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Time Decay for Fermion Systems with Persistent Vacuum

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Abstract. We consider the relaxation to equilibrium of a system coupled to an infinite fermion reservoir at zero temperature. In the weak coupling limit, we obtain an exponential decay law for the system. In contrast to earlier solutions of such problems, we allow for certain self-interactions of the particles of the reservoir which do not polarise the vacuum but which may influence the exponential decay law.

1. Introduction

We consider a system represented by a Hilbert space $\mathfrak{S}$ coupled to an infinite fermion reservoir at zero temperature represented by a Fermion Fock space $\mathfrak{F}$ with single particle space $\mathscr{H}$ which may typically be $L^2(\mathbb{R}^3)$. For every $g \in \mathscr{H}$ there is a bounded field operator $a(g)$ on $\mathfrak{F}$ satisfying

$$\begin{aligned} a(g)a(h) + a(h)a(g) &= 0, \\ a(g)a^*(h) + a^*(h)a(g) &= \langle h, g\rangle_{\mathscr{H}} 1, \\ a(g)\Omega &= 0, \end{aligned} \tag{1.1}$$

where Ω is the Fock vacuum. The Hamiltonian of the combined system is defined on $\mathfrak{S} \otimes \mathfrak{F}$ and is taken to be

$$H_\lambda = H_{\mathfrak{S}} \otimes 1 + 1 \otimes H_{\mathfrak{F}}^{\alpha} + \lambda H_I, \tag{1.2}$$

where $H_{\mathfrak{S}}$ is the system Hamiltonian, $H_{\mathfrak{F}}^{\alpha}$ is the reservoir Hamiltonian and H_I is the interaction between the two. We study the time evolution under these dynamics. The initial state is taken to be $\psi \otimes \Omega$ where $\psi \in \mathfrak{S}$ is arbitrary. The state of the system $\mathfrak{S}$ at time $t = \lambda^{-2}\tau$ is given in the interaction picture by the density matrix

$$\rho_\lambda(\tau) = \operatorname{tr}_{\mathfrak{F}}[e^{iH_0t}\, e^{-iH_\lambda t}|\psi \otimes \Omega\rangle\langle\psi \otimes \Omega|\, e^{iH_\lambda t}\, e^{-iH_0t}], \tag{1.3}$$

where $H_0 = H_\lambda|_{\lambda=0}$. Obviously $\rho_\lambda(\tau)$ is a non-negative operator on $\mathfrak{S}$ with $\operatorname{tr}_{\mathfrak{S}}[\rho_\lambda(\tau)]$ equal to $\|\psi\|^2$.

Our main theorems, 2.3 and 2.5, show that under certain conditions the limit of weak coupling at long times, $\lim_{\lambda\to 0} \rho_\lambda(\tau)$, with the rescaled time $\tau = \lambda^2 t$ being fixed, exists and is given by a closed expression. The proof consists of two parts, the main one being the determination of a renormalised Hamiltonian H_λ^{REN} of the same form

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as H_λ but acting on a new Hilbert space $\mathfrak{S} \otimes \mathfrak{F}^{\mathrm{REN}}$ such that the reservoir Hamiltonian on $\mathfrak{F}^{\mathrm{REN}}$ is quadratic ('free') and

$$\lim_{\lambda \to 0} \{\rho_\lambda(\tau) - \rho_\lambda^{\mathrm{REN}}(\tau)\} = 0. \tag{1.4}$$

Using the results of an earlier paper [1] we are then able to show that

$$\lim_{\lambda \to 0} \rho_\lambda^{\mathrm{REN}}(\tau) = e^{K^\natural \tau} \rho(0), \tag{1.5}$$

where $K^\natural$ is an operator acting on the space of trace class operators on $\mathfrak{S}$, which in principle can be explicitly computed from H_λ.

In contrast to earlier work the reservoir Hamiltonian is taken to be

$$H_{\mathfrak{F}}^\alpha = H_{\mathfrak{F}} + \alpha V, \tag{1.6}$$

where $H_{\mathfrak{F}}$ is the free reservoir Hamiltonian whose single particle restriction is called S, α is a small (non-zero) constant and V is a polynomial in the field operators which annihilates the vacuum. Our technique is to expand in powers of α, which necessitates some regularity of V such as a space cut-off, which we shall discuss in Section 3. We leave as open problems the case with vacuum polarisation and the generalisation to positive temperatures. These problems involve essentially the same difficulty, namely that it is not clear how to combine the Gellman-Low formula with the trick of partial contractions which is essential to our method.

We shall take the system-reservoir interaction term to be

$$H_I = Q \otimes \{a(g^0) + a^*(g^0)\}, \tag{1.7}$$

where g^0 is suitably smooth and Q is bounded, but our methods can be extended to cover the case

$$H_I = Q \otimes a^\#(f_1) \cdots a^\#(f_n) + \text{adjoint}, \tag{1.8}$$

where $a^\# = a$ or a^*, with only slight extra complications. The Lee type hypothesis of [1] that $f_1, \ldots, f_n$ have disjoint supports in momentum space is not necessary. If, however,

$$\langle a^\#(f_1) \cdots a^\#(f_n)\Omega, \Omega\rangle_{\mathfrak{F}} \neq 0, \tag{1.9}$$

then the operator A_{00} of [1] is non-zero and one should use the modified form of the theory of master equations in [2] to solve the renormalized problem.

2. The Main Estimates

The Hamiltonian of the system is taken to be

$$H_\lambda = H_{\mathfrak{S}} \otimes 1 + 1 \otimes H_{\mathfrak{F}} + \alpha 1 \otimes \sum_{r,s=1}^{M} V_{rs} + \lambda Q \otimes \{a(g^0) + a^*(g^0)\}, \tag{2.1}$$

where all terms except V_{rs} have been defined in Section 1. We suppose $V_{rs} = V_{sr}^*$ and that

$$V_{rs} = \int_{\mathbb{R}^3} dx a^*(g_x^1) \cdots a^*(g_x^r) a(g_x^{r+1}) \cdots a(g_x^{r+s}), \tag{2.2}$$

where g_x^r is the translate of g^r by a distance x in space multiplied by $h(x)$, where h is a space cut-off function of compact support. The vectors $(g_x^1, \cdots, g_x^{r+s})$ may depend on r, s. Because fermion field operators are bounded, V_{rs} are bounded operators and it is easy to prove the self-adjointness of H_λ. The conditions $r, s \geqslant 1$ ensure that

$$V_{rs}\Omega = 0, \quad V_{rs}^*\Omega = 0. \tag{2.3}$$

Our estimates are based on the *main assumption* that the test functions g_x^k which occur in the Hamiltonian satisfy inequalities of the type

$$|\langle e^{iSt} g_x^k, g_y^l \rangle| \leqslant G(x - y, t), \tag{2.4}$$

where $G \in L^1(\mathbb{R}^4)$. Conditions under which (2.4) is satisfied are discussed in Section 3.

For any bounded operator B on $\mathfrak{F}$ we define

$$\langle B \rangle = \langle B\Omega, \Omega \rangle_{\mathfrak{F}}, \tag{2.5}$$

$$B_t = e^{itH} B e^{-itH}, \tag{2.6}$$

while if B is an operator on $\mathfrak{S} \otimes \mathfrak{F}$ we define

$$B_t = e^{it(H_{\mathfrak{S}} \otimes 1 + 1 \otimes H_{\mathfrak{F}})} B e^{-it(H_{\mathfrak{S}} \otimes 1 + 1 \otimes H_{\mathfrak{F}})}. \tag{2.7}$$

Lemma 2.1. There exists an expansion

$$\rho_\lambda(\tau) = \sum_{m, n_a, n_c = 0}^{\infty} \lambda^{n_a + n_c} \alpha^m \rho_{m n_a n_c}(t), \tag{2.8}$$

with bounds

$$\|\rho_{m n_a n_c}(t)\| \leqslant C^{m + n_a + n_c} t^{m + n_a + n_c} / (m + n_a + n_c)!, \tag{2.9}$$

and

$$\|\rho_{m n_a n_c}(t)\| \leqslant C^{m + n_a + n_c} t^{n_0} / n_0!, \tag{2.10}$$

where

$$n_0 = \min(n_a, n_c). \tag{2.11}$$

Proof. We start with the formula

$$e^{iH_0 t} e^{-iH_\lambda t} = \sum_{m, n_a, n_c \geqslant 0} \sum_{(m, n_a, n_c)}' (-i)^{m + n_a + n_c} \lambda^{n_a + n_c} \alpha^m \times \int_{0 \leqslant t_1 \leqslant \cdots \leqslant t_{m + n_a + n_c} \leqslant t} \prod_{j=1}^{m + n_a + n_c} dt_j B_{t_j}^j. \tag{2.12}$$

Here,

$$B^j = Q^j \otimes V^j$$

$$\left.\begin{array}{l} \text{where for } m \text{ values of } j, \\ \quad Q^j \otimes V^j = 1 \otimes V_{r(j)s(j)}, \qquad 1 \leqslant r(j),\ \ s(j) \leqslant M, \\ \text{and for } n_c \text{ values of } j, \\ \quad Q^j \otimes V^j = Q \otimes a^*(g^0), \\ \text{and for } n_a \text{ values of } j, \\ \quad Q^j \otimes V^j = Q \otimes a(g^0). \end{array}\right\} \tag{2.13}$$

The summation $\sum'$ is over the at most $A = M^2 + 2$ possible choices for the B^j. Substituting (2.12) into (1.3), we get

$$\rho_\lambda(\tau) = \sum_{\substack{m_1, n_{a1}, n_{c1} \geqslant 0 \\ m_2, n_{a2}, n_{c2} \geqslant 0}} \sum_{(m_1, n_{a1}, n_{c1})}' \sum_{(m_2, n_{a2}, n_{c2})}' (-i)^{\cdots} \lambda^{n_{a1}+n_{c1}+n_{a2}+n_{c2}} \alpha^{m_1+m_2}$$
$$\times \int_{\substack{0 \leqslant t_1 \leqslant \cdots \leqslant t_{d_1} \leqslant t \\ 0 \leqslant s_1 \leqslant \cdots \leqslant s_{d_2} \leqslant t}} dt_j\, ds_j\, Q^{d_1}_{t_{d_1}} \cdots Q^1_{t_1} |\psi\rangle\langle\psi| Q^{d_1+1}_{s_1} \cdots Q^{d_1+d_2}_{s_{d_2}} \tag{2.14}$$
$$\times \langle V^{d_1}_{s_1} \cdots V^{d_1+d_2}_{s_{d_2}} V^{d_1}_{t_{d_1}} \cdots V^1_{t_1} \rangle.$$

where $d_j = n_{aj} + n_{cj} + m_j$, $j = 1, 2$.

The coefficients $\rho_{mn_an_c}(t)$ of equation (2.8) are defined to be the finite partial sums in equation (2.14) with $m_1 + m_2 = m$, $n_{a1} + n_{a2} = n_a$, $n_{c1} + n_{c2} = n_c$.

The estimate equation (2.9) is then a consequence of the boundedness of all operators Q^j and V^j in equation (2.14). We use this bound only to establish that the series (2.8) converges.

We now proceed to the bound equation (2.10) which is the first main estimate of this paper. Since Q is bounded, we have

$$\|\rho_\lambda(\tau)\| \leqslant \sum_{m, n_a, n_c \geqslant 0} |\lambda|^{n_a+n_c} |\alpha|^m C^{m+n_a+n_c} \sum_{k=0}^{m+n_a+n_c} \sum{}''$$
$$\times \int_{\Delta^t_{k, m+n_a+n_c-k}} |\langle V^1_{t_1} \cdots V^{m+n_a+n_c}_{t_{m+n_a+n_c}} \rangle|, \tag{2.15}$$

where $\sum''$ is the sum over all choices of the V^j subject to the conditions (2.13) and

$$\Delta^t_{k,l} = \{t \in \mathbb{R}^{k+l} | 0 \leqslant t_1 \leqslant \cdots \leqslant t_k \leqslant t;\ t \geqslant t_{k+1} \geqslant \cdots \geqslant t_{k+l} \geqslant 0\}. \tag{2.16}$$

The main idea of the proof of equation (2.10) is a resummation of (2.15) based on an indexing by the *type of vertex occurring*. We introduce some notation.

Let $\mathscr{V}, \mathscr{V}'$ be two finite unordered collections of vertices V^j. We define the following functions of $\mathscr{V}, \mathscr{V}'$:

$$\begin{aligned} &n_a(n_c) \text{ the number of } V^j = a(g^0)(V^j = a^*(g^0)) \text{ in the set } \mathscr{V} \cup \mathscr{V}'; \\ &n_0 = \inf(n_a, n_c), \\ &v = |\mathscr{V}|, \quad v' = |\mathscr{V}'|. \end{aligned} \tag{2.17}$$

We rewrite equation (2.15) as

$$\|\rho_\lambda(\tau)\| < \sum_{m, n_a, n_c \geqslant 0} |\lambda|^{n_a+n_c} |\alpha|^m \sum_{\mathscr{V}, \mathscr{V}'}{}''' I(\mathscr{V}, \mathscr{V}'), \tag{2.18}$$

where $\sum'''$ is over those choices of $\mathscr{V}, \mathscr{V}'$ for which n_a, n_c take the values given by the first summation, $v + v' - n_a - n_c = m$, and $I(\mathscr{V}, \mathscr{V}')$, is the corresponding integral and summation of (2.15).

The proof of inequality (2.10) is then an immediate consequence of the following:

Lemma 2.2. With the notation (2.17), one has the bound

$$I(\mathscr{V}, \mathscr{V}') \leqslant C^{v+v'} t^{n_0}/n_0!. \tag{2.19}$$

Indeed,

$$\|\rho_{m,n_a,n_c}\| \leqslant C^{m+n_a+n_c} \sup I(\mathscr{V}, \mathscr{V}'), \tag{2.20}$$

where the sup is over $\mathscr{V}, \mathscr{V}'$ with fixed n_a, n_c and $m = v + v' - n_a - n_c$, and the factor comes from arranging the $M + 2$ types of vertices into the sets $\mathscr{V}, \mathscr{V}'$.

Proof. We suppose for definiteness that $n_a \geqslant n_c$, the opposite case being obtained by a left-right annihilation-creation symmetry. Since $H_{\mathfrak{F}}$ is a free Hamiltonian

$$a_t^{\#}(g) = a^{\#}(e^{iSt}g). \tag{2.21}$$

Therefore the vacuum expectation values can be expanded in terms of two-point functions in the usual way [3]. We shall use the language of Hepp freely but our estimates depend on *doing only some of the contractions* in the calculation of vacuum expectation values. We 'draw' creators to the left and annihilators to the right.

Definitions. For every type of vertex (V_{rs} or $a(g^0)$ or $a^*(g^0)$) which has at least one annihilation leg we pick one of them and call it the *selected* annihilation leg of that vertex. A *graph* is a linear arrangement of vertices of specified type labelled by times indexed in increasing order from left to right; each selected annihilation leg is contracted to some creation leg in such a way that there are no loops. A *diagram* is a

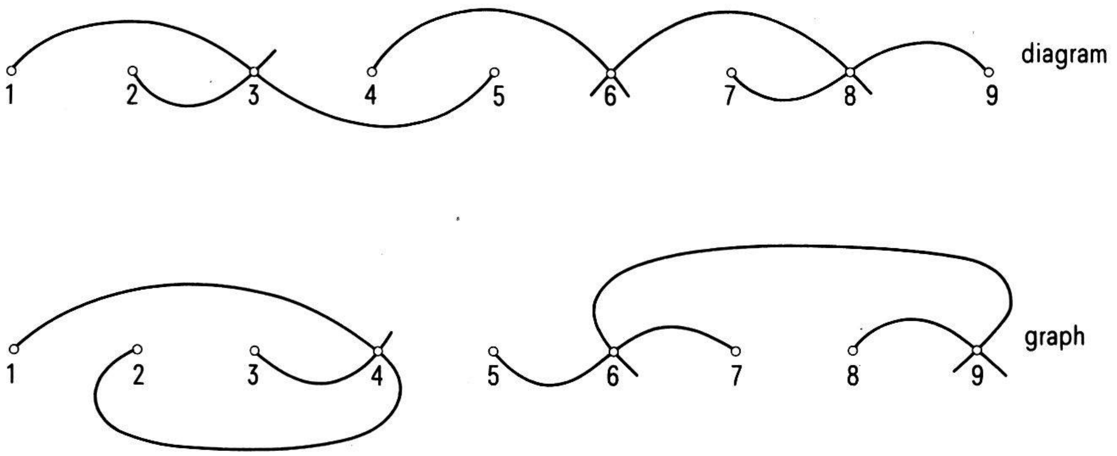

Figure 1

graph in which the condition of absence of loops is replaced by the stronger condition that each selected annihilation leg is contracted to a creation leg on its right. Each connected component of a graph contains exactly one $a^*(g^0)$ and therefore there are exactly n_c connected components in a graph.

By contracting the selected annihilation legs and by Wick's theorem, we obtain

$$|I(\mathscr{V}, \mathscr{V}')| \leqslant C^{v+v'} \sum_{d \in \mathscr{D}} \int_{\Delta} d\mathbf{t} \int d\mathbf{x} F_d(\mathbf{x}, \mathbf{t}), \tag{2.22}$$

where $\mathscr{D}$ is the set of all diagrams, $\mathbf{x} = x_1, \ldots, x_{v+v'}$ and $\mathbf{t} = t_1, \ldots, t_{v+v'}$. The constant term arises by estimating the uncontracted legs using the norm boundedness of fermi field operators,

$$\|a^{\#}(e^{iS\tau}g_x^k)\| < \|g^k\|_2 \|h\|_\infty. \tag{2.23}$$

It also accounts for the number of orderings of the vertices in $\mathscr{V}$ and $\mathscr{V}'$. For each contraction in a diagram d there is a factor

$$|\langle e^{iS(t_i-t_j)}g_{x_i}^k, g_{x_j}^l\rangle|, \tag{2.24}$$

where we adopt the convention $g_x^0 = g^0 \cdot \delta(x)$. Finally F_d is the product of the bounds (2.4) on the factors (2.24) for all contractions occurring in d.

We now define $\mathfrak{G}$ with v, v', n_{aj}, n_{cj} fixed to be the set of all graphs and $\mathscr{P}$ to be the set of all permutations of $1,\ldots, v$, and of $v+1,\ldots, v+v'$. We define a one-one map $(\pi, d) \rightarrow (\pi, g)$ of $\mathscr{P} \times \mathscr{D} \rightarrow \mathscr{P} \times \mathfrak{G}$ by permuting the time indices of d with π and then rearranging the vertices and links simultaneously so that the time indices are again in increasing order from left to right, to obtain the graph g. (See Fig. 1 where $\pi = \binom{123456789}{134829567}$.)

For each $\pi \in \mathscr{P}$ we consider the region

$$\Delta_\pi = \{0 \leqslant t_{\pi(1)} \leqslant \cdots \leqslant t_{\pi(v)} \leqslant t, 0 \leqslant t_{\pi(v+v')} \leqslant \ldots \leqslant t_{\pi(v+1)} \leqslant t\} \tag{2.25}$$

to be a subset of the hypercube

$$\mathscr{C} = [0, t]^{v+v'}.$$

Finally we associate a function F_g to each graph g in the same way as for diagrams, so that

$$\int_{\Delta_\pi} dt \int d\mathbf{x} F_d(\mathbf{x}, \mathbf{t}_\pi) = \int_{\Delta_\pi} dt \int d\mathbf{x} F_g(\mathbf{x}, \mathbf{t}). \tag{2.26}$$

Then

$$\begin{aligned} |I(\mathscr{V}, \mathscr{V}')| &\leqslant C^{m+n} \sum_{d\in\mathscr{D}} \sum_{\pi\in\mathscr{P}} (v!\, v'!)^{-1} \int_{\Delta_\pi} dt \int d\mathbf{x} F_d(\mathbf{x}, \mathbf{t}_\pi) \\ &\leqslant C^{m+n} \sum_{g\in\mathfrak{G}} \sum_{\pi\in\mathscr{P}} (v!\, v'!)^{-1} \int_{\Delta_\pi} dt \int d\mathbf{x} F_g(\mathbf{x}, \mathbf{t}) \\ &= C^{m+n} \sum_{g\in\mathfrak{G}} (v!\, v'!)^{-1} \int_{\mathscr{C}} dt \int d\mathbf{x} F_g(\mathbf{x}, \mathbf{t}) \\ &\leqslant C^{m+n} (v!\, v'!)^{-1} \sup_{g\in\mathfrak{G}} \int_{\mathscr{C}} dt \int d\mathbf{x} F_g(\mathbf{x}, \mathbf{t}) \\ &\leqslant C^{m+n} (v!\, v'!)^{-1} C^{m+n} t^{n_c}, \end{aligned} \tag{2.27}$$

by the integrability of G, equation (2.4); integration over each of the n_c components of $g \in \mathfrak{G}$ contributes one time factor t but no space factor because each component contains a contraction to $a^*(g^0)$.

Now

$$\begin{aligned} |\mathfrak{G}| &\leqslant (M^2+2)^{v+v'}(mM+n_c)(mM+n_c-1)\cdots(mM+n_c-m-n_a+1) \\ &\leqslant C^{v+v'}(m+n_a)! \end{aligned} \tag{2.28}$$

The factors $M^2 + 2$ count the number of choices of vertices, the remaining factors the number of possible contractions. Therefore

$$I(\mathscr{V}, \mathscr{V}') \leqslant C^{m+n}(v!\, v'!)^{-1}(m + n_a)!\, t^{n_c} \leqslant C^{m+n} t^{n_c}/n_c! \tag{2.29}$$

This completes the proof of Lemma 2.2.

Returning to equation (2.14) we see that by completely contracting the vacuum expectation value we obtain

$$\rho_{m,n_a,n_c}(t) = \sum_k \rho^k_{m,n_a,n_c}(t) \tag{2.30}$$

where ρ_k is the sum of all terms whose completely contracted graphs have k connected components. Our first main theorem is then as follows:

Theorem 2.3. There is an $\alpha_0 > 0$ such that for $|\alpha| < \alpha_0$ and $\tau < \infty$,

$$\lim_{\lambda\to 0} \rho_\lambda(\tau) = \sum_{m,n=0}^{\infty} \lim_{\lambda\to 0} \lambda^{2n}\alpha^m \rho^n_{mnn}\,(\lambda^{-2}\tau). \tag{2.31}$$

Proof. The proof is in two steps. By equation (2.10)

$$\begin{aligned}\|\rho_\lambda(\tau)\| &\leqslant \sum |\lambda|^{n_a+n_c}|\alpha|^m \|\rho_{m,n_a,n_c}(t)\| \\ &\leqslant 2\sum_{m=0}^{\infty} \sum_{0\leqslant n_c \leqslant n_a < \infty} |\lambda|^{n_a+n_c}|\alpha|^m C^{m+n_a+n_c} t^{n_c}/n_c! \\ &= 2\sum_{m,n,p=0}^{\infty} (C|\lambda|)^p (C|\alpha|)^m (C\lambda^2 t)^n/n!\end{aligned} \tag{2.32}$$

Therefore the series (2.30) converges uniformly in λ in regions $|\lambda|, |\alpha| < 1/2C$, $0 \leqslant \tau \leqslant \tau_0$, where $\tau_0 < \infty$ is arbitrary. Therefore

$$\lim_{\lambda\to 0} \rho_\lambda(\tau) = \sum_{m,n_a,n_c=0}^{\infty} \lim_{\lambda\to 0} \lambda^{n_a+n_c}\alpha^m \rho_{mn_an_c}(\lambda^{-2}\tau). \tag{2.33}$$

It remains to prove that for each fixed m, n_a, n_c,

$$\lim_{\lambda\to 0} \lambda^{n_a+n_c}\rho_{m,n_an_c}(\lambda^{-2}\tau) = \delta_{n_an_c} \lim_{\lambda\to 0} \lambda^{2n_a}\rho_{mn_an_a}(\lambda^{-2}\tau), \tag{2.34}$$

or equivalently, if $2k < n_a + n_c$, then

$$\lim_{\lambda\to 0} \lambda^{n_a+n_c}\rho^k_{mn_an_c}(\lambda^{-2}\tau) = 0. \tag{2.35}$$

By repeating the arguments of Lemma 2.2 but with fully contracted graphs and without trying to control the dependence of the constants on m, n_a, n_c one obtains

$$\|\rho^k_{mn_an_c}(t)\| \leqslant Ct^k. \tag{2.36}$$

Therefore $\|\lambda^{n_a+n_c}\rho^k_{mn_an_c}(\lambda^{-2}\tau)\| \leqslant C|\lambda|^{n_a+n_c-2k}\tau^k$ which converges to zero unless $2k = n_a + n_c$, and this proves the theorem.

We now claim that there is a renormalized problem which produces exactly the terms on the right hand side of (2.31). The renormalised reservoir Hamiltonian is quadratic. The renormalised problem is therefore exactly soluble by the techniques of Davies [1, 2]. In general the Hamiltonian is slightly complicated to write down so we consider just a few simple but typical cases.

The simplest case is described in the following corollary, where we explicitly exhibit the dependence of $\rho_\lambda(\tau)$ on α, which has previously been suppressed.

Corollary 2.4. If $V_{rs} = 0$ unless $r \geqslant 2$ and $s \geqslant 2$ then

$$\lim_{\lambda\to 0} \rho_\lambda^\alpha(\tau) = \lim_{\lambda\to 0} \rho_\lambda^0(\tau). \tag{2.37}$$

In other words the reservoir self-interaction has no effect on the system in the weak coupling limit.

Proof. If $m > 0$ then every completely contracted graph has fewer than n_0 components. Therefore the only contributing terms are those with $m = 0$, in other words the same as the terms for the Hamiltonian obtained by putting $\alpha = 0$.

In the next case a new phenomenon, that of a mass renormalisation, is introduced. We take

$$V_{12}^* = V_{21} = \int dx a^*(g_x^1)a^*(g_x^2)a(g_x^3) \tag{2.38}$$

and all the other $V_{rs} = 0$. We also make an assumption of Lee type, that g^1, g^2 have supports in momentum space which are disjoint from the support of g^0 and g^3.

Theorem 2.5. There exists a renormalised Hamiltonian

$$H^{\mathrm{REN}} = H_{\mathfrak{S}} \otimes 1 + 1 \otimes H_{\mathfrak{F}\mathrm{REN}} + \alpha 1 \otimes W^{\mathrm{REN}} + \lambda Q\{b(g^0 \oplus 0) + b^*(g^0 \oplus 0)\} \tag{2.39}$$

such that W^{REN} is quadratic in the field operators and

$$\lim_{\lambda\to 0} \rho_\lambda(\tau) = \lim_{\lambda\to 0} \rho^{\mathrm{REN}}(\tau). \tag{2.40}$$

Proof. The renormalised Fock space is defined as the fermion Fock space whose single particle space is

$$\mathscr{H}^{\mathrm{REN}} = \mathscr{H} \oplus \mathscr{H} \underset{\mathrm{anti}}{\otimes} \mathscr{H}, \tag{2.41}$$

which is a subspace of the original Fock space. Then H_{REN} is the free Hamiltonian on $\mathfrak{F}^{\mathrm{REN}}$ whose restriction to $\mathscr{H}^{\mathrm{REN}}$ is

$$S^{\mathrm{REN}} = S \oplus (1 \otimes S + S \otimes 1). \tag{2.42}$$

If $g \in \mathscr{H}^{\mathrm{REN}}$ we denote the corresponding field operator on $\mathfrak{F}^{\mathrm{REN}}$ by $b(g)$. Defining $g_x \in \mathscr{H}^{\mathrm{REN}}$ by

$$g_x = 0 \oplus a^*(g_x^1)a^*(g_x^2)\Omega, \tag{2.43}$$

the operator W is defined as

$$W = \int_{\mathbb{R}^3} dx b^*(g_x)b(g_x^3) + \text{adjoint}. \tag{2.44}$$

The Hamiltonian H_λ^{REN} is of the general type to which Theorem 2.3 applies. The only observation needed to prove the theorem is that

$$\rho_{m,n,n}^n(t) = \rho_{m,n,n}^n(t)^{\mathrm{REN}}, \tag{2.45}$$

because the corresponding completely contracted graphs are those for which each component is a chain of the form

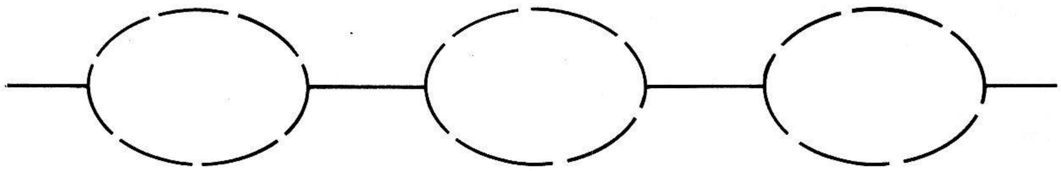

Figure 2

We stress that the value of the above reductions depends upon the fact that the renormalised problem is exactly soluble by the methods of [1]. This is seen by writing

$$H_\lambda^{\mathrm{REN}} = H_{\mathfrak{S}} \otimes 1 + 1 \otimes H_{\mathfrak{F}}^{\mathrm{REN}}(\alpha) + \lambda Q \otimes \{b(g^0 \oplus 0) + b^*(g^0 \oplus 0)\},$$

where $H_{\mathfrak{F}}^{\mathrm{REN}}(\alpha)$ is a free Hamiltonian on $\mathfrak{F}^{\mathrm{REN}}$ determined by its restriction to $\mathscr{H}^{\mathrm{REN}}$,

$$S_\alpha^{\mathrm{REN}} = S \oplus (S \otimes 1 + 1 \otimes S) + \alpha \int_{\mathbb{R}^3} dx h(x) |g_x^3\rangle\langle g_x| + \text{adjoint}.$$

The existence and form of

$$\lim_{\lambda \to 0} \rho_\lambda^{\mathrm{REN}}(\tau)$$

is given in [1] in terms of Q, $H_{\mathfrak{S}}$, g^0 and S_α^{REN} under certain conditions which may be verified without too much difficulty in our case.

3. Discussion and Extensions of the Model

All our calculations have been based heavily on the estimate (2.4) where $G \in L^1(\mathbb{R}^4)$, which is unfortunately too strong for most applications. If the single particle space is $\mathscr{H} = L^2(\mathbb{R}^3)$ and $S = -\Delta$, and if $g_x^r(y) = g^r(x+y)$ then in momentum space variables

$$\langle e^{iSt} g_x^1, g^2 \rangle = \int dk \hat{g}^1(k) \hat{g}^2(k) \exp i\{k^2 t + k \cdot x\}, \tag{3.1}$$

which cannot lie in $L^1(\mathbb{R}^4)$ because it is the Fourier transform of a distribution on $\mathbb{R}^4$ supported on the surface

$$\{(k, E) : k^2 = E\}. \tag{3.2}$$

We now give a few examples of how the necessary regularisation can be achieved.

Example 1. We show that the conditionn $G \in L^1(\mathbb{R}^4)$ can be satisfied if one includes a space cut-off function h of compact support as assumed so far. A direct computation shows that if $S = -\Delta$ and all $g^r \in L^1 \cap L^2$ then

$$\begin{aligned} |\langle e^{iSt} g_x^r, g^s \rangle| &\leqslant Ch(x)(1 + t^2)^{-3/4} \\ &= G(x, t) \end{aligned} \tag{3.3}$$

so $G \in L^1(\mathbb{R}^4)$.

Example 2. Another method of circumventing the infinite volume difficulties consists of attaching a second reservoir independently to each point in space [4]. In the crudest possible form this amounts to choosing the single particle space to be $L^2(\mathbb{R}^4)$, the fourth variable representing the degrees of freedom of the second reservoir. The Hamiltonian is

$$(S\psi)(x, E) = -\sum \frac{\partial^2}{\partial x_i^2} \psi(x, E) + E\psi(x, E), \tag{3.4}$$

and each of the g_x^r defining the Hamiltonian H_λ is taken to be

$$g_x^r(y, E) = f^r(y + x) h^r(E). \tag{3.5}$$

where f^r and h^r lie in Schwartz space. Then

$$\begin{aligned} \langle e^{iSt} g_x^1, g^2 \rangle &= \int \hat{f}^1(k) \overline{\hat{f}^2(k)} h^1(E) \overline{h^2(E)} \exp i\{k^2 t + k \cdot x + Et\}\, dk\, dt \\ &= u(t) v_t(x), \end{aligned} \tag{3.6}$$

where u lies in Schwartz space and

$$\hat{v}_t(k) = \hat{f}^1(k) \overline{\hat{f}^2(k)} e^{ik^2 t}. \tag{3.7}$$

Now

$$\|v_t\|_1 \leqslant \|(1 - \Delta)\hat{v}_t\|_2 \leqslant C(1 + t^2), \tag{3.8}$$

since f^1 and f^2 lie in Schwartz space. Therefore

$$\begin{aligned} \int dx\, dt |\langle e^{iSt} g_x^1, g^2 \rangle| &< C \int dt |u(t)|(1 + t^2) \\ &< \infty. \end{aligned} \tag{3.9}$$

Therefore our theory applies to this model even in the case without space cut-off but it is clear that the dissipation is caused by the second reservoir.

Example 3. To illustrate the difficulties of the infinite volume problem without regularisation, we consider a simple exactly soluble case. We put $\mathscr{H} = L^2(\mathbb{R}^3)$, $S = -\Delta$ and suppose the only non-zero V_{rs} is

$$V_{11} = \int_{\mathbb{R}^3} a^*(g_x) a(g_x)\, dx, \tag{3.10}$$

where g lies in Schwartz space. Then formally

$$V_{11} = \int_{\mathbb{R}^3} |\hat{g}(k)|^2 a^*(k) a(k)\, dk \tag{3.11}$$

where $\hat{g}$ is the Fourier transform of g.

One may take $\mathfrak{F}^{\mathrm{REN}} = \mathfrak{F}$ and

$$H_{\mathscr{F}}^{\mathrm{REN}}(\alpha) = \int_{\mathbb{R}^3} \{k^2 + \alpha|\hat{g}(k)|^2\} a^*(k)a(k)\, dk, \tag{3.12}$$

so that the renormalised single particle Hamiltonian is

$$(S_\alpha^{\mathrm{REN}}\psi)(k) = \{k^2 + \alpha|\hat{g}(k)|^2\}\psi(k). \tag{3.13}$$

A prerequisite for decay is that the two-point function

$$G_\alpha(t) = \langle e^{iS_\alpha{}^{\mathrm{REN}}t} f, f\rangle \tag{3.14}$$

converges to zero as $t \to \infty$. In this connection the following lemma, which may be proved by integration by parts, is relevant.

Lemma 3.1. If $f, g \in \mathscr{S}$ and $\hat{f}$ vanishes in some neighbourhood of zero in momentum space then for sufficiently small real α,

$$\lim_{t\to 0} G_\alpha(t) = 0. \tag{3.16}$$

The existence of an infra-red problem is however revealed by the fact that the above lemma may fail without the condition on f. By careful choice of g, in Schwartz space, one may in fact arrange that S^{REN} has non-empty point spectrum for a set of real α with an accumulation point at zero.

An entirely different example suggests that it may be impossible to construct a successful infinite volume theory which does not make serious use of the reality of the coupling constant α.

Indeed suppose that f, g are spherically symmetric, and write all expressions in terms of the radial variable. Then one has the

Lemma 3.2. Suppose that f, g are C^∞ with compact support in $(0, \infty)$ and that $g(k) = k$ on $\mathrm{supp}(f) \neq \varnothing$. Then there exist arbitrarily small complex α such that $G_\alpha(t)$ does not lie in $L^1(\mathbb{R})$.

Proof. Suppose on the contrary that for all $|\alpha| < \delta$, $G_\alpha(t)$ lies in $L^1(\mathbb{R})$. Then $\alpha \to G_\alpha$ is an analytic $L^1(\mathbb{R})$-valued function. If $\hat{}$ is the Fourier transform it follows that $\alpha \to G_\alpha$ is an analytic function with values in the space of continuous functions on $\mathbb{R}$, so for all $y \in \mathbb{R}$, $\alpha \to \hat{G}_\alpha(y)$ is an analytic function.

Now if α is real,

$$\begin{aligned} G_\alpha(t) &= \int_0^\infty 4\pi k^2|f(k)|^2 \exp\{itk^2(1+\alpha)\}\, dk \\ &= \int_0^\infty \frac{4\pi y}{1+\alpha}\left| f\left(\frac{y^{1/2}}{(k+\alpha)^{1/2}}\right)\right|^2 e^{ity} \frac{dy}{2y^{1/2}(k+\alpha)^{1/2}}, \end{aligned} \tag{3.17}$$

so

$$\hat{G}_\alpha(y) = \frac{y^{1/2}}{(1+\alpha)^{3/2}}\left| f\left(\frac{y^{1/2}}{(1+\alpha)^{1/2}}\right)\right|^2. \tag{3.18}$$

Taking $y^{1/2}$ to be the smallest number in supp f, one has $\hat{G}_\alpha(y) = 0$ for small $\alpha > 0$ but $\hat{G}_\alpha(y) \neq 0$ for small $\alpha < 0$, so $\hat{G}_\alpha(y)$ cannot be analytic around $\alpha = 0$.

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