On the Anomalous Skin Effect in Metals Part 2: Some Aspects of Superconductors, Sum-Rule Applications and Limits of this Technique

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(16. VI. 75)

Abstract. Based on previous work applying to normal conducting metals [1], an analog network model for superconducting metals is developed. The reciprocal surface impedance associated with this model is used as a response function in connection with the Kramers-Kronig relations and the sum rule to compute electrodynamic properties of superconductors.

Introduction

This paper discusses electrodynamic effects which are rather straightforward consequences of results established in Part 1 [1] from developing an analog model for normal conducting metals. Firstly, the close resemblance between the electrodynamics of normal and superconductors is used to set up an analog model for a superconducting metal. We shall point out that it must be incomplete because it tolerates unlimited propagation velocities, and that taking into account the parallel flow of normal electrons with superelectrons in principle eliminates this difficulty. Such a completed two-fluid model is however no longer a linear model. Secondly, an approach for an approximate two-fluid model is made, assuming discontinuous switching between the pure state models at a typical frequency (gap frequency). In a similar way to Tinkham and Ferrell [2] we compute, by means of the sum-rule technique, the effective penetration depth of a magnetic field into a superconductor as a function of mean-free electron path in the normal conducting state, and the coherence length in the superconducting state as a function of mean-free electron path in the normal conductor. Generally, the results look reasonable, and we find the fact attractive that a macroscopic solution to a microscopic problem is derivable in this way. By the same technique, we further compute for the case of pure metals whose individual normal-state material parameters are given, the relationship between an assumed gap frequency ω_g and the corresponding coherence length ξ_0 in the superconducting state, and from this $\alpha = v_F/(\xi_0 \omega_g)$. ($v_F$ = Fermi velocity.) We find a physically acceptable value around unity for α, which however is almost independent of the assumed ω_g value, and thus conclude that any value for the energy gap of an individual metal fits with Kramers-Kronig relations and sum rule. Thirdly, we emphasize the unsatisfactory aspects revealed by the analysis. It turns out that numerical results computed from the approximate two-fluid model are no better than semi-quantitative. While it is obvious that improvement should be sought by working out the approximate model nearest to a complete two-fluid model,

it is also obvious that a complete model will have non-linear elements and will not, therefore, generate a linear response function. Thus, we conclude that the sum-rule technique cannot be developed to a quantitative method.

1. Analog Model for the Superconducting State

There is a close resemblance between the current-density versus electric-field relation in a normal conductor and the current-density versus magnetic-vector potential correspondence in a superconductor, which was first recognized and discussed by Pippard [3] and – based on very different arguments – confirmed by BCS (see, e.g., de Gennes [4]). The formal procedure is: substitute for the electron mean free path l a coherence length ξ, and for the specific resistance ρ an inductance $j\omega\Lambda_{\mathrm{L}}\xi_0/\xi$, where Λ_{L} stands for the (London) superconductivity constant, ξ_0 means the coherence length in an ideal lattice, and ξ the coherence length as reduced in a real lattice by contamination or distortion.

This procedure yields, for low enough frequencies, out of the analog model designed in Part 1 for normal conductors, the superconductor model plotted in Figure 1. We should note that not only transmission line 1a is the Maxwellian-fields transmission line 2a of Part 1, but that also the transmission line 1d represents the same electrodynamic interaction mechanism as line 2d of Part 1. Its visually different appearance results merely from the need to adapt the coupling facilities to the transformed line 1c. All material data are contained in this line. The inductive impedances which replace corresponding dissipative impedances of line 2c of Part 1, reflect the step of making potentials (in superconductors) rather than fields (in normal conductors) responsible for current flow. We cannot perceive this phenomenon classically and, therefore, do not attempt to interpret line 1c.

The model corresponds to the set of equations

$$\frac{\partial E}{\partial x} = -j\omega\mu_0 H \tag{1}$$

$$-\frac{\partial H}{\partial x} = i \tag{1'}$$

$$(j\omega\Lambda_{\mathrm{L}}\xi_0/2)\cdot\frac{\partial i^*}{\partial x} = E - (j\omega\Lambda_{\mathrm{L}}\xi_0/\xi)\cdot i \tag{2}$$

$$(j\omega\Lambda_{\mathrm{L}}\xi_0/2)\cdot\frac{\partial i}{\partial x} = E^* - (j\omega\Lambda_{\mathrm{L}}\xi_0/\xi)\cdot i^* \tag{2'}$$

$$\frac{\partial E^*}{\partial x} = j\omega\mu_0 H^* \tag{3}$$

$$\frac{\partial H^*}{\partial x} = i^*, \tag{3'}$$

and to the differential equation

$$\frac{\partial^6 i}{\partial x^6} - \frac{1}{(\xi/2)^2}\cdot\frac{\partial^4 i}{\partial x^4} + \frac{2\cdot\mu_0}{(\Lambda_{\mathrm{L}}\cdot\xi_0/2\cdot\xi/2)}\cdot\frac{\partial^2 i}{\partial x^2} - \left(\frac{\mu_0}{\Lambda_{\mathrm{L}}\cdot\xi_0/2}\right)^2\cdot i = 0 \tag{4}$$

for the supercurrent density i. Equations (1) to (4) also follow from equations (11) to (14) of Part 1, if we make the appropriate substitutions there.
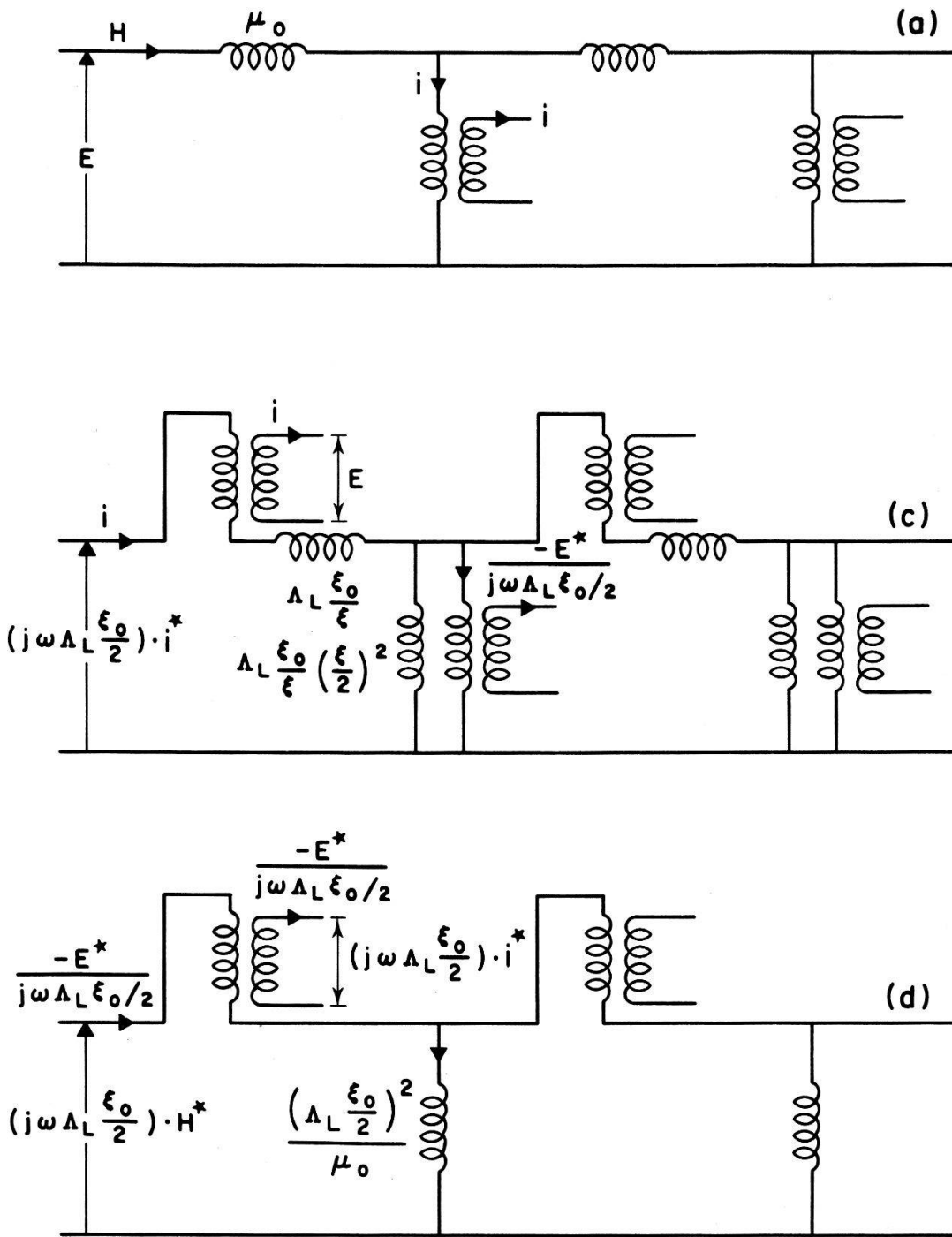
Figure 1
Analog model of superconducting metal. Transmission lines represent (a) Maxwellian field (E, H); (c) Current densities (i, i^*); (d) Electromagnetic field (E^*, H^*).

Boundary conditions, as before, may either be diffuse or specular, or a mixture of both types. Diffuse surface scattering is associated with reversible storage of energy in the surface, while this phenomenon does not occur with specular surface scattering.

Solutions of equation (4) yield the shape of a penetrating magnetic field as shown in Figure 2. Plots refer to diffuse surface scattering and to the limiting case of a very pure or very impure metal.

A characteristic of pure metals is a region of reversed field [3, 5] while impure metals show a tendency towards an exponential penetration curve. In a pure-metal

Figure 2

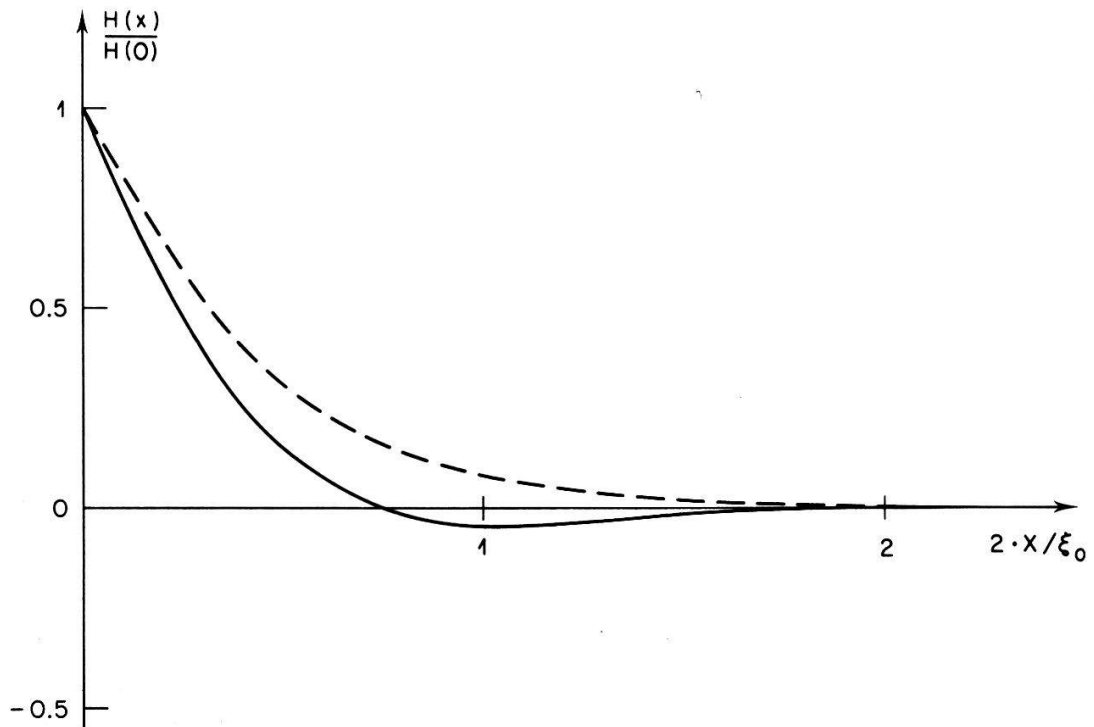Shape of a penetrating magnetic field. Full line: pure metal ($\xi = \xi_0 = 4760$ Å, $l = \infty$); Broken line: dirty metal ($\xi_0 = 4760$ Å, $\xi = 400$ Å, $l = 464$ Å). Surface scattering is diffuse.

situation with $\xi = \xi_0 \gg \lambda_L$ ($\lambda_L = \sqrt{\Lambda_L/\mu_0}$ = London penetration depth), the above-mentioned substitutions yield, out of equation (18) of Part 1, the expression for the surface impedance Z_∞ at extreme anomalous conditions

$$Z_\infty = j\omega\mu_0(\lambda_L^2\xi_0/2)^{1/3},\tag{5}$$

or an effective penetration depth

$$\lambda_{\text{eff}\,\infty} = (\lambda_L^2\xi_0/2)^{1/3},\tag{6}$$

which comes very close to the corresponding BCS value

$$\lambda_{\text{eff}} = (0.62\lambda_L^2\xi_0)^{1/3}.$$

In the dirty-metal limit $\xi \ll \xi_0$ the effective penetration depth becomes

$$\lambda_{\text{eff}} = \lambda_L(\xi_0/\xi)^{1/2},\tag{7}$$

which also corresponds to the BCS value.

It is obvious that the analog model (Fig. 1) is incomplete, due to unlimited signal propagation velocities along the transmission lines. A complete model should include elements which establish finite propagation velocities in the transmission lines at high frequencies. We know from phenomenology that superconductivity goes over to normal conductivity at high frequencies, so that these velocities are those of the normal conducting-state model. However, it is very unlikely that a linear complete (two-fluid) model can be made up from the low-frequency model. The condition for a finite velocity is that the longitudinal and transverse elements of the transmission lines are dual [6]. Now a possible longitudinal impedance element z_L, representing parallel
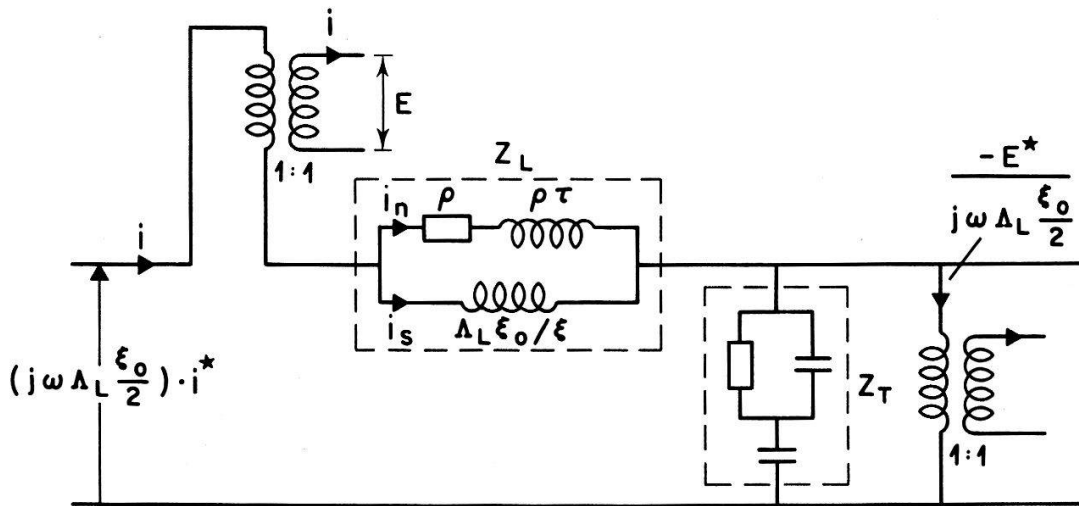
Figure 3
Longitudinal impedance z_L, representing parallel flow of normal and supercurrent densities (i_n, i_s, respectively) in transmission line 1c, and associated transverse dual impedance z_T.

flow of normal and supercurrent densities (i_n, i_s, respectively) in line 1c, should be something like that plotted in Figure 3. The associated transverse dual impedance element z_T, which is also shown qualitatively in Figure 3, is not however an inductance at low frequencies as requested by the model Figure 1c.

2. Correspondence between Normal and Superconducting Properties from Kramers-Kronig and Sum-Rule Equations

In this section, we introduce an approximate two-fluid model for superconductors, and compute correspondence between electrodynamic properties of normal and superconductors by exploiting Kramers-Kronig equations and sum rule [7, 8]. The idea of proceeding in this way is due to Tinkham and Ferrell [2], but the detailed properties of, and requirements for, a response function were not emphasized as much in the earlier work as here.

Let us first briefly sketch the technique. The Kramers-Kronig equations

$$\sigma_1(\omega) = \frac{2}{\pi} \cdot \int_0^\infty \frac{\omega_1 \cdot \sigma_2(\omega_1) \cdot d\omega_1}{\omega_1^2 - \omega^2} + \text{const} \tag{8}$$

$$\sigma_2(\omega) = -\frac{2\omega}{\pi} \cdot \int_0^\infty \frac{\sigma_1(\omega_1) \cdot d\omega_1}{\omega_1^2 - \omega^2}, \tag{8'}$$

claim that the real and imaginary parts of any complex function of frequency

$$\sigma(\omega) = \sigma_1(\omega) - j\sigma_2(\omega)$$

depend on each other as expressed by equations (8) and (8') provided that σ as a function of time generates a linear response to a physical cause, so that no effect (response) precedes its cause. Among the wide variety of possible response functions for a metal we choose the reciprocal surface impedance (surface admittance).

We have shown in Part 1 that the respective analog model generates for normal conducting metals a surface admittance

$$\sigma^n(\omega) = \sigma_1^n(\omega) - j\sigma_2^n(\omega), \tag{9}$$

which is phenomenologically realistic at all frequencies, and which satisfies the mathematical conditions for a response function.

For superconductors, the situation is less satisfactory because there is no linear model for all frequencies. We have defined as our response function for $\omega < \omega_g$ the surface admittance

$$\sigma^s(\omega) = \sigma_1^s(\omega) - j\sigma_2^s(\omega) \tag{9'}$$

which is generated by the superconductor analog model of the preceding section, and for $\omega > \omega_g$ the surface admittance generated by the normal conductor model of Part 1. This corresponds to supposing a discontinuous transition from superconductivity to normal conductivity at the gap frequency, and is an approximate approach in view of experiments by Tinkham et al. [9] indicating that this transition is continuously smeared out on the frequency scale.

The sum rule, claiming that

$$\int_0^\infty [\sigma_1^n(\omega_1) - \sigma_1^s(\omega_1)] \cdot d\omega_1 = 0, \tag{10}$$

is an implication of equation (8'), and of $\sigma^s(\omega) \to \sigma^n(\omega)$ for sufficiently high frequencies, because if we choose ω so large ($\omega \gg \omega_g$), that a superconducting metal becomes practically normal conducting, then $\sigma_2^s(\omega) - \sigma_2^n(\omega) = 0$ may with equation (8') be written as

$$0 = \int_0^\infty \frac{\sigma_1^n(\omega_1) - \sigma_1^s(\omega_1)}{\omega_1^2 - \omega^2} \cdot d\omega_1. \tag{11}$$

Since also $\sigma_1^n(\omega_1) - \sigma_1^s(\omega_1) = 0$ for $\omega_1 \gg \omega_g$, the integration in equation (11) extends only to some upper limit for ω_1. This permits to choose ω so large that $\omega \gg \omega_1$. After that we find equation (10) by treating the denominator of the integrand as a constant and shifting it in front of the integral sign.

Like the Kramers-Kronig equations (8) and (8'), the sum rule expresses a very generally valid correspondence, because it holds for any particular dependence of σ^n and σ^s on frequency. This view point does not, of course, make sense if expressions for σ^n or σ^s are used which do not fulfill all presuppositions for a response function. In our case, we know that a phenomenologically realistic transition from superconductivity to normal conductivity around gap frequency conflicts with the requirement of a linear response function. The sum rule is thus not strictly valid. Nevertheless, we shall subsequently take it literally and confirm that it is at least a very useful and handy tool.

We proceed in the following manner: because of the presumed sharp transition at $\omega = \omega_g$, the integration in equation (10) extends to the range $\omega < \omega_g$ only. We are able to compute the contribution of the normal conductor, and to conclude that there is no contribution from the superconductor. This result would damage the sum rule and is therefore not acceptable. The superconductor has, in fact, its contribution concentrated in a pole at $\omega = 0$, of strength

$$A = \int_0^{\omega_g} \sigma_1^n(\omega_1) \cdot d\omega_1. \tag{12}$$

The pole is associated, according to equation (8), with a low-frequency susceptance

$$\sigma_2^s(\omega) = \frac{2}{\pi} \cdot \frac{A}{\omega}. \tag{13}$$

This result is more conveniently expressed in terms of effective penetration depths; since $\sigma_2^s(\omega) = 1/(\omega \cdot \mu_0 \cdot \lambda_{eff})$

$$\lambda_{eff} = \pi/(2 \cdot \mu_0 \cdot A). \tag{14}$$

We have computed from equation (12) A as a functional of l, and subsequently λ_{eff} from equation (14). The result, corresponding to the data of tin [$\rho \cdot l = 1.05 \cdot 10^{-15}$ Ωm^2, $v_F = 1.31 \cdot 10^6$ $m \cdot sec^{-1}$ and $\lambda_{eff} = 600$ Å for the pure-metal case limit $(l \to \infty)$], yields the overall behavior correctly but gives only a semi-quantitative fit to the experimental data of Pippard [3] (see Fig. 4). $\lambda_{eff} = 600$ Å is obtained with $\omega_g =$
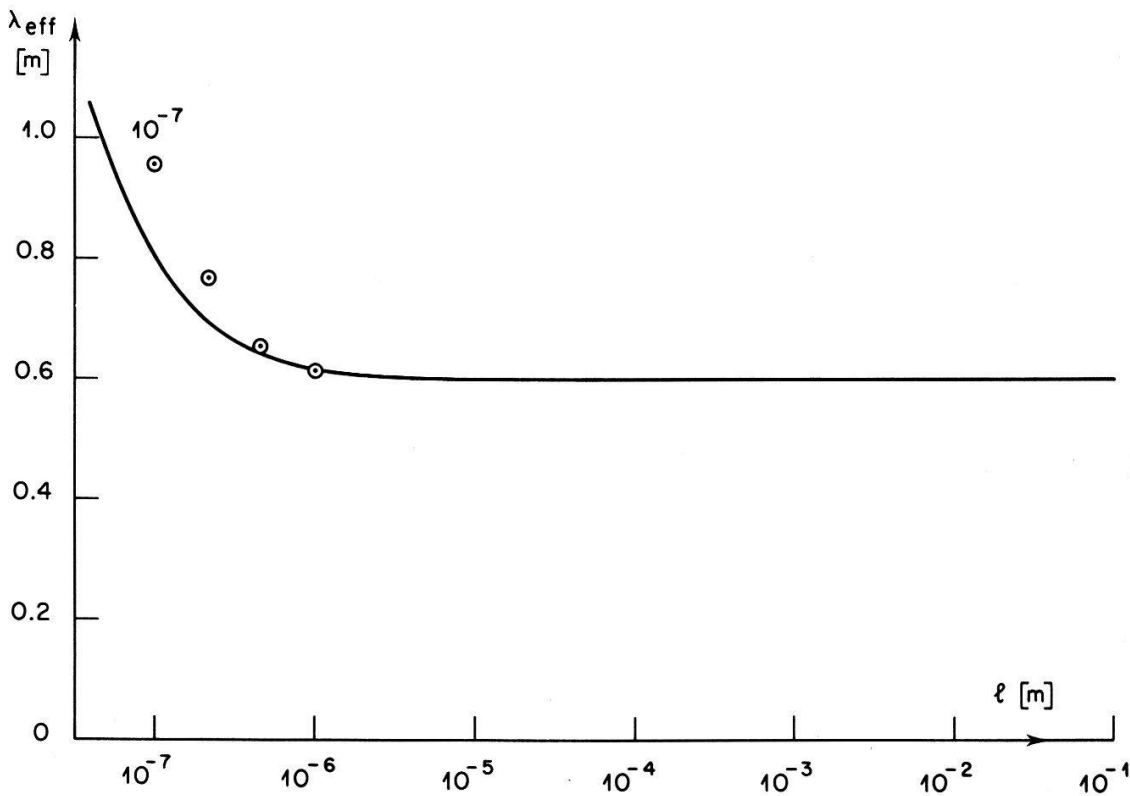

Figure 4
Effective penetration depth λ_{eff} as a function of l. Circles refer to the data by Pippard [3].

$2.75 \cdot 10^{12}$ sec^{-1}, which is higher than found in tunneling experiments, but not unreasonable. From $\xi_0 = v_F/(\alpha \cdot \omega_g)$ we determine with $\alpha = 1$[1]) a corresponding $\xi_0 = 4760$ Å. Some discrepancy seems natural. An abrupt phase change at $\omega = \omega_g$ is an over-simplification. The same pole strength A would be feasible with a smaller ω_g if the phase transition were continuous above ω_g.

The result expressed by equation (14) does not make direct reference to the superconductor analog model. We can, therefore, in principle go a little bit further and involve its properties more explicitly.

[1])   BCS gives $\alpha = \pi/2$, experiments by Tinkham et al. [9] $\alpha \approx 1.06$, by Faber and Pippard [10] $\alpha \approx 2$.

Let us first mention the case $l \to \infty$. Then the superconductor model contains the material parameters $\Lambda_L = \rho l / v_F$ and $\xi = \xi_0$. If for ρl, v_F and ξ_0 the afore-mentioned values are used, the model-predicted effective penetration depth should agree with that computed from equation (14). The impact of the comparison is unfortunately somewhat affected by uneasiness with the boundary conditions: while the Kramers-Kronig and sum-rule procedure yields penetration depths without a need to specify boundary conditions, the model does not permit such calculation unless boundary conditions have been properly defined. We have chosen either completely diffuse or completely specular surface scattering and determined the ratio Λ/Λ_L for both conditions. The result is $\Lambda/\Lambda_L = 3.85$ for specular, and $\Lambda/\Lambda_L = 1.13$ for diffuse scattering, and thus seems to be compatible with the latter condition. However, bearing in mind the surface energy storing effect mentioned in Section 1 of this paper, we feel that diffuse surface-scattering is not necessarily a natural physical boundary condition, and therefore the discrepancy might actually be larger than only the computed 13%. Secondly, let us consider the case of a finite l. Then the superconductor model contains the three parameters Λ, ξ_0, ξ. We may accept ξ_0 from before and the Λ values just determined, and compute the coherence length ξ which corresponds to the chosen l. An example of such curves $\xi(l)$ is plotted in Figure 5 for specular and diffuse surface scattering. Again the overall tendency is in agreement with BCS, except that with both types of surface scattering ξ does not approximate l for extremely small l's, but becomes greater than l for sufficiently small l's.

In a series of additional numerical computations, the following different approach was made to set up the expression of the surface admittance in a superconducting metal: first the ratio $\sigma_1^s(\omega)/\sigma_1^n(\omega)$ was fitted to a shape similar to that proposed by
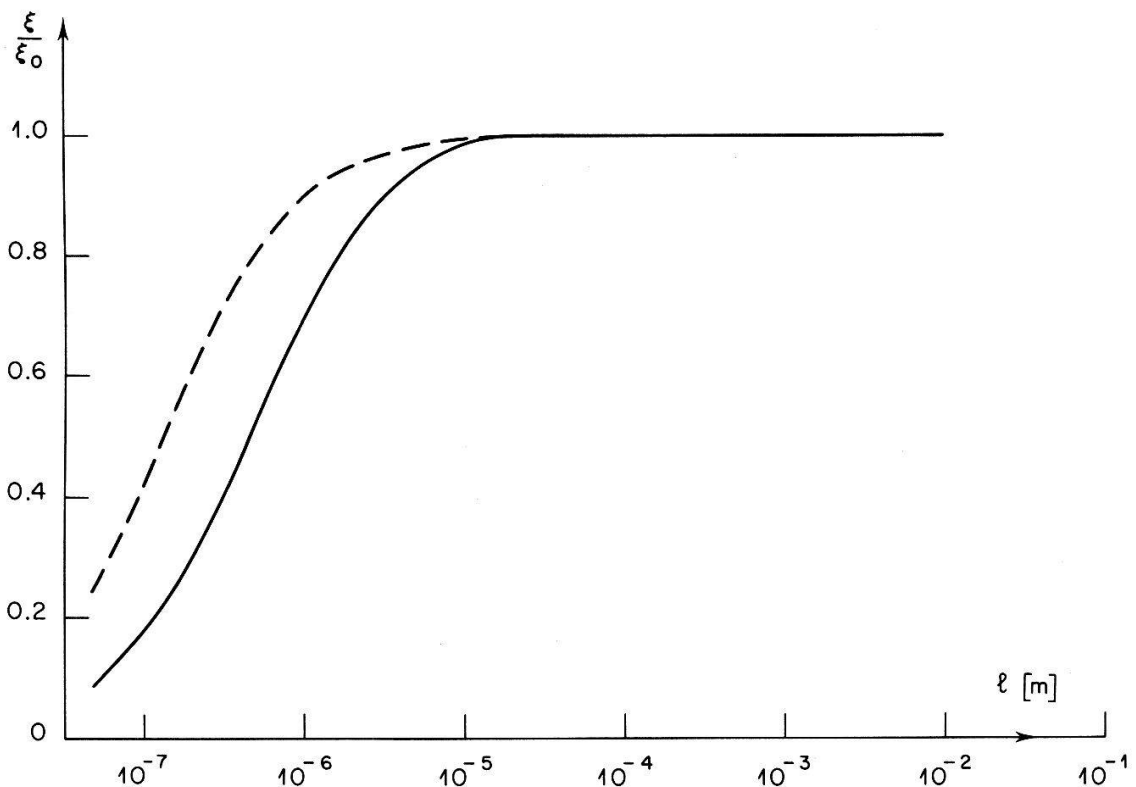

Figure 5

Coherence length ξ as a function of l. Surface scattering is diffuse (full line) or specular (broken line).

Tinkham et al. [9]. Then $\sigma_1^s(\omega)$ was determined by multiplying the ratio with the model-generated $\sigma_1^n(\omega)$, and $\sigma_2^s(\omega)$ in accordance with the Kramers-Kronig relations. Numerical results for $\lambda_{\text{eff}}(l)$ and $\xi(l)$ obtained by this procedure, with various parameters ρl, v_F and ξ_0, were not more convincing than those obtained with our models and the suggestion of a discontinuous transition from one to the other model at $\omega = \omega_g$. We believe in fact that abrupt switching from super to normal conductance at the gap frequency is the most general approach that can be made at all to combine our low and high-frequency models, because only with a vanishing transition width on the frequency scale may we achieve that the non-linearities involved with the response functions in the transition-frequency range are not perceived in the integrals of equations (8), (8′) and (10).

An additional series of computations was made to check whether the gap frequency of an individual pure metal, e.g., tin, could be determined by the sum rule, if its material parameters $(\rho l, v_F, \epsilon, l \to \infty)$ associated with the normal-conductor analog model are known. The superconducting state was characterized by the material parameters $(\Lambda_L = \rho \cdot l/v_F, \xi_0)$ associated with the superconductor model. If now any value ω_g is chosen for the transition frequency between the normal and superconducting states, the sum rule permits to compute the corresponding coherence length ξ_0, and from this $\alpha = v_F/(\xi_0 \cdot \omega_g)$, where α is a numerical factor around unity if the gap frequency meets with a realistic choice. With the data of tin ($\rho l = 1.05 \cdot 10^{-15}$ Ωm^2, $v_F = 1.31 \cdot 10^6$ m/sec, $1 < \epsilon < 12$, $l \to \infty$; $\Lambda_L = 8.04 \cdot 10^{-22}$ Ω m/sec, $\omega_g = 1.71 \cdot 10^{12}$ sec^{-1}) we found $\alpha \approx 0.85$, which is definitely less than the BCS value $\pi/2$, but still reasonable[2]). Then ω_g was chosen smaller or larger, but it turned out that α is
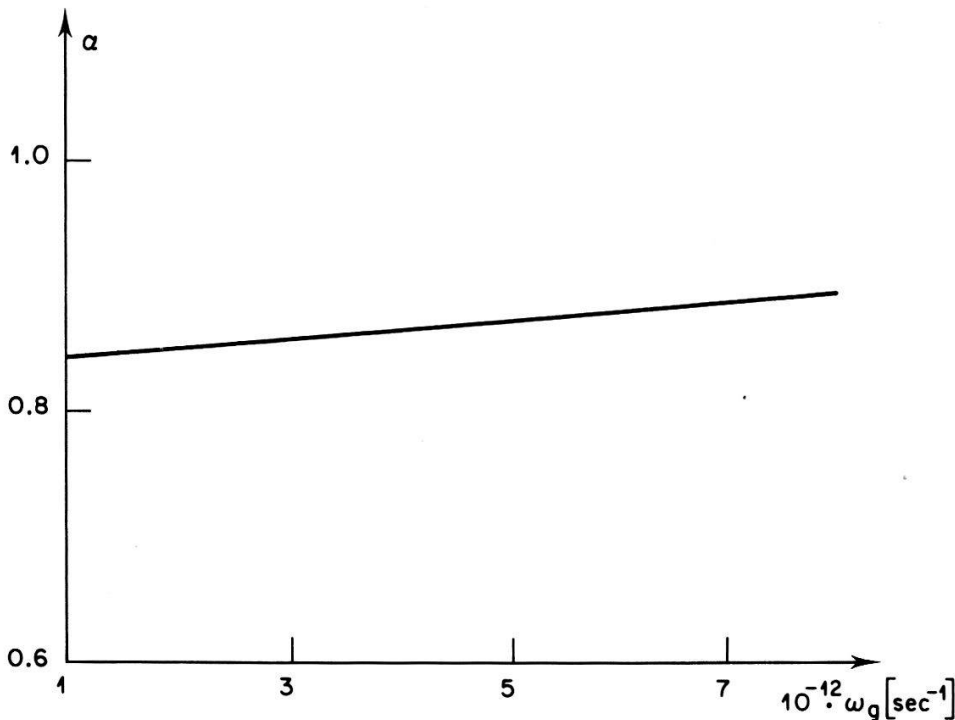

Figure 6
$\alpha = v_F/(\xi_0 \cdot \omega_g)$ as a function of ω_g for tin.

[2]   Essentially the same situation with numerical factors was found by R. Sommerhalder, K. E. Drangeid and W. Bächtold [12] when deriving the maximum Josephson tunneling current density with the sum-rule technique.

almost independent of ω_g. This finding is plotted as Figure 6. Only very slightly deviating curves were obtained when $\rho \cdot l$, or v_F, or ϵ were varied within the limits given by typical metals. We are thus left with the result that our attempt to estimate the gap frequency was not successful. If we take into account that ω_g is a parameter which is typical for the nonlinear transition region between superconductance and normal conductance, the failure seems due to having reached the borders of linear analysis.

3. Conclusions

We have investigated, within the frame of linear models and linear analysis, the electrical conductance properties of normal and superconducting metals. Very generally valid concepts – Kramers-Kronig equations and sum rule – have been applied to find correspondence between the material parameters involved with either the normal or superconducting states.

We arrive at the conclusion that interesting relations such as the coherence length in a superconductor as a function of mean-free electron path in the normal-conducting state, are derivable in this way. The appeal of this technique is that a simple macroscopic procedure is being used to bypass much more sophisticated quantum-mechanical considerations[3]). The accuracy of our procedure is, nevertheless, limited. We suggest that nonlinearity which must be associated with a properly designed two-fluid model of superconductors, marks this limit.

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[3])   Miller [11] e.g., used the equation giving the current density as a functional of the vector potential for an impurity superconductor derived by Mattis and Bardeen [13], which is a quantum approach, to derive curves $\lambda_{eff}(l)$.