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# The Linear Boltzmann Operator–Spectral Properties and Short-Wavelength Limit

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*Abstract.* The spectrum of the linear Boltzmann operator for hard spheres is investigated for large wave numbers $k$ ($k = 2\pi/\lambda$, $\lambda =$ wavelength of the initial disturbance). This is done by converting the eigenvalue problem for the Boltzmann operator into an analogous problem for a related but more tractable operator. These spectral properties are of some consequence with regard to the time evolution generated by the linear Boltzmann operator – it is shown that the contribution of the collision operator is reduced to that of the collision frequency in the limit $k \to \infty$.

## 1. Introduction

In order to test the linearized Boltzmann equation, the propagation of sound in monoatomic gases is very intensively studied at present [15, 16]. One distinguishes between two regimes according to the ratio $\lambda/l$ ($\lambda =$ wavelength of sound, $l =$ mean free path of the gas molecules). The case $\lambda \gg l$ corresponds to the Clausius gas, which is well described by the Navier–Stokes approximation, whereas for $\lambda \ll l$ we have the Knudsen regime. It is generally assumed that the latter and the transition from one regime to the other are adequately described by the linearized Boltzmann equation.

Most experiments on sound propagation are of the source problem type. The sound wave is generated by an emitter, roughly speaking, an oscillating wall [9]. Theoretically, at high frequencies, there seems to occur an interesting phenomenon, namely the absorption of point eigenvalues of the linear Boltzmann operator in its continuous spectrum. This event characterizes the transition region $\lambda \approx l$. In this connection, mathematical questions, such as the analytic continuation of eigenvalues into the continuum and the possible occurrence of spectral concentration, arise and have been treated recently by several authors [16, 17].

In this work we look at the initial value problem for the linearized Boltzmann equation in an infinite medium. Suppose, at time $t = 0$, a spatially periodic initial disturbance $h(\mathbf{x}, \mathbf{v}, 0)$ which evolves in time according to the linearized Boltzmann equation

$$\frac{\partial h}{\partial t} + \mathbf{v}\frac{\partial h}{\partial \mathbf{x}} = -Ih. \tag{1.1}$$

With the ansatz $h(\mathbf{x}, \mathbf{v}, t) = f(\mathbf{v})\exp(i\mathbf{k}\mathbf{x} - i\omega t)$ we get

$$-i\omega f = -i(\mathbf{k}\mathbf{v})f - If \equiv -B_{\mathbf{k}}f \tag{1.2}$$

$I$ denotes the linearized collision operator.

We notice that we are also led to equation (1.2) when we consider a general initial disturbance $f(\mathbf{x},\mathbf{v},0)$. By a spatial Fourier transformation we see that the transformed function $\hat{f}(\mathbf{k},\mathbf{v},0)$ must satisfy (1.2).

For fixed $\mathbf{k}$ the allowed frequencies $\omega$ are eigenvalues of the operator $-iB_{\mathbf{k}}$. If $\mathbf{k}$ varies, the complex-valued function $\omega(\mathbf{k})$ describes a so-called dispersion law.

In the sequel we will study the spectral properties of $B_{\mathbf{k}}$. After a review of some known results, the spectrum $\sigma(B_{\mathbf{k}})$ will be investigated in the limit $k \to \infty$, that is, in the short-wavelength limit (Corollary 6.3). Our method is to consider the eigenvalue problem for an operator closely related to $B_{\mathbf{k}}$ in the space $L_1(\mathbb{R}^3)$ instead of $L_2(\mathbb{R}^3)$. But, as we shall see, certain other $L_p$-spaces serve for the same purpose as well. We show that the operator-$B_{\mathbf{k}}$ generates not only a contraction semigroup, but even a bounded semigroup of negative type, at least for sufficiently large $k$ (Theorem 9.1). Furthermore, it is seen in this connection that the semigroup generated by $-B_{\mathbf{k}}$ has, in a certain sense, a limit as $k \to \infty$ (Theorems 9.2 and 9.3).

Throughout the whole work we shall confine ourselves to a hard sphere interaction of the molecules and we shall consider the full linear Boltzmann operator.

## 2. The Operators $I$ and $B_{\mathbf{k}}$

The linearization procedure which leads to (1.1) suggests the introduction of a Hilbert space $L_2(\mathbf{v};\varphi_0\,d^3v)$ where the scalar product is given by

$$\langle f,g\rangle = \int \varphi_0 f\bar{g}\,d^3v \qquad \varphi_0 = \frac{1}{(2\pi)^{3/2}}\,e^{-v^2/2}$$

(the Maxwellian $\varphi_0$ occurs as a weight function).

Introducing the new functions

$$\hat{f} = \varphi_0^{1/2} f$$

we can go over to the ordinary Hilbert space of square-integrable functions, $L_2(\mathbb{R}^3)$, and we have

$$(\hat{f},\hat{g}) = \int \hat{f}\bar{\hat{g}}\,d^3v = \langle f,g\rangle.$$

We mainly refer to this Hilbert space in the following. The $\wedge$ will be dropped. All quantities have been made dimensionless and are normalized such that $\nu(0)=1$ (see b) below). This can always be achieved by choosing suitable scale factors [3].

The propagation vector $\mathbf{k}$ is considered as a (real) parameter.

The operator $I$ has the following properties:

a) It can be decomposed into two parts

$$I = \nu(v) - K \qquad v = |\mathbf{v}| \tag{2.1}$$

where $K$ is a sum of two integral operators

$$K = K_2 - K_1 \tag{2.2}$$

$$K_1(\mathbf{v},\mathbf{v}') = \frac{1}{8\pi}|\mathbf{v}-\mathbf{v}'|\exp[-\tfrac{1}{4}(v^2+v'^2)] \tag{2.3}$$

$$K_2(\mathbf{v}, \mathbf{v}') = \frac{1}{2\pi} \frac{1}{|\mathbf{v} - \mathbf{v}'|} \exp\left[ -\frac{1}{8}\left( |\mathbf{v} - \mathbf{v}'|^2 + \frac{(v^2 - v'^2)^2}{|\mathbf{v} - \mathbf{v}'|^2} \right) \right] \tag{2.4}$$

$$\nu(v) = \tfrac{1}{2} e^{-v^2/2} + \frac{1}{2}\left( v + \frac{1}{v} \right) \int_0^v dx\, e^{-x^2/2}. \tag{2.5}$$

b) The collision frequency $\nu(v)$ is a monotonically increasing function with an asymptote for $v \to \infty$. Denoting the slope of the asymptote by $b > 0$, we have

$$\nu(v) > b \cdot v \quad v \geqslant 0. \tag{2.6}$$

Furthermore $\nu(0) = 1$.

c) $K$ is a compact, positive, self-adjoint operator on $L_2(\mathbb{R}^3)$ [3, 2].

d) $I = \nu(v) - K$ is a self-adjoint operator on $D(I) = D(\nu(v))$.

e) $I \geqslant 0$; zero being a five-fold degenerate eigenvalue with the normalized eigenfunctions

$$\varphi_0^{1/2}; \quad v_i \varphi_0^{1/2} \quad (i = 1, 2, 3); \quad \frac{v^2}{\sqrt{15}} \varphi_0^{1/2}$$

corresponding to the five additive constants of the motion in a binary collision.

f) The spectrum of $I$, $\sigma(I)$, consists of a discrete and an essential part. The latter, or 'continuum', is identical with the set of values which $\nu(v)$ assumes as $v$ takes on all possible values, i.e. the interval $[1, \infty)$. The discrete part consists of infinitely many eigenvalues in the interval $[0, 1)$ which accumulate at 1 [12]. It is possible that some eigenvalues lie in the continuum (Fig. 1).

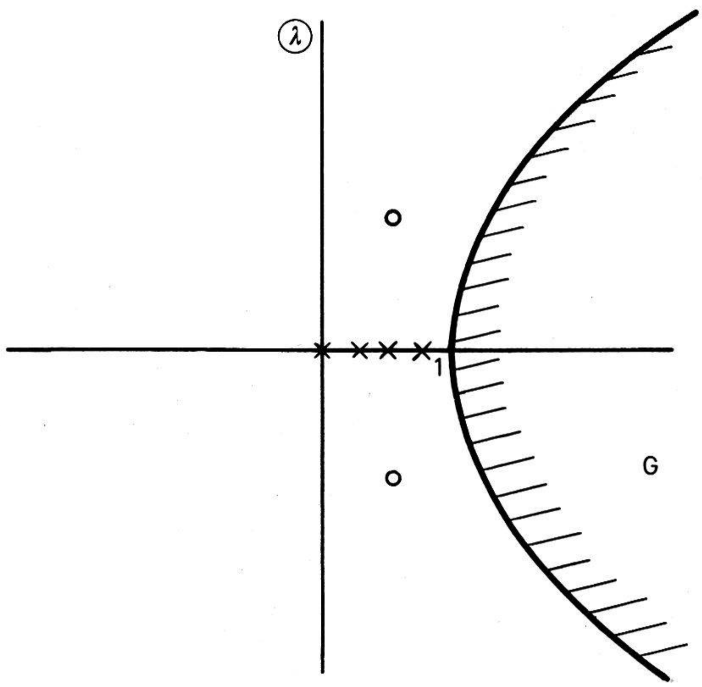


Figure 1
The spectra of $I$ and $B_{\mathbf{k}}$, $\times$ eigenvalues of $I$, $\circ$ eigenvalues of $B_{\mathbf{k}}$.

We now consider the operator $B_{\mathbf{k}}$:

g) It follows from relation (2.6) that we can take $D(B_{\mathbf{k}}) = D(I) = D(\nu(v))$. Then $B_{\mathbf{k}}$ is closed, unbounded and not self-adjoint.

h) $\mathrm{Re}(B_{\mathbf{k}} f, f) = (If, f) \geqslant 0$ (see e)). Therefore $\sigma(B_{\mathbf{k}})$ lies in the right half-plane.

i) $\sigma(B_{\mathbf{k}})$ is symmetric with respect to the real axis, and we have: if $f(\mathbf{v})$ is an eigenfunction belonging to the eigenvalue $\lambda$ then $\overline{f(-\mathbf{v})}$ is an eigenfunction for $\bar{\lambda}$.

j) $\sigma(B_{\mathbf{k}})$ consists of the continuum $G = (i\mathbf{kv} + \nu(v)/\mathbf{v} \in \mathbb{R}^3)$ and a discrete part of isolated eigenvalues with finite multiplicity which can only accumulate on the boundary of $G$ (Fig. 1). It is, in principle, possible that some eigenvalues lie in the continuum.[1])

k) The residual spectrum of $B_{\mathbf{k}}$ is empty [17].

For later use, we list some further properties of the integral operator $K$. We no longer restrict ourselves to the space $L_2(\mathbb{R}^3)$.

l) $K$ is a bounded operator in $L_p(\mathbb{R}^3)$, $1 \leqslant p \leqslant \infty$. This follows from the symmetry of the kernel $K(\mathbf{v}, \mathbf{v}')$ together with the boundedness of [4, p. 527]

$$\int |K(\mathbf{v}, \mathbf{v}')| d^3 v' \leqslant C \tag{2.7}$$

We have $\|K\|_p \leqslant C$ independent of $p$.

m) $K$ is also bounded as an operator from $L_p(\mathbb{R}^3)$ to $L_\infty(\mathbb{R}^3)$, if $\frac{3}{2} < p \leqslant \infty$. This is evident if we observe that $(1 = (1/p) + (1/p'))$:

$$|Kf| \leqslant \left( \int |K(\mathbf{v}, \mathbf{v}')|^{p'} d^3 v' \right)^{1/p'} \|f\|_p \equiv C_p(\mathbf{v}) \|f\|_p. \tag{2.8}$$

In view of (2.3) and (2.4) the integral exists if $1 \leqslant p' < 3$, i.e. $\frac{3}{2} < p \leqslant \infty$, and $C_p(\mathbf{v})$ is a bounded function with respect to $\mathbf{v}$. Thus we have

$$\|Kf\|_\infty \leqslant \left( \sup_{\mathbf{v} \in \mathbb{R}^3} C_p(\mathbf{v}) \right) \|f\|_p \equiv C_p \|f\|_p. \tag{2.9}$$

n) The function $C_p(\mathbf{v})$ introduced in (2.8) satisfies the inequality

$$C_p(\mathbf{v}) \leqslant \gamma_p (1 + v^2)^{-1/2p'} \quad \gamma_p > 0 \tag{2.10}$$

as can be shown by an explicit evaluation of the integral.

The statements in m) and n) indicate that $Kf$ in general satisfies stronger bounds than $f$ does. Slightly generalizing Grad's results, we obtain as a further 'smoothing property' of $K$ [3]:

o) If $|f| \leqslant (1 + v'^2)^r$ then

$$\int |K(\mathbf{v}, \mathbf{v}')|^q |f(\mathbf{v}')| d^3 v' \leqslant \beta_{q,r} (1 + v^2)^{r-1/2} \tag{2.11}$$

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[1]) B. Nicolaenko [17, p. 146] claims this to be impossible except for $\lambda = \nu(0) = 1$. We submit that this is not proven.

$r$ may be any real number, and $q$ is restricted to $1 \leqslant q < 3$. Of course, the statements above are also valid for each one of the kernels $K_1(\mathbf{v},\mathbf{v}')$ and $K_2(\mathbf{v},\mathbf{v}')$ separately. If the constants $C_p$, $\gamma_p$ and $\beta_{q,r}$ only refer to one of these kernels, we distinguish them by upper indices, $C_p^{(1)}$, $\gamma_p^{(2)}$ ... etc.

p) Let us denote by $M_R = \{\mathbf{v}/v \leqslant R\}$ the ball of radius $R > 0$ in $\mathbf{v}$-space. We consider the space $\mathrm{Lip}(\beta, M_R)$ of continuous functions defined on $M_R$ which satisfy a Lipschitz condition with exponent $\beta$ there. The following lemma shows that $K$ maps certain $L_p$-spaces into $\mathrm{Lip}(\beta, M_R)$.

*Lemma 2.1:* Let $f \in L_p(\mathbb{R}^3)$, $\frac{3}{2} < p \leqslant \infty$. Then $g = Kf \in \mathrm{Lip}(\beta, M_R)$, that is

$$|g(\mathbf{v}+\Delta\mathbf{v}) - g(\mathbf{v})| \leqslant C(R,p',\beta)|\Delta\mathbf{v}|^\beta \|f\|_p \tag{2.12}$$

uniformly in $\mathbf{v}$;

$$v \leqslant R, \quad |\mathbf{v}+\Delta\mathbf{v}| \leqslant R; \quad 0 < \beta < 1 \quad \text{and} \quad \beta < \frac{3-p'}{p'}.$$

*Proof:* To prove this lemma, we follow Kantorowitsch and Akilow [6, p. 293]. It is sufficient to consider only the kernel $K_2(\mathbf{v},\mathbf{v}')$ which is not as well-behaved as $K_1(\mathbf{v},\mathbf{v}')$. It will follow immediately from the proof for $K_2$ that $K_1$ satisfies (2.12) with $\beta = 1$ and $C(R,p',\beta)$ independent of $R$. This is due to the exponential factors occurring in $K_1$. Using Hölder's inequality we have

$$|g(\mathbf{v}+\Delta\mathbf{v}) - g(\mathbf{v})| \leqslant \left(\int |K_2(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}') - K_2(\mathbf{v},\mathbf{v}')|^{p'} d^3v'\right)^{1/p'} \|f\|_p. \tag{2.13}$$

We split the integral into two parts corresponding to the domains $D_1$ and $D_2$. $D_1$ is the interior of a sphere with radius $2|\Delta\mathbf{v}|$ and centre at $\mathbf{v}$, $D_2$ is the exterior part. Integrating first over $D_1$, we get

$$\left(\int_{D_1} K_2^{p'}(\mathbf{v},\mathbf{v}')\,d^3v'\right)^{1/p'} = \left(\int_{D_1} \frac{K_2^{p'}(\mathbf{v},\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^{\beta p'}} |\mathbf{v}-\mathbf{v}'|^{\beta p'}\,d^3v'\right)^{1/p'} \leqslant 2^\beta |\Delta\mathbf{v}|^\beta A \tag{2.14}$$

and

$$\left(\int_{D_1} K_2^{p'}(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}')\,d^3v'\right)^{1/p'} = \left(\int_{D_1} \frac{K_2^{p'}(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}')}{|\mathbf{v}+\Delta\mathbf{v}-\mathbf{v}'|^{\beta p'}} |\mathbf{v}+\Delta\mathbf{v}-\mathbf{v}'|^{\beta p'}\,d^3v'\right)^{1/p'}$$
$$\leqslant 3^\beta |\Delta\mathbf{v}|^\beta A \tag{2.15}$$

where $A$ is defined by

$$\left(\int \frac{K_2^{p'}(\mathbf{v},\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^{\beta p'}}\,d^3v'\right)^{1/p'} \leqslant A \tag{2.16}$$

$A$ is finite if $p'(\beta+1) < 3$, that is $\beta < (3-p')/p'$. This is one of the conditions imposed on $\beta$ in Lemma 2.1. By means of Minkowski's inequality, we get

$$I_1 \equiv \left(\int_{D_1} |K_2(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}') - K_2(\mathbf{v},\mathbf{v}')|^{p'} d^3v'\right)^{1/p'} \leqslant (2^\beta + 3^\beta)|\Delta\mathbf{v}|^\beta A \tag{2.17}$$

The integral over $D_2$ can be estimated as follows

$$I_2 \equiv \left( \int_{D_2} |K_2(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}') - K_2(\mathbf{v},\mathbf{v}')|^{p'} d^3 v' \right)^{1/p'}$$

$$\leqslant |\Delta\mathbf{v}| \left( \int_{D_2} \left( \int_0^1 |(\mathrm{grad}_{\mathbf{v}} K_2)(\mathbf{v}+\lambda\Delta\mathbf{v},\mathbf{v}')| d\lambda \right)^{p'} d^3 v' \right)^{1/p'}. \tag{2.18}$$

Now we take into account that in $D_2$

$$|\mathbf{v}+\lambda\Delta\mathbf{v}-\mathbf{v}'| \geqslant |\mathbf{v}-\mathbf{v}'| - \lambda|\Delta\mathbf{v}| \geqslant |\mathbf{v}-\mathbf{v}'| - |\Delta\mathbf{v}| > \tfrac{1}{2}|\mathbf{v}-\mathbf{v}'|. \tag{2.19}$$

Thus

$$I_2 \leqslant 2^{1-\beta}|\Delta\mathbf{v}| \left\{ \int_{D_2} \left[ \int_0^1 |(\mathrm{grad}_{\mathbf{v}} K_2)(\mathbf{v}+\lambda\Delta\mathbf{v},\mathbf{v}')|\,|\mathbf{v}+\lambda\Delta\mathbf{v}-\mathbf{v}'|^{1-\beta} d\lambda \right]^{p'} \right.$$

$$\left. \times |\mathbf{v}-\mathbf{v}'|^{p'(\beta-1)} d^3 v' \right\}^{1/p'}$$

$$\leqslant 2^{1-\beta}|\Delta\mathbf{v}| \left\{ \int_{D_2} \left[ \int_0^1 |(\mathrm{grad}_{\mathbf{v}} K_2)(\mathbf{v}+\lambda\Delta\mathbf{v},\mathbf{v}')|^{p'}\,|\mathbf{v}+\lambda\Delta\mathbf{v}-\mathbf{v}'|^{(1-\beta)p'} d\lambda \right] \left[ \int_0^1 d\lambda \right]^{p'/p} \right.$$

$$\left. \times |\mathbf{v}-\mathbf{v}'|^{p'(\beta-1)} d^3 v' \right\}^{1/p'}$$

$$\leqslant |\Delta\mathbf{v}|^{\beta} \int_0^1 d\lambda \int_{D_2} |(\mathrm{grad}_{\mathbf{v}} K_2)(\mathbf{v}+\lambda\Delta\mathbf{v},\mathbf{v}')|^{p'}\,|\mathbf{v}+\lambda\Delta\mathbf{v}-\mathbf{v}'|^{(1-\beta)p'} d^3 v'. \tag{2.20}$$

The step from the second to the last inequality is only allowed if we require that $\beta < 1$. This gives the additional condition on $\beta$. To proceed further, we note that $K_2(\mathbf{v},\mathbf{v}')$ has the form (2.4) $|\mathbf{v}-\mathbf{v}'|\cdot B(\mathbf{v},\mathbf{v}')$ where $B(\mathbf{v},\mathbf{v}')$ is a bounded (positive) function. We have

$$|\mathrm{grad}_{\mathbf{v}} K_2(\mathbf{v},\mathbf{v}')| \leqslant \frac{|\mathrm{grad}_{\mathbf{v}} B(\mathbf{v},\mathbf{v}')|}{|\mathbf{v}-\mathbf{v}'|} + \frac{B(\mathbf{v},\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|^2}. \tag{2.21}$$

A direct calculation which makes strong use of property o) from above shows that on $M_R$

$$\sup_{v\leqslant R} \left( \int \frac{|\mathrm{grad}_{\mathbf{v}} B(\mathbf{v},\mathbf{v}')|^{p'}}{|\mathbf{v}-\mathbf{v}'|^{\beta p'}} d^3 v' \right)^{1/p'} \leqslant B(R) \tag{2.22}$$

with $B(R) \sim R^{1/p}$ for $R \to \infty$.

Using Minkowski's inequality again we obtain from equations (2.21) and (2.22)

$$\left( \int_{D_2} (|(\mathrm{grad}_{\mathbf{v}} K_2)(\mathbf{v}+\lambda\Delta\mathbf{v},\mathbf{v}')|\,|\mathbf{v}+\lambda\Delta\mathbf{v}-\mathbf{v}'|^{1-\beta})^{p'} d^3 v' \right)^{1/p'} \leqslant B(R)+A. \tag{2.23}$$

Therefore $I_2$ is estimated by

$$I_2 \leqslant |\Delta \mathbf{v}|^{\beta}(B(R) + A) \tag{2.24}$$

and finally

$$\left(\int |K_2(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}') - K_2(\mathbf{v}, \mathbf{v}')|^{p'} d^3 v'\right)^{1/p'}$$

$$\leqslant I_1 + I_2 \leqslant [(2^{\beta} + 3^{\beta} + 1)A + B(R)]|\Delta\mathbf{v}|^{\beta} \equiv C(R, p', \beta)|\Delta\mathbf{v}|^{\beta}. \tag{2.25}$$

Furthermore, we have

$$\lim_{p' \to 3} C(R, p', \beta) = \infty, \qquad \lim_{\beta \to (3-p')/p'} C(R, p', \beta) = \infty \quad (p' \geqslant \tfrac{3}{2})$$

and

$$C(R, p', \beta) \sim R^{1/p} \to \infty \quad \text{as} \quad R \to \infty (p \neq \infty).$$

If $p > 3$ we can take $\beta = 1$.

## 3. Results of Perturbation Theory

### *a) The analytic perturbation* $i\mathbf{kv}$

Some conclusions about the local behaviour of the eigenvalues may be drawn from the theory of analytic perturbation. In this context, we refer to the book of Kato [8, p. 365]. If we consider $i\mathbf{kv} = ikvz = xvz (z = \cos \sphericalangle (\mathbf{k}, \mathbf{v}))$ as a perturbation of the operator $I$, where now the parameter $x$ is allowed to take on any complex value, $B_x$ forms a holomorphic family of type $(B)$ [14, 18]. Therefore, the isolated eigenvalues depend holomorphically on $x$ with only algebraic singularities (branch points). That means: if $\lambda_0$ is an eigenvalue of $B_{k_0}$, then at least one eigenvalue $\lambda(k)$ of $B_k$ with $\lim_{k \to k_0} \lambda(k) = \lambda_0(k_0)$ lies in a certain neighbourhood of $\lambda_0(k_0)$. Thus, an isolated eigenvalue cannot 'suddenly appear' or 'disappear'. But it is possible that an eigenvalue tends to infinity or becomes absorbed by the continuum.

In the first case, i.e. $|\lambda(k)| \xrightarrow[k \to k_0]{} \infty$, we will see in b) that $\lambda(k)$ is confined to a strip of width $\|K\|$ around $G$. The second case is especially striking if we choose $x$ real. Then the eigenvalues of $B_x = xvz + \nu(v) - K$ are real. The continuum covers the interval from $\inf\{(xvz + \nu(v)) | v \geqslant 0, -1 \leqslant z \leqslant 1\}$ to $+\infty$. The lower bound decreases monotonically with increasing $|x|$, as long as $|x| \leqslant b$ (2.6). But for $|x| > b$ the continuum covers the whole real axis and the eigenvalues have been absorbed by the continuum. Responsible for this is the sudden extension of the continuum for real $x$. In the true Boltzmann operator (where $x = ik$) the continuum does not share this property. But, nevertheless, we shall see in the following section that the eigenvalues have a 'tendency to the continuum'.

*Note 3.1:* For Grad's hard power-law potentials [3] with angular cut-off, the considerations above are not valid. In this case $\nu(v)$ behaves like $v^{\alpha}$, $0 \leqslant \alpha < 1$ for $v \to \infty$ $(\alpha = 1 - (4/s)$, interaction potential $\sim 1/r^s$, $s \geqslant 4)$. As a consequence, the perturbation $xvz$ is not relatively bounded with respect to $I$ and $D(I) = D(\nu(v))$ is not contained in $D(vz)$. Thus $B_x$ does not form a holomorphic family of type $(B)$. For any real $x \neq 0$

the continuum covers the whole real axis and for $|x| \to 0$ an analytic perturbation series does not necessarily exist. The perturbation should rather be considered within the framework of asymptotic perturbation theory [8] (see also [13]).

*b) K as a perturbation*

Apart from the 'local' results described above, only little can be said about the location of the eigenvalues of $B_{\mathbf{k}}$ by means of perturbation theory alone. We merely want to mention a very rough result which can be obtained by regarding the bounded operator $K$ as a perturbation in $B_{\mathbf{k}}$.

Setting

$$A = i\mathbf{kv} + \nu(v) \tag{3.1}$$

$$A_\lambda = (A - \lambda)^{-1} = (i\mathbf{kv} + \nu(v) - \lambda)^{-1}, \quad \lambda \in \mathbb{C} \tag{3.2}$$

we obtain for the resolvent of $B_{\mathbf{k}}(\lambda \notin \sigma(B_{\mathbf{k}}))$

$$(B_{\mathbf{k}} - \lambda)^{-1} = (A - K - \lambda)^{-1} = (\mathbb{1} - A_\lambda K)^{-1} A_\lambda = \sum_{n=0}^{\infty} (A_\lambda K)^n A_\lambda.$$

The series converges whenever $\|A_\lambda K\| < 1$.

Because $\|A_\lambda\| = 1/\mathrm{dist}[\lambda, G]$ the convergence is guaranteed if $\mathrm{dist}[\lambda, G] > \|K\|$, and therefore

$$\mathrm{dist}[\lambda, G] > \|K\| \quad \text{implies} \quad \lambda \in \rho(B_{\mathbf{k}}). \tag{3.3}$$

We note that this result becomes meaningless in the limit $k \to \infty$. In this limit, it is even weaker than statement 2.h), because $\|K\| > 1$ (which follows from the fact that $(Kf,f) = (\nu(v)f,f) > 1$ if $f$ is an eigenfunction associated with the eigenvalue zero of $I$).

## 4. A First Proposition Concerning the Global Behaviour of the Eigenvalues

We begin by converting the eigenvalue problem for $B_{\mathbf{k}}$

$$(B_{\mathbf{k}} - \lambda)f = (A - K - \lambda)f = (i\mathbf{kv} + \nu(v) - K - \lambda)\, f = 0 \tag{4.1}$$

into a more convenient form. As $(A - \lambda)$ is invertible, we see that (4.1) is equivalent to

$$A_\lambda Kf = f. \tag{4.2}$$

In the following we assume that $\lambda \notin G$.

Then $A_\lambda K$ is a bounded operator on $L_2(\mathbb{R}^3)$ (each of the two factors is bounded). Therefore (4.2) and consequently (4.1) have no solution if $\|A_\lambda K\| < 1$. This gives us a practical criterion to discuss the regions where eigenvalues of $B_{\mathbf{k}}$ can occur: we only need to find out the $\lambda$- and $k$-dependence of $\|A_\lambda K\|$. The following theorem shows indeed that the eigenvalues move towards the continuum as $k$ increases.

*Theorem 4.1:* For any $\varepsilon > 0$ there exists a $k_\varepsilon > 0$ so that for $k \geqslant k_\varepsilon$ the half-plane $\mathrm{Re}\,\lambda \leqslant 1 - \varepsilon$ lies within the resolvent set $\rho(B_{\mathbf{k}})$.

*Proof:* We have, by means of the Schwarz inequality,

$$\begin{aligned}\|A_\lambda Kf\|^2 &= (A_\lambda Kf, A_\lambda Kf) = (Kf, A_\lambda^* A_\lambda Kf)\\ &\leqslant \|Kf\|\,\|A_\lambda^* A_\lambda Kf\| \leqslant \|K\|\,\|A_\lambda^* A_\lambda Kf\|\,\|f\|.\end{aligned} \tag{4.3}$$

Now we concentrate on the expression $\|A_\lambda^* A_\lambda Kf\|$.

Remembering the property 2.m) of the operator $K$ we can write

$$\|A_\lambda^* A_\lambda Kf\|^2 = \int \frac{|Kf|^2\, d^3 v}{|i\mathbf{kv} + \nu(v) - \lambda|^4} \leqslant C_2^2 \|f\|^2 \int \frac{d^3 v}{|i\mathbf{kv} + \nu(v) - \lambda|^4}. \tag{4.4}$$

We set $\lambda = \lambda_1 + i\lambda_2$ and obtain

$$\begin{aligned}\int \frac{d^3 v}{|i\mathbf{kv} + \nu(v) - \lambda|^4} &= 2\pi \int_0^\infty \int_{-1}^{1} \frac{v^2\, dv\, dz}{[(kvz - \lambda_2)^2 + (\nu(v) - \lambda_1)^2]^2}\\ &< \frac{2\pi}{k} \int_0^\infty \int_{-\infty}^{\infty} \frac{v\, dv\, du}{[u^2 + (\nu(v) - \lambda_1)^2]^2}\\ &= \frac{\pi^2}{k} \int_0^\infty \frac{v\, dv}{(\nu(v) - \lambda_1)^3} \equiv \frac{Q(\lambda_1)}{k}.\end{aligned} \tag{4.5}$$

In the transformations made above, we have introduced the new variable $u = kvz - \lambda_2$. By extending the integration over $u$ from $-\infty$ to $+\infty$ the $\lambda_2$-dependence has dropped out. Therefore $Q(\lambda_1)$ is only defined for $\lambda_1 < 1$. We observe that $Q(\lambda_1)$ increases monotonically for $\lambda_1 \to 1$.

Together with (4.3) and (4.5) we finally get

$$\|A_\lambda K\| < C_2^{1/2} \|K\|^{1/2} Q^{1/4}(\lambda_1) k^{-1/4}. \tag{4.6}$$

Relation (4.6) shows that if we choose $k_\varepsilon = C_2^2 \|K\|^2 Q(1 - \varepsilon)$ the assertion of Theorem 4.1 is proved.

Since $\varepsilon \to 0$ implies $k_\varepsilon \to \infty$, Theorem 4.1 only leads to the modest result that the eigenvalues tend towards the line $\operatorname{Re}\lambda = 1$ for $k \to \infty$ which becomes the boundary of the continuum in this limit.

## 5. The Operator $A_\lambda K$ in the Space $L_1(\mathbb{R}^3)$ and a Preliminary Lemma

In order to investigate further the behaviour of the eigenvalues of $B_{\mathbf{k}}$, it is necessary to study the operator $A_\lambda K$ and in particular its norm for $\lambda \in G$. For this purpose, it has proved to be an advantage to consider the operator in certain $L_p$-spaces. Here we shall deal in detail with the space $L_1(\mathbb{R}^3)$ because then the considerations are simplified in many respects. Other spaces will also be treated later on.

We prove the following theorem:

*Theorem 5.1:* $A_\lambda K$ is a bounded operator on $L_1(\mathbb{R}^3)$ for $\lambda \in G$.

*Proof:* To prove Theorem 5.1 we follow a standard method, namely we first show the boundedness of $A_\lambda K$ on the dense subset $L_1(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3)$. Then, since $A_\lambda K$

is closed, it is bounded on the whole space $L_1(\mathbb{R}^3)$. Suppose $f$ to be an element of $L_1(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3)$. It follows from 2.*l*) that $Kf \in L_1(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3)$. Furthermore, we will show that $Kf \in D(A_\lambda)$. Let us consider

$$\|A_\lambda K\|_1 \leqslant \left( \sup_{\mathbf{v} \in \mathbb{R}^3} \int \frac{|K(\mathbf{v}, \mathbf{v}')| d^3 v'}{|i\mathbf{k}\mathbf{v}' + \nu(v') - \lambda|} \right) \|f\|_1 \equiv \left( \sup_{\mathbf{v} \in \mathbb{R}^3} I(\mathbf{v}) \right) \|f\|_1$$

$$\leqslant \left[ \sup_{\mathbf{v} \in \mathbb{R}^3} (I_1(\mathbf{v}) + I_2(\mathbf{v})) \right] \|f\|_1. \tag{5.1}$$

The integrals $I_i(\mathbf{v})$ refer to the corresponding operators $K_i$ ($i = 1,2$) (note that $K_1$ and $K_2$ are positive functions and that $|K| \leqslant K_1 + K_2$)

a) $I_1(\mathbf{v})$: the integrand becomes singular for $\lambda \in G$, and the denominator vanishes on a circle in $\mathbf{v}'$-space. The circle lies in a plane perpendicular to $\mathbf{k}$ with centre at $\lambda_2 k^{-2}\mathbf{k}$ and radius $= (\nu^{[-1]}(\lambda_1)^2 - \lambda_2^2 k^{-2})^{1/2}$, where $\lambda = \lambda_1 + i\lambda_2$, $|\lambda_2| \leqslant k\nu^{[-1]}(\lambda_1)$, $\nu^{[-1]} =$ inverse function of $\nu(v)$. Setting $\alpha = i\mathbf{k}\mathbf{v}' + \nu(v')$ and observing that $K_1(\mathbf{v}, \mathbf{v}')$ is uniformly bounded with respect to both variables, $K_1(\mathbf{v}, \mathbf{v}') \leqslant a$, we get

$$I_1(\mathbf{v}) \leqslant a \int\limits_{|\alpha - \lambda| \leqslant 1} \frac{d^3 v'}{|\alpha - \lambda|} + \int\limits_{|\alpha - \lambda| \geqslant 1} K_1(\mathbf{v}, \mathbf{v}') \, d^3 v' < a \int \frac{d^3 v'}{|\alpha - \lambda|} + C_\infty^{(1)} \tag{5.2}$$

where

$$C_\infty^{(1)} = \sup_{\mathbf{v} \in \mathbb{R}^3} \int K_1(\mathbf{v}, \mathbf{v}') \, d^3 v'$$

see (2.9).

The remaining integral exists and is a function of $\lambda$ and $k$. In order to evaluate it, we substitute the new variables $w = \nu(v')$ and $u = kv'z'$ for $v'$ and $z'$. The differentials are transformed according to

$$d^3 v' = v'^2 \, dv' \, dz' \, d\varphi = \frac{\nu^{[-1]}(w)}{k} \cdot \frac{\partial \nu^{[-1]}(w)}{\partial w} \, du \, dw \, d\varphi. \tag{5.3}$$

We get

$$\int\limits_{|\alpha - \lambda| \leqslant 1} \frac{d^3 v'}{|\alpha - \lambda|} = \frac{2\pi}{k} \int\limits_{|\alpha - \lambda| \leqslant 1} \frac{\nu^{[-1]}(w) \dfrac{\partial \nu^{[-1]}(w)}{\partial w}}{|\alpha - \lambda|} \, du \, dw \qquad \alpha = w + iu. \tag{5.4}$$

The expression

$$h(w) = \nu^{[-1]}(w) \frac{\partial \nu^{[-1]}(w)}{\partial w} = v \left( \frac{\partial \nu}{\partial v} \right)^{-1}$$

remains finite as $v \to 0$ ($w \to 1$) though $(\partial \nu / \partial v)^{-1}$ becomes infinite. Moreover, the function $h(w)$ is monotonically increasing, hence $h(w) \leqslant h(\lambda_1 + 1)$ in (5.4). The range of integration in (5.4) covers the unit circle or part of it in the $(u, w)$-plane, depending on the proximity of the point $\lambda$ to the boundary of $G$.

Therefore, we can estimate (5.4) from above by integrating over the whole unit circle, irrespective of the exact position of $\lambda$:

$$\int_{|\alpha-\lambda|\leqslant 1} \frac{d^3 v'}{|\alpha-\lambda|} < \frac{2\pi}{k} h(\lambda_1+1) \int_0^1 \int_0^{2\pi} \frac{r dr\, d\varphi}{r} = \frac{(2\pi)^2}{k} h(\lambda_1+1). \tag{5.5}$$

Therefore, in view of (5.1), $A_\lambda K$ is bounded on $L_1(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3)$ and the argument given at the beginning of the proof applies. Of course, it is not really necessary to introduce a dense subset. All we need is to show that $I_1(\mathbf{v})$ is a bounded function. Thanks to Fubini's theorem [7], this already allows us to conclude that $R(K) \subset D(A_\lambda)$. ($R(K)$ denotes the range of $K$.)

b) $I_2(\mathbf{v})$: Here we should notice that the integrand has an additional singularity at $\mathbf{v}=\mathbf{v}'$. It originates from the kernel $K_2(\mathbf{v},\mathbf{v}')$ (2.4) and can coincide with that of $A_\lambda$. First we proceed in the same way as in a):

$$I_2(\mathbf{v}) \leqslant \int_{|\alpha-\lambda|\leqslant 1} \frac{K_2(\mathbf{v},\mathbf{v}')\, d^3 v'}{|\alpha-\lambda|} + C_\infty^{(2)} \tag{5.6}$$

(for $C_\infty^{(2)}$ see (2.9)).

We choose two Hölder exponents $p$ and $p'(1=(1/p)+(1/p'))$ which are subjected to the conditions $\frac{3}{2}<p<2$, that is, $2<p'<3$. We obtain

$$\int_{|\alpha-\lambda|\leqslant 1} \frac{K_2(\mathbf{v},\mathbf{v}')}{|\alpha-\lambda|} d^3 v' < \left( \int_{|\alpha-\lambda|\leqslant 1} \frac{d^3 v'}{|\alpha-\lambda|^p} \right)^{1/p} \left( \int_{\mathbb{R}^3} K_2^{p'}(\mathbf{v},\mathbf{v}')\, d^3 v' \right)^{1/p'} \tag{5.7}$$

$$< \frac{(2\pi)^{2/p}}{k^{1/p}} h(\lambda_1+1)^{1/p} \left(\frac{1}{2-p}\right)^{1/p} \beta_{q,0}^{1/p'} \tag{5.8}$$

(for $\beta_{q,0}$ see (2.11)). This means that $A_\lambda K_2$ is also bounded. So Theorem 5.1 is proved.

Since we know that $A_\lambda K$ is a bounded operator on $L_1(\mathbb{R}^3)$ even for $\lambda \in G$, it is now possible to discuss the $\lambda$- and $k$-dependence of $A_\lambda K$ similarly as in Section 4, but for arbitrary $\lambda$. Yet, since we are dealing with the $L_1$-norm here, we should first set up a connection between the $L_1$- and $L_2$-spectrum of $B_{\mathbf{k}}$.

*Lemma 5.2:* If $f \in L_2(\mathbb{R}^3)$ is an eigenfunction of $B_{\mathbf{k}}$, then $f \in L_1(\mathbb{R}^3)$.

*Proof:* We distinguish between two cases:

a) $\lambda \notin G$: In this case, $A_\lambda = (i\mathbf{k}\mathbf{v} + \nu(v) - \lambda)^{-1}$ is bounded as a function of $\mathbf{v}$. This fact, together with property 2.m) implies that every eigenfunction $f$ (which is a solution of $A_\lambda K f = f$) is bounded, that is, by (2.10):

$$|f| = |A_\lambda K f| \leqslant \frac{\gamma_2}{d(1+v^2)^{1/4}} \qquad d = \mathrm{dist}[\lambda, G] > 0. \tag{5.9}$$

Since $f$ also satisfies the equation $(A_\lambda K)^2 f = f$ we can conclude by means of property 2.o) that

$$|f| = |(A_\lambda K)^2 f| \leqslant \frac{\gamma_2 \beta_{1/4}}{d^2} \frac{1}{(1+v^2)^{3/4}}.$$

Proceeding in this way, we obtain for $f = (A_\lambda K)^4 f_\zeta$

$$|f| \leqslant \text{const.} \cdot (1 + v^2)^{-7/4} \tag{5.10}$$

which shows that $f \in L_1(\mathbb{R}^3)$.

b) $\lambda \in G$: In this case, every eigenfunction $f \in L_2(\mathbb{R}^3)$ of $B_{\mathbf{k}}$ satisfies the equation $f = A_\lambda K f$, but $f$ may or may not be bounded as before. $f$ has the form $f = A_\lambda g$, where $g = Kf$ is bounded. The singularity of $A_\lambda g$ is locally absolutely integrable. So we only have to show that $f$ vanishes rapidly enough at infinity to ensure that $f$ belongs to $L_1(\mathbb{R}^3)$.

First considering the function $KA_\lambda g$, we have (5.1)

$$|KA_\lambda g| \leqslant \|g\|_\infty (I_1(\mathbf{v}) + I_2(\mathbf{v})) \tag{5.11}$$

Therefore $KA_\lambda g$ is bounded, yet, for $v \to \infty$, we can say more about the behaviour of this function. We take a sphere with radius $r_\lambda > 0$ in $\mathbf{v}'$-space, so that $|\alpha - \lambda| \geqslant 1 (\alpha = i\mathbf{k}\mathbf{v}' + v(v'))$ if $v' \geqslant r_\lambda$. We obtain for $K_2 A_\lambda g$

$$|K_2 A_\lambda g| < \|g\|_\infty \left( \int\limits_{v' \leqslant r_\lambda} \frac{K_2(\mathbf{v}, \mathbf{v}')}{|\alpha - \lambda|} d^3 v' + \frac{\gamma_\infty^{(2)}}{(1 + v^2)^{1/2}} \right). \tag{5.12}$$

Remembering the explicit form of $K_2(\mathbf{v}, \mathbf{v}')$ (2.4) we get

$$|K_2 A_\lambda g| < \|g\|_\infty \left( \frac{\exp[-\frac{1}{8}(v - r_\lambda)^2]}{2\pi |v - r_\lambda|} \int\limits_{v' \leqslant r_\lambda} \frac{d^3 v'}{|\alpha - \lambda|} + \frac{\gamma_\infty^{(2)}}{(1 + v^2)^{1/2}} \right). \tag{5.13}$$

It follows that there exists a sphere with radius $R_\lambda > r_\lambda$ so that for $v \geqslant R_\lambda$ we have

$$|K_2 A_\lambda g| < \|g\|_\infty \frac{b(\lambda)}{(1 + v^2)^{1/2}}. \tag{5.14}$$

The function $K_1 A_\lambda g$ can be treated analogously. This allows us to choose the radius $R_\lambda$ giving

$$|KA_\lambda g| < \|g\|_\infty \frac{\tilde{b}(\lambda)}{(1 + v^2)^{1/2}} \quad v \geqslant R_\lambda. \tag{5.15}$$

But for $v \leqslant R_\lambda$, $KA_\lambda g$ is bounded according to (5.11). Therefore (5.15) is valid for all $\mathbf{v}$ with a suitable constant $\tilde{b}(\lambda)$. By iteration as in a) we arrive at the conclusion that $(KA_\lambda)^4 g$ is absolutely bounded by a function of the form $\sim(1 + v^2)^{-2}$. So we see that $A_\lambda (KA_\lambda)^4 g = (A_\lambda K)^5 f = f$ belongs to $L_1(\mathbb{R}^3)$.

## 6. Discussion of $\|A_\lambda K\|_1$. Further Properties of $\sigma(B_{\mathbf{k}})$

The preceding investigations enable us to prove

*Theorem 6.1:* To a given $N > 0$ there exists a $k_N > 0$ so that, for $k \geqslant k_N$, there exist no eigenvalues of $B_{\mathbf{k}}$ in the half-plane $\operatorname{Re} \lambda \leqslant N$.

This means that for $N > 1$ also a certain subset of the continuum is free from eigenvalues.[2])

*Proof:* We show that $\|A_\lambda K\|_1 < 1$ if $k \geqslant k_N$ and $\mathrm{Re}\,\lambda \leqslant N$. It is therefore our task to make explicit the $\lambda$- and $k$-dependence of $\|A_\lambda K\|_1$.

a) $\|A_\lambda K\|_1$: From (5.1) we obtain, by inserting additional powers of $(1 + v'^2)$ into the integrals,

$$\|A_2 K\|_1 \leqslant \sup_{\mathbf{v}\in\mathbb{R}^3} \int\limits_{|\alpha-\lambda|\leqslant 1} \frac{K_1(\mathbf{v},\mathbf{v}')(1+v'^2)^{1/2}}{|\alpha-\lambda|(1+v'^2)^{1/2}}\, d^3v' + \sup_{\mathbf{v}\in\mathbb{R}^3} \int\limits_{|\alpha-\lambda|\geqslant 1} \frac{K_1(\mathbf{v},\mathbf{v}')(1+v'^2)^{\gamma}}{|\alpha-\lambda|\,(1+v'^2)^{\gamma}}\, d^3v'. \tag{6.1}$$

In the first integral the nominator is bounded, i.e. $K_1(\mathbf{v},\mathbf{v}')(1+v'^2)^{1/2} \leqslant b_0$. The second integral can be estimated by means of Hölder's inequality. We choose the Hölder exponents $p$ and $p'$ so that $1 = (1/p) + (1/p')$, $2 < p \leqslant \infty$ and in addition we set $\gamma = 1/2p$. With the notation

$$\sup_{\mathbf{v}\in\mathbb{R}^3} \int K_1^{p'}(\mathbf{v},\mathbf{v}')(1+v'^2)^{\gamma p'}\, d^3v' \equiv b_1 \quad (b_1 = \beta^{(1)}_{p',\gamma p'}, \text{ see (2.11)})$$

we obtain

$$\|A_\lambda K_1\|_1 < b_0 \int\limits_{|\alpha-\lambda|\leqslant 1} \frac{d^3v'}{|\alpha-\lambda|(1+v'^2)^{1/2}} + b_1^{1/p'}\left(\int\limits_{|\alpha-\lambda|\geqslant 1} \frac{d^3v'}{|\alpha-\lambda|^p(1+v'^2)^{1/2}}\right)^{1/p}. \tag{6.2}$$

To evaluate these integrals, we make the same transformation of variables which led from (5.2) to (5.4). But now this produces a factor $h(w)(1+(v^{[-1]}(w)^2)^{-1/2}$ which is bounded by a constant $c$. We obtain

$$\|A_\lambda K_1\|_1 < \frac{4\pi^2}{k} b_0 c + \frac{(2\pi)^{2/p}}{k^{1/p}} b_1^{1/p'} c^{1/p} \left(\frac{1}{p-2}\right)^{1/p}. \tag{6.3}$$

The Hölder exponent $p > 2$ is arbitrary. For fixed $k$ the expression on the right side could be minimized with respect to $p$. By (6.3), $\|A_\lambda K_1\|_1$ is uniformly estimated for $\lambda \in \mathbb{C}$. For $\lambda \notin G$ and $\mathrm{dist}[\lambda, G] \geqslant 1$ the first integral in (6.1) simply drops out.

b) $\|A_\lambda K_2\|_1$: First we estimate that part of $I_2(\mathbf{v})$ which corresponds to the integration over $|\alpha - \lambda| \geqslant 1$. It will be denoted by $A_2$. We choose $p, p', \gamma$ and $c$ as in a). The integral

$$b_2 \equiv \sup_{\mathbf{v}\in\mathbb{R}^3} \int K_2^{p'}(v,v')(1+v'^2)^{\gamma p'}\, d^3v'$$

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[2]) This shows that $\lambda = \nu(0) = 1$ is not a particular point with respect to the point spectrum, see footnote p. 102.

exists since $\gamma p' < \frac{1}{2}$. It follows

$$A_2 < (2\pi)^{2/p}\left(\frac{1}{p-2}\right)^{1/p} c^{1/p}\, b_2^{1/p'}\, k^{-1/p}. \tag{6.4}$$

This bound is again independent of $\lambda$.

But the integral over $|\alpha - \lambda| \leqslant 1$ (denoted by $A_1$) yields a $\lambda$-dependence. We choose $q, q'$ such that $\frac{3}{2} < q < 2$, $(1/q) + (1/q') = 1$ (i.e. $2 < q' < 3$) and we set $\delta = 1/2q'$. Multiplying the integrand by $(1 + v'^2)^\delta$ we obtain the estimate

$$A_1 < (2\pi)^{1/q}\, b_3^{1/q'}\left(\int\limits_{|\alpha-\lambda|\leqslant 1} \frac{v^{[-1]}(w)\dfrac{\partial v^{[-1]}(w)}{\partial w}\,du\,dw}{|\alpha-\lambda|^q(1 + v^{[-1]}(w)^2)^{\delta q}}\right)^{1/q} \tag{6.5}$$

where

$$b_3 = \sup_{\mathbf{v}\in\mathbb{R}^3} \int K_2^{q'}(\mathbf{v},\mathbf{v}')(1 + v'^2)^{1/2}\, d^3 v'.$$

The function

$$v^{[-1]}(w)\frac{\partial v^{[-1]}}{\partial w}(1 + v^{[-1]}(w)^2)^{-\delta q} \equiv g(w)$$

tends to infinity like $v^{[-1]}(w)^{1-q/q'}$ when $w \to \infty$. Through $\tilde{g}(w) = \sup_{w' \leqslant w} g(w')$ we define a monotonically increasing (positive) function. Then it follows

$$A_1 < (2\pi)^{2/q}\, b_3^{1/q'}\, \tilde{g}(\lambda_1 + 1)^{1/q}\left(\frac{1}{2-q}\right)^{1/q} k^{-1/q}. \tag{6.6}$$

We observe that $b_3$ tends to infinity as $q \to \frac{3}{2}$.

Summarizing, the sum of expressions (6.3), (6.4) and (6.6) is an upper bound for $\|A_\lambda K\|_1$. For $k \to \infty$ the power $k^{-1/p}$ $(p > 2)$ decreases most slowly. Therefore we can write

$$\|A_\lambda K\|_1 < k^{-1/p}(\tau(p) + \tilde{g}(\lambda_1 + 1)^{1/q}\sigma(q)). \tag{6.7}$$

This is valid for $k \geqslant 1$, $p > 2$ and $\frac{3}{2} < q < 2$. The function $\tau(p)$ resp. $\sigma(q)$ become singular if $p \to 2$ resp. $q \to \frac{3}{2}$ or $q \to 2$. The function $\tilde{g}(\lambda_1 + 1)$ grows like $\sim\lambda_1^{(1/q)-(1/q')}$ as $\lambda_1 \to \infty$. To a given $N > 0$, the number $k_N$, which is required by Theorem 6.1, can be expressed by

$$k_N = (\tau(p) + \tilde{g}(N+1)^{1/q}\sigma(q))^p \tag{6.8}$$

$k_N$ could in principle be minimized with respect to $p$ and $q$.

*Corollary 6.2:* To a given $\varepsilon > 0$ there exists a $k_\varepsilon > 0$ so that, for $k \geqslant k_\varepsilon$, there are no eigenvalues of $B_{\mathbf{k}}$ in the region $\{\lambda | \mathrm{dist}[\lambda, G] \geqslant \varepsilon\}$.

This result is of interest in so far as it is independent of $\lambda_1$, in contrast to Theorem 6.1.

The proof of Corollary 6.2 follows immediately from the estimates (6.3) and (6.4). Relation (6.3) can be taken over unchanged whereas the integral corresponding to $A_2$ has now to be evaluated over the range $|\alpha - \lambda| \geqslant \varepsilon$. Therefore, we obtain

$$A_2 < (2\pi)^{2/p} c^{1/p} b_2^{1/p'} \left(\frac{1}{p-2}\right)^{1/p} \varepsilon^{-1+(2/p)} k^{-1/p} \tag{6.9}$$

independent of $\lambda$.

Combining Theorem 6.1 and Corollary 6.2 we get

*Corollary 6.3:* To given $N > 0$ and $\varepsilon > 0$ there exists a $k_{N,\varepsilon}$ so that, for $k \geqslant k_{N,\varepsilon}$, there exist no eigenvalue of $B_{\mathbf{k}}$ in the region $\{\lambda | \operatorname{Re} \lambda \leqslant N \text{ or } \operatorname{dist}[\lambda, G] \geqslant \varepsilon\}$.

*Note 6.1:* The $\lambda$-dependence of the estimate (6.7) does not allow us to conclude that there exist no eigenvalues at all for sufficiently large $k$. The mere estimation of $\|A_\lambda K\|_1$ does not seem to be sufficient for a proof. An improvement of the estimate (6.7) would be easily possible, say, by integrating over the range $|\alpha - \lambda| \leqslant 1$ instead of $\mathbb{R}^3$ in the integral defining the constant $b_3$ in (6.5). Yet, this does not lead to a $\lambda$-dependence, which would compensate for the growth of $\tilde{g}(\lambda_1 + 1)^{1/q}$ in (6.7). Whether the $\lambda$-dependence will disappear if we use a different function space in place of $L_1(\mathbb{R}^3)$ cannot be excluded off-hand. However, it seems that those $L_p$-spaces in which the operator $A_\lambda K$ can be well discussed are not of much more help than the space $L_1(\mathbb{R}^3)$ (see Section 8).

The $\lambda$-dependence in (6.7) is due to the fact that the operator $K_2(\mathbf{v}, \mathbf{v}')$ does not have suitable 'smoothing properties'. It is, for instance, important to notice that the image $\int K_2^{q'}(\mathbf{v}, \mathbf{v}') f(v') d^3 v'$ $(2 < q' < 3)$ of the function $f = (1 + v'^2)^{\rho/2}$ $(\rho = 1 + \varepsilon, \varepsilon > 0$ arbitrarily small) is unbounded and tends to infinity like $v^\varepsilon$, when $v \to \infty$. If it were bounded (as e.g. $\int K_1(\mathbf{v}, \mathbf{v}') f(v') d^3 v'$) we could find an upper bound of $A_1$, which does not depend on $\lambda$ by a suitable choice of the exponents $q$, $q'$ and $\delta$.

## 7. Further Properties of $A_\lambda K$ in $L_1(\mathbb{R}^3)$

The space $L_1(\mathbb{R}^3)$ has among other things the advantage that its dual space $L_\infty(\mathbb{R}^3)$ is well adapted for our purposes. For any $f \in L_1(\mathbb{R}^3)$ and $g \in L_\infty(\mathbb{R}^3)$ we denote by $\langle f, g \rangle = \int fg\, d^3 v$, the value of the linear continuous functional (represented by $g$) at the point $f$. The dual operator of $K$, $K^T$, is defined through

$$\langle Kf, g \rangle = \langle f, K^T g \rangle \tag{7.1}$$

$K^T$ is an integral operator on $L_\infty(\mathbb{R}^3)$ and we can put $K^T(\mathbf{v}, \mathbf{v}') = K(\mathbf{v}', \mathbf{v})$.

*Lemma 7.1:* The operator $K$ is compact on $L_1(\mathbb{R}^3)$. The proof of this lemma is based on a theorem of Schauder [7, p. 282]:

*Lemma 7.2:* A bounded operator in a Banach space is compact if and only if its dual operator is compact.

We prove Lemma 7.1 in two steps:

a) First we look at the operator $K^T$ (7.1) as a mapping from $L_\infty(\mathbb{R}^3)$ to $C(M_R)$. $C(M_R)$ denotes the Banach space of continuous functions defined on the ball $M_R$, normed by the usual maximum norm. This mapping has already been considered in 2.p) where we found that the image is not only contained in $C(M_R)$ but even in $\operatorname{Lip}(\beta, M_R)$.

Now we show that the mapping $K^T: L_\infty(\mathbb{R}^3) \to C(M_R)$ is compact, i.e. we have to verify that the image of every bounded subset of $L_\infty(\mathbb{R}^3)$ is relatively compact in $C(M_R)$. Since $M_R$ is a compact set in $\mathbb{R}^3$ the relatively compact subsets of $C(M_R)$ can be well characterized by the Ascoli–Arzelà theorem [7]. Accordingly, a subset of $C(M_R)$ is relatively compact if it is equi-bounded and equi-continuous. First of all, it follows from (2.12) that every image of a bounded subset of $L_\infty(\mathbb{R}^3)$ is equi-bounded in $C(M_R)$. Second, since the image functions satisfy a uniform Lipschitz condition, this implies equi-continuity. Therefore the mapping $K^T: L_\infty(\mathbb{R}^3) \to C(M_R)$ is compact.

The next step consists of performing the limit $R \to \infty$.

b) We denote by $P_R$ the projection operator upon $C(M_R)$. $P_R$ is defined on $L_\infty(\mathbb{R}^3)$ according to

$$(P_R f)(\mathbf{v}) = \begin{cases} f(\mathbf{v}) & v \leqslant R \\ 0 & v > R. \end{cases} \tag{7.2}$$

It follows from a) that $P_R K^T$ is compact on $L_\infty(\mathbb{R}^3)$. This is evident because $K^T$ is compact as a mapping from $L_\infty(\mathbb{R}^3)$ to $C(M_R)$ and because the norm on $C(M_R)$ is identical with the $L_\infty$-norm.

Next we show that $\lim_{R\to\infty} \|K^T - P_R K\|_\infty = 0$. Then the compactness of $K^T$ follows from a well-known theorem [7, p. 278]: the limit of a norm-convergent sequence of compact operators is compact. Using property 2.n) with respect to the kernel $K^T(\mathbf{v}, \mathbf{v}')$, we obtain

$$|K^T f| \leqslant \frac{\gamma_\infty}{(1+v^2)^{1/2}} \|f\|_\infty. \tag{7.3}$$

By means of this inequality we can write

$$\begin{aligned} \lim_{R\to\infty} \|K^T - P_R K^T\|_\infty &= \lim_{R\to\infty} \left( \sup_{\|f\|_\infty = 1} \sup_{\mathbf{v}\in\mathbb{R}^3} |K^T f - P_R K^T f| \right) \\ &= \lim_{R\to\infty} \left( \sup_{\|f\|_\infty = 1} \sup_{v > R} |K^T f| \right) \leqslant \lim_{R\to\infty} \frac{\gamma_\infty}{(1+R^2)^{1/2}} = 0. \end{aligned} \tag{7.4}$$

Therefore $K^T$ is compact and, according to Lemma 7.2, $K$ is compact too. This proves Lemma 7.1.

By the same method we can prove the compactness of $A_\lambda K$:

*Theorem 7.3:* On $L_1(\mathbb{R}^3)$, $A_\lambda K$ forms

a) a holomorphic family [11] of compact operators when $\lambda \notin G$,
b) a continuous family of compact operators when $\lambda \in G$.

*Proof:*

a) According to Lemma 7.1 $K$ is compact and since $A_\lambda = (i\mathbf{k}\mathbf{v} + \nu(v) - \lambda)^{-1}$ is bounded when $\lambda \notin G$ it follows that $A_\lambda K$ is compact, too. Furthermore, $A$ is holomorphic with respect to $\lambda$ and this is also true for $A_\lambda K$.

b) First, we show the compactness. According to Theorem 5.1 $A_\lambda K$ is bounded when $\lambda \in G$. Hence, the dual operator $(A_\lambda K)^T$ is uniquely determined and bounded on $L_\infty(\mathbb{R}^3)$. From the definition (7.1) of the dual operator and Fubini's theorem, it follows that

$(A_\lambda K)^T = \widetilde{K^T A_\lambda}$. The tilde means that we have to take the formal extension of the operator $K^T A_\lambda$: we define $A_\lambda$ formally on the whole space $L_\infty(\mathbb{R}^3)$ simply as multiplication by $(i\mathbf{kv} + \nu(v) - \lambda)^{-1}$. Generally, this operation leads out of the space $L_\infty(\mathbb{R}^3)$, but we are assured (for instance from the proof of Lemma 5.2b) that $K$ maps all functions $A_\lambda g (g \in L_\infty(\mathbb{R}^3))$ back into the space $L_\infty(\mathbb{R}^3)$. It is in this sense that we understand the operator $\widetilde{K^T A}$.

1) First we again consider the mapping $\widetilde{K^T A}: L_\infty(\mathbb{R}^3) \to C(M_R)$. Also in this case it is true that the functions $g = K^T A_\lambda f (f \in L_\infty(\mathbb{R}^3))$ are Lipschitz continuous. To see this, we estimate the difference $|g(\mathbf{v} + \Delta\mathbf{v}) - g(\mathbf{v})|$:

$$|g(\mathbf{v} + \Delta\mathbf{v}) - g(\mathbf{v})| \leqslant \left( \int |K^T(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}') - K^T(\mathbf{v}, \mathbf{v}')| \, |\alpha - \lambda|^{-1} d^3 v' \right) \|f\|_\infty. \tag{7.5}$$

As usual, we split the integral into two parts corresponding to $|\alpha - \lambda| \leqslant 1$ resp. $|\alpha - \lambda| \geqslant 1$. The former can be estimated by Hölder's inequality

$$\int_{|\alpha-\lambda|\leqslant 1} |K^T(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}') - K^T(\mathbf{v}, \mathbf{v}')| \, |\alpha - \lambda|^{-1} d^3 v'$$

$$< \left( \int |K^T(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}') - K^T(\mathbf{v}, \mathbf{v}')|^{p'} \right)^{1/p'} \left( \int_{|\alpha-\lambda|\leqslant 1} \frac{d^3 v'}{|\alpha - \lambda|^p} \right)^{1/p} \tag{7.6}$$

where $\frac{3}{2} < p < 2$, $2 < p' < 3$, $1 = (1/p) + (1/p')$. The integral over the difference of the kernels has already been evaluated in 2.p). Relation (2.25) yields the desired Lipschitz condition. The second part is directly estimated by

$$\int_{|\alpha-\lambda|\geqslant 1} |K^T(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}') - K^T(\mathbf{v}, \mathbf{v}')| \, |\alpha - \lambda|^{-1} d^3 v'$$

$$< \int_{\mathbb{R}^3} |K^T(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}') - K^T(\mathbf{v}, \mathbf{v}')| d^3 v'. \tag{7.7}$$

Again, the integral is of the type discussed in 2.p).

Now it follows as in Lemma 7.1 from the Ascoli–Arzelà theorem that

$$\widetilde{K^T A}: L_\infty(\mathbb{R}^3) \to C(M_R)$$

is a compact mapping.

2) The limit $R \to \infty$ can be carried out in the same way as in Lemma 7.1, we merely have to remember (5.15). Knowing $\widetilde{K^T A}$ to be compact on $L_\infty(\mathbb{R}^3)$, Lemma 7.2 implies the compactness of $A_\lambda K$ on $L_1(\mathbb{R}^3)$.

3) Continuity of the family $A_\lambda K$: we show that $A_\lambda K$ is, with respect to $\lambda$, continuous in the operator norm. For this purpose, we estimate

$$\|(A_{\lambda_1} - A_{\lambda_2}) K\|_1 \leqslant |\lambda_1 - \lambda_2| \sup_{\mathbf{v} \in \mathbb{R}^3} \left( \int \frac{|K(\mathbf{v}, \mathbf{v}')| \, d^3 v'}{|\alpha - \lambda_1| \, |\alpha - \lambda_2|} \right) \equiv |\lambda_1 - \lambda_2| \sup_{\mathbf{v} \in \mathbb{R}^3} I(\mathbf{v}; \lambda_1, \lambda_2). \tag{7.8}$$

To treat the integral $I_2(v;\lambda_1,\lambda_2)$ we use a method sketched in Vekua [5, p. 33]. We draw a circle around $\lambda_1$ with radius $\rho = 2|\lambda_1 - \lambda_2|$ in the $\lambda$-plane and distinguish between two regions in $\mathbf{v}'$-space:

$$G_1 = \{\mathbf{v}' \mid |\alpha - \lambda_1| \geqslant \rho\} \quad \text{and} \quad G_2 = \{\mathbf{v}' \mid |\alpha - \lambda_1| \leqslant \rho\}.$$

When $\mathbf{v}' \in G_1$, we have $|\alpha - \lambda_2| \geqslant \frac{1}{2}|\alpha - \lambda_1|$, and therefore, by using Hölder's inequality with $\frac{3}{2} < p < 2$, $2 < p' < 3$, $1 = (1/p) + (1/p')$, we get

$$\int_{G_1} \frac{|K(\mathbf{v},\mathbf{v}')|\,d^3v'}{|\alpha - \lambda_1|\,|\alpha - \lambda_2|} \leqslant 2 \int_{G_1} \frac{|K(\mathbf{v},\mathbf{v}')|}{|\alpha - \lambda_1|^2}\, d^3v'$$

$$< 2\left(\int_{\mathbb{R}^3} |K(\mathbf{v},\mathbf{v}')|^{p'}\,d^3v'\right)^{1/p'} \left(\int_{G_1} \frac{d^3v'}{|\alpha - \lambda_1|^{2p}}\right)^{1/p}. \tag{7.9}$$

The first factor represents a bounded function with respect to $\mathbf{v}$. Making the substitution (5.3), the second can be estimated by

$$\int_{G_1} \frac{d^3v'}{|\alpha - \lambda_1|^{2p}} = \frac{2\pi}{k} \int \frac{h(w)\,du\,dw}{|\alpha - \lambda_1|^{2p}} \leqslant a_1|\lambda_1 - \lambda_2|^{2-2p} + a_2|\lambda_1 - \lambda_2|^{3-2p} \tag{7.10}$$

where we used an estimate of the form $h(w) \leqslant b_1 + b_2 r, r = |\alpha - \lambda_1|$, $\alpha = w + iu$. The quantities $a_1, a_2, b_1$ and $b_2$ depend on $\lambda_1$. In case of $\mathbf{v}' \in G_2$, we have (with the same Hölder exponents)

$$\int_{G_2} \frac{|K(\mathbf{v},\mathbf{v}')|\,d^3v'}{|\alpha - \lambda_1|\,|\alpha - \lambda_2|} < \left(\int_{\mathbb{R}^3} |K(\mathbf{v},\mathbf{v}')|^{p'}\,d^3v'\right)^{1/p'} \left(\int_{G_2} \frac{d^3v'}{|\alpha - \lambda_1|^p|\alpha - \lambda_2|^p}\right)^{1/p}. \tag{7.11}$$

Again the first factor is bounded with respect to $\mathbf{v}$. For the second we get

$$\int_{G_2} \frac{d^3v'}{|\alpha - \lambda_1|^p|\alpha - \lambda_2|^p} < \frac{2\pi}{k} h(\bar{w}) \int_{G_2} \frac{du\,dw}{|\alpha - \lambda_1|^p|\alpha - \lambda_2|^p}, \quad \bar{w} = \operatorname{Re}\lambda_1 + \rho. \tag{7.12}$$

To investigate the singularity of this integral in the limit $|\lambda_1 - \lambda_2| \to 0$ we replace the variable $\alpha = w + iu$ by a new one, $\zeta$, according to

$$\zeta = \xi + i\eta = \frac{\alpha - \lambda_1}{|\lambda_1 - \lambda_2|}; \quad du\,dw = |\lambda_1 - \lambda_2|^2\,d\xi\,d\eta \tag{7.13}$$

$$\frac{\alpha - \lambda_2}{|\lambda_1 - \lambda_2|} = \frac{\alpha - \lambda_1}{|\lambda_1 - \lambda_2|} + \frac{\lambda_1 - \lambda_2}{|\lambda_1 - \lambda_2|} = \zeta + e^{i\theta}. \tag{7.14}$$

We obtain

$$\int_{G_2} \frac{du\,dw}{|\alpha - \lambda_1|^p\,|\alpha - \lambda_2|^p} = |\lambda_1 - \lambda_2|^{2-2p} \int_{|\zeta| \leqslant 2} \frac{d\xi\,d\eta}{|\zeta|^p|\zeta + e^{i\theta}|^p} \leqslant M|\lambda_1 - \lambda_2|^{2-2p}. \tag{7.15}$$

It follows from equations (7.8), (7.10) and (7.15) that

$$\|(A_{\lambda_1} - A_{\lambda_2})K\|_1 < c_1|\lambda_1 - \lambda_2|^{(2/p)-1} + c_2|\lambda_1 - \lambda_2|^{(3/p)-1}\ (\tfrac{3}{2} < p < 2). \tag{7.16}$$

Hence, the continuity of $A_\lambda K$ is obvious.

*Note 7.1:* Since $A_\lambda K$ is compact even for $\lambda \in G$, the eigenvalues possibly existing inside $G$ are of finite multiplicity.[3]) The continuity of $A_\lambda K$ with respect to $\lambda$ is not used in the following. It merely shows that the operator-valued function $A_\lambda K$, being a holomorphic family outside $G$, is continuously extended into the domain $G$.

## 8. The Operator $A_\lambda K$ in Certain $L_p$-Spaces

In our previous considerations, the space $L_1(\mathbb{R}^3)$ was merely a tool for the discussion of the behaviour of the $L_2$-spectrum of $B_{\mathbf{k}}$ in which we were mainly interested. However, there are other spaces serving this purpose.

First we consider $L_p$-spaces with $1 < p < 2$, which can be treated similarly to the space $L_1(\mathbb{R}^3)$. We estimate the $L_p$-norm of $A_\lambda K$:

$$|A_\lambda Kf| \leqslant \int \left(\frac{|K(\mathbf{v},\mathbf{v}')|}{|\alpha-\lambda|^p}|f(\mathbf{v}')|^p\right)^{1/p} |K(\mathbf{v},\mathbf{v}')|^{1-(1/p)}\, d^3v'$$

$$|A_\lambda Kf|^p \leqslant \left(\int \frac{K(\mathbf{v},\mathbf{v}')}{|\alpha-\lambda|^p}|f(v')|^p\, d^3v'\right)\cdot\left(\int |K(\mathbf{v},\mathbf{v}')|\, d^3v'\right)^{p/p'} \tag{8.1}$$

and with (2.7)

$$\|A_\lambda Kf\|_p \leqslant C^{1/p'}\|f\|_p \sup_{\mathbf{v}'\in\mathbb{R}^3}\left(\int \frac{|K(\mathbf{v},\mathbf{v}')|\, d^3v}{|\alpha-\lambda|^p}\right)^{1/p} \qquad \alpha = i\mathbf{k}\mathbf{v} + \nu(v) \tag{8.2}$$

Splitting the integral into two parts as it is already familiar to us, we see that only the integral over $|\alpha-\lambda| \leqslant 1$ requires a closer inspection. By using Hölder's inequality, we get

$$\int_{|\alpha-\lambda|\leqslant 1} \frac{|K(\mathbf{v},\mathbf{v}')|}{|\alpha-\lambda|^p}\, d^3v < \left(\int_{\mathbb{R}^3} |K(\mathbf{v},\mathbf{v}')|^{q'}\, d^3v\right)^{1/q'} \left(\int_{|\alpha-\lambda|\leqslant 1} \frac{d^3v}{|\alpha-\lambda|^{pq}}\right)^{1/q} \tag{8.3}$$

We must have $2 < q' < 3$, $\frac{3}{2} < q < 2$ and $pq < 2$. These conditions can only be simultaneously fulfilled if $1 \leqslant p < \frac{4}{3}$. When $p > \frac{4}{3}$ the above conclusions are not justified.

We have thus proved the first part of

*Theorem 8.1:* $A_\lambda K$ is a bounded operator on $L_p(\mathbb{R}^3)$ with $1 \leqslant p < \frac{4}{3}$ and $\frac{3}{2} < p < 2$.

The second part, that is the boundedness for $\frac{3}{2} < p < 2$, can be shown in a different way. We know from 2.m) that, for these values of $p$, $Kf(f \in L_p(\mathbb{R}^3))$ is bounded. This fact implies $Kf \in D(A_\lambda)$ and therefore $A_\lambda K$ is defined on the whole space $L_p(\mathbb{R}^3)$. It follows then from the closed graph theorem [7, p. 79] that $A_\lambda K$ is a bounded operator.

3) See footnote p. 102.

Explicitly we have, with (2.9),

$$\|A_\lambda Kf\|_p^p = \int \frac{|Kf|^p}{|\alpha-\lambda|^p}\, d^3 v \leqslant \int\limits_{|\alpha-\lambda|\leqslant 1} \frac{|Kf|^p\, d^3 v}{|\alpha-\lambda|^p} + \|K\|_p^p \|f\|_p^p$$

$$\leqslant C_p^p \int \frac{d^3 v}{|\alpha-\lambda|^p} \cdot \|f\|_p^p + \|K\|_p^p \|f\|_p^p. \tag{8.4}$$

*Note 8.1:* In the $L_p$-spaces given by Theorem 8.1, Lemma 5.2 and Corollary 6.3 could be formulated correspondingly.

Next we turn to the case $p \geqslant 2$. Then the singularity of $A_\lambda(\lambda \in G)$ is no longer locally absolutely integrable to the power $p$. Therefore, the domain of $A_\lambda K$ cannot be the whole space and one might believe that the operator would be unbounded. But this is not true as the following theorem shows:

*Theorem 8.2:* $A_\lambda K(\lambda \in G)$ is a bounded operator on $D(A_\lambda K)$, if

a) $\lambda \in G - \partial G$: for $2 \leqslant p \leqslant \infty$,
b) $\lambda \in \partial G$ for $2 \leqslant p < 4$
($\partial G$ is the boundary of $G$).

We emphasize that boundedness is claimed to hold on the (natural) domain $D(A_\lambda K)$ of $A_\lambda K$, that will turn out to be a proper subspace of $L_p(\mathbb{R}^3)$.

*Proof:* We begin by characterizing the domain of $A_\lambda K$. Let $f \in L_p(\mathbb{R}^3)$, $2 \leqslant p \leqslant \infty$ be an element of $D(A_\lambda K)$. Since $Kf = g$ is continuous 2.p), we must necessarily have $g(\mathbf{v}_\lambda) = 0\,(i\mathbf{k}\mathbf{v}_\lambda + \nu(v_\lambda) - \lambda = 0)$. Otherwise $A_\lambda g$ would not be absolutely integrable to the power $p$ at $\mathbf{v} = \mathbf{v}_\lambda$. On the other hand, as we shall see below, the condition $g(\mathbf{v}_\lambda) = 0$ is also sufficient for $f \in D(A_\lambda K)$ (for the two cases specified in Theorem 8.2). With regard to Lemma 2.1 we can write

$$\|A_\lambda Kf\|_p^p = \int\limits_{|\alpha-\lambda|\leqslant 1} \frac{|Kf|^p\, d^3 v}{|\alpha-\lambda|^p} + \int\limits_{|\alpha-\lambda|\geqslant 1} \frac{|Kf|^p\, d^3 v}{|\alpha-\lambda|^p} \tag{8.5}$$

$$\leqslant C^p(\nu^{[-1]}(\lambda_1+1), p', \beta)\|f\|_p^p \int\limits_{|\alpha-\lambda|\leqslant 1} \frac{|\mathbf{v}-\mathbf{v}_\lambda^0|^{\beta p}}{|\alpha-\lambda|^p}\, d^3 v + \|K\|_p^p \|f\|_p^p \tag{8.6}$$

$\mathbf{v}_\lambda^0$ denotes that point on the singular circle $\{\mathbf{v} | i\mathbf{k}\mathbf{v} + \nu(v) - \lambda = 0\}$ which lies closest to $\mathbf{v}$. To investigate the remaining integral, we distinguish between the two cases mentioned in Theorem 8.2:

a) Because of the choice of $\mathbf{v}_\lambda^0$ the factor $|\mathbf{v} - \mathbf{v}_\lambda^0|$ in (8.6) only depends on $v$ and $z$, so we have

$$|\mathbf{v}-\mathbf{v}_\lambda^0|^2 = v^2 + v_\lambda^{02} - 2vv_\lambda^0(zz_\lambda^0 + \sqrt{1-z^2}\sqrt{1-z_\lambda^{02}}) \tag{8.7}$$

$\lambda \notin \partial G$ implies $|z_\lambda^0| < 1$ and $v_\lambda^0 > 0$. We expand the expression (8.7) near $v_\lambda^0$, $z_\lambda^0$ in a Taylor series. The leading terms read

$$|\mathbf{v}-\mathbf{v}_\lambda^0|^2 \sim (v - v_\lambda^0)^2 + \frac{v_\lambda^{02}}{1 - z_\lambda^{02}}(z - z_\lambda^0)^2. \tag{8.8}$$

Introducing $w = \nu(v)$ and $u = kvz$ we obtain

$$w - \lambda_1 \sim \left.\frac{\partial \nu}{\partial v}\right|_{v=v_\lambda^0} \cdot (v - v_\lambda^0) \equiv a(v - v_\lambda^0) \quad a \neq 0 \tag{8.9}$$

$$z - z_\lambda^0 \sim \frac{1}{kv_\lambda^0}(u - \lambda_2) - \frac{\lambda_2}{akv_\lambda^{02}}(w - \lambda_1) \tag{8.10}$$

where $\lambda = \lambda_1 + i\lambda_2$. Putting $r^2 = (u - \lambda_2)^2 + (w - \lambda_1)^2$ we see from (8.8), (8.9) and (8.10) that $|\mathbf{v} - \mathbf{v}_\lambda^0|^2 \sim r^2$ as $\mathbf{v} \to \mathbf{v}_\lambda^0$. The integral (8.6) exists if $\beta p - p > -2$. This can always be achieved by a suitable choice of $\beta$. When $p > 3$ we may choose $\beta = 1$, whereas when $2 \leqslant p \leqslant 3$ we have to verify that the two conditions $\beta p - p > -2$ and $\beta < (3 - p'/p')$ (Lemma 2.1) are not in contradiction. But the first one is equivalent to $\beta > (2 - p'/p')$ which is less than $(3 - p'/p')$ so we can choose $\beta$ between these two values. (8.6) shows that $A_\lambda K$ is bounded on its domain. Moreover, the operator is closed and therefore $D(A_\lambda K)$ is also closed.[4]) This proves part a) of Theorem 8.2.

b) Let $\lambda \in \partial G$, i.e. $z_\lambda^0 = \pm 1$. Furthermore let $v_\lambda^0 \neq 0$. Then (8.7) simplifies to

$$|\mathbf{v} - \mathbf{v}_\lambda^0|^2 = v_\lambda^{02} + v^2 - 2vv_\lambda^0 z. \tag{8.11}$$

Near $\mathbf{v}^0$ we have

$$|\mathbf{v} - \mathbf{v}_\lambda^0|^2 \sim -2v_\lambda^{02}(z - z_\lambda^0). \tag{8.12}$$

Therefore $|\mathbf{v} - \mathbf{v}_\lambda^0|^2 \sim r$ as $\mathbf{v} \to \mathbf{v}_\lambda^0$ and the new condition which guarantees the existence of the integral (8.6) reads

$$\frac{\beta p}{2} - p > -2. \tag{8.13}$$

This can only be satisfied if $p < 4$.

Now let $v_\lambda^0 = 0$, $\lambda_1 = 1$, $\lambda_2 = 0$. We obtain

$$|\mathbf{v} - \mathbf{v}_\lambda^0|^2 \sim v^2. \tag{8.14}$$

Since $v_\lambda^0 = 0$ we have to take the next order in (8.9)

$$w - 1 \sim \left.\frac{1}{2}\frac{\partial^2 \nu}{\partial v^2}\right|_{v=0} \cdot v^2. \tag{8.15}$$

Therefore, we see that again $|\mathbf{v} - \mathbf{v}_\lambda^0|^2 \sim r$ as $\mathbf{v} \to \mathbf{v}_\lambda^0$ and (8.13) must be satisfied. This proves part b) of Theorem 8.2.

*Note 8.2:* From Theorem 6.1 or Corollary 6.3 we know that the $L_1$-norm of the operator $A_\lambda K$ is reduced with increasing $k$ uniformly in some region of the $\lambda$-plane, and consequently the eigenvalues of $A_\lambda K$ cannot exceed $\|A_\lambda K\|_1$ in absolute value. We can easily generalize Lemma 5.2 and establish the result that all eigenfunctions $f \in L_p(\mathbb{R}^3)$,

[4]) But in general $R(A_\lambda K)$ is not contained in $D(A_\lambda K)$. One could introduce the new domain $\bigcap_{n=0}^{\infty} D((A_\lambda K)^n)$.

$\frac{3}{2} < p \leqslant \infty$, of $A_\lambda K f = \mu f (\mu \in \mathbb{C})$ are also elements of $L_1(\mathbb{R}^3)$. Therefore, in any one of these $L_p$-spaces, we have $|\mu| \leqslant \|A_\lambda K\|_1$. In view of later applications, we confine ourselves to the spaces $L_2(\mathbb{R}^3)$ and $L_\infty(\mathbb{R}^3)$. In these cases, the spectrum of $A_\lambda K$ for $\lambda \notin G$ is purely discrete since the operator is compact. Introducing the notion of the spectral radius [7, p. 211]

$$r_\sigma \equiv \lim_{n\to\infty} \|(A_\lambda K)^n\|_p^{1/n} = \max_{\mu \in \sigma(A_\lambda K)} |\mu| \quad p = 2, \infty \tag{8.16}$$

we get the result

$$r_\sigma \leqslant \|A_\lambda K\|_1 \quad \lambda \notin G. \tag{8.17}$$

As an example, we consider the space $L_\infty(\mathbb{R}^3)$. According to (8.17) the spectral radius is bounded as $\lambda$ approaches the boundary of $G$ while the operator norm of $A_\lambda K$ tends towards infinity. To be definite, we take $\lambda$ to be real, $-\infty < \lambda < 1$. We consider the operator $KA_\lambda$ in $L_\infty(\mathbb{R}^3)$ which is the dual operator of $A_\lambda K$ with respect to $L_1(\mathbb{R}^3)$. It follows from our previous discussion of $\|A_\lambda K\|_1$, especially from (6.7), that uniformly in $-\infty < \lambda < 1$

$$\|A_\lambda K\|_1 = \|KA_\lambda\|_\infty < \frac{c_q}{k^{1/q}} \quad q > 2. \tag{8.18}$$

Putting $d = |1 - \lambda|$ we get, using $\|A_\lambda\|_\infty = d^{-1}$,

$$r_\sigma = \lim_{n\to\infty} \|A_\lambda (KA_\lambda)^{n-1} K\|_\infty^{1/n} < \lim_{n\to\infty} d^{-1/n} c_q^{1-(1/n)} \|K\|_\infty^{1/n} k^{(1/q)((1/n)-1)} = c_q k^{-1/q} \tag{8.19}$$

We see that the spectral radius is estimated by the same bound as $\|A_\lambda K\|_1$ uniformly in $-\infty < \lambda < 1$.

## 9. Time Evolution in the Limit $k \to \infty$

We know from Scharf [10] that the operator $-B_{\mathbf{k}}$ generates a contraction semigroup $T^t$ in the space $L_2(\mathbb{R}^3)$. This result is mainly due to the fact that $-B_{\mathbf{k}}$ is dissipative, that is 2.h) $\mathrm{Re}(-B_{\mathbf{k}} f, f) \leqslant 0$ [7, p. 250]. The semigroup $T^t$ solves the initial value problem for the linearized Boltzmann equation according to

$$f(t) = T^t f(0) = e^{-B_{\mathbf{k}} t} f(0) \tag{9.1}$$

$T^t$ has the contraction property

$$\|T^t\| \leqslant 1. \tag{9.2}$$

Moreover $T^t$ is a family of bounded operators in $L_2(\mathbb{R}^3)$ satisfying

$$T^t T^s = T^{t+s} \quad (t, s \geqslant 0) \quad \text{(semigroup property)}$$

$$T^0 = \mathbb{1}$$

$$\lim_{t\to t_0} T^t f = T^{t_0} f \quad \text{for each } t_0 \geqslant 0 \quad \text{(strong continuity) and each } f \in L_2(\mathbb{R}^3).$$

Theorem 6.1 tells us that the eigenvalues of $-B_{\mathbf{k}}$ move to the left in the complex plane with increasing $k$. If $k$ is sufficiently large, the half-plane $\operatorname{Re}\lambda > -1$ is contained in the resolvent set of $-B_{\mathbf{k}}$. Since the character of time evolution is determined to a certain degree by the location of the spectrum of the generator $-B_{\mathbf{k}}$, one might expect that estimate (9.2) could be replaced by a sharper one for sufficiently large $k$, for instance one of the form

$$\|T^t\| \leqslant M e^{-\gamma t} \quad M \geqslant 1,\ 0 < \gamma \leqslant 1. \tag{9.3}$$

But $M = 1$ has to be excluded at once, since this would contradict dissipativity. For, if $M = 1$, $0 < \gamma \leqslant 1$,

$$\operatorname{Re}(T^t f - f, f) = \operatorname{Re}(T^t f, f) - \|f\|^2 \leqslant \|T^t f\|\,\|f\| - \|f\|^2 \leqslant e^{-\gamma t}\|f\|^2 - \|f\|^2 \leqslant 0 \tag{9.4}$$

and therefore

$$\operatorname{Re}(-B_{\mathbf{k}} f, f) = \lim_{t\to 0} \operatorname{Re} \frac{1}{t}(T^t f - f, f) \leqslant -\gamma \|f\|^2. \tag{9.5}$$

Thus $-B_{\mathbf{k}} + \gamma$ should be dissipative (at least for large $k$) but this is impossible since for an eigenfunction $f_0(\|f_0\| = 1)$ of $I$ belonging to the eigenvalue 0 we have

$$\operatorname{Re}((-B_{\mathbf{k}} + \gamma) f_0, f_0) = \gamma > 0 \quad \forall \mathbf{k} \tag{9.6}$$

clearly contradicting (9.5).

Now we prove

*Theorem 9.1:* Let $0 < \gamma < 1$ be given. Then there exists a constant $M_\gamma > 1$ so that for sufficiently large $k$

$$\|T^t\| \leqslant M_\gamma e^{-\gamma t}. \tag{9.7}$$

For $k$ large enough, the value of $M_\gamma$ will turn out to be independent of $k$, it merely is a function of $\gamma$. (9.7) means that $T^t$ is a bounded semigroup of negative type [8] for sufficiently large $k$.

Before proving this theorem, some preparations are necessary. First, consider the operator $K$ as a perturbation in $-B_{\mathbf{k}} = -A + K(A = i\mathbf{k}\mathbf{v} + \nu(v))$. Denoting by $T_0^t = e^{At}$ the unperturbed semigroup generated by $-A$ we can write down an integral equation for $T^t$ [8, p. 495]:

$$T^t = T_0^t + \int_0^t T_0^{t-s} K T^s \, ds. \tag{9.8}$$

We can solve (9.8) by successive approximation

$$T^t = \sum_{n=0}^{\infty} T_n^t \tag{9.9}$$

$$T_{n+1}^t = \int_0^t T_0^{t-s} K T_n^s \, ds \tag{9.10}$$

It can be proved by induction that

$$\|T_n^t\| \leqslant \|K\|^n \frac{t^n}{n!} e^{-t} \tag{9.11}$$

where we used $\|T_0^t\| = e^{-t}$. We see from (9.11) that (9.9) is absolutely convergent and

$$\|T^t\| \leqslant e^{-t} e^{\|K\| t}. \tag{9.12}$$

Since $\|K\| > 1$ this result is not of much help with regard to Theorem 9.1 we are going to prove. We will therefore truncate the series (9.9) after a finite number of terms and treat the remainder by the following method. The semigroup $T^t$ can be expressed in closed form by an inverted Laplace integral [1, p. 349]

$$T^t f = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{a-i\omega}^{a+i\omega} e^{\lambda t} (B_{\mathbf{k}} + \lambda)^{-1} f d\lambda. \tag{9.13}$$

The limit exists uniformly with respect to $t$ in any interval $[1/\varepsilon, \varepsilon]$, $\varepsilon > 0$ (for $t = 0$ the limit is $\frac{1}{2}f$). Since $a > 0$, the path of integration, $\Gamma$, runs entirely in the resolvent set of $-B_{\mathbf{k}}$ (Fig. 2).

The disadvantage of the representation (9.13) is that in general the integral is not absolutely convergent. A customary technique in dealing with this integral is to deform the path of integration such that it runs within a sector $|\arg \lambda| > (\pi/2) + \varepsilon$ $(\varepsilon > 0)$ for $|\lambda| \to \infty$. This would produce an exponentially decaying integrand in (9.13). But in our case, this method does not work because $\sigma(-B_{\mathbf{k}})$ is, for $k \to \infty$, not confined to a fixed sector of the complex plane. The series (9.9) can be recovered from (9.13) by expanding $(B_{\mathbf{k}} + \lambda)^{-1}$ in terms of $A_{-\lambda} K$, that is (see 3.b))

$$(B_{\mathbf{k}} + \lambda)^{-1} = \sum_{n=0}^{n} (A_{-\lambda} K)^n A_{-\lambda}. \tag{9.14}$$

The series certainly converges if we choose $\operatorname{Re} \lambda > 0$ large enough. Inserting (9.14) in (9.13) and evaluating the integrals successively by means of the residue theorem, we arrive at (9.9).

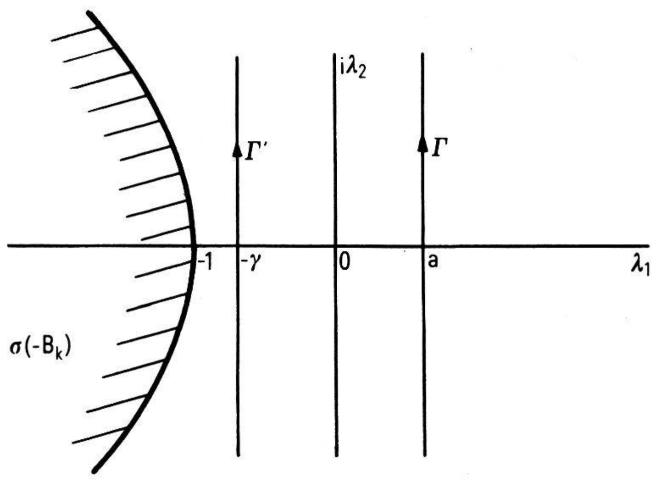


Figure 2
Paths of integration.

*Proof* (Theorem 9.1): First we consider the series (9.9) which we truncate after four terms

$$\tilde{T}^t = \sum_{n=0}^{4} T_n^t. \tag{9.15}$$

Let $\gamma$, $0 < \gamma < 1$ be given. We have with a suitable constant $\tilde{M}_\gamma$

$$\|\tilde{T}^t\| \leqslant e^{-t} \sum_{n=0}^{4} \|K\|^n \frac{t^n}{n!} = e^{-\gamma t} \sum_{n=0}^{4} \|K\|^n \frac{t^n}{n!} e^{\gamma t - t} \leqslant \tilde{M}_\gamma e^{-\gamma t} \tag{9.16}$$

(9.16) is valid for all $\mathbf{k} \in \mathbb{R}^3$.

To estimate the remainder

$$\tilde{\tilde{T}}^t = \sum_{n=5}^{\infty} T_n^t \tag{9.17}$$

of the series (9.9), we take representation (9.13) together with (9.14)

$$\tilde{\tilde{T}}^t = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \sum_{n=5}^{\infty} (A_{-\lambda} K)^n A_{-\lambda} \, d\lambda = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (A_{-\lambda} K)^5 (\mathbb{1} - A_{-\lambda} K)^{-1} A_{-\lambda} \, d\lambda. \tag{9.18}$$

It is our intention to show the absolute convergence of this integral on the path of integration $\Gamma'$ (running from $-\gamma - i\infty$ to $-\gamma + i\infty$) lying on the left of the original path $\Gamma$ (Fig. 2). We discuss the operator $(\mathbb{1} - A_{-\lambda} K)^{-1}$ first. Until now its existence is obvious only when $\operatorname{dist}[\lambda, -G] > \|K\|$. But it follows from (8.17) that we can fix a value $k_\eta$ such as that for $k \geqslant k_\eta$

$$r_\sigma(A_{-\lambda} K) \leqslant \eta < 1 \tag{9.19}$$

uniformly in $-1 < \operatorname{Re}\lambda < \infty$. $\eta$ denotes an arbitrary fixed number between 0 and 1. Consequently, the half-plane $-1 < \operatorname{Re}\lambda < \infty$ is contained in $\rho(-B_{\mathbf{k}})$ or equivalently

$$1 \in \rho(A_{-\lambda} K), \quad -1 < \operatorname{Re}\lambda < \infty, k \geqslant k_\eta \tag{9.20}$$

and the series

$$R(1; A_{-\lambda} K) \equiv (\mathbb{1} - A_{-\lambda} K)^{-1} = \sum_{n=0}^{\infty} (A_{-\lambda} K)^n \tag{9,21}$$

converges in the operator norm. Moreover, we can show that the resolvent $R(1; A_{-\lambda} K)$ is bounded in norm uniformly in $-\gamma \leqslant \operatorname{Re} < \infty$, $k \geqslant k_\eta$ by

$$\|R(1; A_{-\lambda} K)\| \leqslant \max_{\substack{k \geqslant k_\eta \\ -\gamma \leqslant \operatorname{Re}\lambda < \infty}} \|(\mathbb{1} - A_{-\lambda} K)^{-1}\| \equiv C_\gamma. \tag{9.22}$$

Once $k_\eta$ is fixed, the constant $C_\gamma$ is merely a function of $\gamma$. (9.22) holds for the following reasons: $R(1; A_{-\lambda} K)$ is a continuous function with respect to $\lambda$ and $k$ because $A_{-\lambda} K$ is continuous (even holomorphic) in both variables. This is a consequence of a theorem on the continuity of the resolvent given for example by Kato [8, p. 212]. Furthermore, it follows from (4.6) that for sufficiently large $k$, say $k \geqslant k_\gamma$ (suppose $k_\gamma > k_\eta$)

$$\|A_{-\lambda} K\| \leqslant b < 1 \tag{9.23}$$

uniformly in $-\gamma \leqslant \operatorname{Re}\lambda < \infty$. Then (9.21) is absolutely convergent and thus $R(1; A_{-\lambda}K)$ is uniformly bounded for $-\gamma \leqslant \operatorname{Re}\lambda < \infty$ and $k \geqslant k_\gamma$. In addition, for $k_\eta \leqslant k \leqslant k_\gamma$ and sufficiently large $|\lambda|$, $\|A_{-\lambda}K\| < 1$ in view of (3.3). Hence the continuous function $R(1; A_{-\lambda}K)$ is bounded at infinity with respect to both variables. This fact implies that it is bounded everywhere in $-\gamma \leqslant \operatorname{Re}\lambda < \infty$, $k \geqslant k_\eta$ and assumes its maximum $C_\gamma$ at some point.

To estimate (9.18), we actually would only need an upper bound of $R(1; A_{-\lambda}K)$ with respect to the straight line $\Gamma'$ which would obviously be less or equal to $C_\gamma$.

To show the absolute convergence of (9.18), we investigate the $\lambda_2$-dependence of $\|A_{-\lambda}K\|$ $(\lambda = \lambda_1 + i\lambda_2)$. For this purpose, we revert to (4.3), (4.5) and (4.6). We calculate the integral (4.5) by using the following estimate of $\nu(v) + \lambda_1$ valid uniformly in $-\gamma \leqslant \lambda_1 < \infty$

$$\nu(v) + \lambda_1 > \sqrt{b_1 v^2 + b_2} \quad b_1 b_2 > 0 \tag{9.24}$$

where $b_1$ and $b_2$ are two suitable constants. When $\gamma \to 1$ we must require that $b_1 \to 0$ and $b_2 \to 0$. Putting $v_1 = vz$, we see, by (4.5), that

$$\int \frac{d^3 v}{|i\mathbf{kv} + \nu(v) + \lambda|^4} = \int \frac{d^3 v}{[(kv_1 + \lambda_2)^2 + (\nu(v) + \lambda_1)^2]^2} < \int \frac{d^3 v}{[(kv_1 + \lambda_2)^2 + b_1 v^2 + b_2]^2}$$

$$= 2\pi \int_{-\infty}^{\infty} dv_1 \int_0^{\infty} \frac{\rho\, d\rho}{[(kv_1 + \lambda_2)^2 + b_1 v_1^2 + b_1 \rho^2 + b_2]^2}$$

$$= \frac{\pi}{b_1} \int_{-\infty}^{\infty} dv_1 \frac{1}{(kv_1 + \lambda_2)^2 + b_1 v_1^2 + b_2}$$

$$= \frac{\pi^2}{b_1 \sqrt{(k^2 + b_1) b_2 + b_1 \lambda_2^2}}. \tag{9.25}$$

Instead of (4.6), we obtain the sharper result

$$\|A_{-\lambda} K\| < \frac{D_\gamma}{[(k^2 + b_1) b_2 + b_1 \lambda_2^2]^{1/8}} \quad D_\gamma > 0. \tag{9.26}$$

This allows us to estimate (9.18) on the integration path $\Gamma'$ where the integral is absolutely convergent. We get

$$\|\tilde{\tilde{T}}^t\| < \frac{1}{2\pi(1-\gamma)} C_\gamma D_\gamma^5 e^{-\gamma t} \int_{-\infty}^{\infty} \frac{d\lambda_2}{[(k^2 + b_1) b_2 + b_1 \lambda_2^2]^{5/8}} \equiv \tilde{\tilde{M}}_\gamma e^{-\gamma t}. \tag{9.27}$$

Together with (9.16)

$$\|T^t\| \leqslant \tilde{T}^t\| + \|\tilde{\tilde{T}}^t\| < (\tilde{M}_\gamma + \tilde{\tilde{M}}_\gamma) e^{-\gamma t} \equiv M_\gamma e^{-\gamma t}. \tag{9.28}$$

By construction, the constant $M_\gamma$ tends to infinity as $\gamma \to 1$. Thus Theorem 9.1 is proved.

This theorem shows that the Fourier components $\hat{f}(\mathbf{k}, \mathbf{v}, t)$ of a solution of the Boltzmann equation (1.1) decay exponentially in time for large absolute values of the argument $\mathbf{k}$.

Next we are concerned with the equation whether the semigroup $T^t$ approaches a limit as $k \to \infty$. Of course, the operator $B_{\mathbf{k}}$ does not converge to a well-defined operator, but we expect from our discussion of its spectrum (Section 6) that the time evolution according to $T^t$ approaches in some sense that generated by the operator $-A = -i\mathbf{kv} - \nu(v)$ in the short-wavelength limit ($k \to \infty$). We call this semigroup $V^t = e^{-At}$. Then the following theorem holds:

*Theorem 9.2:* For each $f \in L_2(\mathbb{R}^3)$ and for arbitrary $\gamma$, $0 < \gamma < 1$, we have $\lim_{k\to\infty} e^{\gamma t}\|T^t f - V^t f\| = 0.$

The limit exists uniformly in $0 \leqslant t < \infty$.

*Proof:* Remembering that $V^t = T_0^t$ we get from (9.9)

$$T^t - V^t = \sum_{n=1}^{\infty} T_n^t \tag{9.29}$$

The first term of this series reads (with the notation $\alpha(\mathbf{v}) = i\mathbf{kv} + \nu(v)$)

$$T_1^t f = \int_0^t e^{-\alpha(\mathbf{v})(t-s)} \left( \int K(\mathbf{v}, \mathbf{v}')\, e^{-\alpha(\mathbf{v}')s} f(\mathbf{v}')\, d^3 v' \right) ds. \tag{9.30}$$

Therefore

$$\|T_1^t f\| \leqslant \int_0^t e^{-t} \left\| \int K(\mathbf{v}, \mathbf{v}')\, e^{(1-\alpha(\mathbf{v}'))s} f(\mathbf{v}')\, d^3 v' \right\| ds. \tag{9.31}$$

We consider the inner integral for fixed $\mathbf{v}$ and $s$

$$g_{\mathbf{k}}(\mathbf{v}, s) \equiv \int K(\mathbf{v}, \mathbf{v}') e^{(1-\nu(v'))s} e^{-i\mathbf{kv}'s} f(\mathbf{v}')\, d^3 v'. \tag{9.32}$$

We know that $K(\mathbf{v}, \mathbf{v}') \in L_2(\mathbb{R}^3)$ with respect to $\mathbf{v}'$. Since $f \in L_2(\mathbb{R}^3)$, it follows that

$$K(\mathbf{v}, \mathbf{v}')\, e^{(1-\nu(v'))s} f(\mathbf{v}') \in L_1(\mathbb{R}^3)$$

with respect to $\mathbf{v}'$.

Thus the integral (9.32) can be interpreted as the Fourier transform of an $L_1$-function, at least for $s > 0$. Hence, by the Riemann–Lebesgue lemma

$$\lim_{k\to\infty} |g_{\mathbf{k}}(\mathbf{v}, s)| = 0 \quad s > 0. \tag{9.33}$$

Furthermore $g_{\mathbf{k}}(\mathbf{v}, s) \in L_2(\mathbb{R}^3) (s \geqslant 0)$ and

$$|g_{\mathbf{k}}(\mathbf{v}, s)| \leqslant \int |K(\mathbf{v}, \mathbf{v}')|\, |f(\mathbf{v}')| d^3 v' \in L_2(\mathbb{R}^3). \tag{9.34}$$

Using Lebesgue's principle of dominated convergence, we conclude that

$$\lim_{k\to\infty} \|g_{\mathbf{k}}(\mathbf{v}, s)\| = 0 \quad s > 0. \tag{9.35}$$

Since the estimate (9.34) is uniform with respect to $s$ we see, by the same argument (the point $s = 0$ has measure zero), that

$$\|T_1^t f\| \leqslant e^{-t} \int_0^t \|g_{\mathbf{k}}(\mathbf{v}, s)\| ds \underset{k \to \infty}{\rightarrow} 0. \tag{9.36}$$

(9.15), (9.10) and (9.16) imply

$$\|(\tilde{T}^t - V^t) f\| \leqslant \sum_{n=1}^{4} \|T_n^t f\| \leqslant \tilde{N}_\gamma e^{-\gamma t} \int_0^t \|g_{\mathbf{k}}(\mathbf{v}, s)\| ds \underset{k \to \infty}{\rightarrow} 0. \tag{9.37}$$

The remainder of the series (9.9) has already been estimated in (9.27). The constant $\tilde{\tilde{M}}_\gamma$ introduced there is in fact reduced with increasing $k$ as we see from the integral occurring in (9.27). Hence

$$\|T^t f - V^t f\| \leqslant \|\tilde{T}^t f - V^t f\| + \|\tilde{\tilde{T}}^t f\| < \left( \tilde{N}_\gamma \int_0^t \|g_{\mathbf{k}}(\mathbf{v}, s)\| ds + \tilde{\tilde{M}}_\gamma(k) \right) e^{-\gamma t} \underset{k \to \infty}{\rightarrow} 0. \tag{9.38}$$

This proves Theorem 9.2. The uniformity mentioned there follows from the uniform convergence on every finite interval (which is obvious from (9.38)) together with (9.7) which implies $\|T^t - V^t\| \leqslant \|T^t\| + \|V^t\| \leqslant (M_{\gamma'} + 1) e^{-\gamma' t}$ for some $\gamma' > \gamma$. Theorem 9.2 is similar to a theorem given by Kato [8, p. 502] but with the difference that in our case the semigroup $T^t$ does not converge to a well-defined semigroup which is independent of $k$.

In the space $L_\infty(\mathbb{R}^3)$ an analogous theorem holds, but moreover, we have:

*Theorem 9.3:* Let $f \in L_\infty(\mathbb{R}^3)$ and $\gamma$ be any number $0 < \gamma < 1$. Then

$$\lim_{k \to \infty} e^{\gamma t} \|T^t - V^t\|_\infty = 0.$$

The limit exists uniformly in any interval $t \in [\varepsilon, \infty)$, $\varepsilon > 0$.

Before we turn to the proof, we look at the general properties of the semigroups $T^t$ and $V^t$ in the space $L_\infty(\mathbb{R}^3)$. First, the spectrum of $B_{\mathbf{k}}$ is essentially the same in the two spaces. In particular, there is the essential spectrum $G$ and the isolated eigenvalues, both being identical in $L_\infty(\mathbb{R}^3)$ and $L_2(\mathbb{R}^3)$. Moreover, the operator $K$ is compact in $L_\infty(\mathbb{R}^3)$ (Lemma 7.2). Second, the operator $-B_{\mathbf{k}}$ generates a so-called quasi-bounded semigroup (8, p. 495) in $L_\infty(\mathbb{R}^3)$ but not necessarily a contraction semigroup. It follows from the perturbation theory for semigroups [8, p. 495] that we have

$$\|T^t\|_\infty \leqslant e^{(-1 + \|K\|_\infty) t}, \quad \|K\|_\infty > 1. \tag{9.39}$$

This rather modest result can be improved similarly as in Theorem 9.1. The method is the same as before and is shown in the proof of Theorem 9.3 below.

*Proof:* We revert to (9.30) and obtain by changing the order of the two integrals

$$T_1^t f = e^{-\alpha(\mathbf{v}) t} \int d^3 v' \, K(\mathbf{v}, \mathbf{v}') f(\mathbf{v}') \int_0^t ds \, e^{(\alpha(\mathbf{v}) - \alpha(\mathbf{v}')) s}$$

$$= \int d^3 v' \, K(\mathbf{v}, \mathbf{v}') f(\mathbf{v}') \frac{e^{-\alpha(\mathbf{v}') t} - e^{-\alpha(\mathbf{v}) t}}{\alpha(\mathbf{v}) - \alpha(\mathbf{v}')}. \tag{9.40}$$

Let $t \geqslant \varepsilon > 0$ and $0 < \gamma < 1$. We estimate (9.40) by

$$e^{\gamma t}|T_1^t f| \leqslant \|f\|_\infty \left( \int d^3 v' |K(\mathbf{v},\mathbf{v}')| \frac{e^{(\gamma-\nu(\mathbf{v}'))\varepsilon}}{|\alpha(\mathbf{v})-\alpha(\mathbf{v}')|} + e^{(\gamma-\nu(\mathbf{v}))\varepsilon} \int d^3 v' \frac{|K(\mathbf{v},\mathbf{v}')|}{|\alpha(\mathbf{v})-\alpha(\mathbf{v}')|} \right)$$

$$\text{uniformly in } t \geqslant \varepsilon > 0. \tag{9.41}$$

The integrals in (9.41) can be estimated by the same method as we treated $\|A_\lambda K\|_1$ in Section 6, that is, by means of suitable Hölder inequalities (the $\lambda$ there is replaced by $\alpha(\mathbf{v})$). Thanks to the exponential factors, the integrals represent bounded functions with respect to $\mathbf{v}$ and as $k$ increases the integrals vanish. Therefore

$$e^{\gamma t}\|T_1^t\|_\infty \underset{k\to\infty}{\to} 0 \quad \text{uniformly in } t \geqslant \varepsilon. \tag{9.42}$$

Using (9.10) we see that

$$\|\tilde{T}^t - V^t\|_\infty \leqslant \sum_{n=1}^{5} \|T_n^t\|_\infty \leqslant A_\gamma(k) e^{-\gamma t} \underset{k\to\infty}{\to} 0 \quad (A_\gamma(k) \underset{k\to\infty}{\to} 0). \tag{9.43}$$

Here we summed up five terms in $T^t$ instead of only four as in (9.37). The remaining integral reads

$$\tilde{\tilde{T}}^t = \frac{1}{2\pi i} \int_\Gamma d\lambda\, e^{\lambda t} (A_{-\lambda} K)^6 (\mathbb{1} - A_{-\lambda} K)^{-1} A_{-\lambda}. \tag{9.44}$$

Again a relation corresponding to (9.22) exists. But the reason for this is not the same as in (9.22), for the $L_\infty$-norm of $A_{-\lambda} K$ is not reduced with increasing $k$, that is, the analogue of (9.23) does not hold. Yet we know from Note 8.2, especially (8.19), that the spectral radius of $A_{-\lambda} K$ is small for large $k$, precisely

$$r_\sigma(A_{-\lambda} K) \leqslant \eta < 1 \quad \text{for } k \geqslant k_\eta, \quad \text{uniformly in } -1 < \operatorname{Re}\lambda < \infty. \tag{9.45}$$

Furthermore, considering the half-plane $-\gamma \leqslant \operatorname{Re} < \infty$, $0 < \gamma < 1$, by (8.18) and (8.19)

$$\|(A_{-\lambda} K)^n\|_\infty^{1/n} < d^{-1/n} c_q^{1-(1/n)} \|K\|_\infty^{-1/n} k^{-1/2q} \quad d = 1-\gamma, \quad q > 2 \tag{9.46}$$

for $n \geqslant 2$ uniformly in $k \geqslant k_\eta$ and $-\gamma \leqslant \operatorname{Re}\lambda < \infty$, assuming $k_\eta \geqslant 1$.

If we choose $k_\eta$ so large that $k_\eta > c_q^{2q}$, then beyond a certain value of $n$, say $n \geqslant N$, the right-hand side in (9.46) is smaller than 1, uniformly in $k \geqslant k_\eta$ and $-\gamma \leqslant \operatorname{Re}\lambda < \infty$, that is

$$\|(A_{-\lambda} K)^n\|_\infty^{1/n} \leqslant 1 - \varepsilon \quad \varepsilon > 0 \tag{9.47}$$

where $\varepsilon$ is a sufficiently small number. Thus we obtain from (9.21)

$$\|(\mathbb{1} - A_{-\lambda} K)^{-1}\|_\infty \leqslant \sum_{n=0}^{N-1} \|(A_{-\lambda} K)\|^n + \sum_{n=N}^{\infty} (1-\varepsilon)^n$$

$$\leqslant \sum_{n=0}^{N-1} \frac{\|K\|_\infty^n}{d^n} + (1-\varepsilon)^N \cdot \frac{1}{\varepsilon} \equiv C'_\gamma. \tag{9.48}$$

Finally, we consider the operator $KA_{-\lambda}$ with regard to its $\lambda_2$-dependence ($\lambda = \lambda_1 + i\lambda_2$). We have

$$|KA_{-\lambda}f| \leqslant \|f\|_\infty \int \frac{K(\mathbf{v},\mathbf{v}')}{|\alpha(\mathbf{v}') + \lambda|}\, d^3 v'$$

$$\leqslant \|f\|_\infty \left( \int |K(\mathbf{v},\mathbf{v}')|^{4/3}\, d^3 v' \right)^{3/4} \left( \int \frac{d^3 v'}{|\alpha(\mathbf{v}') + \lambda|^4} \right)^{1/4}. \tag{9.49}$$

The second integral on the right-hand side has already been discussed in (9.25) and (9.26). Therefore, by integrating over $\Gamma'$

$$\|\tilde{\tilde{T}}^t\|_\infty \leqslant \frac{C_\gamma' \|K\|_\infty}{2\pi(1-\gamma)^2} e^{-\gamma t} \int_{-\infty}^{\infty} d\lambda_2 \|KA_{-\lambda}\|^5 \underset{k \to \infty}{\to} 0. \tag{9.50}$$

Hence Theorem 9.3 follows from (9.43) and (9.50).

Theorems 9.2 and 9.3 show that the influence of the integral operator $K$ on the time evolution is negligible for $k \to \infty$. The collision operator $I$ only appears through the collision frequency $\nu(v)$ which acts as a damping factor on the amplitude of the initial disturbance. In the linearized Boltzmann equation, the collision frequency describes the damping due to the collisions between the 'disturbed molecules' and the surroundings which are assumed to be in equilibrium. We see that this effect is dominant if we consider a strongly inhomogeneous disturbance in space ($k$ large). On the other hand, it is well-known that for smooth initial disturbances ($k$ small), the full collision operator $I$ becomes important.

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