ, σ_+ and σ_-. We find that these EGF's are moments of a Gaussian measure $d\mu_{C_m}(\varphi)$ on $\mathscr{S}'_{\text{real}}(\mathbb{R}^2)$ with mean 0 and covariance C_m, the momentum space representation of which is given by $[k^2(1 + e^2/(k^2 + m^2))]^{-1}$, $k \neq 0$. It is easy to show that, as $m \searrow 0$, these EGF's converge in $\mathscr{S}'$, and their limits are moments of a Gaussian measure $d\mu_{C_0}(\varphi)$ with mean 0 and covariance $C_0 = (-\Delta + e^2)^{-1}$. In this limit A_μ loses its dynamical degrees of freedom. This *proves* (1.15)–(1.16).)

Let us now comment on the meaning of the angle θ. The formal Lagrangian (1.1) with $M = g = 0$ seems to be chirally invariant. But in fact, the different versions of the explicit solution do not possess chiral invariance (see [29, 43, 45]). There is always a one-parameter family of solutions connected by chiral transformations which transform

$$N(\bar\psi\psi) \to \cos\theta\, N(\bar\psi\psi) + \sin\theta\, N(i\bar\psi\gamma_5\psi)$$
$$N(i\bar\psi\gamma_5\psi) \to -\sin\theta\, N(\bar\psi\psi) + \cos\theta\, N(i\bar\psi\gamma_5\psi)$$

Specifying one particular solution fixes the angle θ in the equivalence; if we require a parity symmetric vacuum it can be seen that θ has to be $n\pi (n \in \mathbb{Z})$, since this requires $(\Omega, N(\bar\psi\gamma_5\psi)\Omega) = 0$ (cf. [29]). Coleman, Jackiw and Susskind [4] have given a physical interpretation of the parameter θ: It corresponds to specifying charges at infinity or equivalently a constant electric field. This can easily be read off the equivalence (0.3): A chiral transformation corresponds to a shift of φ by a constant, and φ is linearly related to the electric field.

These considerations also make clear that we are *not* dealing with a spontaneously broken symmetry in the standard sense: Adding a constant to φ does *not* induce an automorphism of the algebra of observables which commutes with space-time translations; the corresponding current $j^5_\mu = (1/\sqrt{\pi})\,\partial_\mu\varphi$ is *not* conserved. We could of course restrict our algebra of observables to the one generated by the chirally invariant current j_μ; then we have such an automorphism, but it is the trivial one.

So there is no conflict with the general theorem asserting that a continuous symmetry cannot be spontaneously broken in two dimensions, ([5]; D. Maison (private communication) gave a very short and elegant proof; the result is also implicit in the work of Ezawa and Swieca [14]).

Next we consider the case where $e \neq 0$, $g > 0$, $M = 0$, and here we present a complete argument: We introduce an ultraviolet cutoff in the current j^μ:

$$\begin{aligned} j_\kappa^0(\mathbf{x}) &= -\frac{1}{\sqrt{\pi}} \int d\mathbf{y} h_\kappa(\mathbf{x} - \mathbf{y})\, \partial_\mathbf{y} \varphi(\mathbf{y}), \\ j_\kappa^1(\mathbf{x}) &= \frac{1}{\sqrt{\pi}} \int d\mathbf{y} h_\kappa(\mathbf{x} - \mathbf{y}) \pi(\mathbf{y}), \end{aligned} \tag{1.18}$$

where π is the momentum operator canonically conjugate to φ and $\{h_\kappa\}$ is a sequence of symmetric test functions converging in $\mathscr{S}'$ to δ, as $\kappa \to \infty$. Obviously

$$\tfrac{1}{2} \int :j_\kappa^\mu j_{\kappa,\mu}:(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x} = \frac{1}{2\pi} \int [:(\partial_\mathbf{x} \varphi_\kappa)^2:(\mathbf{x}) + :\pi_\kappa^2:(\mathbf{x})] g(\mathbf{x})\, d\mathbf{x},$$

and the r.h.s. is *quadratic* in φ and π. Therefore the total Hamiltonian of the theory with $e \neq 0, g > 0$ and $M = 0$ is quadratic in φ and π and hence it can be diagonalized by a *Bogoliubov transformation.* The EGF's of the currents j^μ and $\sigma_\pm$ can then be calculated explicitly. One readily finds that they are the expectation values of products of $(1/\sqrt{\pi})\, \partial_\mathbf{x}\varphi, (i/\sqrt{\pi})\, \partial_t \varphi, :e^{\pm i(2\sqrt{\pi}\varphi + \theta)}:_1$ with respect to the Gaussian measure $d\mu_{C_{\kappa,g}}$ on $\mathscr{S}'_{\text{real}}(\mathbb{R}^2)$ with mean 0 and covariance

$$C_{\kappa,g} = (-\Delta + K_{\kappa,g} + m_0^2)^{-1},$$

where $\mathrm{K}_{\kappa,g}$ is the integral operator with kernel

$$-\frac{1}{\pi}[\partial_t^2 \int h_\kappa(\mathbf{x} - \mathbf{y}) g(\mathbf{y}) h_\kappa(\mathbf{y} - \mathbf{z})\, d\mathbf{y} + \int \partial_\mathbf{x} h_\kappa(\mathbf{x} - \mathbf{y}) g(\mathbf{y})\, \partial_\mathbf{z} h_\kappa(\mathbf{y} - \mathbf{z})\, d\mathbf{y}].$$

As $\kappa \to \infty$ and $g(\mathbf{x}) \to g$ (=const.)

$$\int_{\mathscr{S}'} d\mu_{C_{\kappa,g}}(\varphi) \prod_{i=1}^n \varphi(x_i) \to \int_{\mathscr{S}'} d\mu_C(\varphi) \prod_{i=1}^n \varphi(x_i),$$

where

$$C = (-(1 + g/\pi)\Delta + m_0^2)^{-1}, \qquad \text{in} \quad \mathscr{S}'(\mathbb{R}^{2n}).$$

This completes the discussion of the model for $e \neq 0$, $g > 0$, and $M = 0$. Next we must investigate the case where $M \neq 0$.

(*b*) *Step 2* of the proof: From (1.6), (1.16) and (1.17) we have

$$MN(\bar{\psi}\psi)(x) = M{:}\cos(2\sqrt{\pi}\, \varphi(x) + \theta){:}_1 \tag{1.19}$$

From [16, 17] we know that the theory with Lagrangian density

$$\mathscr{L}(\varphi) = \tilde{\mathscr{L}}_0(\varphi) - M{:}\cos(2\sqrt{\pi}\, \varphi(x) + \theta){:}_1, \tag{1.20}$$

where $\tilde{\mathscr{L}}_0(\varphi) = \frac{1}{2}(1 + g/\pi){:}\,\partial^\mu \varphi\, \partial_\mu \varphi{:}(x) - \frac{1}{2} m_0^2 {:}\varphi^2{:}(x)$ (that is the free Euclidean field has the two point function $C(x - y) \equiv$ kernel of $(-(1 + g/\pi)\Delta + m_0^2)^{-1}$) has a well-defined perturbation theory in M, *for all* $g > 0$. (All terms in the perturbation series of the EGF's in powers of M are well-defined; [16, 17].) By a finite field strength renormalization the theory obtained from (1.20) is seen to be equivalent to the one with Langrangian density

$$\mathscr{L}(\varphi) = \mathscr{L}_0(\varphi) - M{:}\cos(\varepsilon\varphi(x) + \theta){:}_1, \tag{1.21}$$

where now $\mathscr{L}_0$ is the free Lagrangian of a neutral, *canonical*, scalar field φ of mass $m_0/\sqrt{(1 + g/\pi)}$, and $\varepsilon^2 = 4\pi/(1 + g/\pi) < 4\pi$.

Convergence of perturbation theory for the EGF's in M at $M = 0$ for small $|M|$ is proven in the next section. By other methods it has been proven in [16, 17] that for all $\varepsilon^2 = 4\pi/(1 + g/\pi) < 4\pi$ and all real M the theory obtained from the Lagrangian (1.21) exists in the infinite volume limit and satisfies all Wightman axioms with the possible exception of uniqueness of the vacuum.

The proof of Theorem 1 is now complete.

Since isomorphisms of the type described by Theorem 1 have recently attracted a great deal of attention, [3, 30, 31, 41] we want to add some comments about the motivation for presenting our own proof: We feel that the use of Euclidean methods to prove the isomorphism asserted in Theorem 1 is somewhat novel and yields simple arguments. The observation that the commutation relations (1.12) and (1.13) combined with (1.15) determine the currents σ_+ and σ_- uniquely, up to the choice of θ, (and modulo domain questions), and that the *Schwinger model* may quite easily be obtained as the *limit of massive* QED_2 (for which Theorem 1 is almost obvious), as the mass of the vector boson tends to 0, appear to be new, too. This method seems to be quite powerful. It can be used to prove many more isomorphisms of the type asserted in Theorem 1. As two examples representative for many, we mention: (a) The Gross-Neveu model[3] [23] can be reformulated as a pure Bose theory with charged super-selection sectors. (b) Let χ be a neutral, scalar Bose field; the Yukawa model with interaction $\lambda{:}\bar{\psi}\psi{:}\chi + g/2{:}j^\mu j_\mu{:}$ in two dimensions can be mapped isomorphically onto a pure Bose theory with charged super-selection sectors. This model is, for $g > 0$, a more regular version of the conventional Yukawa model ($g = 0$). It is doubtful, however, whether this isomorphism is of any use in the construction of this model.

2. Convergence of the Perturbation Series of the EGF's in λ, for $m_0^2 > 0$

(*a*) *Introduction: The Main Result*

From now on we formulate our results in the language of the massive sine-Gordon theory formally defined by the Lagrangian (0.2). Theorem 1 can be used as a dictionary to translate these results into the language of the massive Thirring-Schwinger model. In this section we intend to prove that the perturbation expansion of the Euclidean Green's functions in λ converges, provided $|\lambda/m_0^2|$ is sufficiently small. Our proof is based on the Glimm-Jaffe-Spencer cluster expansion [21] and a result of Dimock's [6] which says that for a scalar Bose field with Lagrangian self-interaction for which the cluster expansion converges the n[th] derivative of an EGF in the coupling constant λ at $\lambda = 0$ is given by standard perturbation theory. (By the results of [16, 17] all terms in the perturbation series of an arbitrary EGF in λ are well-defined.)

Convergence of the perturbation series of scattering amplitudes does not follow from our present methods, though it does follow from the results of this paper and of ref. [13] (cf. also [34]) that perturbation theory is asymptotic to the scattering

[3]) For a suitable range of bare coupling constants.

amplitudes at $\lambda = 0$. More information about scattering in the sine-Gordon theory is contained in Sections 3, 4.

The main result of this section is

Theorem 2.1 (= Theorem 2 of the Introduction): Given $m_0^2 > 0$ and $\varepsilon^2 < 4\pi$ there exists a $\lambda_0(\varepsilon)$ such that for $|\lambda| < \lambda_0(\varepsilon)m_0^2$ the Feynman perturbation series in λ for the EGF's of the massive sine-Gordon theory converges. For real λ, the EGF's obtained by summation of the perturbation series uniquely determine a relativistic quantum field theory satisfying *all* Wightman axioms, and the energy-momentum spectrum has a positive mass gap.

Remarks. The basic ingredient for the proof of Theorem 2.1 is the cluster expansion, [21]. It was originally developed for the well known $P(\varphi)_2$-models discussed, e.g., in [11, 37]. Here we show that the estimates of [16, 17] permit us to apply the cluster expansion to the massive sine-Gordon theory, as well. In the following we briefly sketch the main strategy of the cluster expansion and then describe the modifications relative to the $P(\varphi)_2$ models allowing us to apply it to the massive sine-Gordon theory. It is known (see e.g. [16, 17]) that the massive sine-Gordon theory in the Euclidean region (for $\theta = 0$) is isomorphic to the *classical, two component Yukawa gas* in the grand canonical ensemble. The perturbation expansion of EGF's in λ is the same as the activity expansion of the correlation functions of this classical gas. This suggests that it should be possible to find a (nontrivial!) modification of Ruelle's elegant fixed point methods, see, e.g., [35], to prove convergence of the perturbation expansion in λ. This is not attempted in the present paper.

Throughout the following we assume $m_0 \geqslant 1$ (this is no loss of generality because we can always perform a scaling transformation).

Definition. The Euclidean action of the sine-Gordon theory in a bounded space-time region Λ is defined by

$$U_\Lambda(\varphi) = -\int_\Lambda d^2x \, {:}\cos(\varepsilon\varphi(x) + \theta){:}_{m_0} \tag{2.1}$$

The space-time cutoff EGF's of this model are given by

$$S_\Lambda^\lambda(x_1, \ldots, x_n) = \frac{\langle \prod_{i=1}^n \varphi(x_i) e^{-\lambda U_\Lambda(\varphi)} \rangle_0}{\langle e^{-\lambda U_\Lambda(\varphi)} \rangle_0}, \tag{2.2}$$

where $\langle \text{——} \rangle_0$ denotes expectation with respect to the Gaussian measure $d\mu_0(\varphi)$ on $\mathscr{S}' = \mathscr{S}'_{\text{real}}(\mathbb{R}^2)$ with mean 0 and covariance $(-\Delta + m_0^2)^{-1}$. From [16, 17] we infer that these definitions are meaningful, if Λ is a bounded, open set in $\mathbb{R}^2$ (e.g., a rectangle) and λ is real. Convergence of the cluster expansion for some (real or complex) λ leads to the existence of the infinite volume limit (independent of boundary conditions)

$$\lim_{\Lambda \nearrow \mathbb{R}^2} S_\Lambda^\lambda(x_1, \ldots, x_n) = S^\lambda(x_1, \ldots, x_n) \tag{2.3}$$

(e.g., in the sense of complex measures on $\mathbb{R}^{2n}$; see [21] Theorem 2.2.2 and below). Furthermore the convergence of the cluster expansion transfers analyticity properties of the finite volume Schwinger functions to the infinite volume objects. It is here that the basic difference between $P(\varphi)_2$ and sine-Gordon interaction comes in: In the sine-Gordon model numerator and denominator of (2.2) are entire in λ; the quotient is analytic in a disc around the origin; in $P(\varphi)_2$, on the other hand, numerator and denominator of (2.2) only exist as they stand for *Re* $\lambda \geqslant 0$; (2.2) is *not* analytic at $\lambda = 0$. In fact perturbation theory is known to diverge [25].

One essential ingredient for the convergence of the cluster expansion is an inequality of the form

$$\tfrac{1}{2} \leqslant |Z^\lambda(\Delta)| \leqslant \tfrac{3}{2} \tag{2.4}$$

where

$$Z^\lambda(\Delta) = \langle e^{-\lambda U_\Delta(\varphi)} \rangle_0 \tag{2.5}$$

(Δ is a unit square in $\mathbb{R}^2$). For sufficiently large m_0 (2.4) is sufficient to prove

$$e^{-a|\Lambda|} \leqslant |Z^\lambda(\Lambda)| \leqslant e^{b|\Lambda|} \tag{2.6}$$

for some positive constants a, b ($|\Lambda|$ is the volume of Λ).

In the sine-Gordon model (2.4) holds for all λ with $|\lambda| < \lambda_1(\varepsilon)$, for some $\lambda_1(\varepsilon) > 0$, uniformly in m_0, for $m_0 \geqslant 1$. We only have to choose $\lambda_1(\varepsilon)$ such that

$$\sum_{n=1}^{\infty} \frac{\lambda_1(\varepsilon)^n}{n!} \langle U_\Delta(\varphi)^n \rangle_0 \leqslant \tfrac{1}{2}, \quad \text{for } m_0 = 1, \tag{2.7}$$

which is possible since $\langle e^{-\lambda U_\Delta} \rangle_0$ is entire in λ as shown in [16, Section 3] and [17, Section IV]. For $m_0 \geqslant 1$ (2.7) is a consequence of conditioning (see [24], also [16, 17]).

(*b*) *The Cluster Expansion: Basic Ideas*

First we introduce a new set of Euclidean fields in terms of which the cluster expansion can be formulated more conveniently and the EGF's of which determine the sine-Gordon theory completely: We define

$$c_\vartheta(x) = :\cos(\varepsilon\varphi(x) + \vartheta):_{m_0}, \quad \vartheta \in [0, 2\pi). \tag{2.8}$$

From the point of view of the massive Thirring-Schwinger model these fields are actually much more natural than the field φ.

A standard application of integration by parts on function space, see, e.g., [9], shows that the EGF's S_Λ^λ of the field φ can be expressed in terms of EGF's of the fields c_ϑ and that, as a consequence, the convergence asserted in (2.3) follows from the convergence of

$$S_\Lambda^\lambda(\{x, \vartheta\}_N) = \frac{\langle \prod_{j\in N} c_{\vartheta_j}(x_j)\, e^{-\lambda U_\Lambda(\varphi)} \rangle_0}{\langle e^{-\lambda U_\Lambda(\varphi)} \rangle_0}, \tag{2.9}$$

as $\Lambda \nearrow \mathbb{R}^2$, (in the sense of complex measures on $\mathbb{R}^{2|N|}$). Indeed (2.9) is more general than (2.3): It would suffice to set $\vartheta_j = \theta$, or $\theta + \pi/2$, for all $j \in N$.

For the proof of convergence of (2.9), as $\Lambda \nearrow \mathbb{R}^2$, the cluster expansion proposes the following procedure:

Cover $\mathbb{R}^2$ with a cubic lattice $\mathscr{L}$ with lattice constant 1. Let $\mathscr{B}$ denote the set of all bonds of $\mathscr{L}$. Let $C_\mathscr{B}$ be the operator $(-\Delta_\mathscr{B} + m_0^2)^{-1}$, where $\Delta_\mathscr{B}$ is the Laplacian with 0-Dirichlet data on $\mathscr{B}$. The Gaussian measure on $\mathscr{S}'$ with mean 0 and covariance $C_\mathscr{B}$ is denoted $d\mu_{C_\mathscr{B}}$ and $\langle \text{——} \rangle_{C_\mathscr{B}}$ is expectation with respect to $d\mu_{C_\mathscr{B}}$. The measure $d\mu_{C_\mathscr{B}}$ decouples regions that are separated by the bonds of $\mathscr{B}$ completely. We set

$$S^{\lambda}_{\Lambda, C_{\mathscr{B}}}(\{x, \vartheta\}_N) = \frac{\langle \prod_{j \in N} c_{\vartheta_j}(x_j) e^{-\lambda U_\Lambda(\varphi)} \rangle_{C_{\mathscr{B}}}}{\langle e^{-\lambda U_\Lambda(\varphi)} \rangle_{C_{\mathscr{B}}}}, \tag{2.10}$$

Obviously $S^{\lambda}_{\Lambda, C_{\mathscr{B}}}$ is *independent* of Λ as soon as Λ contains the smallest union of lattice squares of $\mathscr{L}$ containing $\{x\}_N$, and taking the limit $\Lambda \nearrow \mathbb{R}^2$ is therefore trivial.

The cluster expansion removes the 0-Dirichlet data on the bonds of $\mathscr{B}$ step by step and estimates, after a partial resummation, the terms in the final series.

Removing 0-Dirichlet data on some bond $b \in \mathscr{B}$ introduces a convergence factor proportional to $m_0^{-\eta}(\eta > 0)$ or, in a term localized near x, a convergence factor proportional to $\exp(-m_0 \operatorname{dist}(b, x))$. These factors yield convergence of the expansion.

Notations. A collection of bonds b in $\mathscr{B}$ is denoted by Γ; $\Gamma^c = \mathscr{B} \backslash \Gamma$ (complement of Γ in $\mathscr{B}$). Let $w(\{x\}_N)$ be a function of compact support in $L^p(\mathbb{R}^{2|N|})$ with $(p/p - 1)\varepsilon^2 < 4\pi$. We set $X_0 = \operatorname{supp} w$. Let X range over finite unions of closed lattice squares and let Γ range over the set of finite collections of bonds in $\mathscr{B}$ such that

$$\left.\begin{array}{l} \text{(i) each connected component of } X\backslash\Gamma^c \text{ meets } X_0 \\ \text{(ii) } \Gamma \subset \operatorname{Int} X \end{array}\right\} \tag{2.11}$$

To each bond $b \in \mathscr{B}$ attach a real number $s_b \in [0, 1]$, and define a covariance

$$C(\{s\}_{\mathscr{B}}) = \sum_{\Gamma \subset \mathscr{B}} \prod_{b \in \Gamma} s_b \prod_{b \in \Gamma^c} (1 - s_b) C_{\Gamma^c},$$

where C_{Γ^c} is the operator $(-\Delta_{\Gamma^c} + m_0^2)^{-1}$, and Δ_{Γ^c} is the Laplacian with 0-Dirichlet data at Γ^c. (Clearly $C(1, 1, \ldots) = (-\Delta + m_0^2)^{-1}$.) Let

$$s(\Gamma) = \{s(\Gamma)_b\}_{b \in \mathscr{B}} \quad \text{with}$$

$$s(\Gamma)_b = \begin{cases} s_b, \, b \in \Gamma, & (0 \leqslant s_b \leqslant 1) \\ 0, \, b \notin \Gamma \end{cases}$$

Expectations with respect to the Gaussian measure on $\mathscr{S}'$ with mean 0 and covariance $C(\cdot)$ are denoted $\langle \text{——} \rangle_{C(\cdot)}$, and

$$Z^{\lambda}_{\Gamma}(\Lambda) = \langle e^{-\lambda U_\Lambda(\varphi)} \rangle_{C_\Gamma}.$$

The *cluster expansion* is summarized in the equation

$$\begin{aligned} S^{\lambda}_{\Lambda}(w, \{\vartheta\}_N) = \sum_{\substack{X, \Gamma \\ \text{(as in (2.11))}}} \int_0^1 \cdots \int_0^1 \prod_{b \in \Gamma} ds_b \frac{\partial}{\partial s_b} \langle \int \prod_{j \in N} c_{\vartheta_j}(x_j)\, d^2x_j \\ \times w(\{x\}_N) e^{-\lambda U_{\Lambda \cap X}(\varphi)} \rangle_{C(s(\Gamma))} Z_{\partial X}(\Lambda \backslash X) \cdot Z(\Lambda)^{-1} \end{aligned} \tag{2.12}$$

This expansion is derived in [21, Section 3, Eq. (3.15)]. It is *model-independent.*[4])

The basic result of [21] is

Theorem, [21, Section 4]: The convergence of the cluster expansion (2.12) yields convergence of (2.9) and exponential cluster properties.

For a proof of this theorem in the case of the $P(\varphi)_2$-models we refer the reader to [21]. Fortunately the main reasons why this theorem is true are largely model-

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[4]) Throughout this paper *Wickordering* is always *matched* to the covariance $C(s(\Gamma))$; in contradistinction to [21].

independent. We sketch them here briefly for the example of the two-point function of an even interaction: Apply the cluster expansion (2.12) to the difference

$$S_\Lambda^\lambda\left(w \otimes w', \left\{\frac{\pi}{2}, \frac{\pi}{2}\right\}\right) - S_\Lambda^\lambda\left(w, \left\{\frac{\pi}{2}\right\}\right) S_\Lambda^\lambda\left(w', \left\{\frac{\pi}{2}\right\}\right) = S_\Lambda^\lambda\left(w \otimes w', \left\{\frac{\pi}{2}, \frac{\pi}{2}\right\}\right), \tag{2.13}$$

We let $X_0' = \operatorname{supp} w'$. It is quite easy to see that all terms in the cluster expansion of (2.13) vanish, unless there exists at least one path joining X_0 and X_0' that has the property that each bond hit by the path belongs to Γ. If $|\Gamma|$ denotes the number of bonds in Γ we conclude that for all such terms $\neq 0$

$$|\Gamma| \geqslant \text{const. dist}\,(X_0, X_0'). \tag{2.14}$$

The number of differentiations in $\prod_{b\in\Gamma} \partial/\partial s_b$ on the r.h.s. of (2.13) is equal to $|\Gamma|$.

Each differentiation introduces a convergence factor < 1, (provided m_0 is large enough and under the assumption that some combinatoric estimates on certain path integrals with measure $d\mu_{C(s(\Gamma))}$ which we prove below for the massive sine-Gordon theory are true). Therefore such a term is bounded by

$$O(e^{-a|\Gamma|}), \text{ uniformly in } \Lambda; \tag{2.15}$$

the number of terms with given $|\Gamma|$ satisfying (2.11) is bounded by $O(e^{b|\Gamma|})$, and $b < a$, provided m_0 is sufficiently large. Therefore inequality (2.14) yields *exponential clustering*.

To prove convergence of (2.9) as $\Lambda \nearrow \mathbb{R}^2$ we notice that for $\tilde{\Lambda} \supset \Lambda$

$$S_{\tilde{\Lambda}}^\lambda(w, \{\vartheta\}_N) - S_\Lambda^\lambda(w, \{\vartheta\}_N)$$

is precisely of the form (2.13) with $X_0' = \tilde{\Lambda}\backslash\Lambda$. As a consequence it is bounded by

$$O(e^{-\text{const. dist.}\,(X_0, \tilde{\Lambda}\backslash\Lambda)})$$

which proves convergence, as $\Lambda \nearrow \mathbb{R}^2$ ([21, part 1, Theorem 2.2.2).

Next we prove the input estimates for the proof of convergence of the cluster expansion (2.12) and of the bound (2.15). Up to combinatoric estimates *only depending* on the number of space-time dimensions and estimates on the covariances $C(s(\Gamma))$ which are of course *independent* of the choice of the action $U_\Lambda(\varphi)$ the only basic estimate is

Proposition 2.2 [21; Proposition 5.3]. Given any positive constant K there exists a positive number $\mu \geqslant 1$ such that for any Λ and λ with $|\lambda| < \lambda_1(\varepsilon)$

$$\left|\left\langle \int \partial^\Gamma \int \prod_{i=1}^n c_{\vartheta_i}(x_i) e^{-\lambda U_\Lambda(\varphi)}\, d\varphi_{C(s(\Gamma))}\, ds(\Gamma), w\right\rangle\right| \leqslant e^{-K|\Gamma| + K'|\Lambda|}|w| \tag{2.16$^{5)}$}$$

for all $m_0 > \mu$, some constant K', independent of m_0 and λ and some norm $|\cdot|$ continuous on $\mathscr{S}(\mathbb{R}^2)$ independent of m_0 and λ.

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[5]) Our notations are as in [21]:

$$\left\langle \int \partial^\Gamma \int \prod_{i=1}^n c_{\vartheta_i}(x_i)\, e^{-\lambda U_\Lambda(\varphi)}\, d\varphi_{C(s(\Gamma))}\, ds(\Gamma), w\right\rangle$$

$$= \int_0^1 \cdots \int_0^1 \prod_{b\in\Gamma} ds_b \frac{\partial}{\partial s_b} \left\langle \int \prod_{i=1}^n c_{\vartheta_i}(x_i)\, d^2x_i w(x_1, \ldots, x_n)\, e^{-\lambda U_\Lambda(\varphi)}\right\rangle_{C(s(\Gamma))}.$$

If w has compact support and $\|w\|_r < \infty$ with $r > (1-\alpha)^{-1}$, $\alpha \equiv \varepsilon^2/4\pi$, we can replace $|w|$ in (2.16) by $K_r\|w\|_r$ for some finite constant K_r.

Proof of Proposition 2.2. Without loss of generality let us assume

$$w = w \prod_{i=1}^{n} \chi_{\Delta_i}(x_i) \tag{2.17}$$

where Δ_i are unit lattice squares. In this case we choose some fixed $r_0 > (1+\alpha)/(1-\alpha)$ and replace $|w|$ by $\|w\|_{r_0}$. (A choice for $|\cdot|$ follows by summing over all possible localizations of a general w). Let $\mathscr{P}(\Gamma)$ be the set of all partitions π of Γ. By Leibniz' rule we obtain

$$\left|\left\langle \int \partial^\Gamma \int \prod_{i=1}^{n} c_{\vartheta_i}(x_i) e^{-\lambda U_\Lambda}\, d\varphi_{C(s(\Gamma))}\, ds(\Gamma), w\right\rangle\right|$$

$$= \sum_{\pi \in \mathscr{P}(\Gamma)} \left\langle \int ds(\Gamma) \int d\varphi_{C(s(\Gamma))} \left[\prod_{\gamma\in\pi} \tfrac{1}{2}\, \partial^\gamma C(s(\Gamma)) \cdot \Delta_\varphi\right] \prod_{i=1}^{n} c_{\vartheta_i}(x_i) e^{-\lambda U_\Lambda}, w\right\rangle \tag{2.18}$$

We define

$$\partial^\lambda C(j_\gamma) = \chi_{\Delta_{j_{\gamma,1}}}\, \partial^\gamma C(s(\Gamma)) \chi_{\Delta_{j_{\gamma,2}}}$$

where $j_\gamma = (j_{\gamma,1}, j_{\gamma,2}) \in \mathbb{Z}^4$ is a localization index. Then

$$\partial^\gamma C(s(\Gamma)) = \sum_{j_\gamma \in \mathbb{Z}^4} \partial^\gamma C(j_\gamma) \tag{2.19}$$

We substitute (2.19) into (2.18) and expand: each term in the resulting sum is then indexed by $\pi \in \mathscr{P}(\Gamma)$ and $\{j_\gamma, \gamma \in \pi\}$. After applying Δ_φ and using Leibniz' rule we obtain a sum of terms of the form $\int Re^{-\lambda U_\Lambda}\, d\varphi_{C(s(\Gamma))}$ where

$$R = \int \prod_{j=1}^{m} d^2\, x_j c_{\vartheta_j}(x_j) w'(x_1, \ldots, x_m) \tag{2.20}$$

and

$$w' = w \int \prod_{\gamma \in \pi} \partial^\gamma C(j_\gamma) \delta \tag{2.21}$$

(where the integration is over the contracted variables in the δ-functions).

The following two lemmas, combined with Schwarz's inequality, are sufficient to estimate the individual terms:

Lemma 2.3.

$$\langle e^{-\lambda U_\Lambda}\rangle_{C(s(\Gamma))} \leqslant e^{K_0|\Lambda|}$$

where the constant K_0 depends only on $|\lambda|$ and ε.

Lemma 2.4. (analogue of Theorem 9.4 of [21])
Let $w: \mathbb{R}^{2n} \to \mathbb{C}$ be a function in $L^p(\mathbb{R}^{2n})(p = (1+\alpha)/(1-\alpha))$ with support in a product $\Delta_1 \times \cdots \times \Delta_n$ of lattice squares of $\mathscr{L}$. Let R be of the form (2.20) and let $N(\Delta)$ be the number of indices $i \in \{1, \ldots, m\}$ with $\Delta_i = \Delta$. Then

$$\langle R\rangle_{C(s(\Gamma))} \leqslant \|w'\|_p \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_1^{N(\Delta)}$$

Corollary 2.5.

$$\langle Re^{-\lambda U_\Lambda}\rangle_{C(s(\Gamma))} \leqslant \|w'\|_{2p} \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_2^{N(\Delta)} e^{K_3|\Lambda|} \tag{2.22}$$

From (2.21) we see that by Hölder's inequality

$$\|w'\|_{2p} \leqslant \|w\|_{p_0} \left\| \int \prod_{\gamma\in\pi} \partial^\gamma C(j_\gamma)\delta \right\|_r$$

with

$$r^{-1} + p_0^{-1} = \tfrac{1}{2}p^{-1} = \tfrac{1}{2}(1-\alpha)/(1+\alpha)$$

For $\| \int \prod_{\gamma\in\pi} \partial^\gamma C(j_\gamma)\,\delta \|_r$ we will give an estimate only depending on $(\pi, \{j_\gamma\})$, (not on the way the variables are contracted by the δ-functions, that is, not on the way the φ-derivatives in (2.18) are distributed among the different factors according to Leibniz' rule). Of course we need an estimate on the total number of terms arising from the φ-differentiations; let us call it $M = M(\pi, \{j_\gamma\})$; furthermore we need an estimate on $\prod_\Delta N(\Delta)!$. This is supplied by

Lemma 2.6 (replacing Lemma 10.1 of [21]). Let $M(\Delta)$ be the number of $j_{\gamma,1}$ or $j_{\gamma,2}$ such that $\Delta_{j_{\gamma,1}} = \Delta$ or $\Delta_{j_{\gamma,2}} = \Delta$. Then there is a constant K_4 such that

$$M \leqslant n^n e^{K_4|\Gamma|} \prod_{\Delta} M(\Delta)! \tag{2.23}$$

$$\prod_{\Delta} N(\Delta)! \leqslant n^n e^{K_4|\Gamma|} \prod_{\Delta} M(\Delta)! \tag{2.24}$$

Our estimate on the differentiated covariance has the following form

Lemma 2.7.

(1) $$\left\| \int \prod_{\gamma\in\pi} d^\gamma C(j_\gamma)\delta \right\|_r \leqslant e^{-K|\Gamma|} \prod_{\gamma\in\pi} e^{-(m_0/K_5)d(j_\gamma,\gamma)} K_6(r,\gamma)$$

where $d(j_\gamma, \gamma) = \max_{b\in\gamma}\{\text{dist}(\Delta_{j_1}, b) + \text{dist}(\Delta_{j_2}, b)\}$, and $K \to \infty$, as $m_0 \to \infty$.

(2) $$\sum_{\pi\in\mathscr{P}(\Gamma)} \prod_{\gamma\in\pi} K_6(r,\gamma) \leqslant \exp(K_7(r)|\Gamma|)$$

Lemmas 2.3 to 2.6 are sufficient to estimate each term in the sum over $(\pi, \{j_\gamma\})$. This final sum is controlled by Lemma 10.2 of [21] which we just quote without repeating the proof:

Lemma 2.8 (=Lemma 10.2 of [21]). Given $\pi \in \mathscr{P}(\Gamma)$ and $r' > 0$, there exists a constant K_8 (independent of m_0) such that

$$\sum_{\{j_\gamma\}} \prod_{\gamma\in\pi} e^{-(m_0/K_5)d(j_\gamma,\gamma)} \prod_{\Delta} (M(\Delta)!)^{r'} \leqslant e^{K_8|\Gamma|}.$$

Proposition 2.2 follows from Corollary 2.5 and Lemmas 2.6 through 2.8 in a straightforward way (see [21]: Let us denote the left hand side of (2.16) by X. Then by Corollary 2.5 and the remarks made thereafter

$$\begin{aligned} X &\leqslant \sum_{\pi\in\mathscr{P}(\Gamma)} \sum_{\{j_\gamma\}} \|w'\|_{2p} \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_2^{N(\Delta)} \cdot e^{K_3|\Lambda|} M(\pi, \{j_\gamma\}) \\ &\leqslant \|w\|_{p_0} e^{K_3|\Lambda|} \sum_{\pi\in\mathscr{P}(\Gamma)} \sum_{\{j_\gamma\}} \left\| \int \prod_{\gamma\in\pi} \partial^\gamma C(j_\gamma)\delta \right\|_r \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_2^{N(\Delta)} \cdot M(\pi, \{j_\gamma\}) \end{aligned} \tag{2.25}$$

Using Lemma 2.7, (1) we obtain

$$\begin{aligned} X \leqslant & \sum_{\pi \in \mathscr{P}(\Gamma)} \sum_{\{j_\gamma\}} \|w\|_{p_0} e^{K_3|\Lambda| - K|\Gamma|} \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_2^{N(\Delta)} \prod_{\gamma \in \pi} e^{-(m_0/K_5) d(j_\gamma, \gamma)} K_6(r, \gamma) \\ & \times M(\pi, \{j_\gamma\}) \leqslant \|w\|_{p_0} e^{K_3|\Lambda| - K|\Gamma|} \left[\sum_{\pi \in \mathscr{P}(\Gamma)} \prod_{\gamma \in \pi} K_6(r, \gamma) \right] \\ & \times \sum_{\{j_\gamma\}} \max_{\pi \in \mathscr{P}(\Gamma)} \left[M(\pi, \{j_\gamma\}) \prod_{\gamma \in \pi} \exp\left[-\frac{m_0}{K_5} d(j_\gamma, \gamma) \right] \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_2^{N(\Delta)} \right] \end{aligned} \tag{2.26}$$

Now we use Lemma 2.7, (2) and Lemma 2.6, together with the fact

$$\sum_{\Delta} N(\Delta) \leqslant n + |\Gamma| \tag{2.27}$$

to obtain

$$\begin{aligned} X \leqslant & \|w\|_{p_0} e^{K_3|\Lambda| - K|\Gamma| + K_7|\Gamma| + 2K_4|\Gamma| + |\Gamma| \ln K_2} (n^2 K_2)^n \\ & \times \sum_{\{j_\gamma\}} \max_{\pi \in \mathscr{P}(\Gamma)} \left[\prod_{\gamma \in \pi} \exp\left[-\frac{m_0}{K_5} d(j_\gamma, \gamma) \right] \prod_{\Delta} (M(\Delta)!)^{(1+3\alpha)/(1+\alpha)} \right] \end{aligned} \tag{2.28}$$

Using now Lemma 2.8 to bound the remaining sum and redefining K (and setting $K_3 = K'$), Proposition 2.2 follows. Q.E.D.

Next let us turn to the proofs of Lemmas 2.3 through 2.7.

Proof of Lemma 2.3. On the quadratic form domain of $(-\Delta + 1)^{-1}$ (the Sobolev space $\mathscr{H}_{-1}$)

$$C(s(\Gamma)) \leqslant (-\Delta + 1)^{-1}, \quad \text{for all } m_0 \geqslant 1$$

and for all $s(\Gamma)$, $(0 \leqslant s_b \leqslant 1)$. Therefore we obtain from 'conditioning' [24]; see also [16], Theorem 2.4 or [17], Lemma IV.3):

$$|\langle e^{-\lambda U_\Lambda} \rangle_{C(s(\Gamma))}| \leqslant \langle e^{-Re\lambda U_\Lambda} \rangle_{C(s(\Gamma))} \leqslant \langle e^{-Re\lambda U_\Lambda} \rangle_0 \text{ [6]}) \tag{2.29}$$

It was shown in [16], Cor. 3.5 (see also [17], Theorem IV.8) that

$$\langle e^{-Re\lambda U_\Lambda} \rangle_0 \leqslant A_r e^{B_r |Re\lambda|^r |\Lambda|} \tag{2.30}$$

where A_r, B_r are finite constants independent of λ and $|\Lambda|$, provided $r > \alpha/(1-\alpha)$. Q.E.D.

Proof of Lemma 2.4. Let $C = C(s(\Gamma))$. We first discuss instead of R

$$\tilde{R} = \int \prod_{j=1}^{k} :e^{i\varepsilon\varphi}:_C(x_j) \prod_{j=k+1}^{m} :e^{-i\varepsilon\varphi}:_C(x_j) w'(x_1, \ldots, x_m)\, d^2x_1 \cdots d^2x_m \tag{2.31}$$

By explicit computation

$$\begin{aligned} \int \tilde{R}\, d\varphi_C = & \int \exp\left[-\frac{\varepsilon^2}{2} U_C(x_1, \ldots, x_k; x_{k+1}, \ldots, x_m) \right] \\ & \times w'(x_1, \ldots, x_m)\, d^2x_1 \cdots d^2x_m, \end{aligned} \tag{2.32}$$

[6]) We recall that Wick ordering is matched to the covariance of the underlying Gaussian measure.

where

$$U_C(x_1, \ldots, x_k; x_{k+1}, \ldots, x_m) = \sum_{\substack{i \neq j \\ 1 \leqslant i, j \leqslant k}} C(x_i, x_j) + \sum_{\substack{i \neq j \\ k+1 \leqslant i, j \leqslant m}} C(x_i, x_j) - 2 \sum_{\substack{1 \leqslant i \leqslant k \\ k+1 \leqslant j \leqslant m}} C(x_i, x_j) \tag{2.33}$$

(2.32) can be estimated by Hölder's inequality, using the fact that

$$w'(x_1, \ldots, x_m) = \prod_{i=1}^{m} \chi_{\Delta_i}(x_i) w'(x_1, \ldots, x_m)$$

We obtain

$$\left| \int \tilde{R}\, d\varphi_C \right| \leqslant \|w'\|_p \left[\int \prod_{i=1}^{m} \chi_{\Delta_i}(x_i) \exp\left(-\frac{\varepsilon^2}{2} p' U_C(x_1, \ldots, x_k; x_{k+1}, \ldots, x_m) \right) \right.$$

$$\left. \times\, d^2x_1 \cdots d^2x_m \right]^{1/p'} = \|w'\|_p \left[\int d\varphi_C \prod_{j=1}^{k} :e^{i\varepsilon'\varphi}:_C(\chi_{\Delta_j}) \prod_{j=k+1}^{m} :e^{-i\varepsilon'\varphi}:_C(\chi_{\Delta_j}) \right]^{1/p'} \tag{2.34}$$

where $\varepsilon' = \varepsilon\sqrt{p'}$; $p' = p/(p-1)$; (note that $\alpha p' = (1+\alpha)/2 < 1$ because $p = (1+\alpha)/(1-\alpha)$). We now apply the Schwarz inequality to the $d\varphi_C$ integral thus obtaining an integrand of the form $|u|^2$. Then by a conditioning inequality (see [24], [17]) we can replace $d\varphi_C$ by $d\varphi_0$. We first get

$$\left| \int \tilde{R}\, d\varphi_C \right| \leqslant \|w'\|_p \left[\int d\varphi_0 \prod_{i=1}^{m} |:e^{i\varepsilon'\varphi}:(\chi_{\Delta_i})|^2 \right]^{1/2p'}$$

This integral can then be controlled by the so-called *checkerboard estimate* of [24], and we obtain.

$$\left| \int \tilde{R}\, d\varphi_C \right| \leqslant \|w'\|_p \prod_{\Delta} \| |:e^{i\varepsilon'\varphi}:(\chi_\Delta)|^{2N(\Delta)} \|_s^{1/2p'} \tag{2.35}$$

for some $s > 4$.

Now it has been shown in [16] that

$$\int d\varphi_0 |:e^{i\varepsilon'\varphi}:(\chi_\Delta)|^q \leqslant C^q_{\varepsilon', \beta} \Gamma(q+1)^\beta, \tag{2.36}$$

for $\beta > \varepsilon'^2/4\pi$; if we pick $\beta = 1$ we get from (2.35) and (2.36)

$$\left| \int \tilde{R}\, d\varphi_C \right| \leqslant \|w'\|_p \prod_{\Delta} (N(\Delta)!)^{1/p'} K_1^{N(\Delta)} = \|w'\|_p \prod_{\Delta} (N(\Delta)!)^{2\alpha/(1+\alpha)} K_1^{N(\Delta)} \tag{2.37}$$

It is trivial to obtain the same bound for $\left| \int R\, d\varphi_C \right|$. Q.E.D.

Corollary 2.5 follows from Lemmas 2.3 and 2.4 and Schwarz's inequality.

Proof of Lemma 2.6. $M(\Delta)$ differentiations of $\prod_{i=1}^n c_{\vartheta_i}(x_i)\, e^{-\lambda U_\Lambda}$ produce no more than

$$(n_\Delta + 1)(n_\Delta + 2) \cdots (n_\Delta + M(\Delta)) \leqslant (n_\Delta + M(\Delta))!, \quad n_\Delta \equiv |\{i : x_i \in \Delta\}|$$

terms. If we use the inequality $(a+b)! \leqslant (a+b)^a b!$, we obtain

$$M \leqslant \prod_{\Delta} (n_\Delta + M(\Delta))! \leqslant \prod_{\Delta} (n_\Delta + M(\Delta))^{n_\Delta} M(\Delta)! \tag{2.38}$$

Using

$$(a + b)^a = a^a\left(1 + \frac{b}{a}\right)^a \leqslant a^a e^b \tag{2.39}$$

and the fact that

$$\sum M(\Delta) = 2|\Gamma| \tag{2.40}$$

we obtain

$$M \leqslant \prod_{\Delta} n_\Delta{}^{n_\Delta}\; e^{M(\Delta)} M(\Delta)! \leqslant n^n\, e^{2|\Gamma|} \prod_{\Delta} M(\Delta)!. \tag{2.41}$$

Furthermore

$$N(\Delta) \leqslant n_\Delta + M(\Delta)$$

and therefore

$$\prod_{\Delta} N(\Delta)! \leqslant \prod_{\Delta} (n_\Delta + \; M(\Delta))^{n_\Delta} M(\Delta)! \leqslant n^n\, e^{K_4|\Gamma|} \prod_{\Delta} M(\Delta)! \qquad \text{Q.E.D.}$$

as above.

Proof of Lemma 2.7. This proof is somewhat technical, although not difficult; we divide it into several steps.

Definition 2.1. Given $\gamma \in \pi$, let $L(\gamma)$ be the set of all possible linear orderings of the bonds in γ. Given $l \in L(\gamma)$ let $b_1, b_2, \ldots$ be the bonds of γ ordered as prescribed by l.

Set $b'_1 = b_1$, let b'_2 be the first element of $\{b_2, b_3, \ldots\}$ *not* touching b'_1, b'_3 the first element of $\{b'_2, \ldots\}$ not touching b'_2, etc. We then set $a_j = \mathrm{dist}(b'_{j+1}, b'_j)$, $1 \leqslant j \leqslant m(l) < |\gamma|$, and define $|l| = \sum_{i=1}^{m(l)} a_i$.

Lemma 2.9.

(i) $\inf_{\{\gamma : |\gamma| \leqslant 7\}} \min_{l \in L(\gamma)} |l| = 0$

(ii) For $|\gamma| > 7$, $\min_{l \in L(\gamma)} |l| \geqslant \frac{1}{7}|\gamma|$.

The proof is obvious.

Definition 2.2.

$$\pi_7 = \{\gamma | \gamma \in \pi, |\gamma| \leqslant 7\}$$
$$\pi_7^C = \{\gamma | \gamma \in \pi, |\gamma| > 7\}$$

Definition 2.3.

$$\pi_7^0 = \{\gamma | \gamma \in \pi_7, d(j_\gamma, \gamma) = 0\}$$
$$\pi_7^> = \{\gamma | \gamma \in \pi_7, d(j_\gamma, \gamma) \geqslant 1\}$$

Lemma 2.10. For $\gamma \in \pi_7^C \cup \pi_7^>$

$$\|\partial^\gamma C(j_\gamma)\|_\infty \leqslant K_9^{|\gamma|} \exp\left[-\, d(j_\gamma, \gamma)\frac{m_0}{K_5}\right] \exp\left[-\frac{m_0}{14K_{10}} \cdot |\gamma|\right] \sum_{l \in L(\gamma)} \exp\left[-\frac{m_0|l|}{2K_{10}}\right]$$

Proof. The proof is essentially contained in [21, 38].

First consider π_7^C. Based on the Wiener representation of the covariance it is shown in [21, 38] that for $|\gamma| > 7$

$$\|\partial^\gamma C(j_\gamma)\|_\infty \leqslant \sum_{l\in L(\gamma)} K_9^{|\gamma|} \exp\left[-\frac{m_0|l|}{K_{10}}\right]\exp\left[-m_0\frac{d(j_\gamma,\gamma)}{K_{10}}\right] \tag{2.42}$$

Since $|l| \geqslant \frac{1}{7}|\gamma|$, we can bound (2.42) by

$$\sum_{l\in L(\gamma)} K_9^{|\gamma|} \exp\left[-\frac{m_0|l|}{2K_{10}} - \frac{m_0|\gamma|}{14K_{10}} - \frac{m_0\, d(j_\gamma,\gamma)}{K_{10}}\right] \tag{2.43}$$

In the second case we have $\gamma \in \pi_7^{>}$, that is $|\gamma| \leqslant 7$, but $d(j_\gamma, \gamma) \geqslant 1$. The primary estimate from the Wiener representation is in this case

$$\|\partial^\gamma C(j_\gamma)\|_\infty \leqslant K_9^{|\gamma|} \exp\left[-\frac{m_0}{K_{11}} d(j_\gamma,\gamma)\right] \tag{2.44}$$

Since $|\gamma| \leqslant 7d(j_\gamma, \gamma)$ and $|l| \leqslant 7d(j_\beta, \gamma)$ we obtain again an estimate of the form (2.43) Q.E.D.

To prove Lemma 2.7 we have to estimate

$$\left\|\int \prod_{\gamma\in\pi} \partial^\gamma C(j_\gamma)\delta\right\|_r.$$

This is done by Hölder's inequality (see Lemma 9.2 of [21]) and the following trivial observation: Given a localization index $j \in \mathbb{Z}^4$, there are at most $q(=31)$ sets of bonds γ in π_7^0 such that $j_\gamma = j$. This leads to

Lemma 2.11.

$$\left\|\int \prod_{\gamma\in\pi_7^0} C(j_\gamma)\delta\right\|_r \leqslant K_{12}^{|\gamma|}\, m_0^{-|\gamma|/rqK_{13}} \sum_{l\in L(\gamma)} \exp\left[-\frac{m_0|l|}{2K_{10}}\right] \tag{2.45}$$

Proof. First we use Lemma 9.2 of [21] (repeated use of Hölder's inequality) to bound the left hand side of (2.45) by

$$\prod_{\gamma\in\pi_7^0} \|\partial^\gamma C(j_\gamma)\|_{rq}$$

Now from Proposition 7.2 of [21] (scaling) we get

$$\|\partial^\gamma C(j_\gamma)\|_{rq} \leqslant K_{14} m_0^{-2/rq} \tag{2.46}$$

On the other hand we still have the bound

$$\|\partial^\gamma C(j_\gamma)\|_{rq} \leqslant \sum_{l\in L(\gamma)} K_9^{|\gamma|} \exp\left[-\frac{m_0|l|}{K_{10}}\right] \tag{2.47}$$

(Eq.(8.8) of [21]). Taking geometric means of (2.46) and (2.47) and using $|\gamma| \leqslant 7$ for $\gamma \in \pi_7^0$ proves Lemma 2.11 Q.E.D.

Finally we can now estimate $\|\int \prod_{\gamma\in\pi} \partial^\gamma C(j)\,\delta\|_r$ by using the ∞-norm (Lemma 2.10) wherever possible and the rq-norm (Lemma 2.11) wherever necessary:

$$\left\|\int \prod_{\gamma\in\pi} \partial^\gamma C(j)\delta\right\|_r \leqslant \prod_{\gamma\in\pi} \sum_{l\in L(\gamma)} m_0^{-|\gamma|/K_{13}rq} K_{15}^{|\gamma|} \exp\left[-\frac{m_0}{K_5}(d(j_\gamma,\gamma) + |l|)\right] \tag{2.48}$$

for some constant K_5.

With the definition

$$K_6(r,\gamma) \equiv \sum_{l\in L(\gamma)} K_{15}^{|\gamma|} \exp\left[-\frac{m_0|l|}{K_5}\right] \tag{2.49}$$

and using $\sum_{\gamma \in \pi} |\gamma| = |\Gamma|$, for all $\pi \in \mathscr{P}(\Gamma)$, we get assertion (1) of Lemma 2.7; the assertion (2) is just Proposition 8.2 and equ. (8.5) of [21] (up to a trivial change $3 \to K_5$). This completes the proof of Lemma 2.7. Q.E.D.

We have now completed the proof of the convergence of the cluster expansion and therefore we have established analyticity of the Schwinger functions in λ in a circle around the origin.[7])

To identify the expansion coefficients with the usual expressions of Feynman perturbation theory we appeal to a result of Dimock's [6] which states that if the cluster expansion converges the n$^{\text{th}}$ derivative of a Schwinger function with respect to λ at $\lambda = 0$ is given by the n$^{\text{th}}$ order term of the Feynman perturbation series.

This completes the proof of Theorem 2.1.

Let us conclude with some comments on the physical importance of this result: Since the theory has a *unique vacuum*, there is *no way* to construct charged superselection sectors, as it was done for the massless sine-Gordon model (the massive Thirring model) in [18]. This is an expression of the *confinement* of charges which results from the extreme long range character of the two-dimensional Coulomb force.

3. Particle Structure

In this section we propose to prove the existence of isolated one-particle shells in the energy-momentum spectrum of the sine-Gordon theory for small $|\lambda/m_0^2|$.

The main tool of our analysis is a very efficient expansion devised by Spencer [39] that yields decay estimates for r-particle irreducible Euclidean Green's functions. Spencer has developed this expansion in the context of the $P(\varphi)_2$ models. For the convenience of the reader we shall outline the basic ideas of Spencer's techniques and then present the necessary modifications of some of his estimates that permit us to apply the expansion to the sine-Gordon theory. These modifications consist mainly of combinatorial refinements that are required by the non-polynomial nature of the sine-Gordon action.

We now present some definitions and then explain the connections between decay estimates of one particle irreducible Green's functions and the existence of one-particle states. In the following expectation values are taken with respect to a physical measure on $\mathscr{S}'$, (e.g. the sine-Gordon measure constructed in Section 2).

Definitions.

3.1 Let ψ denote any one of the fields φ, c_ϑ; $\vartheta \in [0, 2\pi)$. Let Q_1 and Q_2 be polynomials in ψ. We define the '*channel-connected*' expectation $\langle Q_1; Q_2\rangle^c$ by

$$\langle Q_1; Q_2\rangle^c = \langle Q_1 Q_2\rangle - \langle Q_1\rangle\langle Q_2\rangle.$$

3.2 The '*channel-one particle irreducible*' expectation of Q_1, Q_2 is defined by

$$\langle Q_1; Q_2\rangle^1 = \langle Q_1; Q_2\rangle^c - \int \langle Q_1; \varphi(z_1)\rangle^c \Gamma(z_1, z_2)\langle \varphi(z_2); Q_2 \rangle^c d^2 z_1\, d^2 z_2,$$

where Γ is the Euclidean two point *vertex function*, i.e. the kernel of the inverse of the integral operator whose kernel is the connected two point EGF.

[7]) It is a standard result that, for real λ with $|\lambda|$ so small that the cluster expansion converges, the Schwinger functions are the moments of a unique probability measure on $\mathscr{S}'$ (defining an infinite volume expectation $\langle$——$\rangle$).

3.3 The *self energy part* (one particle irreducible kernel [20, 39]) is given by

$$k_C(x, y) = (\Gamma - C^{-1})(x, y),$$

where C is the *free*, connected two point EGF, i.e. the (kernel of the) covariance of the free, Gaussian measure on $\mathscr{S}'$.

3.4. A polynomial Q in ψ is *localized* if it is a polynomial in fields smeared with functions of compact support. (Technically a polynomial is said to be localized if it is measurable with respect to the σ-algebra associated with a bounded region of space-time.)

The main *technical* result of this section is

Theorem 3.1. Let $\langle\text{——}\rangle$ denote the expectation with respect to the infinite volume sine-Gordon measure constructed in Section 2, and let Q_1, Q_2 be localized polynomials in ψ; let Q^T denote the translate of Q by the vector $(0, T)$.

Then, given an ε with $\varepsilon^2 < 4\pi$ and any $\delta > 0$, there exist positive numbers λ_0 and M_0 such that

(A) $$|\langle Q_1, Q_2^T\rangle^1| \leqslant K_1(Q_1, Q_2, m_0) \exp(-2m_0(1 - \delta)T)$$

and, for $|x - y| \geqslant 1$

(B) $$|k_C(x, y)| \leqslant K_2 \exp(-2m_0(1 - \delta)|x - y|),$$

for all coupling constants $\lambda \in [-\lambda_0, \lambda_0]$ and bare masses $m_0 > M_0$.

The proof of Theorem 3.1 is the (technical) core of this section.

Next we interpret the decay estimates (A) and (B) in terms of the *particle structure* of the sine-Gordon theory:

(1) Let Q_1 and Q_2 be arbitrary polynomials in φ localized at strictly *positive* times, and let $\bar{Q}_1^\theta$ denote the complex conjugate of the reflection of Q_1 at $t = 0$, [33]. Let $P_1^{\mathscr{K}}$ denote the selfadjoint projection (with respect to the Osterwalder-Schrader scalar product, [33]) onto the *orthogonal complement* of *all states in the physical Hilbert space* spanned by

$$\{\Omega, \varphi(f)\Omega\colon \operatorname{supp} f \subseteq \{t \geqslant 0\}\}; \qquad \text{(see [20,33])}$$

Let m denote the physical mass (*mass gap*). Suppose now that for all such Q_1 and Q_2 the exponential decay rate of $\langle \bar{Q}_1^\theta P_1^{\mathscr{K}} Q_2^T\rangle$ in T is at least $m(1 + \Delta)$, for some $\Delta > 0$. Then the space of all vectors in the physical Hilbert space of energy between m and $m(1 + \Delta/2)$ is contained in the span of

$$\{\varphi(f)\Omega\colon \operatorname{supp} f \subseteq \{t \geqslant 0\}\}.$$

(2) It then follows from regularity properties of the EGF's, an exponential decay rate of k_C of at least $m(1 + \Delta)$ and the spectral representations of the two point EGF S_2^c and of Γ that the spectrum of the mass operator in the interval $[m, m(1 + \Delta/2)]$ consists of the single eigenvalue m.

(3) The irreducibility of the polynomial algebra generated by φ then proves that there is only *one* species of particles of mass m.

(4) It now suffices to notice that

(i) if $k_C(x, y)$ decays like $e^{-m(1+\Delta)|x-y|}$ and $\langle \bar{Q}_1^\theta ; Q_2^T\rangle^1$ like $e^{-m(1+\Delta)T}$ the exponential decay rate of

$$\langle \bar{Q}_1^\theta P_1^{\mathscr{K}} Q_2^T \rangle - \langle \bar{Q}_1^\theta ; Q_2^T \rangle^1$$

in T is at least $m(1 + \Delta)$; see [20, Theorem 5.4].

(ii) Assuming Theorem 3.1, the results of Section 2, and choosing $|\lambda/m_0^2|$ so small that $\delta < \frac{1}{2}$

$$0 < m < m_0(1 + \delta) < 2m_0(1 - \delta)$$

which proves that the exponential decay rate of $\langle \bar{Q}_1^\theta ; Q_2^T \rangle^1$ in T and of $k_C(x, y)$ in $|x - y|$ is at least $m(1 + \Delta)$, for some $\Delta > 0$, and, using (i), yields the required decay of $\langle \bar{Q}_1^\theta P_1^{\mathscr{K}} Q_2^T \rangle$.

This completes our (intuitive) argument. We supplement it by the following nice, rigorous result recently proven by C. Burnap [1] which we state in a form adapted to the context of this paper.

Theorem 3.2. Consider a relativistic quantum field theory fulfilling the following additional conditions on the EGF's:

(a) Existence of Euclidean fields (i.e. Nelson-Symanzik positivity)[8]), and e.g.

$$\hat{S}_2^c(p) = \int \frac{d\rho(a)}{p^2 + a} \text{ with } \int \frac{d\rho(a)}{a} \textit{ finite.}$$

(b) There exists a positive mass gap, i.e. for some $m > 0$ and arbitrary localized polynomials Q_1 and Q_2

$$|\langle Q_1 ; Q_2^T \rangle^c| \leqslant K(Q_1, Q_2) e^{-mT}$$

(c) There exists an upper (mass) gap, i.e. there is some $\Delta > 0$ such that, with $m_2 \equiv m(1 + \Delta)$,

(C1) $\quad |k_C(x, y)| \leqslant K' \, e^{-m_2|x-y|}$, for $|x - y| \geqslant 1$

(C2) $\quad |\langle Q_1 ; Q_2^T \rangle^1| \leqslant K''(Q_1, Q_2) \, e^{-m_2 T}$

(C3) $\quad m_1 = -\overline{\lim_{|x-y|\to\infty}} \dfrac{1}{|x - y|} S_2^c(x, y) < m_2$

Then the spectrum of the mass operator $M = (H^2 - P^2)^{1/2}$ in the interval $[0, m_2)$ consists only of the two isolated eigenvalues 0 and m_1, (and $m = m_1$).

Remarks.

(i) For the proof of Theorem 3.2 we refer to [1]. Unexpected multiplicity of the eigenvalue m of the mass operator is ruled out in Remark (3) above.

(ii) Assuming Theorems 3.1 and 3.2 we can now prove *Theorem 3* of the Introduction, (i.e., the existence of an isolated one particle shell in the energy-momentum spectrum of the sine-Gordon theory): We simply verify hypotheses (a)–(c) of Theorem 3.2.

Clearly the sine-Gordon theory is Nelson-Symanzik positive, and the EGF's of φ are locally integrable, by Section 2. From Section 2 we also infer that the connected two point EGF $S_2^c(0, x)$ is integrable. Therefore, and since all Wightman axioms are satisfied in the sine-Gordon theory, S_2^c has a spectral representation, and

$$\int d^2x S_2^c(0, x) = \int \frac{d\rho(a)}{a} \text{ is } \textit{finite.}$$

8) This condition is not essential, but it is convenient.

(As a matter of fact $\int d\rho(a) = 1$, since the field φ is a *canonical* field in the sine-Gordon theory). The proof of (b) is the main result of Section 2, and the physical mass m satisfies $m < m_0(1 + \delta)$, for $\delta < \frac{1}{2}$, provided $|\lambda/m_0^2|$ is sufficiently small. Hypotheses (C1) and (C2) follow from Theorem 3.1 and Remark (4) (ii) above, by choosing $\delta < \frac{1}{2}$. Finally (C3) follows from Section 2 and Remark (4) (ii), by choosing $\delta < \frac{1}{2}$. This completes the proof.

We now turn to the technical part of this section: the proof of Theorem 3.1. We follow closely Spencer's analysis [39]. Without loss of generality our considerations may be limited to channel one particle irreducible expectations of the form $\langle Q_1; Q_2\rangle^1$ with Q_1 localized in $\{t \geqslant T/2\}$, since the self energy part k_C can be expressed in terms of expectations of this form, see [39].

Following Spencer [39] we introduce a family of 'horizontal' (lattice) lines

$$l_k = \{(\mathbf{x}, t) \in \mathbb{R}^2 : t = k, k \in \mathbb{Z}\}, \tag{3.1}$$

and corresponding interpolating parameters $t_k \in [0, 1]$ (just as we did in Section 2 for the bonds of $\mathbb{Z}_*^2$).

The general expression for the covariances of the Gaussian measures we are using is

$$C(t, s; x, y) = \sum_{\Gamma \subset \mathbb{Z}^2} \sum_{L \subset \mathbb{Z}} \prod_{b \in \Gamma} s_b \prod_{b \notin \Gamma} (1 - s_b) \prod_{k \in L} t_k \prod_{k \notin L} (1 - t_k) C_{\Gamma^c, L^c}(x, y) \tag{3.2}$$

where C_{Γ^c, L^c} is the kernel of $(-\Delta_{\Gamma^c, L^c} + m_0^2)^{-1}$, and Δ_{Γ^c, L^c} is the Laplacian with 0-Dirichlet data on $\Gamma^c \cup L^c$.

In the following $\langle \text{——} \rangle^c (t)$, $\langle \text{——} \rangle^1(t)$ and $k(t, \cdot)$ are defined as previously $\langle \text{——} \rangle^c$, $\langle \text{——} \rangle^1$, k_c, respectively, but with the covariance $C(x - y)$ replaced by $C(t, 1; x, y)$ (later by $C(t, s; x, y)$). In our analysis *Wick ordering is always matched to* $C(t, s; \cdot)$.

Spencer's analysis is inspired by the fact that in the perturbation expansion of $\langle Q_1; Q_2\rangle^1$ each line l_k that separates the supports of Q_1 and Q_2 is crossed by at least two internal lines of each non-vanishing Feynman diagram contributing to $\langle Q_1; Q_2\rangle^1$. Therefore one expects that $\langle Q_1; Q_2\rangle^1$ has two particle decay. In spite of the convergence of perturbation theory proven in Section 2 for the sine-Gordon theory such an argument is of course not rigorous. An analytical, non-perturbative approach towards proving two particle decay of $\langle Q_1; Q_2\rangle^1$ is based on the following

Lemma 3.3. (Spencer [39])

$$\frac{\partial^r}{\partial t_i^r} k(t; x, y)\Big|_{t_i = 0} = 0, \quad \text{for } r = 0, 1, \tag{3.3}$$

if l_i separates x from y. If l_i separates the supports of Q_1 and Q_2

$$\frac{\partial^r}{\partial t_i^r} \langle Q_1; Q_2\rangle^1(t)\Big|_{t_i = 0} = 0, \quad \text{for } r = 0, 1. \tag{3.4}$$

Proof. We only prove (3.4), since a detailed proof of (3.3) is contained in [39]. The case $r = 0$ is of course trivial. For $r = 1$

$$\frac{\partial}{\partial t_i} \langle Q_1; Q_2\rangle^1(t)\Big|_{t_i = 0} = \frac{\partial}{\partial t_i} \langle Q_1; Q_2\rangle^c(t)\Big|_{t_i = 0}$$

$$- \frac{\partial}{\partial t_i} \int \langle Q_1; \varphi(z)\rangle^c(t) \Gamma(t; z, z') \langle \varphi(z'); Q_2\rangle^c(t)\, d^2z\, d^2z'\Big|_{t_i = 0}$$

The first term gives (see [39])

$$-\int \langle Q_1;\varphi(z)\rangle^c(t)\dot{C}(t)^{-1}(z,z')\langle\varphi(z');Q_2\rangle^c(t)\,d^2z\,d^2z'\big|_{t_i=0}$$

(where the dot denotes $\partial/\partial t_i$). If we now use

$$\frac{\partial}{\partial t_i}\langle Q_1;\varphi(z)\rangle^c(t)\big|_{t_i=0} = \int \langle Q_1;\varphi(x)\rangle^c\dot{C}(t)^{-1}(x,y)\langle\varphi(y);\varphi(z)\rangle^c(t)\,d^2x\,d^2y\big|_{t_i=0}$$

and the fact that by (3.3)

$$\dot{\Gamma}(t)\big|_{t_i=0} = \dot{C}(t)^{-1}\big|_{t_i=0}$$

the assertion follows immediately. Q.E.D.

Remark. An alternate proof of (3.3) and (3.4) follows from the convergence of the perturbation expansions of $k(t;x,y)$ and $\langle Q_1;Q_2\rangle^1(t)$ in λ about $\lambda = 0$ which follows from a straightforward extension of our results in Section 2.

Let Q_1 be localized in $\{t \leqslant -T/2\}$ and Q_2 in $\{t \geqslant T/2\}$. Let $I(=I(Q_1,Q_2)) \subset \mathbb{Z}$ be the set of all integers i such that l_i separates the supports of Q_1 and Q_2 and let $I^c = \mathbb{Z}\backslash I$. If f is some function of $\{t_k : k\in\mathbb{Z}\}$ we simply write $f(t)$ for $f(\{t\}_I,\{1\}_{I^c})$. We set

$$F(t) \equiv \langle Q_1;Q_2\rangle^1(t) \tag{3.5}$$

Then, by (3.4) and Taylor's theorem,

$$F(1) = \int_0^1 \cdots \int_0^1 \prod_{i\in I} dt_i(1-t_i)\frac{\partial^2}{\partial t_i^2}F(t) \tag{3.6}$$

Theorem 3.1 follows from a careful estimate on the r.h.s. of (3.6).[9])

Spencer uses a modified version of the cluster expansion [21] to derive such an estimate. We briefly review his procedure: First we evaluate the r.h.s. of (3.6) by using Leibniz' rule and integration by parts on function space (see Section 2 and [39]). The final expression then involves derivatives of the covariance $C(t)$ of the form

$$\partial_t^\alpha C(t) \equiv \prod_{i\in I}\frac{\partial^{r_i}}{\partial t_i^{r_i}}C(t), \tag{3.7}$$

where $r_i \in \{0,1,2\}$ and α stands for the family $\{r_i\}_{i\in I}$, i.e. α is a function on I (or the set of horizontal lines separating the supports of Q_1 and Q_2) with values in $\{0,1,2\}$. The set of all such functions is denoted $I^{(2)}$.

Second we express the t-derivatives of $F(t)$ in terms of derivatives of expectations with respect to a collection of auxiliary variables $\{h(\alpha):\alpha\in I^{(2)}\}$ on which these expectations depend holomorphically. We then apply the Cauchy estimate to derive bounds on the $\{h(\alpha)\}$-derivatives. (This is a key trick of Spencer's analysis, [39].)

Let us first consider an unnormalized expectation of the form

$$\int_{\mathscr{S}'} e^{-\lambda U}Q\,d\varphi(t), \tag{3.8}$$

[9]) The intuition behind estimating (3.6) is that each derivative $\partial/\partial t_i$ introduces a decay factor $\propto e^{-m_0}$, whence two-particle decay.

where U is a space-time cutoff sine-Gordon action and $d\varphi(t)$ is the Gaussian measure on $\mathscr{S}'$ with mean 0 and covariance $C(t)$.

For $\beta \in I^{(2)}$, let $\mathscr{P}(\beta)$ be defined to be

$$\{\beta_1, \ldots, \beta_n, \ldots : \beta_l \in I^{(2)}, \forall l, \sum_{l=1}^{\infty} \beta_l = \beta\} \tag{3.9}$$

Moreover, given $j = (j_1, j_2) \in \mathbb{Z}^4$, we define

$$C_j(t; x, y) = \chi_{\Delta_{j_1}}(x) C(t; x, y) \chi_{\Delta_{j_2}}(y),$$

with χ_Δ the characteristic function of Δ; (such localizations are familiar from Section 2). Finally we define

$$C_j(t)\Delta_\varphi \equiv \int C_j(t; x, y) \frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} d^2x\, d^2y.$$

Then the t-derivatives of (3.8) are given by

$$\partial_t^\beta \int e^{-\lambda U} Q\, d\varphi(t) = \sum_{\pi \in \mathscr{P}(\beta)} \left(\prod_{\alpha \in \pi} \frac{\partial}{\partial h(\alpha)} \int \prod_{\alpha \in \pi} \sum_{j \in \mathbb{Z}^4} (1 + h(\alpha)\, \partial_t^\alpha C_j(t)\Delta_\varphi) \right. \tag{3.10}$$
$$\times\, e^{-\lambda U} Q\, d\varphi(t) \Big|_{h=0}$$

Following [39] we define an h-dependent expectation:

$$\langle Q \rangle(t, h) \equiv \frac{\int \prod_{\alpha \in \pi} \sum_{j \in \mathbb{Z}^4} (1 + h(\alpha)\, \partial_t^\alpha C_j \Delta_\varphi)\, e^{-\lambda U} Q\, d\varphi(t)}{\int \prod_{\alpha \in \pi} \sum_{j \in \mathbb{Z}^4} (1 + h(\alpha)\, \partial_t^\alpha C_j \Delta_\varphi)\, e^{-\lambda U}\, d\varphi(t)} \tag{3.11}$$

(and, similarly, $k(t, h)$ by using the series expansion of k in terms of expressions like (3.11)).

The following identity is basic for our analysis:

Lemma 3.4.

$$\partial_t^\beta \langle Q \rangle(t, h) \Big|_{h=0} = \sum_{\pi \in \mathscr{P}(\beta)} \prod_{\alpha \in \pi} \frac{\partial}{\partial h(\alpha)} \langle Q \rangle(t, h) \Big|_{h=0}$$

The proof of this lemma is given in [39].

An important, and model independent, part of Spencer's analysis is that, by means of Lemmas 3.3 and 3.4, the proof of Theorem 3.1 reduces to the following two estimates; (we recommend that the reader consult [39] for these arguments):

Theorem 3.5. Let $\delta > 0$ and $I \subset \mathbb{Z}$ be given. Then there exist finite positive numbers $M_0(\delta)$ and $\lambda_0(\delta, \varepsilon, M_0)$ such that for $m_0 \geqslant M_0$ and $-\lambda_0 \leqslant \lambda \leqslant \lambda_0$

$$\int W(x) \langle \prod \psi(x_i) \rangle(t, h)\, dx \equiv \langle Q_W \rangle(t, h)$$

is *analytic* in h, for h in the region

$$P_{m_0} = \{h(\alpha) : |h(\alpha)| \leqslant e^{m_0(1-\delta)(d(\alpha)+1)}, \alpha \in I^{(2)}\} \tag{3.12}$$

where ψ stands for φ or c_ϑ, $\vartheta \in [0, 2\pi)$, and

$$d(\alpha) = \begin{cases} 0, \ |\alpha| = 1 \\ \infty, \text{ if } \alpha(i) = 2 \text{ for some } i; \ \left(\text{note that } \dfrac{\partial^2}{\partial t^2} C(t;\cdot) = 0, \text{ for all } i!\right) \\ \max\{|i-j| : i, j \in \operatorname{supp} \alpha\}, \text{ otherwise.} \end{cases}$$

Furthermore, if $W = W_1 \cdot W_2$ and

$$T = \operatorname{dist}(\operatorname{supp} W_1, \operatorname{supp} W_2), \ h \in P_{m_0},$$

then there are positive constants a and c such that

$$|\langle Q_{W_1}; Q_{W_2}\rangle^c(t, h)| \leqslant e^{-cT} \ e^{am_0 \operatorname{supp} Q} \ M(\|W\|_p, \deg Q) \tag{3.13}$$

for some $p \in (1, \infty)$, depending on ε.

Remark. Whereas Spencer has a 4/3-norm on the r.h.s. of (3.13) we shall in general have a p-norm with p large, depending on the choice of the parameter ε in the sine-Gordon action; ($p \nearrow \infty$, as $\varepsilon^2 \nearrow 4\pi$).

Corollary 3.5′. Let $\{\zeta_j\}_{j \in \mathbb{Z}^2}$ be a continuous partition of unity; $\zeta_j(x) \equiv \zeta(x - j)$ with ζ supported in $\{x : |x| \leqslant 1\}$. Furthermore let $f(x)$ be a continuous function. Under the hypotheses of Theorem 3.5

$$\int k(t, h; 0, x) f(x) \zeta_j(x) \, d^2x$$

is *analytic* in $h \in P_{m_0}$ and bounded by $C\|f\zeta_j\|_\infty$, for some finite constant C independent of j.

Remarks.

(A) Corollary 3.5′ is a consequence of an expansion of $k(t, h: 0, f\zeta_j)$ in terms of expectation values of the form $\langle Q_W\rangle(t, h)$; see [39], and of Theorem 3.5. Given Theorem 3.5, the proof is essentially the same as in the case of the $P(\varphi)_2$-models which is given in [39].

(B) Given Theorem 3.5 and Corollary 3.5′, Theorem 3.1 follows by model-independent arguments (depending only on the estimates of Theorem 3.5 and Corollary 3.5′, properties of covariances and the Cauchy estimate on the derivatives in h). We note however that we only get *two-particle decay* for k_C rather than three particle decay (as in [39] where an *even* $P(\varphi)_2$ interaction is considered), because for $\theta \neq 0, \pi$ the sine-Gordon interaction is *not* even in φ, and therefore only the first t_i-derivatives of $k(t; \cdot)$ vanish at $t_i = 0$.

(C) The proof of Theorem 3.5 follows by applying a variant of the cluster expansion of Section 2. In a similar way as the convergence of the cluster expansion of Section 2 may be reduced to one basic, model-dependent estimate (Proposition 2.2), the convergence of the cluster expansion for h-dependent expectations can be reduced to such an estimate: Lemma 3.6 below. This follows from the general (model-independent) theory of the cluster expansion; see [39].

We now turn to the proof of this input estimate.

Definition 3.5. We denote by β a pair (α, j) with $\alpha \in I^{(2)}$ and $j = (j_1, j_2) \in \mathbb{Z}^4$. We then define

$$d(\beta) = \max\{d(j, l_{i(\alpha)}), |j_1 - j_2|\},$$

where $i(\alpha)$ is the smallest integer in the support of α and $l_{i(\alpha)}$ the corresponding horizontal line.

Lemma 3.6. For $m_0 > M_0$, $|\lambda| < \lambda_0$, and M_0 sufficiently large, $\lambda_0 = O(e^{-qM_0})$, for some finite q, $h(\alpha) \in P_{m_0}$,

$$\left|\partial_t^\Gamma \int \prod_{\beta=(\alpha,j)\in B} (h(\alpha)\,\partial_t^\alpha C_j(t)\Delta_\varphi) Q_W\, e^{-\lambda U_X}\, d\varphi(t,s)\right| \leqslant D \prod_{\beta\in B} e^{-1/2\, d(\beta)}, \tag{3.14}$$

where $B \subset B_0(X) \equiv I^{(2)} \times (\mathbb{Z}^2 \cap X)^{\times 2}$, X is some bounded rectangle in $\mathbb{R}^2$, supp $W \subset \times\, \Delta_i$ (the Δ_i's are unit squares in $\mathbb{R}^2$), and $d\varphi(t, s)$ is the Gaussian measure on $\mathscr{S}'$ with covariance $C(t, s; \cdot)$. Furthermore

$$D = e^{K'|X|}e^{-K|\Gamma|}\|W\|_p K_3(Q)e^{am_0 \operatorname{supp} Q},\ K_3(Q) \leqslant (\deg Q)^{2\deg Q} \tag{3.15}$$

Here again $K \to \infty$ as $M_0 \to \infty$ whereas all the other constants can be chosen independently of M_0.

Remark. Spencer's proof of Lemma 3.6 for $P(\varphi)_2$ depends in two places on the fact that the interaction is given by a polynomial:

(1) In the use of Hölder's inequality for an expression of the form $\|\prod \partial_s^\lambda\, \partial_t^\alpha C_j\|_q$. This can be cured in a similar way as in Section 2 (proof of Lemma 2.7), using ∞-norms wherever possible. Note that there are at most two t_i-derivatives for each $i \in \mathbb{Z}$ and at most one s_b-derivative for each $b \in \mathbb{Z}_*^2$.

(2) In the argument that each differentiation Δ_φ produces a factor $\lambda^{2/p}$ (p is the degree of the interaction polynomial). In our case this is obviously not true in this form: the interaction density can be differentiated an arbitrary number of times without giving zero. Since each differentiation produces a factor ε we could require ε to be small to obtain an upper mass gap; this would permit us to take over Spencer's analysis essentially unchanged and thus yield a 'simple' proof. But as it turns out we do not need a restriction on ε. By taking into account a bound on the number of φ-derivatives in a given square we get the upper mass gap for $\varepsilon^2 < 4\pi$ and λ/m_0^2 sufficiently small.

Now let us turn to the proof of Lemma 3.6. We apply the Leibniz rule to compute the left hand side of (3.14) and obtain

$$\sum_{\Gamma_1+\Gamma_2=\Gamma}\left\{\sum_{\cup\gamma_\beta'=\Gamma_1}\ \sum_{\pi\in\mathscr{P}(\Gamma_2)} \int h(\alpha) \prod_{\beta\in B} \partial_s^{\gamma'_\beta}\, \partial_t^\alpha C_j\cdot\Delta_\varphi \prod_{\gamma\in\pi} \partial_s^\gamma C\cdot\Delta_\varphi Q_W\, e^{-\lambda U_X}\, d\varphi(t,s)\right\}, \tag{3.16}$$

(see [39]).

We note that there are at most $2^{|\Gamma|}$ terms in the sum over Γ_1, Γ_2. For fixed Γ_1, Γ_2 we substitute again $C = \sum_{j\in\mathbb{Z}^4} C_{j_\gamma}$: we then obtain a sum over $\pi \in \mathscr{P}(\Gamma_2)$, $\{j_\gamma\}_{\gamma\in\pi}$, $\{\gamma'_\beta\}_{\cup\gamma'_\beta=\Gamma_1}$. The procedure is now similar as in Section 2: we apply the differentiations Δ_φ and estimate each resulting term using Corollary 2.5: this gives a bound

$$|\lambda|^g \|w'\|_q\, e^{M_1|X|} \prod_\Delta N(\Delta)!^{2\alpha/(1+\alpha)}\, M_2^{N(\Delta)} \tag{3.17}$$

for each term, where $g \equiv g(B)$ will be estimated later;

$$w' = W\cdot v\cdot \prod_{\beta\in B} h(\alpha) \tag{3.18}$$

where

$$v = \int \prod_{\beta \in B} \partial_s^{\gamma_\beta'} \partial_t^\alpha C_j \prod_{\gamma \in \pi} \partial_s^\gamma C_{j_\gamma} \prod \delta \tag{3.19}$$

(the integration is again over the variables contracted by δ-functions). By Hölder's inequality we have

$$\|w'\|_q \leqslant \|W\|_p \|v\|_r \Big| \prod_{\beta \in B} h(\alpha) \Big| \tag{3.20}$$

with $q^{-1} = p^{-1} + r^{-1}$. For $\|v\|_r$ we prove an estimate analogous to Lemma 2.7:

Lemma 3.7.

(1)
$$\|v\|_r \leqslant \exp[-K(|\Gamma| + |B|)] \prod_{\beta \in B} M_3(r, \gamma_\beta')$$
$$\exp\left[-\frac{m_0}{M_4}(2d(\beta) + d(j_{\gamma_\beta'}, \gamma_\beta'))\right]$$
$$\exp[-m_0(1-\delta)d(\alpha)] \prod_{\gamma \in \pi} \exp\left[-\frac{m_0}{M_4} d(j_\gamma, \gamma)\right] M_3(r, \gamma) \tag{3.21}$$

where $K \to \infty$, as $m_0 \to \infty$;

(2)
$$\sum_{\pi \in \mathscr{P}(\Gamma_2)} \prod_{\gamma \in \pi} M_3(r, \gamma) \leqslant \exp[M_5|\Gamma_2|] \leqslant \exp[M_5|X|]; \tag{3.22}$$

(3)
$$\sum_{\cup \gamma_\beta' = \Gamma_1} \prod_{\beta \in B} M_3(r, \gamma_\beta')$$
$$\exp\left[-\frac{m_0}{M_4}(d(\beta) + d(j_{\gamma_\beta'}, \gamma_\beta'))\right] \leqslant \exp[M_5|X|] \tag{3.23}$$

We postpone the proof and turn to estimating $g(B)$:

Lemma 3.8. For $C > 0$, B given, let $f_C(B)$ be the number of $\beta \in B$ with $d(\beta) \leqslant C$. Then there is a finite constant q_C such that

$$q_C(g(B) + \operatorname{supp} Q_W) \geqslant f_C(B) \tag{3.24}$$

Proof. We produce a factor λ each time we apply a φ-differentiation to $e^{-\lambda U_X}$. Thus $g(B) \geqslant$ number of different localizations j ($\in \mathbb{Z}^2$) in B outside supp $Q_W \geqslant$ number of different localizations $j \in \mathbb{Z}^2$ in $B \sim$ supp Q_W. Now for each localization $j \in \mathbb{Z}^2$ there is only a finite number q_C of $\beta \in B$ with $d(\beta) \leqslant C$ and $j = j_1$ or $j = j_2$ ($\beta = (\alpha, j_1, j_2)$). Therefore

$$f_C(B) \leqslant q_C(g(B) + \operatorname{supp} Q_W).$$

Q.E.D.

Let us combine what we have:

Lemma 3.9.

$$|\lambda|^g \|w'\|_q \leqslant \|W\|_p \exp[-K(|\Gamma| + |B|)] \exp[am_0 \operatorname{supp} Q_W] \prod_{\beta \in B} M_3(r, \gamma_\beta')$$
$$\times \exp\left[-\frac{m_0}{M_4}(d(\beta) + d(j_{\gamma_\beta'}, \gamma_\beta'))\right] \prod_{\gamma \in \pi} \exp\left[-\frac{m_0}{M_4} d(j_\gamma, \gamma)\right] M_3(r, \gamma) \tag{3.25}$$

(with the same constants M_3, M_4 as in Lemma 3.7).

Proof. Recall that h is in the region P_{m_0} defined by (3.12). Thus, using (3.21) and (3.12), we get

$$|\lambda|^g \prod_{\beta\in B} |h(a)| \, \|v\|_r \leqslant \exp[-K(|\Gamma| + |B|)] \prod_{\beta\in B} M_3(r, \gamma'_\beta)$$
$$\exp\left[-\frac{m_0}{M_4}(d(\beta) + d(j_{\gamma'_\beta}, \gamma'_\beta))\right] \prod_{\gamma\in\pi} \tag{3.26}$$
$$\exp\left[-\frac{m_0}{M_4} d(j_\gamma, \gamma)\right] M_3(r, \gamma)\left(\prod_{\beta\in B} \exp\left[m_0 - \frac{m_0}{M_4} d(\beta)\right]\right) |\lambda|^g$$

We estimate the last factors using Lemma 3.8:

$$|\lambda|^g \prod_{\beta\in B} \exp[m_0(1 - M_4^{-1} d(\beta))] \leqslant |\lambda|^g \prod_{\substack{\beta\in B\\ d(\beta)\leqslant M_4}} \exp(m_0)$$
$$= |\lambda|^g \exp(f_{M_4} m_0) \leqslant |\lambda \exp(q_{M_4} m_0)|^g \exp(m_0 q_{M_4} \operatorname{supp} Q_W) \tag{3.27}$$

If we choose $|\lambda| \leqslant \exp(-m_0 q_{M_4})$, Lemma 3.9 follows from (3.27) and (3.26); $(a = q_{M_4})$. Q.E.D.

To obtain Lemma 3.6, we have to control the remaining sum over $\pi \in \mathscr{P}(\Gamma_2)$, $\{j_\gamma\}_{\gamma\in\pi}$, $\{\gamma'_\beta\}_{\cup\gamma'_\beta=\Gamma_1}$. This is done by two simple estimates similar to bounds proven in Section 2.

Lemma 3.10. (analogue of Lemma 2.6; Lemma 5.2 of [39]). Let $M \equiv M(\pi, \{j\}, \{\gamma'_\beta\})$ be the number of terms arising from the differentiations Δ_φ

$$M(\Delta) = \operatorname{card}\{\beta\in B \mid \Delta_{j_i} = \Delta, i = 1 \text{ or } 2\} + \operatorname{card}\{\gamma\in\pi \mid \Delta_{j_i,\gamma} = \Delta, i = 1 \text{ or } 2\}$$

Then there is a constant M_5 such that

$$M \leqslant e^{M_5(|\Gamma|+|B|)} n^n \prod_\Delta M(\Delta)! \tag{3.28}$$

$$\prod_\Delta N(\Delta)! \leqslant e^{M_5(|\Gamma|+|B|)} n^n \prod_\Delta M(\Delta)!, \tag{3.29}$$

where $n =$ degree Q_W.

We omit the proof which is similar to the proof of Lemma 2.6.

Lemma 3.11. (analogue of Lemma 2.8; Lemma 5.3 of [39]). There is a constant $M_6(r)$ such that

$$\prod_{\beta\in B} \exp\left[-\frac{m_0}{M_4} d(\beta)\right] \sum_{\{j_\gamma\}_{\gamma\in\pi}} \prod_{\gamma\in\pi} \exp\left[-\frac{m_0}{M_4} d(j_\gamma, \gamma)\right] \prod_\Delta (M(\Delta)!)^r \leqslant \exp(M_6(r)|X|) \tag{3.30}$$

for any $r > 0$.

The proof can be found in [39].

Clearly Lemmas 3.7–3.11 yield Lemma 3.6, with $K_3(Q) \leqslant (\deg Q)^{2\deg Q}$; see Sect. 2.

Let us now turn to the proof of Lemma 3.7. The idea is again, as in Section 2, to use the ∞-norm for as many factors as possible and then to use Hölder's inequality in the form of Proposition 9.2 of [21] for the remaining factors. We use the following estimates on the differentiated covariances:

Lemma 3.12. For $d(j, \alpha) \geqslant 1$, $d(j, \gamma) \geqslant 1$

$$\|\partial_s^\gamma \partial_t^\alpha C_j\|_\infty \leqslant M_3(q, \gamma) \exp^{-m_0(1-\delta)d(\alpha)}$$
$$\exp\left[-\frac{m_0}{M_4}(2d(j, \alpha) + d(j, \gamma))\right] \exp\left[-\frac{m_0}{M_7}(|\gamma| + 1)\right] \tag{3.31}$$

where the $M_3(q, \gamma)$ are the same as in Lemma 3.7 and also fulfill (3.22) and (3.23); $q \in [1, \infty)$ is here arbitrary, (but see Section 2, Lemma 2.10).

Lemma 3.13.

$$\|\partial_s^\gamma \partial_t^\alpha C_j\|_q \leqslant M_3(q, \gamma) \exp[-m_0 d(\alpha)(1 - \delta)]$$
$$\exp\left[-\frac{m_0}{M_4}(2d(j, \alpha) + d(j, \gamma))\right] \exp\left[-\frac{m_0}{qM_7}(|\gamma| + 1)\right] \tag{3.32}$$

again with the $M_3(q, \gamma)$ of Lemma 3.7.

The *proofs* of these two lemmas are easy exercises if one uses the technique of the Wiener integral representation developed in [21, 38]; cf. also Section 2. We use

$$|\partial_s^\gamma \partial_t^\alpha C_j| \leqslant |\partial_s^\gamma C_j| \qquad \text{and} \qquad |\partial_s^\gamma \partial_t^\alpha C_j| \leqslant |\partial_t^\alpha C_j|$$

and multiply bounds on $|\partial_s^\gamma C_j|^\eta$ and $|\partial_t^\alpha C_j|^{1-\eta} (0 < \eta < 1)$. Lemma 3.13 is essentially contained in [39], except for the extra factor $e^{-m_0/qM_7(|\gamma|+1)}$ which can be obtained easily (see the proof of Lemma 2.7). As in [39] we obtain $M_3(q, \gamma) = K_6(q, \gamma)^\eta$, where $K_6(q, \gamma)$ is defined in eq. (2.49). Equations (3.22) and (3.23) are proven in Spencer's paper [39]. Q.E.D.

Lemma 3.7 can now be proven similar to Lemma 2.7 if we note that for a given localization $i \in \mathbb{Z}^2$ there is only a finite maximal number q_0 of $\alpha \in I^{(2)}$ or $\gamma \in \pi$ such that $d(j, \alpha) = 0$ or $d(j, \gamma) = 0$ and $j_1 = i$ or $j_2 = i$; so we never need a Hölder index bigger than q_0 to bound $\|w'\|_q$.

This completes the proof of Lemma 3.7.

Thus Lemma 3.6, hence Theorem 3.5, Cor. 3.5′, hence Theorem 3.1 – see [39] – are proven.

This concludes the proof of the upper mass gap. Let us remark that Spencer's method to study the Bethe–Salpeter kernel can also be applied to the massive sine-Gordon model; it gives four particle decay of the Bethe–Salpeter kernel in the case $\theta = 0$ or π (even theory) for sufficiently weak coupling.

4. Scattering in QED$_2$

In this section we discuss the scattering theory of QED$_2$ or, in other words, of the massive sine-Gordon model. We propose to prove the *existence* and *non-triviality* of an isometric scattering matrix and we investigate the dependence of scattering amplitudes on the charges at infinity, i.e., on the angle θ in the sine-Gordon action.

First we note that Theorem 2.1 (existence of the theory and Wightman axioms) and Theorem 3.2 (existence of one-particle states; see also Theorem 3 of the Introduction) guarantee the existence of a Haag–Ruelle scattering theory and of an isometric scattering matrix [26].

Next we want to show that the scattering matrix is non-trivial. For this purpose we prove that perturbation theory in λ about $\lambda = 0$ is asymptotic to the scattering amplitudes of the sine-Gordon model. This result is a rather straightforward consequence of ref. [13]; see also [34].

In order to be able to apply the general (axiomatic) results of [13] we must only verify two basic properties of the Euclidean Green's functions of the sine-Gordon theory that go slightly beyond what we have already proven in Sections 2 and 3 (Lemmas 4.1 and 4.2).

Let $\gamma = (m_0, \lambda, \varepsilon, \theta)$ denote the four bare parameters and $\langle - \rangle_\gamma^c$ the truncated (connected) expectation defined by the infinite volume interacting measure of the massive sine-Gordon model; (see Section 2). Throughout the following Wick ordering is always done with respect to a fixed bare mass, set, e.g., $= 1$. We have to consider the following Euclidean fields:

$$\begin{aligned} \psi_1 &= \varphi, & \psi_2 &= \tfrac{1}{2}{:}\varphi^2{:}, \\ \psi_3 &= {:}\cos(\varepsilon\varphi + \theta){:}_1, & \psi_4 &= {:}\sin(\varepsilon\varphi + \theta){:}_1, \\ \psi_5 &= {:}\varphi\sin(\varepsilon\varphi + \theta){:}_1, & \psi_6 &= {:}\varphi^2\cos(\varepsilon\varphi + \theta){:}_1, \ldots \end{aligned} \tag{4.1}$$

The first lemma we need in order to apply the results of [13] is

Lemma 4.1. Let $|\lambda/m_0^2|$ be so small that the cluster expansion of Section 2 converges. Let $Q(\{x, v\})$ denote $\prod_{i=1}^n \{\psi_{v_i}(x_i); \}$ where $v_i = 1, 2, \ldots, 5, \ldots$. Then (in the sense of distributions)

$$(1) \quad \frac{\partial}{\partial\lambda}\langle Q(\{x, v\})\rangle_\gamma^c = -\int d^2x' \langle Q(\{x, v\}); \psi_3(x')\rangle_\gamma^c$$

$$(2) \quad \frac{\partial}{\partial m_0^2}\langle Q(\{x, v\})\rangle_\gamma^c = -\int d^2x' \langle Q(\{x, v\}); \psi_2(x')\rangle_\gamma^c$$

$$(3) \quad \frac{\partial}{\partial\theta}\langle Q(\{x, v\})\rangle_\gamma^c = \left\langle \frac{\partial}{\partial\theta} Q(\{x, v\})\right\rangle_\gamma^c + \lambda\int d^2x' \langle Q(\{x, v\}); \psi_4(x')\rangle_\gamma^c$$

$$(4) \quad \frac{\partial}{\partial\varepsilon}\langle Q(\{x, v\})\rangle_\gamma^c = \left\langle \frac{\partial}{\partial\varepsilon} Q(\{x, v\})\right\rangle_\gamma^c + \lambda\int d^2x' \langle Q(\{x, v\}); \psi_5(x')\rangle_\gamma^c.$$

Similar equations can be derived for

$$(5) \quad \frac{\partial^{r_1+r_2+r_3+r_4}}{\partial\lambda^{r_1}\,\partial m_0^{2r_2}\,\partial\theta^{r_3}\,\partial\varepsilon^{r_4}}\langle Q(\{x, v\})\rangle_\gamma^c, \quad \text{for arbitrary, finite } r_1, \ldots, r_4.$$

Proof. For suitably space-time cutoff sine-Gordon expectations and all $\varepsilon \in (-2\sqrt{\pi}, 2\sqrt{\pi})$, $m_0 > 0$ the proof of Lemma 4.1 is an easy exercise in applying Corollary 2.5 (see also [16, 17] and [13, Appendix]) which we leave to the reader.

We then apply integration by parts on function space, [9], in order to reduce the r.h.s. of the space-time cutoff versions of equations (1) – (5) to (partially) truncated expectations of polynomials in the fields ψ_1, ψ_3, and ψ_4. (Some of the factors in these polynomials are smeared out with functions of the form $\prod_{l=1}^p C_{x_l}$ where $C_x(y) = C_{m_0}(x - y)$ is the kernel of $(-\Delta + m_0^2)^{-1}$, and these functions are admissible in the estimates of Section 2, since $C_{m_0}(x - y)$ is exponentially decreasing in $|x - y|$ and has only a logarithmic singularity at $x = y$!). Using the fact that

$$\frac{\delta}{\delta\varphi(y)}\psi_1(x) = \delta(x - y)(!),$$

$$\frac{\delta}{\delta\varphi(y)}\psi_3(x) = -\varepsilon\psi_4(x)\,\delta(x - y), \ldots,$$

it is now straightforward to check that the cluster expansion of Section 2 applies to such expectations (after they have been smeared out with some test function). Using the convergence of the cluster expansion and some geometric arguments due to Dimock [6] one verifies that these expectations converge as the space-time cutoffs are removed.

Since in this way convergence of the space-time cutoff version of the r.h.s. of (5), as the cutoffs are removed, is proven for arbitrary, finite $r_1 + \cdots + r_4$, Lemma 4.1 follows. Information on the parameter intervals on which Equations (1) – (5) hold follows of course by inspecting the conditions for the convergence of the cluster expansion derived in Section 2. Note that 0 is an interior point of the corresponding λ- and ε-intervals. Differentiability in θ holds on the whole interval $[0, 2\pi]$. Given λ and ε, differentiability in m_0^2 holds on (M_0^2, ∞) for some finite $M_0 = M_0(\lambda, \varepsilon)$. Also note that the various derivatives of the EGF's at $\lambda = 0$ are free field EGF's and therefore correspond to standard perturbation theory, (a fact already used in Section 2 to identify the derivatives in λ at $\lambda = 0$). Q.E.D.

We remark that the results of Section 2 not only yield analyticity of the EGF's in λ near $\lambda = 0$ but also analyticity of the EGF's in θ, (which strengthens Lemma 4.1, (3)).

The next property of the EGF's we must establish in order to apply the results of [13] is a regularity property.

Lemma 4.2. Suppose that $\varepsilon^2 < 2\pi$. Let $f_1, \ldots, f_n$ be test functions in $\mathscr{S}(\mathbb{R})$, and $\mathring{x}_1 < \mathring{x}_2 < \cdots < \mathring{x}_n$. Then

(1) the EGF's $\left\langle \prod_{i=1}^{n} \psi_{v_i}(f_i \otimes \delta_{\mathring{x}_i}) \right\rangle_\gamma$ are locally integrable functions of $\mathring{x}_1, \ldots, \mathring{x}_n$, and .

(2) $\left| \left\langle \prod_{i=1}^{n} \psi_{v_i}(f_i \otimes \delta_{\mathring{x}_i}) \right\rangle_\gamma \right| \leqslant n! \prod_{i=1}^{n} \|f_i\|_{v_i}$ for some Schwartz norms $\| \cdot \|_{v_i}$.

Proof. It suffices to prove this lemma for the case where $\operatorname{supp} f_i \subset [j_i, j_i + 1]$ some integers j_i, $i = 1, \ldots, n$, with bounds that are uniform in j_i. The general case then follows by summing over all possible localizations, using a C^∞ partition of unity. We now prove (2): By Hölder's inequality

$$\left| \left\langle \prod_{i=1}^{n} \psi_{v_i}(f_i \otimes \delta_{\mathring{x}_i}) \right\rangle_\gamma \right| \leqslant \prod_{i=1}^{n} \langle |\psi_{v_i}(f_i \otimes \delta_{\mathring{x}_i})|^n \rangle_\gamma^{1/n}. \tag{4.2}$$

From Corollary 2.5 we know that

$$\langle |\psi_v(f \otimes \delta_{\mathring{x}})|^q \rangle_0^{1/q} \tag{4.3}$$

is finite if $\varepsilon^2 < 2\pi$ and $f \in L^r$ with $r > (1 - \varepsilon^2/2\pi)^{-1}$, $\operatorname{supp} f$ compact ($v = 1, 2, \ldots$).

We may therefore apply the technique of [36] for proving bounds on Schwinger functions (which extends the 'Glimm–Jaffe φ-bounds'; see [36, Section 8] and [19]) to conclude that

$$\langle |\psi_{v_i}(f_i \otimes \delta_{\mathring{x}_i})|^n \rangle_\gamma^{1/n} \leqslant K(\gamma) \langle |\psi_{v_i}(f_i \otimes \delta_{\mathring{x}_i})|^{pn} \rangle_0^{1/pn} \tag{4.4}$$

for some $p \in (2, \infty)$ only depending on the bare mass m_0 (for all γ for which $\langle - \rangle_\gamma$ exists as a limit of cutoff expectations with free boundary conditions). Here

$$K(\gamma) \leqslant \langle e^{p\lambda U_\Delta} \rangle_0^{1/p},$$

where U_Δ is the sine-Gordon action integrated over a unit square. The second factor on the r.h.s. of (4.4) can be estimated as follows:

$$\langle|\psi_{\nu_i}(f_i \otimes \delta_{x_i})|^{pn}\rangle_0^{1/np} \leqslant np\alpha_r^{(\nu_i)}\|f_i\|_r, \tag{4.5}$$

with $\alpha_r^{(\nu)}$ finite, for all $r < \infty$, if $\nu = 1, 2$, and $\alpha_r^{(\nu)}$ finite, for all $r > (1 - \varepsilon^2/2\pi)^{-1}$, if $\nu = 3, 4, \ldots$. By complex interpolation it suffices to prove (4.5) for the case where pn is an even integer. The bound (4.5) for $\nu = 1, 2$ is standard; (a simple proof follows from hypercontractivity). For $\nu = 3, 4$ the bound (4.5) is contained in [16, 17], and for $\nu = 5, \ldots$ it follows from the one for $\nu = 3, 4$ by using analyticity of $\langle\psi_\nu(f \otimes \delta_x)^{pn}\rangle_0$ in ε in a complex neighbourhood of $(0, \sqrt{2\pi})$ and the Cauchy estimate. Estimates (4.4) and (4.5) yield Lemma 4.2, (2) from which (1) follows immediately. Q.E.D.

Remarks. The reader may wonder why we have assumed that $\varepsilon^2 < 2\pi$ in Lemma 4.2: If $\varepsilon^2 > 2\pi$ estimates (4.3) – (4.5) break down, since $\langle\psi_\nu(f \otimes \delta_{\mathring{x}})^q\rangle_0$ is infinite, for all $\nu \geqslant 3$, $q \geqslant 2$, and $\psi_\nu(f \otimes \delta_{\mathring{x}})$ is not a well-defined random variable. Nevertheless Lemma 4.2, (1) and a generalized version of Lemma 4.2, (2) (see [13, Theorem 3]; this is all that is needed in order for the methods of [13] to be applicable) still seem to be true in the region of convergence of the cluster expansion. A complete proof of this statement would however require some substantial refinements of some of the technical estimates in the cluster expansion and is therefore not attempted here. Let us finally emphasize that for $\nu_i = 1$ or 2, all $i = 1, \ldots, n$, Lemma 4.2 holds without change, for all $\varepsilon^2 < 4\pi$. Combining Lemma 4.1, (1) and (2) and Lemma 4.2 with [13, Theorem 8] we conclude that given a physical mass $\bar{m} > 0$ there exists a positive $\lambda_0(\varepsilon, \bar{m})$ such that the equation $m(\lambda, m_0) = \bar{m}$ has a unique solution $m_0 = m_0(\bar{m}, \lambda)$, for all $\lambda \in (-\lambda_0, \lambda_0)$. We may therefore fix the physical mass throughout most of the following.

Corollary 4.3. For $\varepsilon^2 < 2\pi$ Feynman perturbation theory in λ about $\lambda = 0$ is asymptotic to the scattering amplitudes of the massive sine-Gordon model, and the scattering matrix is non-trivial, provided λ is small enough. The scattering amplitudes are C^∞ in θ and ε.

Proof. The assertion about Feynman perturbation theory follows from Theorem 2.1, the existence of one particle states established in Section 3 and Lemmas 4.1 and 4.2, by Theorems 10 and 12 of [13]. From these results it moreover follows that the scattering amplitudes are C^∞ in θ and ε. To show that the scattering matrix is non-trivial for small enough λ it now suffices to compute, e.g., first order contributions to the scattering amplitudes: To first order in λ the $2 \to 2$ amplitude is given by

$$i\lambda \frac{\varepsilon^4}{4!} \cos\theta\, \delta(p_1 + p_2 - p_3 - p_4), \tag{4.6}$$

where $p = ((\mathbf{p}^2 + \bar{m}^2)^{1/2}, \mathbf{p})$. (We note that in two space-time dimensions formula (4.6) alone does *not* imply non-triviality of the scattering matrix: (4.6) is consistent with $S = \exp i\lambda\, (\varepsilon^4/4!) \cos\theta$.) The $2 \to 4$ amplitude is given by

$$-i\lambda \frac{\varepsilon^6}{6!} \cos\theta\, \delta(p_1 + p_2 - p_3 - p_4 - p_5 - p_6), \tag{4.7}$$

and the $2 \to 3$ amplitude by

$$-i\lambda \frac{\varepsilon^5}{5!} \sin\theta\, \delta(p_1 + p_2 - p_3 - p_4 - p_5). \tag{4.8}$$

Formulas (4.6) – (4.8) not only prove that scattering is non-trivial for small λ but also show that it depends non-trivially on the angle θ, or, in QED_2 language, on the *charges at infinity*; [4]. Q.E.D.

Apart from the scattering amplitudes of the fundamental (lightest) particles of the theory the *bound state spectrum* of the sine-Gordon theory depends on θ, as well:

Combining Lemmas 4.1 and 4.2 and Theorems 2.1, 3.1, and 3.2 with Theorems 7 and 10 and (I.11), (I.12) of [13] one concludes that for *fixed* m_0 perturbation theory in λ about $\lambda = 0$ is asymptotic to $m^2(\lambda, m_0)$. This and the results of [40] and [22, §3.4] prove that for $\theta = \pi$ and small enough $\lambda > 0$ there exists a *two particle bound state* with a mass in the interval $[m, 2m]$. This is because the interaction between two particles is attractive for $\theta = \pi$ and very small $\lambda > 0$ (see (4.6)). However for $\theta = 0$ and very small $\lambda > 0$ there is *no* such bound state! (See [40]; $\theta = 0$ and $\lambda > 0$ is of course the same as $\theta = \pi$ and $\lambda < 0$).

Next we briefly discuss scattering for small, but not necessarily very small coupling constants λ and $\varepsilon^2 < 4\pi$:

(1) Applying Equations (I.11) – (I.13) of [13] and combining them with Theorem 2.1 and the Malgrange–Zerner [12, 28] theorem we conclude that there exist positive numbers $\lambda_1(\varepsilon, \bar{m})$ and $\delta_1(\varepsilon, \bar{m})$ such that for $|\mathrm{Re}\,\lambda| < \lambda_1$, $|\mathrm{Im}\,\lambda| < \delta_1$ the equation

$$m(\lambda, m_0) = \bar{m}$$

has a *unique* solution $m_0^2 = m_0^2(\bar{m}, \lambda) \in \mathbb{C}$ that is analytic in λ for $|\mathrm{Re}\,\lambda| < \lambda_1$, $|\mathrm{Im}\,\lambda| < \delta_1$.

The cluster expansion of Section 2 yields joint analyticity of the EGF's in λ and m_0^2, for $|\lambda| < \lambda_0$ and m_0 in some complex neighborhood of (M_0, ∞); see [21]. Thus, as functions of λ and $\bar{m}$, the EGF's are analytic in λ, for $|\mathrm{Re}\,\lambda| < \lambda_2$ and $|\mathrm{Im}\,\lambda| < \delta_2$, for some positive $\lambda_2 \leqslant \lambda_1$ $(\leqslant \lambda_0)$ and $\delta_2 \leqslant \delta_1$.

Using now results of [13] and refs. given there and combining them with the Malgrange–Zerner theorem one shows that real analyticity in λ, for fixed, real $\bar{m}$ holds for the restrictions of the amputated, generalized Green's (H-) functions to the whole complex mass shell, for all real λ at which the EGF's are analytic. The scattering amplitudes on (sub-) regions of the physical points can be obtained as various boundary values of the generalized Green's functions; see [2].

Applying now the edge of the wedge theorem we conclude that the scattering amplitudes as functions of λ cannot vanish in any interval contained in the domain of holomorphy of the EGF's (which contains in particular the interval $(-\lambda_2, \lambda_2)$).

(2) Similar arguments imply that the scattering amplitudes of the massive sine-Gordon model that are non-trivial for small ε and some given, real λ cannot vanish in any ε-interval contained in $(-2\sqrt{\pi}, 2\sqrt{\pi})$. The outline of the proof is as follows: For $v = 1, 2$ and all $\varepsilon^2 < 4\pi$ both Lemmas 4.1 *and* 4.2 hold. It is easy to check (by inspection of the explicit expressions derived in [16, 17]) that for bounded functions $h_1, \ldots, h_n$ of compact support and $v_i = 3$ or 4, $i = 1, \ldots, n$, the expectations

$$\langle \prod_{i=1}^{n} \psi_{v_i}(h_i) \rangle_0$$

are analytic in ε in some complex neighborhood of $(0, 2\sqrt{\pi})$. The same is then true for $\langle e^{\lambda U_\Lambda} \rangle_0$, and $\frac{1}{2} \leqslant |\langle e^{\lambda U_\Lambda} \rangle|_0 \leqslant 3/2$ for such ε, provided $|\lambda|$ is small enough. Estimates on such expectations are uniform in ε in compact subsets of some complex neighborhood of $(0, 2\sqrt{\pi})$. Therefore we obtain convergence of the expansions of Sections 2 and 3 for all ε in some complex neighborhood of $(0, 2\sqrt{\pi})$ and small λ. This combined with Lemma 4.2 (for $v_1 = 1$ or 2) and formulas (I.11) – (I.13) of [13] proves that for fixed bare mass $m_0 > M_0$ the physical mass $\bar{m}$ is jointly analytic in λ and ε in some complex domain in $\mathbb{C}^2$. Restricting the amputated, generalized Green's functions to the complex $\bar{m}$-mass shell, applying analytic completion in the complex mass shell variables, λ and ε and repeating then the arguments outlined in (1) (but this time with m_0 rather than $\bar{m}$ fixed) proves our assertion.

Theorems 2.1, 3.1 and 3.2 combined with the strong results of [40] concerning the properties of the energy-momentum spectrum (for small energies, in models with a Bethe–Salpeter kernel that has four-particle decay) and with the Malgrange–Zerner theorem suggest that for energies between $2m + \delta$ and $4m - \delta$ (with $\delta > 0$ arbitrarily small) the Feynman perturbation series for the two-particle scattering amplitude in λ about $\lambda = 0$ has a *finite, positive radius of convergence* (that presumably will depend on δ – perturbation theory can of course not be expected to converge at the thresholds).

The results we have proven are not strong enough, however, to draw this conclusion. Since complete results concerning these questions have not been worked out yet we omit further details.

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