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Hamiltonian Dynamics for Einstein Relativistic Particles: The Classical Particle and the Quantal Particles of Spin 0 and $\frac{1}{2}$

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Abstract. In this paper we present a new Hamiltonian dynamics for the description of Einstein relativistic particles. The theory is based on the idea of passivity in the Einstein relativistic case and is constructed by following a scheme similar to that followed by Piron in his study of the Galilei relativistic particles. The results obtained and presented indicate that the present theory may be considered as a natural generalization of the Galilean–Hamilton dynamics, as well as the Wigner theory for free Einstein relativistic quantal particles.

1. Introduction

In this paper we propose a Hamiltonian formalism for the description of Einstein relativistic particles in interaction with external fields, treating in particular the classical particle and the quantal particles of spin 0 and '$\frac{1}{2}$'.

In constructing the theory, we have followed a programme similar to the one followed by C. Piron [1, 2] in his analysis of the Galilean–Hamilton dynamics:

i) first to try to understand what is the passive point of view in Einstein relativity [3] and to construct the passive action of the Lorentz group on the 'measuring apparatuses' such that each of the observables characterizing the Einstein particle is defined by a system of imprimitivity;
ii) then to construct solutions of these imprimitivity systems corresponding to the classical particle and the quantal particles;
iii) and finally to calculate the most general Hamiltonian compatible with the given definitions of observables.

In addition to constructing the general theory, we also study the free particles of the type mentioned, showing that in this case we obtain (except for notation and interpretation) the usual description. We also show that the Galilean limit ($c \to \infty$) for the free classical Einstein relativistic particle is the free classical Galilei relativistic particle from the passive point of view, i.e. compatible with being in interaction. The theory is being written out as a one-particle theory only, however, it is generalizable to an n-particle theory in the same way as the corresponding Galilean theory. Thus, the theory

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presented may be considered as a generalization of the present theory for free Einstein relativistic particles and of the Galilean–Hamilton dynamics.

Remarks

In the following, p^μ and q^μ must not be confused with the four-momentum and four-position of special relativity; both mathematically and conceptually they are different from these (pp. 182–183). One should also be aware of the fact that the time τ, being the only notion of time entering the theory, is an observable, and that the Hamiltonian generating the time evolution of the particle in the time τ has the same relation to energy as in the Galilean case. Moreover, q^μ is a fictive 'spatial' observable in which the interactions are local; the position observable denoting the 'real' position of the particle in three-dimensional space is $\vec{q}_s = \vec{q} - [\dot{\vec{q}}/(\dot{q}^0 + c)]q^0$ for the classical particle, and $\vec{q}_s = \vec{q} - \frac{1}{2}[\dot{\vec{q}}/(\dot{q}^0 + c), q^0]_+$ for the quantal particles.

2. Einstein Relativistic Particles

An Einsteinian particle is by definition a physical system characterized by a constant $m > 0$ (the rest mass), and for which the observables 'momentum' $\mathbf{p}^\mu$, 'position' $\mathbf{q}^\mu$ and time τ are defined.

The study of the symmetry properties of the measuring apparatuses for these observables lead us to define the following action of the symmetry group G (which is the inhomogeneous Lorentz group G_0 times (direct product) the translations of $\mathbb{R}$, T, i.e. $G = G_0 \times T = \{(\vec{u}, \vec{\theta}, a^\mu, \tau')\}$) on their spectra, i.e. on the nine-dimensional space [4]:

$$\{(p^\mu, q^\mu, \tau) | p^\mu \in \mathbb{R}^4, q^\mu \in \mathbb{R}^4, \tau \in \mathbb{R}\};$$

the 'special Lorentz transformations' $\{\vec{u}\}$:

$$p^\mu \mapsto \Lambda(\vec{u})^\mu{}_\nu p^\nu + mu^\mu$$

$$q^\mu \mapsto \Lambda(\vec{u})^\mu{}_\nu q^\nu$$

$$\tau \mapsto \tau,$$

where $\Lambda(\vec{u})$ is the usual representation of the special Lorentz transformations on $\mathbb{R}^4$ and $\vec{u}^\mu = (\gamma - 1, \gamma\vec{u})$ for $\gamma = (1 - \vec{u}^2/c^2)^{-1/2}$; the rotations $\{\vec{\theta}\}$:

$$p^\mu \mapsto \Lambda(\vec{\theta})^\mu{}_\nu p^\nu = (p^0/c, R(\vec{\theta})\vec{p})$$

$$q^\mu \mapsto \Lambda(\vec{\theta})^\mu{}_\nu q^\nu = (q^0/c, R(\vec{\theta})\vec{q})$$

$$\tau \mapsto \tau;$$

the spatial translations $\{a^\mu\}$:

$$p^\mu \mapsto p^\mu$$

$$q^\mu \mapsto q^\mu + a^\mu$$

$$\tau \mapsto \tau;$$

the time translations $\{\tau'\}$:

$$p^\mu \mapsto p^\mu$$

$$q^\mu \mapsto q^\mu$$

$$\tau \mapsto \tau + \tau'.$$

The Boolean CROCs $\mathscr{B}$, simulating the measuring apparatuses for momentum and position, are both constructed by starting from subsets of $\mathbb{R}^4$, the one for time is constructed from the subsets of $\mathbb{R}$. In each case we have two possibilities, either $\mathscr{B}$ contains all the subsets in question, or $\mathscr{B}$ is the CROC of Borel sets modulo the subsets of measure zero. In any case, G acts in a natural way on the elements $\Delta \in \mathscr{B}$ and we obtain the following systems of imprimitivity:

for the momentum:

$$S(\vec{u})\,\mathbf{p}^\mu(\Delta^\mu) = \mathbf{p}^\mu(\Lambda(\vec{u})^\mu{}_\nu\,\Delta^\nu + mu^\mu)$$

$$S(\vec{\theta})\,\mathbf{p}^\mu(\Delta^\mu) = \mathbf{p}^\mu(\Lambda(\vec{\theta})^\mu{}_\nu\,\Delta^\nu)$$

$$S(a^\mu)\,\mathbf{p}^\mu(\Delta^\mu) = \mathbf{p}^\mu(\Delta^\mu)$$

$$S(\tau')\,\mathbf{p}^\mu(\Delta^\mu) = \mathbf{p}^\mu(\Delta^\mu);$$

for the position:

$$S(\vec{u})\,\mathbf{q}^\mu(\Delta^\mu) = \mathbf{q}^\mu(\Lambda(\vec{u})^\mu{}_\nu\,\Delta^\nu)$$

$$S(\vec{\theta})\,\mathbf{q}^\mu(\Delta^\mu) = \mathbf{q}^\mu(\Lambda(\vec{\theta})^\mu{}_\nu\,\Delta^\nu)$$

$$S(a^\mu)\,\mathbf{q}^\mu(\Delta^\mu) = \mathbf{q}^\mu(\Delta^\mu + a^\mu)$$

$$S(\tau')\,\mathbf{q}^\mu(\Delta^\mu) = q^\mu(\Delta^\mu);$$

for the time:

$$S(\vec{u})\,\tau(\Delta) = \tau(\Delta)$$

$$S(\vec{\theta})\,\tau(\Delta) = \tau(\Delta)$$

$$S(a^\mu)\,\tau(\Delta) = \tau(\Delta)$$

$$S(\tau')\,\tau(\Delta) = \tau(\Delta + \tau').$$

We can now make the definition of an Einsteinian particle more precise:

Definition 1. By an *Einsteinian particle* we mean every propositional system $\mathscr{L}$ for which there is defined a representation S of G of Aut $\mathscr{L}$, which admits observables $\mathbf{p}^\mu$, $\mathbf{q}^\mu$ and τ satisfying the above systems of imprimitivity.

2.1. *The classical particle*

The propositional system $\mathscr{L}$ for the classical particle is realized by

$$\mathscr{L} = \mathscr{P}(\Omega) = \mathscr{P}(\{(p^\mu, q^\mu, \tau) \mid p^\mu \in \mathbb{R}^4, q^\mu \in \mathbb{R}^4, \tau \in \mathbb{R}\}),$$

the set of subsets of Ω, in which $G = \{(\vec{u}, \vec{\theta}, a^\mu, \tau')\}$ acts in a canonical way. Moreover, since all classical observables have a purely discrete spectrum [5], one has to identify $\mathscr{B}$ with the set of subsets of $\mathbb{R}^4$ or $\mathbb{R}$. With these choices, it is easy to verify that the following observables satisfy the imprimitivity relations:

$$\mathbf{p}^\mu(\Delta) = \{(p^\mu, q^\mu, \tau) \mid p^\mu \in \Delta \subset \mathbb{R}^4\}$$

$$q^\mu(\Delta) = \{(p^\mu, q^\mu, \tau) \mid q^\mu \in \Delta \subset \mathbb{R}^4\}$$

$$\tau(\Delta) = \{(p^\mu, q^\mu, \tau) \mid \tau \in \Delta \subset \mathbb{R}\}.$$

These observables may be defined as the inverse images of the following functions:

$$(p^\mu, q^\mu, \tau) \mapsto p^\mu$$

$$(p^\mu, q^\mu, \tau) \mapsto q^\mu$$

$$(p^\mu, q^\mu, \tau) \mapsto \tau;$$

i.e. by the functions:

$$p^\mu(p^\mu, q^\mu, \tau) = p^\mu$$

$$q^\mu(p^\mu, q^\mu, \tau) = q^\mu$$

$$\tau(p^\mu, q^\mu, \tau) = \tau.$$

2.2. The quantal particle of spin 0

In the quantum case time is a superselection rule [6, 7] and the propositional system $\mathscr{L}$ is to be constructed from a family of isomorphic Hilbert spaces H_τ, indexed with respect to $\tau \in \mathbb{R}$, i.e.:

$$\mathscr{L} = \bigvee_{\tau \in \mathbb{R}} \mathscr{P}(H_\tau)$$

Under these circumstances, the symmetry $S(\tau')$ defines for each value of τ a unitary transformation $U_\tau(\tau')$ between the spaces H_τ and $H_{\tau+\tau'}$, and one can identify the spaces H_τ in such a way as to have $U_\tau(\tau') = I$. In this way the representation $S(g)$ of G is reduced in each H_τ to a representation up to a phase of the subgroup $G_0 = \{(\vec{u}, \vec{\theta}, a^\mu)\}$.

For a given τ we let the Hilbert space H_τ be represented by the space $L^2(\mathbb{R}^4, d^4x)$ of square-integrable functions Φ defined on $\mathbb{R}^4$, with the Lorentz invariant scalar product

$$(\Phi, \Psi) = \int_{\mathbb{R}^4} d^4x \, \Phi^*(x^\mu)\, \Psi(x^\mu),$$

and the group $G_0 = \{(\vec{u}, \vec{\theta}, a^\mu)\}$ by the unitary representation U

$$(U(\vec{u})\, \Phi)(x^\mu) = \exp\left(i \frac{m}{\hbar} \tilde{u}^\nu x_\nu \right) \Phi(\Lambda^{-1}(\vec{u})^\mu{}_\nu x^\nu)$$

$$(U(\vec{\theta})\, \Phi)(x^\mu) = \Phi(\Lambda^{-1}(\vec{\theta})^\mu{}_\nu x^\nu)$$

$$(U(a^\mu)\, \Phi)(x^\mu) = \Phi(x^\mu - a^\mu),$$

for $\tilde{u}^\mu = (\gamma - 1, -\gamma\vec{u})$; or by the space $L^2(\mathbb{R}^4, d^4p)$, the Fourier transform F of $L^2(\mathbb{R}^4, d^4x)$, i.e.

$$F: L^2(\mathbb{R}^4, d^4x) \to L^2(\mathbb{R}^4, d^4p)$$

is defined by

$$\Phi(p^\mu) = \frac{1}{(2\pi\hbar)^2} \int_{\mathbb{R}^4} d^4x \exp\left(-\frac{i}{\hbar} p^\nu x_\nu \right) \Phi(x^\mu),$$

and the group G_0 by the representation $V = FUF^{-1}$,

$$(V(\vec{u})\, \Phi)(p^\mu) = \Phi(\Lambda^{-1}(\vec{u})^\mu{}_\nu p^\nu + m\tilde{u}^\mu)$$

$$(V(\vec{\theta})\,\Phi)(p^\mu) = \Phi(\Lambda^{-1}(\theta)^\mu{}_\nu p^\nu)$$

$$(V(a^\mu)\,\Phi)(p^\mu) = \exp\left(-\frac{i}{\hbar}\,a^\nu p_\nu\right)\Phi(p^\mu).$$

The two representations thus defined will be referred to as the x-representation and the p-representation, respectively.

For the given choices, the following observables satisfy the imprimitivity relations: for the momentum:

i) in the p-representation:

$$\mathbf{p}^\mu(\Delta) = \{P_\tau = \chi_\Delta(p^\mu)\},$$

with the corresponding family of Hermitean operators,

$$\{A_\tau^\mu = p^\mu I\};$$

ii) in the x-representation the family of Hermitean operators is:

$$\{A_\tau^\mu = -i\hbar\partial_{x_\mu}\};$$

for the position:

i) in the x-representation:

$$\mathbf{q}^\mu(\Delta) = \{P_\tau = \chi_\Delta(x^\mu)\},$$

with the corresponding family of Hermitean operators,

$$\{A_\tau^\mu = x^\mu I\};$$

ii) in the p-representation the family of Hermitean operators is:

$$\{A_\tau^\mu = i\hbar\partial_{p_\mu}\};$$

for the time:

$$\boldsymbol{\tau}(\Delta) = \{P_\tau = \chi_\Delta(\tau)\},$$

with the corresponding family of Hermitean operators,

$$\{A_\tau = \tau I\}.$$

It follows from the preceding that $\mathbf{p}^\mu$ and $\mathbf{q}^\mu$ satisfy the commutation relations

$$[q^\mu, p^\nu] = i\hbar g^{\mu\nu}.$$

The imprimitivity system for $\mathbf{q}^\mu$ thus constructed is the 'canonical imprimitivity system' for the action of the inhomogeneous Lorentz group G_0 on $\{\underset{\to}{x^\mu} \in \mathbb{R}^4\}$. In fact this action is transitive, with the homogeneous Lorentz group $\{(\vec{u}, \vec{\theta})\}$ as isotropy group (little group), such that the representation U of G_0 in $L^2(\mathbb{R}^4, d^4x)$ is the representation induced from the trivial representation $((\vec{u}, \vec{\theta}) \mapsto I)$ of the Lorentz group. Thus, the imprimitivity theorem [8, 9] applies, and the representation of the imprimitivity system for $\mathbf{q}^\mu$ is unique up to a unitary transformation. Moreover, it is irreducible, only the identity commutes with $(U, \mathbf{q}^\mu)$.

For $\mathbf{p}^\mu$ the situation is somewhat different because in this case the action of the inhomogeneous Lorentz group is non-transitive on $\{p^\mu \in \mathbb{R}^4\}$ and the imprimitivity theorem cannot be applied directly. However, one can decompose $\{p^\mu \in \mathbb{R}\}$ into orbits of G_0, apply the inducing construction on each of the orbits and take the direct integral over the orbits thus obtained. This is the sense of V. It thus follows that the imprimitivity system constructed for $\mathbf{p}^\mu$ is reducible. However, as for the non-uniqueness involved in taking direct integrals, this is of minor importance since what we are searching for is the simultaneous specification of the imprimitivity systems for $\mathbf{p}^\mu$ and $\mathbf{q}^\mu$. Therefore, since one of them is irreducible, by definition the simultaneous specification is irreducible, and moreover, since one is essentially unique, it follows in this case that the other is essentially unique.

2.3. *The quantal particle of spin $\frac{1}{2}$*

To construct the imprimitivity systems for the spin $\frac{1}{2}$ particle we proceed as for the spin 0 particle. For the x-representation we choose the Hilbert space $L^2(\mathbb{R}^4, d^4x) \otimes l^2$ with scalar product:

$$(\Phi, \Psi) = \sum_{i=1}^{\infty} \int_{\mathbb{R}^4} d^4x \Phi_i^*(x^\mu)\Psi^i(x^\mu).$$

In this space the unitary representation U of $G_0 = \{(\vec{u}, \vec{\theta}, a^\mu)\}$ is defined by

$$(U(\vec{u})\,\Phi)^i(x^\mu) = \exp\left(i\frac{m}{\hbar}\tilde{u}^\nu x_\nu\right) D(\vec{u})^i{}_j\,\Phi^j(\Lambda^{-1}(\vec{u})^\mu{}_\nu x^\nu)$$

$$(U(\vec{\theta})\Phi)^i(x^\mu) = D(\vec{\theta})^i{}_j\,\Phi^j(\Lambda^{-1}(\vec{\theta})^\mu{}_\nu x^\nu)$$

$$(U(a^\mu))^i(x^\mu) = \Phi^i(x^\mu - a^\mu),$$

for D being the irreducible representation $(\frac{1}{2}, ia)$ in the principal series $(a \in \mathbb{R})$ [10].

In the p-representation, the representation of G_0 takes the form:

$$(V(\vec{u})\Phi)^i(x^\mu) = D(\vec{u})^i{}_j\,\Phi^j(\Lambda^{-1}(\vec{u})^\mu{}_\nu p^\nu + m\tilde{u}^\mu)$$

$$(V(\vec{\theta})\Phi)^i(x^\mu) = D(\vec{\theta})^i{}_j\,\Phi^j(\Lambda^{-1}(\vec{\theta})^\mu{}_\nu p^\nu)$$

$$(V(a^\mu)\Phi)^i(x^\mu) = \exp\left(-\frac{i}{\hbar}a^\nu p_\nu\right)\Phi^i(p^\mu).$$

For the observables $\mathbf{p}^\mu$, $\mathbf{q}^\mu$ and $\boldsymbol{\tau}$ we then have the following solutions (we write down the corresponding families of Hermitean operators only):

for the momentum:

x-representation: $\{A^\mu_\tau = -i\hbar\,\partial_{x_\mu} \otimes I_l\}$

p-representation: $\{A^\mu_\tau = p^\mu I_L \otimes I_l\}$;

for the position:

x-representation: $\{A^{\mu}_{\tau} = x^{\mu} I_L \otimes I_l\}$

p-representation: $\{A^{\mu}_{\tau} = i\hbar\, \partial_{p_\mu} \otimes I_l\}$;

for the time:

$\{A_\tau = \tau I_L \otimes I_l\}$.

Similar remarks about irreducibility and uniqueness for the imprimitivity systems as in the spin 0 case apply here. In fact U is induced from the representation D of $G_L = \{(\vec{u}, \vec{\theta})\}$. Moreover D depends only on $(\vec{u}, \vec{\theta})$ because $G_0 = \{(\vec{u}, \vec{\theta})\} x_s \{(a^\mu)\}$ the semi-direct product between the Lorentz group and the Abelian group T^4 of translations of $\mathbb{R}^4$.

The observables for the additional degrees of freedom, the spin, is by definition specified by a set of operators on l^2 (thus commuting with p^μ and q^μ) constituting a second-rank anti-symmetric tensor $S^{\mu\nu}$ transforming according to

$$S^{\mu\nu} \mapsto \Lambda(\vec{u}, \vec{\theta})^{\mu}{}_{\alpha}\, \Lambda(\vec{u}, \vec{\theta})^{\nu}{}_{\beta}\, S^{\alpha\beta}.$$

They are thus given by the generators of the representation D of the Lorentz group.

3. The Evolution of Einstein Relativistic Particles

It is a postulate that the reversible evolution of a physical system during an interval of time is described by a symmetry of the propositional system which induces a representation of the translations of the real line. Thus, for a physical system described by a propositional system $\mathscr{L}$ realized by $\bigvee_{\alpha \in \Omega} P(H_\alpha)$, where the Hilbert spaces H_α are all mutually isomorphic, it follows as a consequence of the generalized Wigner theorem [11] that the evolution is induced by a permutation of the points of Ω

$$\alpha \to \alpha_{\tau'},$$

and a family $\{V_\alpha(\tau')\}$ of unitary operators,

$$V_\alpha(\tau'): H_\alpha \to H_{\alpha_{\tau'}},$$

these together satisfying the relations

$$(\alpha_{\tau'_1})_{\tau'_2} = \alpha_{\tau'_1 + \tau'_2}$$

and

$$V_{\alpha_{\tau'_1}}(\tau'_2)\, V_\alpha(\tau'_1) = V_\alpha(\tau'_1 + \tau'_2).$$

We have set $\omega_\alpha(\tau'_1, \tau'_2) \equiv 1$ because even if the group acts effectively on α the phase-factor is of trivial type.

If we postulate some conditions of continuity and differentiability we can deduce the following equations

$$\partial_{\tau'} \alpha_{\tau'} = \chi(\alpha_{\tau'}) \quad \text{and} \quad i\hbar\, \partial_{\tau'} \Psi_{\alpha_{\tau'}} = \mathscr{H}_{\alpha_{\tau'}} \Psi_{\alpha_{\tau'}},$$

i.e. a Schrödinger equation coupled with a system of differential equations defined by a vector-field χ.

For some observable **A** realized by a family of essentially self-adjoint operators A_α, there may exist a new observable $\mathbf{\dot{A}}$ realized by the family of operators

$$\dot{A}_\alpha = \frac{i}{\hbar}[\mathscr{H}_\alpha, A_\alpha].$$

In particular we will postulate that there exists such $\dot{p}^\mu$ and $\dot{\mathbf{q}}^\mu$.

After these generalities we return to the Einsteinian particle to make the equations explicit for the different models by taking into account Einstein relativity.

Definition 2. We shall call *Einsteinian evolution* a reversible evolution induced by a group of symmetries $V(\tau')$ satisfying the following two conditions:

i) the evolution $V(\tau')$ changes τ into $\tau + \tau'$,

$$V_\alpha(\tau')\,\tau_\alpha\,V_\alpha^{-1}(\tau') = \tau_{\alpha_{\tau'}} - \tau'\,I_{\alpha_{\tau'}},$$

ii) for every special Lorentz transformation $\vec{u}$ the observable $\dot{\mathbf{q}}^\mu$ transform according to (covariance condition)

$$U_\alpha(\vec{u})\,\dot{q}^\mu_\alpha\,U_\alpha^{-1}(\vec{u}) = \Lambda^{-1}(\vec{u})^\mu{}_\nu\,\dot{q}^\nu_{\alpha_{\vec{u}}} + \tilde{u}^\mu\,I_{\alpha_{\vec{u}}}.$$

3.1. The classical particle

For the classical particle we have seen that we can take as state-space the differentiable manifold:

$$\Omega = \{(p^\mu, q^\mu, \tau) \mid p^\mu \in \mathbb{R}^4,\ q^\mu \in \mathbb{R}^4, \tau \in \mathbb{R}\},$$

and that the observables are defined by the functions

$$p^\mu(p^\mu, q^\mu, \tau) = p^\mu$$

$$q^\mu(p^\mu, q^\mu, \tau) = q^\mu$$

$$\tau(p^\mu, q^\mu, \tau) = \tau.$$

For such a system, the evolution is represented by a curve in Ω, naturally parametrized with respect to τ, and it follows from the relations in Definition 2 which imposes the conditions

$$\tau_{\tau'} = \tau + \tau'$$

and

$$\dot{q}^\mu(p^\mu, q^\mu, \tau) = \Lambda^{-1}(\vec{u})^\mu{}_\nu\,\dot{q}^\nu(\Lambda(\vec{u})^\mu{}_\nu p^\nu + mu^\mu,\ \Lambda(\vec{u})^\mu{}_\nu q^\nu, \tau) + \tilde{u}^\mu$$

that the vector field χ specifying the evolution (if it is homogeneous) is the generator of the one-parameter group $\{V(\tau')\}$ translating the points along the curve. Since $V\{(\tau')\}$ is a symmetry group of Ω it must respect the symplectic structure imposed on

$$\Gamma = \{(p^\mu, q^\mu) \mid p^\mu \in \mathbb{R}^4, q^\mu \in \mathbb{R}^4\}$$

by the definition of p^μ and q^μ; thus χ must be a Hamiltonian field, i.e.

$$\chi = g^{\alpha\beta}\{(\partial_{p\alpha}\mathscr{H})\,\partial_{q\beta} - (\partial_{q\alpha}\mathscr{H})\partial_{p\beta}\},$$

where the Hamiltonian $\mathscr{H} = \mathscr{H}(p^\mu,q^\mu)$ is a differentiable function on Γ. It follows from the covariance condition that the most general Hamiltonian (for a conservative system) which transforms covariantly under the Lorentz transformations is of the form

$$\mathscr{H}(p^\mu,q^\mu) = \frac{1}{2m}(p^\mu - A^\mu(q^\mu))(p_\mu - A_\mu(q^\mu)) + V(q^\mu).$$

To prove this assertion, we first observe that the choice of χ implies the Hamilton equations:

$$\dot{p}^\mu = \chi_{p\mu} = -\partial_{q_\mu}\mathscr{H}(p^\mu,q^\mu)$$

$$\dot{q}^\mu = \chi_{q\mu} = \partial_{p_\mu}\mathscr{H}(p^\mu,q^\mu).$$

Then putting the covariance condition into the last of these equations, we get

$$\dot{q}^\mu(p^\mu,q^\mu) = \Lambda^{-1}(\vec{u})^\mu{}_\nu(\partial_{p_\nu}\mathscr{H})(\Lambda(\vec{u})^\mu{}_\nu p^\nu + mu^\mu,\ \Lambda(\vec{u})^\mu{}_\nu q^\nu) + \tilde{u}^\mu$$

or, since

$$\Lambda(\vec{u})^\mu{}_\nu\,\tilde{u}^\nu = -u^\mu,$$

$$\Lambda(\vec{u})^\mu{}_\nu\dot{q}^\nu(p^\mu,q^\mu) = (\partial_{p_\mu}\mathscr{H})(\Lambda(\vec{u})^\mu{}_\nu p^\nu + mu^\mu,\ \Lambda(\vec{u})^\mu{}_\nu q^\mu) - u^\mu,$$

i.e.

$$\Lambda(\vec{u})^\mu{}_\nu(\partial_{p_\nu}\mathscr{H})(p^\mu,q^\mu) = (\partial_{p_\mu}\mathscr{H})(\Lambda(\vec{u})^\mu{}_\nu p^\nu + mu^\mu,\ \Lambda(\vec{u})^\mu{}_\nu q^\nu) - u^\mu$$
$$= (\partial_{p_\mu}\mathscr{H})(0,\ \Lambda(\vec{u})^\mu{}_\nu q^\nu) + \frac{1}{m}\Lambda(\vec{u})^\mu{}_\nu p^\nu.$$

Since Λ operates on q^μ in an isometric way,

$$(\partial_{p_\mu}\mathscr{H})(0,\ \Lambda(\vec{u})^\mu{}_\nu q^\nu) = \Lambda(\vec{u})^\mu{}_\nu\,(\partial_{p_\nu}\mathscr{H})(0,q^\mu) \equiv -\frac{1}{m}\Lambda(\vec{u})^\mu{}_\nu\,A^\nu(q^\mu),$$

and we get

$$(\partial_{p_\mu}\mathscr{H})(p^\mu,q^\mu) = \frac{1}{m}(p^\mu - A^\mu(q^\mu)),$$

which, integrated with respect to p^μ, give us the Hamiltonian already referred to. $A^\mu(q^\mu)$ is any four-vector of differentiable functions of q^μ transforming isometrically under the Lorentz transformations, and $V(q^\mu)$ any differentiable scalar function.

As part of an interpretation we will study the case of the *free particle*; however, before doing so, we will introduce a new observable,

$$\vec{q}_s = \vec{q} - \frac{\vec{p}}{p^0 + mc}q^0,$$

which describes the position of the particle in three-dimensional space [12]. Its transformation properties follow from the transformation properties of p^μ and q^μ.

We then define a free particle to be a particle for which the Hamiltonian is

$$\mathscr{H}(p^\mu, q^\mu) = \frac{1}{2m} p^\nu p_\nu = \frac{1}{2m}(\vec{p}^2 - p^{02})$$

and whose momentum and rest-mass are related by

$$(p^0 + mc)^2 - \vec{p}^2 = m^2 c^2.$$

This last condition, which by the invariance of the form

$$(p^0 + mc)^2 - \vec{p}^2$$

is a Lorentz invariant condition, will be referred to as the free-particle condition. It follows that the submanifold of $\{p^\mu \in \mathbb{R}^4\}$, on which the free-particle condition is satisfied, is an orbit of transitivity for the inhomogeneous Lorentz group, and the condition may be interpreted as saying that for a free particle there always exists a frame of reference in which the particle is at rest and $p^0 = 0$, $\vec{p} = 0$. It follows also from the observation that $(p^0 + mc)^2 - \vec{p}^2$ is an invariant form, and that our representation Λ_G of the Lorentz group is conjugate to the usual isometric representation Λ. In fact, let $T_m \in \mathrm{Aut}(\Omega)$ be defined by

$$T_m : (p^0/c, \vec{p}) \mapsto ((p^0/c) + m, \vec{p})$$

$$T_m : (q^0/c, q) \mapsto (q^0/c, \vec{q})$$

$$T_m : \tau \mapsto \tau,$$

then

$$\Lambda_G = T_m^{-1} \Lambda T_m.$$

With the given choice of Hamiltonian the equations of motion are

$$\dot{\vec{q}}_s = \frac{1}{2m}\left(2\vec{p} - \frac{2\vec{p}p^0}{p^0 + mc}\right) = \frac{\vec{p}}{(p^0/c) + m}$$

$$\dot{p}^\mu = 0$$

$$\dot{q}^\mu = \frac{p^\mu}{m},$$

and applying the free-particle condition on the initial conditions we get the solutions:

$$\vec{q}_s = \vec{v}t + \vec{a}' \qquad \text{for} \quad \vec{a}' = \vec{a} - \frac{\vec{u}}{c} a^0$$

$$p^\mu = m(\gamma_{\vec{v}} - 1, \gamma_{\vec{v}}\vec{v}) \quad \text{for} \quad \gamma_{\vec{v}} = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-1/2}$$

$$q^\mu = \tau(\gamma_{\vec{v}} - 1, \gamma_{\vec{v}}\vec{v}) + (a^0/c, \vec{a}).$$

We also observe that if we apply the free-particle condition to eliminate p^0 in the Hamiltonian, we get

$$\mathscr{H}' = p^0 c = c\sqrt{\vec{p}^2 + m^2 c^2} - mc^2,$$

which is the expression for the kinetic energy T of a free particle, known from special relativity. It is easy to show that the transform of T by a special Lorentz transformation $\Lambda_G(\vec{u})$ gives the kinetic energy $\bar{T}$ of the particle, in the new frame, according to special relativity.

We proceed to study the Galilean limit $c \to \infty$. In this limit,

$$(\Lambda(\vec{u})^\mu{}_\nu) \xrightarrow[c\to\infty]{} \begin{bmatrix} 1 & 0 & 0 & 0 \\ u_1 & 1 & 0 & 0 \\ u_2 & 0 & 1 & 0 \\ u_3 & 0 & 0 & 1 \end{bmatrix},$$

$$u^\mu = (\gamma - 1, \gamma\vec{u}) \xrightarrow[c\to\infty]{} (0, \vec{u})$$

and

$$a^\mu = (a^0/c, \vec{a}) \xrightarrow[c\to\infty]{} (0, \vec{a});$$

moreover, for the free particle,

$$\vec{q}_s \xrightarrow[c\to\infty]{} \vec{q}_{SG} = \vec{v}\tau + \vec{a}$$

$$p^\mu \xrightarrow[c\to\infty]{} (0, m\vec{v}) = (0, \vec{p}_G)$$

$$q^\mu \xrightarrow[c\to\infty]{} (0, \vec{v}\tau + \vec{a}) = (0, \vec{q}_G).$$

The action of the contracted transformations on these objects are then:

$$\vec{q}_{SG} \mapsto \vec{q}_{SG} + \vec{b}$$

$$\vec{p}_G \mapsto \vec{p}_G + m\vec{u}$$

$$\vec{q}_G \mapsto \vec{q}_G + \vec{b},$$

and rotations. It follows that $\vec{q}_{SG}$ and $\vec{q}_G$ can be identified, thus in the Galilean limit the free Einstein particle is the free Galilei particle from the passive point of view [13].

The preceding discussion indicates that the Hamiltonian in our formalism has the same relation to energy as in Galilean–Hamilton dynamics; moreover, in the four-momentum p^μ, p^0 denotes the difference of mass from the rest mass due to the interaction and velocity of the particle, and for the free particle we see that this gives the whole energy of the particle. The observable q^μ is to be interpreted by means of the observable $\vec{q}_s$, being the real *position observable* of the particle.

3.2. *The quantal particle of spin 0*

For this model each Hilbert space H_τ is canonically isomorphic to $L^2(\mathbb{R}^4)$, and Ω identified with $\mathbb{R}$ (the τ-axis). According to the first condition in the definition of the

Einsteinian evolution, the reversible evolution is defined by a family of unitary operators $V_\tau(\tau')$ satisfying

$$V_{\tau+\tau_1}(\tau_2')\, V_\tau(\tau_1') = V_\tau(\tau_1' + \tau_2'),$$

because the condition mentioned implies that the permutation of the points of the τ-axis is

$$\tau \mapsto \tau_{\tau'} = \tau + \tau'.$$

If the evolution is homogeneous in time, the problem is considerably simplified because in this case V_τ is independent of τ, and defines a vector representation of a one-parameter group on $L^2(\mathbb{R}^4)$:

$$V(\tau_2')\, V(\tau_1') = V(\tau_2' + \tau_1').$$

Let us impose the continuity condition

$$\lim_{\delta\tau' \to 0} \|(V(\delta\tau') - I)\, \Phi\| = 0 \quad \Phi \in L^2(\mathbb{R}^4),$$

we can then apply the Stone theorem [14] which affirms the existence of a self-adjoint operator $\mathscr{H}$ such that

$$V(\tau') = \exp\left(-\frac{i}{\hbar}\mathscr{H}\tau'\right).$$

The domain of $\mathscr{H}$ is by definition the set of $\Phi \in L^2(\mathbb{R}^4)$ for which the limit

$$\lim_{\delta\tau' \to 0} \frac{i}{\delta\tau'} (V(\delta\tau') - I)\, \Phi$$

exists. For these vectors the Schrödinger equation

$$i\hbar\, \partial_\tau\, \Phi_\tau = \mathscr{H}\, \Phi_\tau$$

is defined.

According to the definition of the Einsteinian evolution, the Hamiltonian must be such that the covariance condition

$$U(\vec{u})\, \dot{q}^\mu\, U^{-1}(\vec{u}) = \Lambda^{-1}(\vec{u})^\mu{}_\nu \dot{q}^\nu + \tilde{u}^\mu$$

is satisfied for

$$\dot{q}^\mu = \frac{i}{\hbar}[\mathscr{H}, q^\mu].$$

Now since

$$U(\vec{u}) p^\mu\, U^{-1}(\vec{u}) = \Lambda^{-1}(\vec{u})^\mu{}_\nu p^\nu + m\tilde{u}^\mu,$$

it follows that $p^\mu - m\dot{q}^\mu$ transform isometrically as a four-vector under the special Lorentz transformations, i.e. it can at most be a vector-valued function of q^μ,

$$p^\mu - m\dot{q}^\mu = A^\mu(q^\mu),$$

or

$$\dot{q}^\mu = \frac{1}{m}(p^\mu - A^\mu(q^\mu)).$$

By means of the commutation relations between p^μ and q^μ it is then easy to verify that

$$\mathscr{H} = \frac{1}{2m}(p^\mu - A^\mu(q^\mu))(p^\mu - A^\mu(q^\mu)) + V(q^\mu),$$

where $V(q^\mu)$ is a scalar function of q^μ.

As an example we will again treat the free particle but, before doing so, we introduce the position observable $\vec{\mathbf{q}}_s$ represented by

$$\vec{q}_s = \vec{q} - \frac{1}{2}\left[\frac{\vec{p}}{p^0 + mc}, q^0\right]_+,$$

which describes the position of the particle in three-dimensional space. It is Hermitean, it has commuting components, and its transformation properties are given by those of p^μ and q^μ; in the p-representation it has the form

$$\vec{q}_s = i\hbar\,\partial_{\vec{p}} - \frac{1}{2}\left[\frac{\vec{p}}{p^0 + mc}, \quad -i\hbar\,\partial_{p^0}\right]_+ = i\hbar\left(\partial_{\vec{p}} + \frac{\vec{p}}{p^0 + mc}\partial_{p^0} - \frac{\vec{p}}{2(p^0 + mc)^2}\right).$$

A free quantal particle of spin 0 is a particle whose Hamiltonian is

$$\mathscr{H} = \frac{1}{2m}p^\nu p_\nu,$$

and for which the Lorentz invariant operator ΔM,

$$\Delta M = \frac{1}{c}\sqrt{(p^0 + mc)^2 - \vec{p}^2} - m,$$

has value zero.

Consider the restriction of ΔM to the non-negative spectrum; then, to interpret this operator, we observe that the boost L_G^{-1} defined by

$$L_G^{-1}(p^\mu)^\mu{}_\nu = \Lambda_G^{-1}\left(\frac{\vec{p}}{(p^0/c) + m}\right)^\mu{}_\nu$$

takes p^μ to $(\Delta M, \vec{0})$,

$$p^\mu \overset{L_G^{-1}}{\longmapsto} \Lambda^{-1}\left(\frac{\vec{p}}{(p^0/c) + m}\right)^\mu{}_\nu p^\nu + m\tilde{u}\left(\frac{\vec{p}}{(p^0/c) + m}\right)^\mu = (\Delta M, \vec{0}),$$

i.e. to the frame where $\vec{p} = \vec{0}$. Thus it follows that ΔM is the difference of mass from the rest mass in this frame, and the free particle condition may be interpreted as saying that in the rest frame of the particle this difference is zero, thus giving meaning to the notion of rest mass.

To see what all this means let us consider the subspace $H_{\Delta M\geqslant 0}$ of $L^2(\mathbb{R}^4, d^3p\,dp^0)$ on which $\Delta M \geqslant 0$. $H_{\Delta M\geqslant 0}$ is isomorphic to

$$L^2\left(R^3\times R^+, d^3p d\Delta M\ \frac{(\Delta M+m)^{1/4}}{((\Delta M+m+\vec{p}^2/c^2)^{1/2}}\right),$$

with the Lorentz invariant scalar product

$$(g,f)=\int \frac{d^3p d\Delta M(\Delta M+m)^{1/4}}{((\Delta M+m)^2+\vec{p}^2/c^2)^{1/2}}\, g^*(\Delta M,\vec{p})f(\Delta M,\vec{p}),$$

and the isomorphism is explicitly given by

$$\Phi\left(\frac{p^0}{c},\vec{p}\right)\mapsto f(\Delta M,\vec{p})=\Phi(\sqrt{(\Delta M+m)^2+\vec{p}^2/c^2}-m,\vec{p}).$$

In this space the observable $\vec{\mathbf{q}}_s$ is represented by (see also Ref. [15])

$$\vec{q}_s=i\hbar\left(\partial_{\vec{p}}-\frac{1}{2}\frac{\vec{p}}{(\Delta M+m)^2c^2+\vec{p}^2}\right),$$

since

$$\left(i\hbar\left(\partial_{\vec{p}}+\frac{\vec{p}}{p^0+mc}\,\partial_{p^0}-\frac{1}{2}\frac{\vec{p}}{(p^0+mc)^2}\right)\Phi\right)\left(\frac{p^0}{c},\vec{p}\right)\Bigg|_{\frac{p^0}{c}=\sqrt{(\Delta M+m)^2+\vec{p}^2/c^2}-m}$$

$$=\left(i\hbar\left(\partial_{\vec{p}}-\frac{1}{2}\frac{\vec{p}}{(\Delta M+m)^2c^2+\vec{p}^2}\right)\Phi\right)(\sqrt{(\Delta M+m)^2+\vec{p}^2/c^2}-m,\vec{p}),$$

and the Hamiltonian $\mathscr{H}$ reads

$$\mathscr{H}=c\sqrt{\vec{p}^2+(\Delta M+m)^2c^2}-\frac{1}{2m}((\Delta M+m)^2c^2+m^2c^2).$$

As ΔM is invariant under the inhomogeneous Lorentz group, it commutes with $V_{H_{\Delta M\geqslant 0}}$ (i.e. V restricted to $H_{\Delta M>0}$) for any $(\vec{u},\vec{\theta},a^\mu)$ and thus determines an integral decomposition of $V_{H_{\Delta M\geqslant 0}}$,

$$V_{H_{\Delta M\geqslant 0}}=\int_{\mathbb{R}^+}^{\oplus} V^{(\Delta M)}\,d\Delta M,$$

acting in

$$H_{\Delta M\geqslant 0}=\int_{\mathbb{R}^+}^{\oplus} H^{(\Delta M)}\,d\Delta M.$$

Moreover, $\vec{q}_s$ and $\mathscr{H}$ commute with ΔM also, and can be decomposed:

$$\vec{q}_s=\int_{\mathbb{R}}^{\oplus}\vec{q}_s{}^{(\Delta M)}\,d\Delta M$$

$$\mathscr{H} = \int_{\mathbb{R}}^{\oplus} \mathscr{H}^{(\Delta M)} \, d\Delta M.$$

Let us consider $V^{(0)}$, $H^{(0)}$, $\vec{q}_S^{(0)}$, and $\mathscr{H}^{(0)}$: It follows from the preceding that $H^{(0)}$ may be represented by

$$L^2\left(\mathbb{R}^3, d^3p \frac{m^{1/4}}{((\vec{p}^2/c^2) + m^2)}\right)$$

in which $V^{(0)}$ act in the following fashion:

$$(V^{(0)}(\vec{u})f)(p^i) = f(\Lambda^{-1}(\vec{u})^i{}_\nu p^\nu - m\gamma u^i) \quad (i = 1, 2, 3)$$

$$(V^{(0)}(\vec{\theta})f)(p^i) = f(R^{-1}(\vec{\theta})^i{}_j p^j)$$

$$(V^{(0)}(a^\mu)f)(p^i) = \exp\left(-\frac{i}{h} a^\nu p_\nu\right) f(p^i);$$

moreover

$$\vec{q}_s{}^{(0)} = i\hbar\left(\partial_{\vec{p}} - \frac{1}{2}\frac{\vec{p}}{\vec{p}^2 + m^2 c^2}\right)$$

and

$$\mathscr{H}^{(0)} = c\sqrt{\vec{p}^2 + m^2 c^2} - mc^2.$$

$$L^2\left(\mathbb{R}^3, d^3p \frac{m^{1/4}}{((\vec{p}^2/c^2) + m^2)^{1/2}}\right)$$

is the Hilbert space of square integrable functions on the transitive orbit,

$$(p^0 + mc)^2 - \vec{p}^2 = m^2 c^2,$$

of the Lorentz group, with the Lorentz invariant measure

$$d\mu(\vec{p}) = \frac{m^{1/4} d^3p}{((\vec{p}^2/c^2) + m^2)^{1/2}},$$

$V^{(0)}$ is an irreducible representation of G_0 in this space, the representation induced from a representation of the stability group $T^4 x_s SO(3)$ being trivial for $SO(3)$, and $\vec{q}_s^{(0)}$ is the Newton–Wigner position observable [16].

Thus, for the free quantal particle of spin 0, we obtain the same description as Wigner [17], except for notation. In fact, the reader might verify that

$$(\Lambda_G(\vec{u})(\sqrt{(p^2/c^2) + m^2} - m, \vec{p}))^i = (\Lambda(\vec{u})(\sqrt{(\vec{p}^2/c^2) + m^2}, \vec{p}))^i \quad (i = 1, 2, 3),$$

when Λ_G is our representation of the special Lorentz transformations, and Λ is the usual isometric representation.

3.3. The quantal particle of spin ½

The evolution of the spin ½ particle is described in a way similar to that of the spin 0 particle, only the form of the most general covariant Hamiltonian is different. In fact, in this case,

$$p^\mu - m\dot{q}^\mu = A^\mu(q^\mu) + S^{\alpha\beta} B_{\alpha\beta}{}^\mu(q^\mu) + S^{\alpha\beta} S^{\gamma\delta} C_{\alpha\beta\gamma\delta}{}^\mu(q^\mu) + \ldots,$$

the right-hand side being the most general function of observables transforming isometrically as a four-vector under the special Lorentz transformation. Now, instead of writing the most general covariant Hamiltonian, we will be content with giving the one expressing a minimal coupling between the external field and the spin. It takes the form

$$\mathscr{H} = \frac{1}{2m}(p^\mu - A^\mu(q^\mu) - S^{\alpha\beta} B_{\alpha\beta}{}^\mu(q^\mu))(p_\mu - A_\mu(q^\mu) - S^{\alpha\beta} B_{\alpha\beta\mu}(q^\mu))$$

$$+ S^{\alpha\beta} D_{\alpha\beta}(q^\mu) + V(q^\mu).$$

As an example, we again treat the free particle: A free quantal particle of spin ½ is a particle whose Hamiltonian is

$$\mathscr{H} = \frac{1}{2m} p^\nu p_\nu,$$

the operator ΔM has value zero, and for which there exists a frame in which it has spin ½.

Proceeding as for the spin 0 case, $\Delta M = 0$ implies that $H^{(0)}$ may be represented by

$$L^2\left(\mathbb{R}^3, d^3p \frac{m^{1/4}}{((\vec{p}^2/c^2) + m^2)^{1/2}}\right) \otimes l^2,$$

in which the inhomogeneous Lorentz transformations are represented by $V^{(0)}$

$$(V^{(0)}(\vec{u})f)^i(p^i) = D(\vec{u})^i{}_j f^j(\Lambda^{-1}(\vec{u})^i{}_\nu p^\nu - m\gamma u^i)$$

$$(V^{(0)}(\vec{\theta})f)^i(p^i) = D(\vec{\theta})^i{}_j f^j(R^{-1}(\vec{\theta})^i{}_j p^j)$$

$$(V^{(0)}(a^\mu)f)^i(p^i) = \exp\left(-\frac{i}{\hbar} a^\nu p_\nu\right) f^i(p^i).$$

To be able to apply the last condition characterizing the free particle of spin ½ we have to make a change of basis. For this purpose let F be the unitary operator defined by

$$(Ff)^i(\vec{p}) = F(\vec{p})^i{}_j f^j(\vec{p})$$

for

$$F(\vec{p}) = D\left(\frac{\vec{p}}{((\vec{p}^2/c^2) + m^2)^{1/2}}\right).$$

In this new basis, $V^{(0)}$ is defined by

$$(V^{(0)}(\vec{u})f)^i(p^i) = D(\vec{\theta}_{\vec{u}})^i{}_j f^j(\Lambda^{-1}(\vec{u})^i{}_\nu p^\nu - m\gamma u^i)$$

$$(V^{(0)}(\vec{\theta})f)^i(p^i) = D(\vec{\theta})^i{}_j f^j(R^{-1}(\vec{\theta})^i{}_j p^j)$$

$$(V^{(0)}(a^\mu)f)^i(p^i) = \exp\left(-\frac{i}{\hbar}a^\nu p_\nu\right) f^i(p^i),$$

where $\vec{\theta}_{\vec{u}}$ is a rotation (i.e. Wigner rotation) defined by

$$\Lambda(\vec{\theta}_{\vec{u}})^\mu{}_\nu = L^{-1}(\vec{p})^\mu{}_\alpha \Lambda(\vec{u})^\alpha{}_\beta L(\Lambda^{-1}(\vec{u})^i{}_\nu p^\nu - m\gamma u^i)^\beta{}_\nu.$$

It is possible to choose a basis $\{\xi_{s,m}\}$ in l^2 in which [18]

$$D(\vec{\theta}) = \sum_{s=1/2,3/2,\ldots}^{\oplus} D^{(s)}(\vec{\theta}),$$

for $D^{(s)}$ being an irreducible representation of the rotation group; thus

$$D(\vec{\theta}_{\vec{u}}) = \sum_{s=1/2,3/2,\ldots}^{\oplus} D^{(s)}(\vec{\theta}_{\vec{u}}),$$

which shows that the inhomogeneous Lorentz group acts reducibly on

$$L^2\left(\mathbb{R}^3, d^3p\,\frac{m^{1/4}}{((\vec{p}^2/c^2)+m^2)^{1/2}}\right)\otimes l^2.$$

The last condition characterizing the free particle of spin $\frac{1}{2}$ then implies that it is associated with the Hilbert space,

$$L^2\left(\mathbb{R}^3, d^3p\,\frac{m^{1/4}}{((\vec{p}^2/c^2)+m^2)^{1/2}}\right)\otimes \mathbb{C}^2,$$

in which the representation $V^{(0,1/2)}$,

$$(V^{(0,1/2)}(\vec{u})f)^i(p^i) = D^{(1/2)}(\vec{\theta}_{\vec{u}})^i{}_j f^j(\Lambda^{-1}(\vec{u})^i{}_\nu p^\nu - m\gamma u^i)$$

$$(V^{(0,1/2)}(\vec{\theta})f)^i(p^i) = D^{(1/2)}(\vec{\theta})^i{}_j f^j(R^{-1}(\vec{\theta})^i{}_j p^j)$$

$$(V^{(0,1/2)}(a^\mu)f)^i(p^i) = \exp\left(-\frac{i}{\hbar}a^\nu p_\nu\right) f^i(p^i),$$

of G_0 is irreducible.

Again, this is, except for notation, a description of the free particle of spin $\frac{1}{2}$ corresponding to the usual one [19], thus justifying our name.

Remark

It follows from the preceding that in the general case, when the particle is in interaction and the free particle conditions cannot be applied, there is a certain probability that we have states of spin $\frac{3}{2}, \frac{5}{2}, \ldots$. The natural interpretation of this is to say that these states denote the particle + pairs of particle/anti-particle.

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