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Relativistic Gravitational Energy is Probably Localizable

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Abstract. Scherrer's linear theory and the tetradic formalism lead to a covariant representation of gravitational energy-momentum (GEM) in general relativity. Unfortunately, the GEM-density is not invariant under transformations of tetrads, and this defect seems to be as awkward as the well-known difficulties of the Einsteinian GEM-complex. Some authors, Møller for instance, suggested supplementary conditions to determine tetrads without ambiguity, but their considerations are of a purely mathematical nature. In the static case, we formulate on the contrary a physically well-founded variational principle. The resulting field equations plus suitable boundary conditions determine the tetrads unambiguously. In this formalism, gravitational energy is thus quite localizable.

1. Introduction

During the last 20 years, the problem of relativistic gravitational energy-momentum (GEM) localizability has been the subject of a great number of papers, which however, have not led to a definitive clarification of the question. In the Einsteinian formalism of general relativity (GR), it is well known that it is impossible to localize GEM (in other words, we cannot construct a *tensorial* expression of the GEM). This fact has even an elementary physical justification (see Landau–Lifschitz's interpretation [1, p. 401]). But later, Scherrer's linear formalism [2] and the theory of tetrads (more particularly developed by Møller [3]) allowed a different approach of the question. Tensorial expressions of GEM could be defined in the framework of these formalisms. The obtained results are certainly very interesting, but they exhibit an important defect: GEM-density is not invariant under transformations of tetrads. It is thus doubtful to speak of GEM-localizability in this case. Different propositions were formulated to solve this type of difficulty. For instance, Møller [3] gives supplementary conditions for the tetrads. Thus the latter should be defined unambiguously, with the help of suitable boundary conditions. Unfortunately, in spite of the 'good' results obtained with this method, these supplementary conditions are of a purely mathematical nature, and have no physical foundations. According to Møller's own opinion, it is then difficult to speak of a 'definitive' solution in the sense of GEM-localizability.

In his last paper about this subject [4, p. 12], Møller expresses the idea that: '...therefore, unless one can find a good physical argument for fixing the gauge throughout the system, it has no physical meaning to speak about the energy distribution inside the system. This would be in complete agreement with Einstein's own point of view. Actually, nobody has so far been able to give a prescription for measuring the energy of the gravitational field in a small region, in contrast to the total

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energy for which such prescriptions are easily given'. In two recent publications [5, 6], we showed however, that there are, on the contrary, some good physical reasons in favour of localizability. It is therefore reasonable to search for *physically justified* supplementary conditions for the tetrads field. The present paper is devoted to this purpose, in the framework of Scherrer's linear formalism.

2. Scherrer's Linear Formalism

In this paragraph, we briefly recall to mind some essential results of Scherrer's formalism (for a detailed account of this formalism, see [2]). In the 'external' case (gravitational field out of the sources) field equations are obtained, which are equivalent to Einstein's:

$$\frac{\partial t_{\lambda,}{}^{\cdot\mu\nu}}{\partial x^{\nu}} - \mathscr{T}_{\lambda,}{}^{\cdot\mu} = 0 \tag{2.1}$$

$t_{\lambda,}{}^{\cdot\mu\nu}$ being a tensorial density antisymmetric in μ and ν, we deduce from (2.1):

$$\frac{\partial \mathscr{T}_{\lambda,}{}^{\cdot\mu}}{\partial x^{\mu}} = 0 \tag{2.2}$$

i.e. *differential conservation laws*. The $\mathscr{T}_{\lambda,}{}^{\cdot\mu}$ are defined by the formulas of the following table:

$$\mathscr{T}_{\lambda,}{}^{\cdot\mu} \equiv g T_{\lambda,}{}^{\cdot\mu} \tag{2.3}$$

$$g \equiv \text{dét}(g^{\lambda\cdot}{}_{,\mu}) \tag{2.4}$$

$$g_{\mu\nu}(x) = e_{\alpha} g^{\alpha\cdot}{}_{,\mu}(x) g^{\alpha\cdot}{}_{,\nu}(x) \qquad g^{\mu\nu}(x) = e^{\alpha} g_{\alpha,}{}^{\cdot\mu}(x) g_{\alpha,}{}^{\cdot\nu}(x)\,^{1)} \tag{2.5}$$

$$T_{\lambda,}{}^{\cdot\mu} \equiv \tfrac{1}{2}\underset{1}{T}_{\lambda,}{}^{\cdot\mu} + \underset{2}{T}_{\lambda,}{}^{\cdot\mu} - 2\underset{3}{T}_{\lambda,}{}^{\cdot\mu} \tag{2.6}$$

$$\underset{1}{T}_{\lambda,}{}^{\cdot\mu} \equiv -4 f_{\alpha\lambda}{}^{\beta} f^{\alpha\gamma}{}_{\beta} g_{\gamma,}{}^{\cdot\mu} + g_{\lambda,}{}^{\cdot\mu}\underset{1}{H} \tag{2.7}$$

$$\underset{2}{T}_{\lambda,}{}^{\cdot\mu} \equiv 2(f^{\alpha\beta}{}_{\gamma} - f^{\beta\alpha}{}_{\gamma}) f^{\gamma}{}_{\lambda\beta} g_{\alpha,}{}^{\cdot\mu} + g_{\lambda,}{}^{\cdot\mu}\underset{2}{H}$$

$$\underset{3}{T}_{\lambda,}{}^{\cdot\mu} \equiv 2 f^{\alpha}(f^{\beta}{}_{\lambda\alpha} g_{\beta,}{}^{\cdot\mu} - f_{\lambda} g_{\alpha,}{}^{\cdot\mu}) + g_{\lambda,}{}^{\cdot\mu}\underset{3}{H}$$

$$\underset{1}{H} \equiv f^{\alpha\beta\gamma} f_{\alpha\beta\gamma} \qquad \underset{2}{H} \equiv f^{\beta\alpha\gamma} f_{\gamma\alpha\beta} \qquad \underset{3}{H} \equiv f^{\alpha} f_{\alpha} \tag{2.8}$$

$$f^{\lambda\cdot}{}_{,\mu\nu} \equiv \tfrac{1}{2}(\partial_{\mu} g^{\lambda\cdot}{}_{,\nu} - \partial_{\nu} g^{\lambda\cdot}{}_{,\mu}) \tag{2.9}$$

$$f^{\lambda}{}_{\mu\nu} \equiv g_{\mu,}{}^{\cdot\alpha} g_{\nu,}{}^{\cdot\beta} f^{\lambda\cdot}{}_{,\alpha\beta} \qquad f_{\alpha} \equiv f^{\beta}{}_{\alpha\beta} \quad \text{etc.}$$

The conservation laws (2.2) imply that the quantities:

$$P_{\lambda} \equiv -\frac{1}{\kappa}\int \mathscr{T}_{\lambda,}{}^{\cdot 0}\, d^{3}x \tag{2.10}$$

are constants of motion ($\kappa \equiv 8\pi G/c^{4}$). In particular, *for a closed system in the static case*, we interpret P_0 as the *total gravitational energy of this system*:

$$P_{0} \equiv -\frac{1}{\kappa}\int \mathscr{T}_{0,}{}^{\cdot 0}\, d^{3}x \tag{2.11}$$

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1) $e_0 \equiv e_{00} = 1$, $e_i \equiv e_{ii} = -1$, $e_{\alpha\beta} = 0\ (\alpha \neq \beta)$.

3. The Field Equations for the Tetrads

The expression (2.11) is to compare with the integral $E_{\text{pot}} = -1/8\pi \int \vec{G}^2\, dV$ of the classical theory ($\vec{G}$ is the gravitational field). Now, we know that the classical field equation out of the source can be derived from the variational principle $\delta E_{\text{pot}} = 0 \Rightarrow \Delta\varphi = 0$. *By analogy, we postulate that, in the static case of GR, the field equations for the tetrads are also deduced from a corresponding variational principle*:

$$\delta P_0 \sim \delta \int \mathscr{T}_{0,}{}^{,0}\, d^3x = 0 \tag{3.1}$$

On the other hand, the $g^{\lambda\cdot}{}_{,\mu}(x)$ have of course to satisfy the constraints (2.5). Let us define:

$$G_{\mu\nu} \equiv g_{\mu\nu} - e_\alpha g^{\alpha\cdot}{}_{,\mu} g^{\alpha\cdot}{}_{,\nu} = 0 \tag{3.2}$$

$$\mathscr{T}_{0,}{}^{,0*} \equiv \mathscr{T}_{0,}{}^{,0} + \lambda^{\mu\nu}(x) G_{\mu\nu} \tag{3.3}$$

where the ten symmetric Lagrange multipliers $\lambda^{(\mu\nu)}(x)$ have the character of a tensorial density of rank 2. In this variational problem with constraints, the field equations are ('external' case):

$$\frac{\partial}{\partial x^\nu}\left[\frac{\partial \mathscr{T}_{0,}{}^{,0*}}{\partial\left(\dfrac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^\nu}\right)}\right] - \frac{\partial \mathscr{T}_{0,}{}^{,0*}}{\partial g^{\lambda\cdot}{}_{,\mu}} = 0 \tag{3.4}$$

The formulas (2.3)–(2.9), (3.2) and (3.3) show that the field equations (3.4) for the tetrads $g^{\lambda\cdot}{}_{,\mu}(x)$ are of second order. The systems (3.2) and (3.4) give 26 equations for the 26 unknown functions $g^{\lambda\cdot}{}_{,\mu}(x)$ and $\lambda^{(\mu\nu)}(x)$. With suitable boundary conditions, we can then hope that these equations determine the 'good' tetrads unambiguously ('good' tetrads = those which give a correct energy value). In our formalism, the equations (3.4) play the same role as Møller's six supplementary conditions. Nevertheless, in our case, the field equations for the tetrads seem to be correctly physically justified by the above variational principle, according to a well-known classical example.

In order that the situation be as described above, it is necessary that the field equations (3.4) are independent. If this is not the case, we could arbitrarily choose some tetrads, and the GE-density would not necessarily be invariant under transformations of tetrads compatible with (3.2) and (3.4). The problem would thus not be resolved in the sense of localization.

But the detailed form of these field equations is very complicated (we give an outline of the calculations in the appendix), and the problem of independence seems to be rather difficult. In this paper, we prefer to illustrate our formalism by an example.

4. The Schwarzschild–Scherrer Solution

In the problem with spherical symmetry (exterior case), Scherrer proposes the following tetrads (for the reasons of this choice, see [7]):

$$(g^{\lambda\cdot}{}_{,\mu}) = \begin{pmatrix} \sqrt{1-\frac{2a}{r}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\frac{2a}{r}}}\sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ 0 & \frac{1}{\sqrt{1-\frac{2a}{r}}}\sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ 0 & \frac{1}{\sqrt{1-\frac{2a}{r}}}\cos\vartheta & -r\sin\vartheta & 0 \end{pmatrix} \tag{4.1}$$

where $a \equiv GM/c^2$. With the help of formulas (2.5), one easily sees that these $g^{\lambda\cdot}{}_{,\mu}$ indeed give the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{2a}{r}\right)(dx^0)^2 - \frac{dr^2}{1 - \frac{2a}{r}} - r^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \tag{4.2}$$

The tetrads (4.1) further lead to the 'good' total energy:

$$P_0 = -Mc^2 \tag{4.3}$$

The diagonal tetrads:

$$(\bar{g}^{\lambda\cdot}{}_{,\mu}) = \begin{pmatrix} \sqrt{1-\frac{2a}{r}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-\frac{2a}{r}}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\vartheta \end{pmatrix} \tag{4.4}$$

are formally simpler than (4.1), and correspond to the same metric. But the total gravitational energy calculated with the help of (4.4) is infinite! The $\bar{g}^{\lambda\cdot}{}_{,\mu}$ are thus not 'good' tetrads.

Now we have verified precisely that *the 'good' tetrads* (4.1) *satisfy the field equations* (3.4), *with the following expressions of the Lagrange multipliers* $\lambda^{(\mu\nu)}(x)$:

$$\lambda^{00} = 0 \qquad \lambda^{11} = -\frac{a\sqrt{1-\frac{2a}{r}}}{r}\sin\vartheta$$

$$\lambda^{22} = \frac{a}{2r^3\sqrt{1-\frac{2a}{r}}}\sin\vartheta \qquad \lambda^{33} = \frac{a}{2r^3\sqrt{1-\frac{2a}{r}}}\cdot\frac{1}{\sin\vartheta} \tag{4.5}$$

$$\lambda^{\lambda\nu} = 0 \quad (\mu \neq \nu)$$

On the other hand, with the total GE given by the tetrads $\bar{g}^{\lambda\cdot}{}_{,\mu}$ being infinite, it is clear that these are not a solution of the equations (3.4), although they satisfy (3.2). *From the point of view of our GE-localization problem, the* $\bar{g}^{\lambda\cdot}{}_{,\mu}$ *have then to be excluded.*

This example thus proves that there is a good chance that our field equations (3.4), physically well justified, are adequate supplementary conditions for the 'good' tetrads.

5. Conclusion

If the equations (3.4) are independent, they give the supplementary conditions (physically justified) which are necessary for an unambiguous determination of the tetrads. At least in the static case, we could then consider our problem as resolved in the sense of the GE-localization.

Even if it is difficult to prove the independence of our field equations, the example of §4 shows nevertheless that they are very restrictive conditions for the tetrads. According to this formalism, the gravitational energy is probably localizable.

Appendix

$\mathscr{T}_{0,}{}^{\cdot 0*}$ contains in particular the term (see (2.6) and (2.7)):

$$\tfrac{1}{2}\underset{1}{\mathscr{T}}_{0,}{}^{\cdot 0} \equiv \tfrac{1}{2}g(-4f_{\alpha 0}{}^{\beta}f^{\alpha\gamma}{}_{\beta}g_{\gamma,}{}^{\cdot 0} + g_{0,}{}^{\cdot 0}\underset{1}{H}) \tag{A.1}$$

As an example, we give the calculation of:

$$\mathscr{U}_{\lambda 0,}{}^{\cdot 0\mu\nu} \equiv \frac{\partial}{\partial\left(\dfrac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^{\nu}}\right)}(gg_{\gamma,}{}^{\cdot 0}f_{\alpha 0}{}^{\beta}f^{\alpha\gamma}{}_{\beta}) \tag{A.2}$$

Let us recall the relation:

$$f^{\alpha}{}_{\beta\gamma} = g_{\beta,}{}^{\cdot\rho}g_{\gamma,}{}^{\cdot\sigma}f^{\alpha\cdot}{}_{,\rho\sigma} = \tfrac{1}{2}g_{\beta,}{}^{\cdot\rho}g_{\gamma,}{}^{\cdot\sigma}\left(\frac{\partial g^{\alpha\cdot}{}_{,\sigma}}{\partial x^{\rho}} - \frac{\partial g^{\alpha\cdot}{}_{,\rho}}{\partial x^{\sigma}}\right) \tag{A.3}$$

from which we deduce:

$$\frac{\partial f^{\alpha}{}_{\beta\gamma}}{\partial\left(\dfrac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^{\nu}}\right)} = \tfrac{1}{2}\delta^{\alpha}{}_{\lambda}(g_{\beta,}{}^{\cdot\nu}g_{\gamma,}{}^{\cdot\mu} - g_{\gamma,}{}^{\cdot\nu}g_{\beta,}{}^{\cdot\mu}) \tag{A.4}$$

On the other hand:

$$f_{\alpha 0}{}^{\beta} = e_{\alpha}e^{\beta}f^{\alpha}{}_{0\beta} \quad \text{etc.} \tag{A.5}$$

(A.2) can be rewritten as follows:

$$\begin{aligned}
\mathscr{U}_{\lambda 0,}{}^{\cdot 0\mu\nu} &= e_{\alpha}e^{\beta}e^{\gamma}gg_{\gamma,}{}^{\cdot 0}\left[\frac{\partial f^{\alpha}{}_{0\beta}}{\partial\left(\dfrac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^{\nu}}\right)}f^{\alpha}{}_{\gamma\beta} + f^{\alpha}{}_{0\beta}\frac{\partial f^{\alpha}{}_{\gamma\beta}}{\partial\left(\dfrac{\partial g^{\lambda\cdot}{}_{,\mu}}{\partial x^{\nu}}\right)}\right] \\
&= \tfrac{1}{2}e_{\alpha}e^{\beta}e^{\gamma}gg_{\gamma,}{}^{\cdot 0}[\delta^{\alpha}{}_{\lambda}(g_{0,}{}^{\cdot\nu}g_{\beta,}{}^{\cdot\mu} - g_{\beta,}{}^{\cdot\nu}g_{0,}{}^{\cdot\mu})f^{\alpha}{}_{\gamma\beta} \\
&\qquad + f^{\alpha}{}_{0\beta}\delta^{\alpha}{}_{\lambda}(g_{\gamma,}{}^{\cdot\nu}g_{\beta,}{}^{\cdot\mu} - g_{\beta,}{}^{\cdot\nu}g_{\gamma,}{}^{\cdot\mu})] \\
&= \tfrac{1}{2}gg_{\gamma,}{}^{\cdot 0}[g_{0,}{}^{\cdot\nu}f_{\lambda}{}^{\gamma,\mu} - g_{0,}{}^{\cdot\mu}f_{\lambda}{}^{\gamma,\nu} + e^{\gamma}f_{\lambda 0,}{}^{\cdot\mu}g_{\gamma,}{}^{\cdot\nu} - e^{\gamma}f_{\lambda 0,}{}^{\cdot\nu}g_{\gamma,}{}^{\cdot\mu}] \\
&= \tfrac{1}{2}g[g_{0,}{}^{\cdot\nu}f_{\lambda,}{}^{\cdot 0\mu} - g_{0,}{}^{\cdot\mu}f_{\lambda,}{}^{\cdot 0\nu} + g^{0\nu}f_{\lambda 0,}{}^{\cdot\mu} - g^{0\mu}f_{\lambda 0,}{}^{\cdot\nu}]
\end{aligned}$$

The $\mathscr{U}_{\lambda 0,}{}^{\cdot 0[\mu\nu]}$ are a $[\mu\nu]$-antisymmetric tensor density under the group of purely spatial coordinate transformations. It follows that $(\partial/\partial x^{\nu})\mathscr{U}_{\lambda 0,}{}^{\cdot 0[\mu\nu]}$ is a contravariant vector density. The left hand sides of the field equations for the tetrads have exactly this character.

The $\partial\mathscr{T}_{0,}{}^{\cdot 0}/\partial g^{\lambda\cdot}{}_{,\mu}$ are calculated in the same way, but here, the development is more tedious.

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