Zeitschrift: Helvetica Physica Acta
Band: 49 (1976)
Heft: 6

Artikel: A general theory of relativistic gravitational energy-momentum localization

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DOI: https://doi.org/10.5169/seals-114793

Helvetica Physica Acta, Vol. 49 (1976), Birkhäuser Verlag, Basel

A General Theory of Relativistic Gravitational Energy-Momentum Localization

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(21. III. 1976)

Abstract. In the framework of Scherrer's linear (tetradic) formalism, field equations are derived from a general variational principle (with the constraints $g_{\mu\nu} = e_\alpha g^{\alpha\cdot}{}_{,\mu} g^{\alpha\cdot}{}_{,\nu}$). By elimination of the Lagrange multipliers, we obtain six antisymmetric equations. With the ten constraints quoted above, these 'supplementary conditions' constitute a system of sixteen equations for the 'tetrads' $g^{\lambda\cdot}{}_{,\mu}(x)$. This system, with appropriate boundary conditions, should determine the $g^{\lambda\cdot}{}_{,\mu}$ unambiguously. It is then possible to express the gravitational energy-momentum (GEM) density at each point of space-time (localization). The study of the weak fields then allows us to give a physical justification of our theory, in contrast to other formalisms, which are more founded on considerations of a mathematical nature. This paper is thus a generalization of [PA 38 481 (1976)] where we gave a physical argument in favour of GEM-localizability in the static case.

1. Introduction

The actual state of research, in the problem of relativistic gravitational energy-momentum (GEM) localization, is as follows. Some physicists (for example: Landau/Lifshitz, Wheeler/Misner/Thorne) think that GEM is not localizable. On the other hand, tetradic formalisms have been developed and have allowed one to solve the problem in favour of localizability, at least from a formal point of view (for example: Scherrer, Møller, Rodichev/Zadonskii). But all these theories have a major defect: they contain specific conditions for the tetrads, which are not physically justified. According to Møller [1] (and we agree with him), only a good physical argument in favour of these 'supplementary conditions' would permit the claim: GEM is localizable.

In a recent paper [2], we showed that such an argument exists for the static case. By analogy with the Newtonian case, we postulated that the total gravitational energy of the system is an extremum: δ (total gravitational energy) $= 0$. This physically natural requirement leads to field equations that are necessary for a precise determination of the 'tetrads'. But these equations are covariant only under the group of coordinate transformations consistent with the static character of the problem, namely the purely spatial coordinate transformations. To obtain covariant equations in the most general case, it is obviously necessary to start with a Lagrangian, the form of which is a scalar density. The present paper is devoted to the development of this idea.

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1) Supported by the Swiss National Research Fund.

2. Construction of the Lagrangian

We construct our theory in the general framework of Scherrer's linear (tetradic) formalism [3]. Because of this, we briefly recall its outlines.

The 'tetrads' $g^{\lambda\cdot}{}_{,\mu}(x)$ are connected with the Einsteinian metric tensor $g_{\mu\nu}(x)$ by the formulas:

$$g_{\mu\nu}(x) = e_\alpha g^{\alpha\cdot}{}_{,\mu}(x) g^{\alpha\cdot}{}_{,\nu}(x). \tag{2.1}$$

We can consider the $g^{\lambda\cdot}{}_{,\mu}$ as covariant (coordinate) vectors. Under a coordinate transformation $x \to \bar{x}$, the $g^{\lambda\cdot}{}_{,\mu}$ change as follows:

$$g^{\lambda\cdot}{}_{,\mu} = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \bar{g}^{\lambda\cdot}{}_{,\alpha}. \tag{2.2}$$

Let us differentiate with respect to x^ν:

$$\frac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^\nu} = \frac{\partial^2 \bar{x}^\alpha}{\partial x^\mu \partial x^\nu} \bar{g}^{\lambda\cdot}{}_{,\alpha} + \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \frac{\partial \bar{g}^{\lambda\cdot}{}_{,\alpha}}{\partial \bar{x}^\beta}. \tag{2.3}$$

Thus the $\partial g^{\lambda\cdot}{}_{,\mu}/\partial x^\nu$ do not constitute a tensor. Interchanging the indices μ and ν, and subtracting (2.3) from the obtained equality, we have:

$$\frac{1}{2}\left(\frac{\partial g^{\lambda\cdot}{}_{,\nu}}{\partial x^\mu} - \frac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^\nu}\right) = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu}\left[\frac{1}{2}\left(\frac{\partial \bar{g}^{\lambda\cdot}{}_{,\beta}}{\partial \bar{x}^\alpha} - \frac{\partial \bar{g}^{\lambda\cdot}{}_{,\alpha}}{\partial \bar{x}^\beta}\right)\right] \tag{2.4}$$

which proves that the quantities:

$$f^{\lambda\cdot}{}_{,\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial g^{\lambda\cdot}{}_{,\nu}}{\partial x^\mu} - \frac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^\nu}\right) \tag{2.5}$$

form a 2-covariant tensor (for each value of λ). It is the simplest tensor that depends on the first derivatives of the $g^{\lambda\cdot}{}_{,\mu}$. We can of course form the tensor $D_\nu g^{\lambda\cdot}{}_{,\mu}$, where D_ν is the usual covariant derivative. But these expressions can be written as follows:

$$D_\nu g^{\lambda\cdot}{}_{,\mu} = f^{\lambda\cdot}{}_{,\nu\mu} + g^{\lambda\cdot}{}_{,\alpha}(f_{,\mu}{}^\alpha{}_\nu + f_{,\nu}{}^\alpha{}_\mu) \tag{2.6}$$

with $f_{;}{}^{\lambda}{}_{\mu\nu} \equiv g_{\alpha,}{}^{\cdot\lambda} f^{\alpha\cdot}{}_{,\mu\nu} (g_{\alpha,}{}^{\cdot\lambda} g^{\alpha\cdot}{}_{,\mu} = \delta^\lambda_\mu)$, namely a linear combination of the $f^{\lambda\cdot}{}_{,\mu\nu}$. The symmetric part of the $\partial g^{\lambda\cdot}{}_{,\nu}/\partial x^\mu$:

$$\gamma^{\lambda\cdot}{}_{,\mu\nu} \equiv \frac{1}{2}\left(\frac{\partial g^{\lambda\cdot}{}_{,\nu}}{\partial x^\mu} + \frac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^\nu}\right) \tag{2.7}$$

is not a tensor. Nevertheless, the expressions:

$$\gamma_{;}{}^{\lambda}{}_{\mu\nu} \equiv g_{\alpha,}{}^{\cdot\lambda} \gamma^{\alpha\cdot}{}_{,\mu\nu} \tag{2.8}$$

play an important role in this formalism, analogous to that of the $\Gamma^\lambda{}_{\mu\nu}$ of the quadratic theory. Indeed it can be proved that the expressions:

$$\partial_{,\mu} v^{\cdot\lambda} \equiv \frac{\partial v^{\cdot\lambda}}{\partial x^\mu} + \gamma_{;}{}^{\lambda}{}_{\alpha\mu} v^{\cdot\alpha} \tag{2.9}$$

are the components of a 1-contravariant and 1-covariant tensor. It is the *covariant derivative* (*in the linear theory*) *of the vector* $v^{\cdot\lambda}$. In particular:

$$\partial_{,\nu} g^{\lambda\cdot}{}_{,\mu} \equiv f^{\lambda\cdot}{}_{,\nu\mu}. \tag{2.10}$$

Our purpose is now to form all the elementary invariants that depend only on the $g^{\lambda}{}_{,\mu}$ and their first derivatives. One can easily verify that there are only three possibilities:

$$\underset{1}{H} \equiv f^{\alpha\beta\gamma}f_{\alpha\beta\gamma} \quad \underset{2}{H} \equiv f^{\alpha\beta\gamma}f_{\gamma\beta\alpha} \quad \underset{3}{H} \equiv f^{\alpha}f_{\alpha} \tag{2.11}$$

with the definitions:

$$\begin{aligned} f^{\lambda}{}_{\mu\nu} &\equiv g_{\mu,}{}^{,\alpha}g_{\nu,}{}^{,\beta}f^{\lambda,}{}_{,\alpha\beta} \quad f_{\lambda} \equiv f^{\alpha}{}_{\lambda\alpha} \\ f^{\lambda\mu\nu} &\equiv \eta^{\mu\alpha}\eta^{\nu\beta}f^{\lambda}{}_{\alpha\beta} \equiv e^{\mu}e^{\nu}f^{\lambda}{}_{\mu\nu} \text{ etc.} \end{aligned} \tag{2.12}$$

$\underset{1}{\Lambda}, \underset{2}{\Lambda}, \underset{3}{\Lambda}$ being three arbitrary real numbers, we construct the Lagrangian density:

$$\mathfrak{H} \equiv \underset{1}{\Lambda}\underset{1}{\mathfrak{H}} + \underset{2}{\Lambda}\underset{2}{\mathfrak{H}} + \underset{3}{\Lambda}\underset{3}{\mathfrak{H}} \tag{2.13}$$

where $\underset{i}{\mathfrak{H}} \equiv g\underset{i}{H}$, $g \equiv \det(g^{\lambda,}{}_{,\mu})$. Scherrer showed [4] that with $\underset{1}{\Lambda} = \frac{1}{2}$, $\underset{2}{\Lambda} = 1$, $\underset{3}{\Lambda} = -2$, the variational principle:

$$\delta \int \mathfrak{H} d^4x = 0 \tag{2.14}$$

leads to the ten Einstein equations for the exterior case. Obviously, these do not allow us to determine the 16 $g^{\lambda,}{}_{,\mu}$ unambiguously. We derive the six necessary supplementary conditions in Section 4.

3. Curvature Tensors of the Linear Formalism

By analogy with the quadratic formalism, one can construct a 'curvature tensor' from the $\gamma_{;}^{;\lambda}{}_{\mu\nu}$ and their first derivatives:

$$r_{;}^{;\lambda}{}_{\rho\sigma\tau} \equiv \frac{\partial \gamma_{;}^{;\lambda}{}_{\rho\sigma}}{\partial x^{\tau}} - \frac{\partial \gamma_{;}^{;\lambda}{}_{\rho\tau}}{\partial x^{\sigma}} + \gamma_{;}^{;\lambda}{}_{\tau\alpha}\gamma_{;}^{;\alpha}{}_{\rho\sigma} - \gamma_{;}^{;\lambda}{}_{\sigma\alpha}\gamma_{;}^{;\alpha}{}_{\rho\tau}. \tag{3.1}$$

By contraction, we get successively:

$$r_{,\rho\tau} \equiv r_{;}^{;\alpha}{}_{\rho\alpha\tau} \qquad (\neq r_{,\tau\rho} \text{ in general!}) \tag{3.2}$$

$$r \equiv r_{;}^{;\alpha}{}_{\alpha} \equiv r_{,\alpha}{}^{\alpha}. \tag{3.3}$$

r is the *curvature scalar of the linear theory*, invariant under the regular coordinate transformations and the (global) Lorentz transformations. But r is not in general invariant under tetrad transformations consistent with (2.1). This scalar plays an important role in our theory (see Section 5).

On the other hand, $\mathfrak{r} \equiv gr$ can be written (see Appendix A):

$$\mathfrak{r} \equiv \frac{\partial}{\partial x^{\alpha}}(gf^{,\alpha}) + g(\underset{2}{H} - 2\underset{3}{H}). \tag{3.4}$$

This form will be useful in what follows.

4. The Field Equations for the Tetrads

At the end of Section 2, we mentioned the variational principle that leads to Einstein's equations for the exterior case. The corresponding Lagrangian density is obtained by putting $\underset{1}{\Lambda} = \frac{1}{2}$, $\underset{2}{\Lambda} = 1$, $\underset{3}{\Lambda} = -2$ in (2.13).

The expression (2.13) being the most general scalar density that depends only on the $g^{\lambda\cdot}{}_{,\mu}$ and their first derivatives[1]), we postulate that the field equations for the $g^{\lambda\cdot}{}_{,\mu}$ be derived from a variational principle analogous to (2.14), but with constants $\underset{i}{\wedge}$ whose ratio is different from $(\frac{1}{2}:1:-2)$. On the other hand, we have to take into consideration the conditions (2.1), so that we treat in fact a *variational principle with constraints* (exterior case):

$$\delta \int \mathfrak{H} d^4x = 0 \qquad (\underset{1}{\wedge}:\underset{2}{\wedge}:\underset{3}{\wedge} \neq \tfrac{1}{2}:1:-2) \tag{4.1}$$

and

$$G_{\mu\nu} \equiv g_{\mu\nu} - e_\alpha g^{\alpha\cdot}{}_{,\mu} g^{\alpha\cdot}{}_{,\nu} = 0. \tag{4.1'}$$

(4.1) and (4.1′) lead to the following field equations:

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial \mathfrak{H}^*}{\partial(\partial g^{\lambda\cdot}{}_{,\mu}/\partial x^\alpha)}\right] - \frac{\partial \mathfrak{H}^*}{\partial g^{\lambda\cdot}{}_{,\mu}} = 0 \tag{4.2}$$

with:

$$\mathfrak{H}^* \equiv \mathfrak{H} + \lambda^{\alpha\beta}(x) G_{\alpha\beta}(x) \tag{4.3}$$

where the $\lambda^{\alpha\beta}(x)$ are ten symmetric Lagrange multipliers (tensor density of rank 2). *The sixteen equations (4.2) and the ten constraints (4.1′), together with adequate boundary conditions, should determine the sixteen tetradic components $g^{\lambda\cdot}{}_{,\mu}(x)$ and the ten Lagrange multipliers* $\lambda^{\mu\nu}(x)$. Furthermore it is obviously necessary to know the ratio of the coefficients $(\underset{1}{\wedge}:\underset{2}{\wedge}:\underset{3}{\wedge})$. We shall return to this point in the next section.

One can in fact considerably simplify this procedure. The characteristic indices λ, μ of the equations (4.2) are ordered according to the scheme $({}_{\lambda,}{}^{\cdot\mu})$. After multiplication of these relations by $e_\nu g^{\nu\cdot}{}_{,\mu}$, the two characteristic indices are below and on the left of the comma: $({}_{\lambda\nu,})$. By symmetrization, the sixteen field equations can then be replaced by an equivalent system of ten symmetric equations containing the $\lambda^{\mu\nu}(x)$ and *six antisymmetric equations not containing the Lagrange multipliers* (indeed one can easily verify that the $\lambda^{\mu\nu}$-contribution of the Lagrangian (4.3) to the equations (4.2) is purely symmetric). To obtain the latter, it is thus sufficient to compute the purely antisymmetric part of the field equations. The antisymmetric contributions of the $\underset{i}{\mathfrak{H}}$ are given by the following expressions (see Appendix B):

$$\begin{aligned}
&\underset{1}{\mathfrak{H}} \to \underset{1}{\vee}_{[\lambda\mu]} \equiv (\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda)\\
&\underset{2}{\mathfrak{H}} \to \underset{2}{\vee}_{[\lambda\mu]} \equiv \tfrac{1}{2}(\partial_\alpha + 2f_\alpha)(f_\mu{}^\alpha{}_\lambda - f_\lambda{}^\alpha{}_\mu) + (\partial_\alpha + 2f_\alpha) f^\alpha{}_{\lambda\mu}\\
&\underset{3}{\mathfrak{H}} \to \underset{3}{\vee}_{[\lambda\mu]} \equiv -\tfrac{1}{2}(\partial_\lambda f_\mu - \partial_\mu f_\lambda)
\end{aligned} \tag{4.4}$$

where $\partial_\lambda \equiv g_{\lambda,}{}^{\cdot\alpha}\partial_{,\alpha}$ (see (2.9)). On the other hand, one can prove the identity:

$$(\partial_\alpha + 2f_\alpha) f^\alpha{}_{\lambda\mu} \equiv -(\partial_\lambda f_\mu - \partial_\mu f_\lambda). \tag{4.5}$$

$\mathfrak{H} \equiv \underset{1}{\wedge}\underset{1}{\mathfrak{H}} + \underset{2}{\wedge}\underset{2}{\mathfrak{H}} + \underset{3}{\wedge}\underset{3}{\mathfrak{H}}$ thus gives the following antisymmetric equations:

$$\underset{1}{\wedge}\underset{1}{\vee}_{[\lambda\mu]} + \underset{2}{\wedge}\underset{2}{\vee}_{[\lambda\mu]} + \underset{3}{\wedge}\underset{3}{\vee}_{[\lambda\mu]} = 0. \tag{4.6}$$

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1) Of course we don't consider more complicated invariants such as $\underset{1}{H}\underset{2}{H}$, $\underset{3}{H}^2$, etc.

If we bear in mind the formulas (4.4) and (4.5), this becomes:

$$(\underset{1}{\Lambda} - \tfrac{1}{2}\underset{2}{\Lambda})(\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda) - (\underset{2}{\Lambda} + \tfrac{1}{2}\underset{3}{\Lambda})(\partial_\lambda f_\mu - \partial_\mu f_\lambda) = 0. \tag{4.7}$$

Notice that the choice $\underset{1}{\Lambda} = \frac{1}{2}$, $\underset{2}{\Lambda} = 1$, $\underset{3}{\Lambda} = -2$ makes (4.7) identically zero. This corresponds to Einstein's equations. Discarding this possibility, we can write quite generally the *six supplementary conditions for the tetrads* as follows:

$$A(\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda) + B(\partial_\lambda f_\mu - \partial_\mu f_\lambda) = 0. \tag{4.8}$$

A and B are two real numbers, whose ratio we shall determine by studying the case of the weak fields. With this reservation, we can claim that the equations (4.8) constitute a very satisfactory mathematical 'solution' of the problem of determining the tetrads. Indeed, *we started from the most general Lagrangian which depends only on the $g^{\lambda\cdot}{}_{,\mu}$ and their first derivatives.*

At this point of the development, it is natural to compare the supplementary conditions (4.8) to those of Møller [5]. Simple but tedious calculations give the result that both the systems are equivalent if we put:

$$\alpha = A \quad \beta = -\frac{A+B}{2} \quad \gamma = -A \quad \delta = -\frac{A+B}{2} \tag{4.9}$$

(for the meaning of the coefficients α, β, γ, δ, see [5]). The role of this remark is important in the case of the weak field (see Section 5).

5. The Case of the Weak Field

In [5], Møller discusses exhaustively the case of the weak field. In particular, he proves that the solutions of his equations are independent of the coefficients α, β, γ, δ. It then results from the final remark of Section 4 that the solution obtained by Møller is also valid for the equations (4.8) in the case of the weak field. In this section we shall simply repeat the essentials.

We say that the gravitational field of an insular system is *weak* if a coordinate system exists in which the $g_{\mu\nu}$ have the form:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \qquad (|h_{\mu\nu}| \ll 1). \tag{5.1}$$

The relations (2.1) then allow us to write the tetrads as follows:

$$g^{\lambda\cdot}{}_{,\mu}(x) = \delta^\lambda{}_\mu + q^{\lambda\cdot}{}_{,\mu}(x) \qquad (|q^{\lambda\cdot}{}_{,\mu}| \ll 1). \tag{5.2}$$

Space-time being asymptotically flat, we introduce a coordinate system, Lorentzian at infinity, in which the $g^{\lambda\cdot}{}_{,\mu}$ have to satisfy the following boundary conditions:

A. $g^{\lambda\cdot}{}_{,\mu}(x) \to \delta^\lambda{}_\mu$ for $r \to \infty$.

B. $g^{\lambda\cdot}{}_{,\mu}(x) - \delta^\lambda{}_\mu$ shows the same asymptotic behaviour as the metric quantities $g_{\mu\nu} - \eta_{\mu\nu}$ for an insular system. This means that, if $\psi(x)$ is any of these quantities, ψ must satisfy the condition of outward radiation:

$$\lim\left(\frac{\partial(r\psi)}{\partial r} + \frac{1}{c}\frac{\partial(r\psi)}{\partial t}\right) = 0$$

for $r \to \infty$ and for all values of $t_0 = t + r/c$ in an arbitrary fixed interval. Moreover, $\psi(x)$ and its first-order derivatives $\partial\psi/\partial x^\mu$ must be everywhere bounded and must go to zero at least as $1/r$ for $r \to \infty$.

If in addition we choose to work in a harmonic coordinate system, characterized by de Donder's conditions:

$$\frac{\partial}{\partial x^\alpha}(h_\mu{}^\alpha - \tfrac{1}{2}\delta_\mu{}^\alpha h) = 0. \tag{5.3}$$

Møller's supplementary conditions then lead to the unique solution:

$$g^{\lambda\cdot}{}_{,\mu}(x) = \delta^\lambda{}_\mu + \tfrac{1}{2}h^\lambda{}_\mu(x). \tag{5.4}$$

The tetradic components $g^{\lambda\cdot}{}_{,\mu}(x)$ are thus expressed very simply as functions of the metric tensor $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$.

Now let us return to the equations (4.8). Is it possible that A is zero? No, for our supplementary conditions would then be reduced to $\partial_\lambda f_\mu - \partial_\mu f_\lambda = 0$, or, in the case of the weak field, $\partial f_\mu/\partial x^\lambda - \partial f_\lambda/\partial x^\mu = 0$. Thus it would be sufficient that the vector f_λ be the gradient of an arbitrary potential function (compatible with the boundary conditions) for these relations to be automatically satisfied. These equations would not be restrictive enough. It follows that A cannot be zero (by similar considerations, Møller showed that $\alpha \neq 0$). On the other hand, suppose quite generally $B \neq 0$ and $B \neq A$ (see (4.9) and [5]) and let $g^{\lambda\cdot}{}_{,\mu}(x)$ be the solution of the equations (4.8) and (2.1) (weak field). The remark at the end of Section 4 allows us to assert that these $g^{\lambda\cdot}{}_{,\mu}(x)$ are also a solution of the equations $(\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda) = 0$. It follows that our $g^{\lambda\cdot}{}_{,\mu}(x)$ are also a solution of $\partial_\lambda f_\mu - \partial_\mu f_\lambda = 0$.

Let us summarize. In the case of the weak field, in harmonic coordinates which are Lorentzian at infinity, the solution of equations (4.8) and (2.1) is simultaneously a solution of the systems:

$$(\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda) = 0 \tag{5.5}$$

$$\partial_\lambda f_\mu - \partial_\mu f_\lambda = 0 \tag{5.5'}$$

with the boundary conditions A, B. It is essential to understand that a solution of (5.5) is necessarily a solution of (5.5′), whereas the converse is not true in general. For that reason, in the case of the weak field, we can consider the equations (5.5) as the desired supplementary conditions.

Starting from these considerations, *we simply postulate that the supplementary conditions (5.5) are still valid in the general case:*

$$\boxed{(\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda) = 0.} \tag{5.6}$$

Thus, there are no more undetermined constant coefficients in the desired conditions.

Our conviction that (5.6) are really the 'good' equations is corroborated by the following fact. In Einstein's theory, the field equations for the $g_{\mu\nu}(x)$ can be derived from the variational principle (exterior case):

$$\delta \int \mathfrak{R}\, d^4x = 0 \tag{5.7}$$

where $\mathfrak{R}$ is the scalar curvature density. But we saw in Section 3 that it is possible to define a curvature scalar of the linear formalism (formulas (3.1)–(3.3)). By analogy with (5.7), let us write the variational principle:

$$\delta \int \mathfrak{r}\, d^4x = 0 \tag{5.8}$$

taking into consideration the constraints (2.1). Because of the formula (3.4) ($(\partial/\partial x^\alpha)(gf^{\cdot\alpha})$ is an ordinary divergence), the Lagrangian density $\mathfrak{r}$ of (5.8) is equivalent to $g(\underset{2}{H} - 2\underset{3}{H})$. (5.8) then leads us to the supplementary conditions for the tetrads:

$$\underset{2}{V}_{[\lambda\mu]} - 2\underset{3}{V}_{[\lambda\mu]} = 0 \tag{5.9}$$

(see (4.4)). If we take into account the identity (4.5), one easily sees that the equations (5.9) are precisely the conditions (5.6). From a purely aesthetic point of view, it seems very satisfactory to us that the latter derive from a variational principle formally analogous to Einstein's one.

The Schwarzschild–Scherrer solution mentioned in [2] is in fact a solution of (5.6) and (2.1). But the diagonal Schwarzschild–Scherrer solution quoted in the same paper does not satisfy the conditions (5.6). Here one could of course object that the relations (5.6) were established with the help of purely mathematical considerations. Nevertheless we think that they are justified by sound physical arguments.

(a) We obtained our supplementary conditions from a *variational principle*. The importance of such principles is well known in physics.

(b) In the case of the *static weak field* (Newton), one easily verifies that the solution obtained from (5.6) also satisfies the extremum principle of gravitational energy developed in [2], which is a quite natural physical requirement.

(c) In the same Newtonian case, the spatial part of the vector $2c^2 f_\mu$ constitutes the gravitational field [6]. The relations (5.5′) (which result from (5.5)) then give rot $\vec{G} = 0$. It is interesting to remark that, in this approximation, our conditions are precisely reduced to one of the classical equations that determine the gravitational field. The equations (5.6) are, so to speak, the relativistic generalization of rot $\vec{G} = 0$.

6. Conclusion

Schematically, one can present the solution of the problem of relativistic GEM-localization as follows:

(1) Solve Einstein's equations for the $g_{\mu\nu}(x)$:

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\kappa_E T_{\mu\nu} \tag{6.1}$$

where $\kappa_E \equiv 8\pi G/c^4$, $T_{\mu\nu} \equiv$ matter tensor ($T_{\mu\nu}$ being symmetric and the supplementary conditions (5.6) antisymmetric, the latter are also valid in the interior case).

(2) Taking into account the constraints:

$$g_{\mu\nu}(x) = e_\alpha g^{\alpha\cdot}{}_{,\mu}(x) g^{\alpha\cdot}{}_{,\nu}(x) \tag{6.2}$$

and the boundary conditions A, B of Section 5, solve the equations for the $g^{\lambda\cdot}{}_{,\mu}(x)$:

$$(\partial_\alpha + 2f_\alpha)(f_\lambda{}^\alpha{}_\mu - f_\mu{}^\alpha{}_\lambda) = 0 \tag{6.3}$$

where $\partial_\alpha \equiv g_{\alpha,}{}^{\cdot\beta}\partial_{,\beta}$ (see Section 2).

(3) Compute the $f^{\lambda\cdot}{}_{,\mu\nu}$ and introduce them into the expression of the GEM-density:

$$\mathfrak{T}_{\lambda,}{}^{\cdot\mu} \equiv \tfrac{1}{2}\underset{1}{\mathfrak{T}}_{\lambda,}{}^{\cdot\mu} + \underset{2}{\mathfrak{T}}_{\lambda,}{}^{\cdot\mu} - 2\underset{3}{\mathfrak{T}}_{\lambda,}{}^{\cdot\mu} \tag{6.4}$$

where

$$\underset{1}{\mathfrak{T}}_{\lambda,}{}^{,\mu} \equiv g[-4f_{\alpha\lambda\beta}f^{\alpha\gamma\beta}g_{\gamma,}{}^{,\mu} + g_{\lambda,}{}^{,\mu}\underset{1}{H}]$$

$$\underset{2}{\mathfrak{T}}_{\lambda,}{}^{,\mu} \equiv g[2(f^{\alpha\beta\gamma} - f^{\beta\alpha\gamma})f_{\gamma\lambda\beta}g_{\alpha,}{}^{,\mu} + g_{\lambda,}{}^{,\mu}\underset{2}{H}] \tag{6.5}$$

$$\underset{3}{\mathfrak{T}}_{\lambda,}{}^{,\mu} \equiv g[2f^{\alpha}(f^{\beta}{}_{\lambda\alpha}g_{\beta,}{}^{,\mu} - f_{\lambda}g_{\alpha,}{}^{,\mu}) + g_{\lambda,}{}^{,\mu}\underset{3}{H}].$$

Gravitational energy-momentum is thus localized.

Appendix A

We give the proof of the relation (3.4).

The covariant derivatives of the linear theory (see (2.9)) do not commute. The commutation relations are formally analogous to those of the quadratic theory:

$$(\partial_{,\sigma}\partial_{,\rho} - \partial_{,\rho}\partial_{,\sigma})T_{;}^{;\lambda}{}_{\mu} = r_{;}^{;\lambda}{}_{\alpha\rho\sigma}T_{;}^{;\alpha}{}_{\mu} - r_{;}^{;\alpha}{}_{\mu\rho\sigma}T_{;}^{;\lambda}{}_{\alpha}. \tag{A.1}$$

Let us apply this formula to the covariant vectors $g^{\lambda,}{}_{,\mu}$:

$$(\partial_{,\sigma}\partial_{,\rho} - \partial_{,\rho}\partial_{,\sigma})g^{\lambda,}{}_{,\mu} = -r_{;}^{;\alpha}{}_{\mu\rho\sigma}g^{\lambda,}{}_{,\alpha} = -r^{\lambda,}{}_{,\mu\rho\sigma}. \tag{A.2}$$

On the other hand, one can prove:

$$\partial_{,\mu}g^{\lambda,}{}_{,\nu} = \frac{\partial g^{\lambda,}{}_{,\nu}}{\partial x^{\mu}} - \gamma_{;}^{;\alpha}{}_{\mu\nu}g^{\lambda,}{}_{,\alpha} = \frac{\partial g^{\lambda,}{}_{,\nu}}{\partial x^{\mu}} - \gamma^{\lambda,}{}_{,\mu\nu}$$

$$= \frac{\partial g^{\lambda,}{}_{,\nu}}{\partial x^{\mu}} - \frac{1}{2}\left(\frac{\partial g^{\lambda,}{}_{,\nu}}{\partial x^{\mu}} + \frac{\partial g^{\lambda,}{}_{,\mu}}{\partial x^{\nu}}\right) = \frac{1}{2}\left(\frac{\partial g^{\lambda,}{}_{,\nu}}{\partial x^{\mu}} - \frac{\partial g^{\lambda,}{}_{,\mu}}{\partial x^{\nu}}\right)$$

namely:

$$\partial_{,\mu}g^{\lambda,}{}_{,\nu} = f^{\lambda,}{}_{,\mu\nu}. \tag{A.3}$$

Taking into account (A.3), (A.2) can be rewritten:

$$\partial_{,\rho}f^{\lambda,}{}_{,\sigma\mu} + \partial_{,\sigma}f^{\lambda,}{}_{,\mu\rho} = r^{\lambda,}{}_{,\mu\rho\sigma} \tag{A.4}$$

and, by cyclic permutation of ρ, σ, μ:

$$\partial_{,\rho}f^{\lambda,}{}_{,\sigma\mu} + \partial_{,\sigma}f^{\lambda,}{}_{,\mu\rho} + \partial_{,\mu}f^{\lambda,}{}_{,\rho\sigma} = 0 \tag{A.5}$$

because of the cyclic character of $r^{\lambda,}{}_{,\mu\rho\sigma}$. Comparing with (A.4), we can write (A.5) as follows:

$$r^{\lambda,}{}_{,\mu\rho\sigma} = -\partial_{,\mu}f^{\lambda,}{}_{,\rho\sigma}. \tag{A.6}$$

We then get successively:

$$r^{\kappa}{}_{\lambda\mu\nu} = g_{\lambda,}{}^{,\alpha}g_{\mu,}{}^{,\beta}g_{\nu,}{}^{,\gamma}r^{\kappa,}{}_{,\alpha\beta\gamma} = -g_{\lambda,}{}^{,\alpha}g_{\mu,}{}^{,\beta}g_{\nu,}{}^{,\gamma}\partial_{,\alpha}f^{\kappa,}{}_{,\beta\gamma}$$
$$= -g_{\lambda,}{}^{,\alpha}[\partial_{,\alpha}(g_{\mu,}{}^{,\beta}g_{\nu,}{}^{,\gamma}f^{\kappa,}{}_{,\beta\gamma}) - \partial_{,\alpha}(g_{\mu,}{}^{,\beta}g_{\nu,}{}^{,\gamma}).f^{\kappa,}{}_{,\beta\gamma}]$$

namely, if we use the simple relation:

$$\partial_{,\mu}g_{\lambda,}{}^{,\nu} = g_{\lambda,}{}^{,\alpha}f_{;}^{;\nu}{}_{\alpha\mu} \tag{A.7}$$

and after some simple calculations:

$$r^{\kappa}{}_{\lambda\mu\nu} = -\partial_{\lambda}f^{\kappa}{}_{\mu\nu} + f^{\kappa}{}_{\nu\alpha}f^{\alpha}{}_{\lambda\mu} - f^{\kappa}{}_{\mu\alpha}f^{\alpha}{}_{\lambda\nu}. \tag{A.8}$$

Contracting successively $\kappa = \mu$ and $\lambda = \nu$, we obtain:

$$r = \partial_\alpha f^\alpha + \underset{2}{H} \qquad (\underset{2}{H} \equiv f^{\alpha\beta\gamma} f_{\gamma\beta\alpha}). \tag{A.9}$$

On the other hand, one can easily prove the following equality:

$$\frac{\partial}{\partial x^\alpha}(g g_{\lambda,}{}^{\cdot\alpha}) = 2 g f_\lambda \qquad (g \equiv \det(g^{\lambda \cdot}{}_{,\mu})). \tag{A.10}$$

Let us multiply (A.9) by g:

$$\mathfrak{r} \equiv g r = g \partial_\alpha f^\alpha + g\underset{2}{H} = g g_{\alpha,}{}^{\cdot\beta} \partial_{,\beta} f^\alpha + g\underset{2}{H}$$

$$= g g_{\alpha,}{}^{\cdot\beta} \frac{\partial f^\alpha}{\partial x^\beta} + g\underset{2}{H} = \frac{\partial}{\partial x^\beta}(g g_{\alpha,}{}^{\cdot\beta} f^\alpha) - \frac{\partial}{\partial x^\beta}(g g_{\alpha,}{}^{\cdot\beta}).f^\alpha + g\underset{2}{H}$$

and, using (A.10):

$$\mathfrak{r} \equiv \frac{\partial}{\partial x^\alpha}(g f^{\cdot\alpha}) + g\underset{2}{H} - 2g\underset{3}{H} \qquad (\underset{3}{H} \equiv f_\alpha f^\alpha) \quad \text{q.e.d.} \tag{A.11}$$

Appendix B

The Lagrangian $\mathfrak{H}^*$ of the formula (4.3) leads to the field equations (4.2). It is rather tedious to calculate the symmetric ($\underset{i}{U}_{(\lambda\mu)}$) and antisymmetric ($\underset{i}{V}_{[\lambda\mu]}$) contributions of $\underset{i}{\mathfrak{H}}$ to the field equations. We therefore give only an example of such a calculation, say $\underset{2}{\mathfrak{T}}_{\lambda,}{}^{\cdot\mu} \equiv \partial\underset{2}{\mathfrak{H}}/\partial g^{\lambda\cdot}{}_{,\mu}$. We start from:

$$\underset{2}{\mathfrak{H}} \equiv g\underset{2}{H} = g f^{\alpha\beta\gamma} f_{\gamma\beta\alpha} = g e^\beta f^\alpha{}_{\beta\gamma} f^\gamma{}_{\beta\alpha}. \tag{B.1}$$

Let us calculate:

$$\frac{\partial g}{\partial g^{\lambda\cdot}{}_{,\mu}} = \min g^{\lambda\cdot}{}_{,\mu} = g g_{\lambda,}{}^{\cdot\mu}. \tag{B.2}$$

On the other hand:

$$f^\alpha{}_{\beta\gamma} = g_{\beta,}{}^{\cdot\rho} g_{\gamma,}{}^{\cdot\sigma} f^{\alpha\cdot}{}_{,\rho\sigma} \to \frac{\partial f^\alpha{}_{\beta\gamma}}{\partial g^{\lambda\cdot}{}_{,\mu}}$$

$$= \frac{\partial g_{\beta,}{}^{\cdot\rho}}{\partial g^{\lambda\cdot}{}_{,\mu}} g_{\gamma,}{}^{\cdot\sigma} f^{\alpha\cdot}{}_{,\rho\sigma} + g_{\beta,}{}^{\cdot\rho} \frac{\partial g_{\gamma,}{}^{\cdot\sigma}}{\partial g^{\lambda\cdot}{}_{,\mu}} f^{\alpha\cdot}{}_{,\rho\sigma}.$$

Let us recall:

$$\frac{\partial g_{\rho,}{}^{\cdot\sigma}}{\partial g^{\lambda\cdot}{}_{,\mu}} = -g_{\rho,}{}^{\cdot\mu} g_{\lambda,}{}^{\cdot\sigma} \quad (\text{from } g_{\rho,}{}^{\cdot\sigma} g^{\rho\cdot}{}_{,\tau} = \delta_\tau{}^\sigma) \to \frac{\partial f^\alpha{}_{\beta\gamma}}{\partial g^{\lambda\cdot}{}_{,\mu}}$$

$$= -g_{\beta,}{}^{\cdot\mu} g_{\lambda,}{}^{\cdot\rho} g_{\gamma,}{}^{\cdot\sigma} f^{\alpha\cdot}{}_{,\rho\sigma} - g_{\beta,}{}^{\cdot\rho} g_{\gamma,}{}^{\cdot\mu} g_{\lambda,}{}^{\cdot\sigma} f^{\alpha\cdot}{}_{,\rho\sigma} \tag{B.3}$$

namely:

$$\frac{\partial f^\alpha{}_{\beta\gamma}}{\partial g^{\lambda\cdot}{}_{,\mu}} = g_{\gamma,}{}^{\cdot\mu} f^\alpha{}_{\lambda\beta} - g_{\beta,}{}^{\cdot\mu} f^\alpha{}_{\lambda\gamma}. \tag{B.4}$$

We then have:

$$\frac{\partial \underset{2}{\mathfrak{H}}}{\partial g^{\lambda,}{}_{,\mu}} = \frac{\partial}{\partial g^{\lambda,}{}_{,\mu}} [e^{\beta} g f^{\alpha}{}_{\beta\gamma} f^{\gamma}{}_{\beta\alpha}]$$

$$= e^{\beta} \frac{\partial g}{\partial g^{\lambda,}{}_{,\mu}} f^{\alpha}{}_{\beta\gamma} f^{\gamma}{}_{\beta\alpha} + e^{\beta} g \frac{\partial f^{\alpha}{}_{\beta\gamma}}{\partial g^{\lambda,}{}_{,\mu}} f^{\gamma}{}_{\beta\alpha} + e^{\beta} g f^{\alpha}{}_{\beta\gamma} \frac{\partial f^{\gamma}{}_{\beta\alpha}}{\partial g^{\lambda,}{}_{,\mu}}$$

and after some calculations:

$$\underset{2}{\mathfrak{T}}_{\lambda,}{}^{,\mu} \equiv \frac{\partial \underset{2}{\mathfrak{H}}}{\partial g^{\lambda,}{}_{,\mu}} = 2(f^{\alpha\beta\gamma} - f^{\beta\alpha\gamma}) f_{\gamma\lambda\beta} g g_{\alpha,}{}^{,\mu} + g g_{\lambda,}{}^{,\mu} \underset{2}{H}. \tag{B.5}$$

The other contributions are computed in the same way. In particular, the antisymmetric $\underset{i}{V}_{[\lambda\mu]}$ are given by the formulas (4.4).

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