Zeitschrift: Helvetica Physica Acta
Band: 47 (1974)
Heft: 6

Artikel: Is there a Euclidean field theory for Fermions?
Autor: Fröhlich, Jürg / Osterwalder, Konrad
DOI: https://doi.org/10.5169/seals-114595

Helvetica Physica Acta
Vol. 47, 1974

Birkhäuser Verlag Basel

Is There a Euclidean Field Theory for Fermions?

by **Jürg Fröhlich**[1]) and **Konrad Osterwalder**[2])

Jefferson Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

(1. X. 74)

Abstract. The possibilities of developing a Euclidean *field* theory including Fermions are systematically studied. Without doubling the number of degrees of freedom free Euclidean Fermi fields are constructed in close analogy to the corresponding construction for bosons. Contact fields are introduced such that the Schwinger functions in points of coinciding arguments have a prescribed, regular behaviour. The Grassmann algebra, the von Neumann algebra and the renormalized, Euclidean currents generated by Euclidean Fermi fields are investigated. A gage space for Euclidean Fermi fields is constructed and different versions of the *Markov property* are proven. The relevance of these Euclidean field structures for the construction of local, relativistic, interacting field models involving Fermions is discussed.

I. Introduction

The purpose of this paper is to study systematically the possibilities to develop a Euclidean *field* theory for Fermions. The motivation for such an attempt of course comes from the great success of the Euclidean field theory approach for bosons in the construction of two- and three-dimensional relativistic Bose field theory models. For an account of this success and reference we refer the reader to the 1973 Erice lectures on Constructive Quantum Field Theory [E 1].

It has been shown in Refs. [OS 1, 2, 3] that a Euclidean formulation of a relativistic quantum field theory is always possible in terms of Schwinger functions, and that the relativistic theory can be reconstructed, if the Schwinger functions satisfy a certain set of conditions. In general there is no a priori reason to expect that the Schwinger functions can be defined as the n-point functions of some Euclidean field operators: however this has been shown to be the case for models involving bosons only [Sy 1], [Ne 1], and this additional structure is very useful for the construction of two- and three-dimensional Bose quantum field theory models. In [OS 4] a set of free Euclidean Fermi fields was constructed, whose n-point functions are the Schwinger functions of the free fermion theory and it was shown how approximate Schwinger functions for non-trivial models could be expressed in terms of these free Euclidean fields. However, other than in the bose case, this did not lead immediately to the notion of an interacting Euclidean Fermi field, in other words the notion of a Euclidean Fermi field does not seem to be stable under 'turning on the interaction'. The main purpose of this paper is to investigate several other possibilities of Euclidean structures which might be

[1]) Supported by the National Science Foundation under Grant GP40354X.
[2]) Supported in part by the National Science Foundation under Grant GP40354X.

stable under 'turning on the interaction' – a property we consider important for a structure to be a useful tool for the construction of a model with interaction in terms of free field theory quantities.

In Chapters II and III we construct new Euclidean Fermi fields in close analogy to the corresponding construction in the Bose case. We use the bispinor formalism of van der Waerden [v.W 1, LL 1] to decompose a Dirac spinor ψ into a two-component spinor u and a two-component antispinor v, which from a group theoretical point of view are the fundamental objects. We consider the spinor u as the fundamental object while v is derived from u by the Dirac equation. Following the Bose case analogy, we then construct a Euclidean Fermi field U, corresponding to the relativistic field u, whereas the Euclidean counterpart of v can be expressed in terms of U. The field U turns out to be a bounded operator valued function on the Sobolev space $\mathscr{H}_{-1} \otimes \mathscr{H}_{-1}$ and therefore it makes sense to talk about sharp time fields, see also [Ne 1]. This somewhat surprising regularity property of the Euclidean fields is achieved by a suitable extension of the free Schwinger functions to points of coinciding arguments – actually at the price of making some of them more singular then necessary. Before we can introduce interactions we must find a way to eliminate such additional singularities. In Chapter IV we show how to do that by introducing *contact fields*. If we set these contact fields equal to 0 then it is impossible to define a formal Euclidean action which at the same time determines a renormalizable theory and also respects the physical positivity condition (or can be cutoff in such a way that physical positivity is still satisfied). However renormalizability and physical positivity are considered to be basic throughout this article and should always be satisfied. Introducing contact fields into the formalism obviously means again doubling of the number of degrees of freedom. We conclude therefore that doubling of the number of degrees of freedom cannot be avoided if we want to construct interacting field models, although it is not necessary as long as we are interested in the free field theory only. In Chapter V we study various Euclidean *field* structures and discuss their stability and usefulness in the construction of local, relativistic interacting field models. In Section V.1 we construct a finite regular gage space in the sense of Segal [Se 1] from Euclidean Fermi fields and in Section V.2 we study the Grassmann algebra generated by the Euclidean Fermi fields and prove a new positivity property. For both cases it is easy to define a local structure and to prove a Markov property. In Section V.3 finally we briefly discuss the Euclidean current algebra and the possibility of applying commutative functional integration. We prove that the Euclidean Green's functions of the (ultraviolet cutoff or renormalized) *scalar currents* associated with a free, spin $\frac{1}{2}$ Dirac field are *not* the moments of a (positive) probability measure. This shows that there is no self adjoint Euclidean scalar current! The results of all three sections do not look very encouraging: none of the structures in question seems to yield a natural framework for the construction of interacting field models.

II. The Bispinor Formalism

In the bispinor formalism [v.W 1] [LL 1] the Dirac equation for a free relativistic spin $\frac{1}{2}$ field of mass m takes the form

$$p^{\alpha\dot{\beta}} v_{\dot{\beta}} = m u^{\alpha} \tag{2.1}$$

$$p_{\dot{\beta}\alpha} u^{\alpha} = m v_{\dot{\beta}} \tag{2.2}$$

where

$$\left\|\begin{matrix} p^{1\dot{1}} & p^{1\dot{2}} \\ p^{2\dot{1}} & p^{2\dot{2}} \end{matrix}\right\| = (p^0\sigma_0 - \vec{p}\vec{\sigma}) = p^0\sigma_0 - \sum_{k=1}^{3} p^k\sigma_k = p_0\sigma_0 + \sum_{k=1}^{3} p_k\sigma_k$$

and

$$\left\|\begin{matrix} p_{\dot{1}1} & p_{\dot{1}2} \\ p_{\dot{2}1} & p_{\dot{2}2} \end{matrix}\right\| = (p^0\sigma_0 + \vec{p}\vec{\sigma}),\ p_\mu = i\frac{\partial}{\partial x^\mu},\ \sigma_\mu \text{ the Pauli matrices.}$$

$(u^\alpha, v_{\dot{\beta}})$ is called a *Dirac bispinor*. We call $(u^\alpha, v_{\dot{\beta}})$ a *Majorana bispinor* if it satisfies the Majorana condition

$$u_\alpha = v^*_{\dot{\alpha}}, \tag{2.3}$$

where $v^*_{\dot{\alpha}} \equiv (v_{\dot{\alpha}})^*$, $u_\alpha = \epsilon_{\alpha\beta}u^\beta$ and $\epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\beta} = i\sigma_2$. For a Majorana bispinor the Dirac equation can be written as

$$\begin{aligned} u^{*\dot{\alpha}} &= -m^{-1}\epsilon^{\dot{\alpha}\dot{\gamma}} p_{\dot{\gamma}\delta} u^\delta, \\ &= m^{-1} p^{\gamma\dot{\alpha}} \epsilon_{\gamma\delta} u^\delta = m^{-1} p^{\gamma\dot{\alpha}} u_\gamma \end{aligned} \tag{2.4}$$

where $u^{*\dot{\alpha}} = (u^\alpha)^*$

For $m = 0$ we may consider only one spinor field, u^α say, satisfying the neutrino equation [We 1]

$$p_{\dot{\beta}\alpha} u^\alpha = 0. \tag{2.5}$$

On the Fock space of the fields u^α and $v_{\dot{\beta}}$ there is a unitary representation $\mathbb{U}(A,a)$ of $SL(2,\mathbb{C})$ Ⓢ $\mathbb{R}^4$, determined by

$$\mathbb{U}(A,a)\, u^\alpha(x)\, \mathbb{U}(A,a)^{-1} = (A^{-1})^\alpha_\beta\, u^\beta(\Lambda x + a)$$

$$\mathbb{U}(A,a)\, v_{\dot{\beta}}(x)\, \mathbb{U}(A,a)^{-1} = (A^{-1})^{\dot{\alpha}}_{\dot{\beta}}\, v_{\dot{\alpha}}(\Lambda x + a),$$

where

$$\left\|\begin{matrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{matrix}\right\| = A \in SL(2,\mathbb{C}),\ A^{\dot{\alpha}}_{\dot{\beta}} = \overline{\epsilon_{\beta\gamma} A^\gamma_\delta \epsilon^{\delta\alpha}} \text{ and } \Lambda = \Lambda(A)$$

is determined by $\sum_\mu (\Lambda x)^\mu \sigma_\mu = A(\sum_\mu x^\mu \sigma_\mu) A^*$.

For a more complete account of the bispinor formalism the reader may consult e.g. Ref. [LL 1], the convention of which we adopt here.

In spite of the symmetric role u^α and $v_{\dot{\beta}}$ play in the Dirac equations (2.1) and (2.2) we will treat them differently in our construction of Euclidean fields. We consider u^α as the fundamental field. It is the solution of the initial value problem

$$\begin{aligned} (\Box + m^2)u^\alpha &= 0, \\ u^\alpha(0,\vec{x}) &= u^{0\alpha}(\vec{x}) \\ (p_{\dot{\beta}\alpha} u^\alpha)(0,\vec{x}) &= m v^0_{\dot{\beta}}(\vec{x}). \end{aligned}$$

The antispinor field v_β is then obtained from u^α by the Dirac equation (2.2). Notice that in this way we obtain a situation similar to the Bose case: the spinor u^α plays the role of the Bose field φ while v_β replaces the conjugate momentum π. Following this analogy, we will construct a Euclidean field U^α corresponding to the relativistic field u_α, while the Euclidean counterpart of v_β is again a derived quantity.

The vacuum expectation value of $u^{*\alpha}(x)\,u^\beta(y)$ – i.e. the free two point Wightman distribution for $u^{*\alpha}$, u^β – is given by

$$\mathfrak{W}^{\dot\alpha\beta}(x,y) \equiv \mathfrak{W}^{\dot\alpha\beta}_M(x,y) \equiv \langle u^{*\dot\alpha}(x)\,u^\beta(y)\rangle$$

$$= m^{-1}\,p^{\beta\dot\alpha}(2\pi)^{-3}\int \frac{d^3p}{2\omega(\vec p)}\exp\{-i[\omega(\vec p)\,(x^0-y^0)-\vec p(\vec x-\vec y)]\} \tag{2.6}$$

where $\omega(\vec p) = (\vec p^{\,2}+m^2)^{1/2}$.

The Dirac equation (2.2) and the Majorana condition (2.3) yield

$$\mathfrak{W}^{\alpha\beta}_M(x,y) \equiv \langle u^\alpha(x)\,u^\beta(y)\rangle$$

$$= m^{-1}\,p^{\alpha\dot\gamma}\,\epsilon_{\dot\gamma\dot\delta}\langle u^{*\dot\delta}(x)\,u^\beta(y)\rangle$$

$$= m^{-2}\,p^{\alpha\dot\gamma}\,\epsilon_{\dot\gamma\dot\delta}\,p^{\beta\dot\delta}(2\pi)^{-3}\int\frac{d^3p}{2\omega(\vec p)}\exp\{-i[\omega(\vec p)\,(x^0-y^0)-\vec p(\vec x-\vec y)]\}$$

$$= \epsilon^{\alpha\beta}(2\pi)^{-3}\int\frac{d^3p}{2\omega(\vec p)}\exp\{-i[\omega(\vec p)\,(x^0-y^0)-\vec p(\vec x-\vec y)]\} \tag{2.7}$$

as the two point function of the spinor field u^α of a *Majorana* bispinor (u^α, v_β). Therefore

$$\mathfrak{W}^{\dot\alpha\dot\beta}_M(x,y) \equiv \langle u^{*\dot\alpha}(x)\,u^{*\dot\beta}(y)\rangle$$

$$\equiv \overline{\langle u^\beta(y)\,u^\alpha(x)\rangle}$$

$$= \epsilon^{\beta\alpha}(2\pi)^{-3}\int\frac{d^3p}{2\omega(\vec p)}\exp\{-i[\omega(\vec p)\,(x^0-y^0)-\vec p(\vec x-\vec y)]\}$$

$$= -\epsilon^{\dot\alpha\dot\beta}(2\pi)^{-3}\int\frac{d^3p}{2\omega(\vec p)}\exp\{-i[\omega(\vec p)\,(x^0-y^0)-\vec p(\vec x-\vec y)]\} \tag{2.8}$$

To get the Wightman distributions for a Dirac bispinor $(u^\alpha, v_{\dot\beta})$ we write it as

$$u^\alpha = \frac{1}{\sqrt{2}}\,(u_1^\alpha + iu_2^\alpha)$$

$$v_{\dot\beta} = \frac{1}{\sqrt{2}}\,(v_{1\dot\beta} + iv_{2\dot\beta}) \tag{2.9}$$

where $(u_1^\alpha, v_{1\dot\beta})$ and $(u_2^\alpha, v_{2\dot\beta})$ are Majorana bispinors determined by

$$u_1^\alpha = \frac{1}{\sqrt{2}}\left(u^\alpha + \frac{1}{m}\,\epsilon^{\alpha\beta}\,p_{\dot\gamma\beta}\,u^{*\dot\gamma}\right),$$

$$u_2^\alpha = \frac{1}{\sqrt{2}i}\left(u^\alpha - \frac{1}{m}\,\epsilon^{\alpha\beta}\,p_{\dot\gamma\beta}\,u^{*\dot\gamma}\right),$$

and equation (2.3). It follows that

$$\mathfrak{W}_D^{\dot{\alpha}\beta}(x,y) = \langle u^{*\dot{\alpha}}(x)\, u^{\beta}(y)\rangle = \mathfrak{W}_M^{\dot{\alpha}\beta},$$
$$\langle u^{\alpha}(x)\, u^{\beta}(y)\rangle = \langle u^{*\dot{\alpha}}(x)\, u^{*\dot{\beta}}(y)\rangle = 0. \tag{2.10}$$

For neutrinos we have instead of (2.10)

$$\mathfrak{W}_n^{\dot{\alpha}\beta}(x,y) = p^{\beta\dot{\alpha}}(2\pi)^{-3}\int \frac{d^3 p}{|\vec{p}|}\exp\{-i[|\vec{p}|(x^0-y^0)-\vec{p}(\vec{x}-\vec{y})]\}. \tag{2.11}$$

The $n+m$ point Wightman functions will be denoted by

$$\mathfrak{W}^{\dot{\alpha}_1\ldots\dot{\alpha}_n\beta_1\ldots\beta_m}(x_1\ldots x_n, y_1\ldots y_m) = \langle u^{*\dot{\alpha}_1}(x_1)\ldots u^{*\dot{\alpha}_n}(x_n)\, u^{\beta_1}(y_1)\ldots u^{\beta_m}(y_m)\rangle$$

and in the case of free fields they are given by sums of products of two point functions.

Now we easily obtain the Schwinger functions $\mathfrak{S}^{\dot{\alpha}_1\ldots\dot{\alpha}_n\beta_1\ldots\beta_m}$ corresponding to the Wightman distributions $\mathfrak{W}^{\dot{\alpha}_1\ldots\dot{\alpha}_n\beta_1\ldots\beta_m}$ by first computing the Wightman functions corresponding to these distributions and then restricting them to Euclidean points of non-coinciding arguments, see e.g. [OS 1].

In particular, from (2.6–11) we get for $x \neq y$

$$\mathfrak{S}_M^{\dot{\alpha}\beta}(x,y) = \mathfrak{S}_D^{\dot{\alpha}\beta}(x,y) = m^{-1}\, P_x^{\beta\dot{\alpha}}\, S(x-y) = -m^{-1}\, P_x^{\gamma\dot{\alpha}}\, \epsilon_{\gamma\delta}\, \mathfrak{S}_M^{\delta\beta}(x,y),$$

where

$$\mathfrak{S}_M^{\alpha\beta}(x,y) = \epsilon^{\alpha\beta}\, S(x-y),$$
$$\mathfrak{S}_M^{\dot{\alpha}\dot{\beta}}(x,y) = -\epsilon^{\dot{\alpha}\dot{\beta}}\, S(x-y) = (m^{-1}\, P_x^{\gamma\dot{\alpha}}\, \epsilon_{\gamma\delta})\,(m^{-1}\, P_y^{\rho\dot{\beta}}\, \epsilon_{\rho\sigma})\, \mathfrak{S}_M^{\delta\sigma}(x,y)$$
$$\mathfrak{S}_n^{\dot{\alpha}\beta}(x,y) = P_x^{\beta\dot{\alpha}}\, S_0(x-y)$$

where

$$S(x-y) = (2\pi)^{-4}\int \frac{d^4 p}{p^2+m^2}\, e^{i(p,x-y)}$$

is the kernel of the operator $(-\Delta + m^2)^{-1}$, S_0 is the same as S but with m set equal to zero, and

$$\|P^{\alpha\dot{\beta}}\| = ip_0\,\sigma_0 - \vec{p}\cdot\vec{\sigma},$$
$$\|P_x^{\alpha\dot{\beta}}\| = \frac{\partial}{\partial x^0}\,\sigma_0 + i\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\,\sigma_k.$$

We write p^2 for $p_0^2 + p^2$, and (p,x) is the Euclidean inner product between the two four vectors p and x.

III. Euclidean Fermi Fields

In this chapter we construct Euclidean Fermi fields the n-point functions of which will be the Schwinger functions $\mathfrak{S}^{\dot{\alpha}_1\ldots\dot{\alpha}_n\beta_1\ldots\beta_m}$. The present construction differs in two ways from the approach in Ref. [OS 4]. First we do not have to double the number of degrees of freedom anymore (at least for $m \neq 0$), because we 'Euclideanize' the field u^{α} only but not $v_{\dot{\beta}}$. Second our Euclidean fields are continuous operator valued distributions on the Sobolev space $\mathscr{H}_{-1}$ – not on $\mathscr{H}_{-1/2}$ as in [OS 4]. It is this additional regularity that allows us to define sharp time fields and to prove a Markov property (Chapter V).

III.1. Majorana fields

We want to construct a Euclidean covariant free spinor field U^{α} such that for $x \neq y$

$$\langle U^{\alpha}(x)\, U^{\beta}(y)\rangle = \mathfrak{S}_M^{\alpha\beta}(x,y) = \epsilon^{\alpha\beta} S(x-y). \tag{3.1}$$

Then we define

$$\tilde{U}^{\dot{\alpha}}(x) = -m^{-1} P_x^{\gamma\dot{\alpha}} \epsilon_{\gamma\delta} U^{\delta}(x) \tag{3.2}$$

so that

$$\langle \tilde{U}^{\dot{\alpha}}(x)\, U^{\beta}(y)\rangle = \mathfrak{S}_M^{\dot{\alpha}\beta}(x,y)$$

$$\langle \tilde{U}^{\dot{\alpha}}(x)\, \tilde{U}^{\dot{\beta}}(y)\rangle = \mathfrak{S}_M^{\dot{\alpha}\dot{\beta}}(x,y).$$

The Schwinger functions $\mathfrak{S}^{\dot{\alpha}_1\ldots\dot{\alpha}_n\beta_1\ldots\beta_m}$ are then of course given by

$\langle \tilde{U}^{\dot{\alpha}_1}(x_1)\ldots U^{\dot{\alpha}_n}(x_n)\, U^{\beta_1}(y_1)\ldots U^{\beta_m}(y_m)\rangle$.

The two point function $\langle U^{\alpha}(x) U^{\beta}(y)\rangle$ given by (3.1) is not sufficient to determine the Fock space of the field U^{α} and the field itself. We supplement (3.1) by the following two point function

$$\langle U_{\alpha}^{*}(x)\, U^{\beta}(y)\rangle = \langle U^{\beta}(y)\, U_{\alpha}^{*}(x)\rangle = \delta_{\alpha}^{\beta} S(x-y) \tag{3.3}$$

which is picked somewhat *at random*, and

$$\langle U_{\alpha}^{*}(x)\, U_{\beta}^{*}(y)\rangle = -\epsilon_{\alpha\beta} S(x-y). \tag{3.4}$$

Notice that (3.4) is *consistent* with (3.1).

Assuming that (3.1) and (3.3/4) hold in the sense of distributions in $\mathscr{S}'(\mathbb{R}^4)$, we now may easily construct the Fock space $\mathscr{E}$ of the field U^{α}. We set (still formally)

$$A^{\alpha}(x) = \tfrac{1}{2}(U^{\alpha}(x) + \epsilon^{\alpha\beta}\, U_{\beta}^{*}(x))$$

$$A_{\alpha}^{*}(x) = \tfrac{1}{2}(U_{\alpha}^{*}(x) + \epsilon_{\alpha\beta}\, U^{\beta}(x)). \tag{3.5}$$

Then by (3.1) and (3.3/4) we have

$$\langle A_{\alpha}^{*}(x)\, A_{\beta}^{*}(y)\rangle = \langle A^{\alpha}(x)\, A^{\beta}(y)\rangle = \langle A_{\alpha}^{*}(x)\, A^{\beta}(y)\rangle = 0$$

$$\langle A^{\alpha}(x)\, A_{\beta}^{*}(y)\rangle = \delta_{\alpha}^{\beta} S(x-y). \tag{3.6}$$

Equations (3.6) show that A^{α} is just an annihilation operator. Let

$$\mathscr{H}_{-1}(\mathbb{R}^4) = \{f \mid (-\Delta + m^2)^{-1/2} f \in L_2(\mathbb{R}^4)\}$$

be the Sobolev space of index -1 and let the one particle Hilbert space $\mathscr{E}^{(1)}$ be

$$\mathscr{E}^{(1)} = \mathscr{H}_{-1}(\mathbb{R}^4) \oplus \mathscr{H}_{-1}(\mathbb{R}^4).$$

A vector in $\mathscr{E}^{(1)}$ will be denoted by $\begin{pmatrix} g \\ h \end{pmatrix}$, $g, h \in \mathscr{H}_{-1}(\mathbb{R}^4)$. Then the Euclidean Fermi Fock space $\mathscr{E}$ is defined as usual to be the Hilbert space completion of the alternating tensor algebra over $\mathscr{E}^{(1)}$

$$\mathscr{E} = \mathbb{C} \oplus \mathscr{E}^{(1)} \oplus (\mathscr{E}^{(1)} \otimes_a \mathscr{E}^{(1)}) \oplus (\mathscr{E}^{(1)} \otimes_a \mathscr{E}^{(1)} \otimes_a \mathscr{E}^{(1)}) \oplus \cdots.$$

The vacuum is denoted by Ω. For $f^\alpha \in \mathscr{H}_{-1}(\mathbb{R}^4)$ we now define

$$A^*_\alpha(f^\alpha) \begin{pmatrix} g_1 \\ h_1 \end{pmatrix} \otimes_a \begin{pmatrix} g_2 \\ h_2 \end{pmatrix} \otimes_a \cdots \otimes_a \begin{pmatrix} g_n \\ h_n \end{pmatrix} = \begin{pmatrix} f^1 \\ f^2 \end{pmatrix} \otimes_a \begin{pmatrix} g_1 \\ h_1 \end{pmatrix} \otimes_a \cdots \otimes_a \begin{pmatrix} g_n \\ h_n \end{pmatrix}, \tag{3.7}$$

and extending $A^*_\alpha(f^\alpha)$ by linearity and continuity to all of $\mathscr{E}$ we get a continuous map $A^*_\alpha(\cdot)$ from $\mathscr{H}_{-1}(\mathbb{R}^4) \oplus \mathscr{H}_{-1}(\mathbb{R}^4)$ into the bounded operators on $\mathscr{E}$. Denoting $(\Omega, \cdot\, \Omega)$ by $\langle \cdot \rangle$ we easily verify that A^α and A^*_α satisfy (3.6). Now we obtain U^α by inverting (3.5)

$$U^\alpha(\cdot) = A^\alpha(\cdot) - \epsilon^{\alpha\beta} A^*_\beta(\cdot). \tag{3.8}$$

$U^\alpha(\cdot)$ is again a continuous linear map from $\mathscr{H}_{-1}(\mathbb{R}^4) \oplus \mathscr{H}_{-1}(\mathbb{R}^4)$ into the bounded operators on $\mathscr{E}$, satisfying (3.1) and (3.3/4). Formula (3.8) is of course simply the decomposition of U^α into a creation and an annihilation part.

A more familiar way of defining $U^\alpha(\cdot)$ would be through its momentum space expansion which turns out to be

$$U^\alpha(x) = (2\pi)^{-2} \int \frac{d^4 p}{\sqrt{p^2 + m^2}}\, v^\alpha_\beta(p) \left[e^{-i(p,x)} a^{*\beta}(p) + e^{i(p,x)} \epsilon^{\beta\gamma} a_\gamma(p) \right] \tag{3.9}$$

where $\|v^\alpha_\beta(p)\| = [(1/|p|)(p_0 \sigma_0 + i\vec{p}\vec{\sigma})]^{1/2} \in SU(2)$ and a_α, $a^{*\beta} \equiv (a_\beta)^*$ are momentum space annihilation and creation operators with a proper transformation law under the Euclidean group.

Finally we define the representation $\mathbb{U}(u_1, u_2, a)$, $u_1, u_2 \in SU(2)$, $a \in \mathbb{R}^4$ of $\overline{iSO_4} = SU(2) \times SU(2)$ Ⓢ $\mathbb{R}^4$ on $\mathscr{E}$ by

$$\mathbb{U}(u_1, u_2, a)\, \Omega = \Omega \tag{3.10}$$

$$\mathbb{U}(u_1, u_2, a)\, A^*_\alpha(x)\, \mathbb{U}(u_1, u_2, a)^{-1} = u^{T\beta}_{1\alpha}\, A^*_\beta(R(u_1, u_2)\, x + a) \tag{3.11}$$

$R(u_1, u_2) \in SO_4$ is defined by $R(u_1, u_2)\, x = x'$ where $(-ix'_0 \sigma_0 + \vec{x}'\sigma) = u_1(-ix_0 \sigma_0 + \vec{x}\vec{\sigma})\, u^T_2$, u^T_k being the transpose of u_k. Notice that (3.10/11) defines $\mathbb{U}$ completely because Ω is cyclic with respect to the polynomial ring generated by A^*_α. Unitarity of $\mathbb{U}$ follows from (3.10/11) and (3.7). Using $\epsilon \bar{u}_1 \epsilon^{-1} = u_1$, $u_1 \in SU(2)$, we find for U^α

$$\mathbb{U}(u_1, u_2, a)\, U^\alpha(x)\, \mathbb{U}(u_1, u_2, a)^{-1} = (u^{-1}_1)^\alpha_\beta\, U^\beta(x' + a) \tag{3.12}$$

and for $\tilde{U}^{\dot\alpha}(x) \equiv -m^{-1} P^{\gamma\dot\alpha}_x \epsilon_{\gamma\delta} U^\delta(x)$:

$$\begin{aligned} \mathbb{U}(u_1, u_2, a)\, \tilde{U}^{\dot\alpha}(x)\, \mathbb{U}(u_1, u_2, a) &= -m^{-1} P^{\gamma\dot\alpha}_x \epsilon_{\gamma\delta} (u^{-1}_1)^\delta_\rho\, U^\rho(x' + a) \\ &= (u^{-1}_2)^{\dot\alpha}_{\dot\beta}\, m^{-1} P^{\gamma\dot\beta}_{x'} \epsilon_{\gamma\delta}\, U^\delta(x' + a) \\ &= (u^{-1}_2)^{\dot\alpha}_{\dot\beta}\, \tilde{U}^{\dot\beta}(x' + a) \end{aligned} \tag{3.13}$$

where we have used $P^{\alpha\dot\beta}_x = (u^{-1}_1)^\alpha_\gamma\, P^{\gamma\dot\delta}_{x'} (\bar{u}_2)^{\dot\beta}_{\dot\delta}$, $(\bar{u}_2)^{\dot\beta}_{\dot\delta} = \epsilon_{\dot\delta\dot\sigma} u^{\dot\sigma}_{2\dot\tau} \epsilon^{\dot\tau\dot\beta}$.

The following theorem summarizes the results of this section.

Theorem 3.1: There exists a free, Euclidean covariant Fermi field U^α such that

1) $\{U^\alpha(x), U^\beta(y)\} = 0, \quad \{U^\alpha(x); U^*_\beta(y)\} = 2\delta^\alpha_\beta S(x-y).$

2) $\|U^\alpha(f_\alpha)\| \leqslant \sqrt{2\sum_\alpha (f_\alpha, (-\Delta + m^2)^{-1} f_\alpha)_{L_2}} \equiv 2\|f\|_{-1}$, for $f_\alpha \in \mathscr{H}_{-1}(\mathbb{R}^4)$.

3) The Schwinger functions $\mathfrak{S}_M^{\dot\alpha_1\ldots\dot\alpha_n\beta_1\ldots\beta_m}(x_1\ldots x_n y_1\ldots y_m)$ of a Majorana bispinor are equal to

$$\langle \tilde U^{\dot\alpha_1}(x_1)\ldots \tilde U^{\dot\alpha_n}(x_n) U^{\beta_1}(y_1)\ldots U^{\beta_m}(y_m)\rangle,$$

where $\tilde U^{\dot\beta}(x) = -m^{-1} P_x^{\alpha\dot\beta} \epsilon_{\alpha\gamma} U^\gamma(x)$.

III.2. Dirac fields

The construction of a Euclidean Dirac field is straightforward if we write the relativistic spinors as sums of two Majorana spinors as in (2.9) and use the results of Section III.1. We construct two Euclidean Majorana spinors U_1^α and U_2^α and their Fock space $\mathscr{E}$, such that

$$\{U_1^\alpha(x), U_2^\beta(y)\} = \{U_1^\alpha(x), U^*_{2\beta}(y)\} = 0$$
$$\langle U_1^\alpha(x)\, U_2^\beta(y)\rangle = \langle U_1^\alpha(x)\, U^*_{2\beta}(y)\rangle = 0.$$

Then we define the Euclidean covariant fields U and W by

$$U^\alpha(x) = \tfrac{1}{2}(U_1^\alpha(x) + iU_2^\alpha(x))$$
$$W^\alpha(x) = \tfrac{1}{2}(U_1^\alpha(x) - iU_2^\alpha(x)). \tag{3.14}$$

Obviously

$$\langle U^\alpha(x)\, U^\beta(y)\rangle = \langle W^\alpha(x)\, W^\beta(y)\rangle = 0$$
$$\langle W^\alpha(x)\, U^\beta(y)\rangle = \epsilon^{\alpha\beta} S(x-y).$$

The following theorem is now an immediate consequence of Theorem 3.1.

Theorem 3.2: There exist free Euclidean covariant Fermi fields U^α and W^α such that

1) $\{U^\alpha(x), U^\beta(y)\} = \{U^\alpha(x), W^\beta(y)\} = \{W^\alpha(x), W^\beta(y)\} = 0$

$\{U^\alpha(x), W^*_\beta(y)\} = 0$

$\{U^\alpha(x), U^*_\beta(y)\} = \{W^\alpha(x), W^*_\beta(y)\} = 2\delta^\alpha_\beta S(x-y).$

2) $\|U^\alpha(f_\alpha)\| \leqslant 2\|f\|_{-1}, \|W^\alpha(f_\alpha)\| \leqslant 2\|f\|_{-1}.$

3) The Schwinger functions $\mathfrak{S}_D^{\dot\alpha_1\ldots\dot\alpha_n\beta_1\ldots\beta_m}(x_1\ldots x_n y_1\ldots y_m)$ of a Dirac bispinor are equal to

$$\langle \tilde U^{\dot\alpha_1}(x_1)\ldots \tilde U^{\dot\alpha_n}(x_n)\, U^{\beta_1}(y_1)\ldots U^{\beta_m}(y_m)\rangle$$

where $\tilde U^{\dot\beta}(x) = -m^{-1} P_x^{\alpha\dot\beta} \epsilon_{\alpha\gamma} W^\gamma(x)$.

III.3. Neutrino fields

We construct Euclidean neutrino fields U^α and W^α by the methods of the previous section. We only have to replace S by S_0, $\mathscr{H}_{-1}$ by $\mathscr{H}^0_{-1} = \{f \mid |k|^{-1}\tilde{f}(k) \in L_2(\mathbb{R}^4)\}$ and we define $\tilde{U}^{\dot\beta}(x)$ to be $-P_x^{\alpha\dot\beta}\epsilon_{\alpha\gamma}W^\delta(x)$.

There is however the new feature that the Euclidean neutrino field has twice as many degrees of freedom as the relativistic neutrino field (which involved only half as many degrees of freedom as the corresponding massive Dirac field). This *cannot* be avoided by just modifying the construction of the Euclidean neutrino field.

IV. The Points of Coinciding Arguments

The Bargmann–Hall–Wightman construction of the analytic continuation of the Wightman functions $\mathfrak{W}(z_1,\ldots z_n)$ leads to a domain of analyticity which does *not* contain Euclidean points of coinciding arguments, i.e. points of the form $(z_1,\ldots z_n)$, $z_i = (ix_i^0, \vec{x}_i)$, x_i^μ real and $z_i = z_j$ for some $i \neq j$. This implies that the Schwinger functions of a Wightman theory are not defined if any two of its arguments coincide. However if we want the Schwinger functions $\mathfrak{S}(x_1,\ldots x_n)$ to be the n-point functions of Euclidean fields, then we must give them some meaning in the points of coinciding arguments as well. This can be achieved by defining distributions $\mathfrak{S}_n \in \mathscr{S}'(\mathbb{R}^{4n})$ such that

$$\mathfrak{S}_n(f) = \int \mathfrak{S}_n(x_1\ldots x_n) f(x_1\ldots x_n)\, d^{4n}x$$

if $f \in \mathscr{S}(\mathbb{R}^{4n})$ and f together with all its derivatives vanishes in points of coinciding arguments, i.e. for $x_i = x_j$, some $i \neq j$. We call the distribution $\mathfrak{S}_n$ an 'extension of the Schwinger function $\mathfrak{S}_n(x_1,\ldots x_n)$ to points of coinciding arguments'. Such extensions always exist, see e.g. [OS 1].

The indeterminacy of the Schwinger functions in points of coinciding arguments was apparent in Chapter III already. We started from the relation (3.1)

$$\mathfrak{S}_M^{\alpha\beta}(x,y) = \epsilon^{\alpha\beta} S(x-y)$$

which was true for $x \neq y$. We then extended $\mathfrak{S}_M^{\alpha\beta}$ to points where $x = y$ by requiring that (3.1) should hold everywhere, i.e. in the sense of distributions in $\mathscr{S}'(\mathbb{R}^8)$. This is a very natural extension, because it leads to the best possible regularity properties for the field U^α: It is with this extension only that U^α can be made $\mathscr{H}_{-1}$ continuous.

The extension given by (3.1) and the definition (3.2) of $\tilde{U}^{\dot\alpha}$ lead automatically to an extension of $\mathfrak{S}_M^{\dot\alpha\beta}$ and of $\mathfrak{S}_M^{\dot\alpha\dot\beta}$. In particular

$$\begin{aligned}
\mathfrak{S}_M^{\dot\alpha\dot\beta}(x,y) &= \langle \tilde{U}^{\dot\alpha}(x)\,\tilde{U}_M^{\dot\beta}(y)\rangle \\
&= (m^{-1}P_x^{\gamma\dot\alpha}\epsilon_{\gamma\delta})\,(m^{-1}P_y^{\rho\dot\beta}\epsilon_{\rho\sigma})\,\langle U^\delta(x)\,U^\sigma(y)\rangle \\
&= -m^{-2}P_x^{\gamma\dot\alpha}\epsilon_{\gamma\delta}P_x^{\rho\dot\beta}\epsilon_{\rho\sigma}\epsilon^{\delta\sigma}S(x-y) \\
&= \epsilon^{\dot\alpha\dot\beta}\left(\frac{p^2}{m^2}\right)S(x-y) \\
&= -\epsilon^{\dot\alpha\dot\beta}[S(x-y) - m^{-2}\delta^4(x-y)]. \qquad (4.1)
\end{aligned}$$

This extension of $\mathfrak{S}_M^{\dot\alpha\dot\beta}$ is not the most regular one. The additional δ-function in (4.1) implies that the fields $\tilde{U}^{\dot\alpha}$ are *not* $\mathscr{H}_{-1}$ continuous – a fact which of course already follows from definition (3.2).

IV.1. Contact fields

As long as we consider the free theory only, we are of course free to choose as an extension of the Schwinger functions whichever suits best our purposes. This is no longer true if we want to write down approximate Schwinger functions for interacting models in terms of free Euclidean fields. Indeed, the δ-function in (4.1) would turn the most innocent looking interactions into non-renormalizable theories, which at the present stage of technology is not necessarily a very helpful feature. Fortunately there is a very simple trick by which we can get rid of the unwanted δ-function and thus come back to the most regular extension of $\mathfrak{S}^{\dot{\alpha}\dot{\beta}}$, without losing the Euclidean field formalism.

The trick is to introduce a *contact field* $\eta^{\dot{\alpha}}$, which is a free Fermi field with the following properties (Majorana case)

$$\langle \eta^*_{\dot{\alpha}}(x)\, \eta^{\dot{\beta}}(y)\rangle = \langle \eta^{\dot{\beta}}(y)\, \eta^*_{\dot{\alpha}}(x)\rangle = \delta^{\dot{\beta}}_{\dot{\alpha}}\, \delta^4(x-y)$$

$$\langle \eta^{\dot{\alpha}}(x)\, \eta^{\dot{\beta}}(y)\rangle = -\epsilon^{\dot{\alpha}\dot{\beta}}\, \delta^4(x-y)\,;\; \langle \eta^*_{\dot{\alpha}}(x)\, \eta^*_{\dot{\beta}}(y)\rangle = \epsilon_{\dot{\alpha}\dot{\beta}}\, \delta^4(x-y)$$

$$\{\eta^*_{\dot{\alpha}}(x), \eta^{\dot{\beta}}(y)\} = 2\delta^{\dot{\beta}}_{\dot{\alpha}}\, \delta^4(x-y), \tag{4.2}$$

all other anticommutators, in particular those with U^α and U^*_α, being zero. Also the two point functions of an η field with a U^α or U^*_α should be zero.

The explicit construction of the fields $\eta^{\dot{\alpha}}$ follows the lines of Section III.1; however, there are some important changes in signs. Again we introduce 'annihilation and creation operators' $B^{\dot{\alpha}}$ and $B^*_{\dot{\alpha}} \equiv (B^{\dot{\alpha}})^*$ by setting

$$B^{\dot{\alpha}}(x) = \tfrac{1}{2}(\eta^{\dot{\alpha}}(x) - \epsilon^{\dot{\alpha}\dot{\beta}}\, \eta^*_{\dot{\beta}}(x)). \tag{4.3}$$

Then by (4.2)

$$\langle B^*_{\dot{\alpha}}(x)\, B^*_{\dot{\beta}}(y)\rangle = \langle B^{\dot{\alpha}}(x)\, B^{\dot{\beta}}(y)\rangle = \langle B^*_{\dot{\alpha}}(x)\, B^{\dot{\beta}}(y)\rangle = 0$$

$$\langle B^{\dot{\alpha}}(x)\, B^*_{\dot{\beta}}(y)\rangle = \delta^{\dot{\alpha}}_{\dot{\beta}}\, \delta^4(x-y).$$

The inversion of (4.3) is

$$\eta^{\dot{\alpha}}(x) = B^{\dot{\alpha}}(x) + \epsilon^{\dot{\alpha}\dot{\beta}}\, B^*_{\dot{\beta}}(x). \tag{4.4}$$

We skip the rest of the construction which is standard. We only mention that

$$B^{\dot{\alpha}} : f_{\dot{\alpha}} \mapsto B^{\dot{\alpha}}(f_{\dot{\alpha}}) \equiv \int B^{\dot{\alpha}}(x) f_{\dot{\alpha}}(x)\, d^4x$$

is a continuous map from $L_2(\mathbb{R}^4) \oplus L_2(\mathbb{R}^4)$ into the bounded operators on $\mathscr{E}$, where $\mathscr{E}$ is now the suitably enlarged Fock space of the fields U^α and $\eta^{\dot{\alpha}}$.

If now we replace $\tilde{U}^{\dot{\alpha}}$ by the field

$$\begin{aligned}\hat{U}^{\dot{\alpha}}(x) &\equiv \tilde{U}^{\dot{\alpha}}(x) - m^{-1}\, \eta^{\dot{\alpha}}(x)\\ &= -m^{-1}(P_x^{\beta\dot{\alpha}}\, \epsilon_{\beta\gamma}\, U^\gamma(x) + \eta^{\dot{\alpha}}(x))\end{aligned} \tag{4.5}$$

then we find by (4.1/2)

$$\langle \hat{U}^{\dot\alpha}(x)\, \hat{U}^{\dot\beta}(y)\rangle = \langle \tilde{U}^{\dot\alpha}(x)\, \tilde{U}^{\dot\beta}(y)\rangle + m^{-2}\langle \eta^{\dot\alpha}(x)\, \eta^{\dot\beta}(y)\rangle$$
$$= -\epsilon^{\dot\alpha\dot\beta} S(x-y). \tag{4.6}$$

Hence the two point function of $\hat{U}$ is again equal to $\mathfrak{S}_M^{\dot\alpha\beta}(x,y)$ for $x \neq y$, and (4.6) defines an extension of $\mathfrak{S}_M^{\dot\alpha\beta}$ to points of coinciding arguments which is more regular than the one defined by (4.1): the unwanted δ-function has disappeared due to the contact field.

It has to be pointed out that the fields $\hat{U}^{\dot\alpha}$ are by no means less singular than the fields $\tilde{U}^{\dot\alpha}$: Introducing contact fields will never improve the singularities of $\langle \tilde{U}^{\dot\alpha}(x)\, \tilde{U}^*_{\dot\beta}(y)\rangle$ unless the contact fields destroy the positivity of the metric in the Hilbert space $\mathscr{E}$. However, as the expressions for the Schwinger functions never involve the fields $\tilde{U}^*$, it suffices for us to have no δ-function in (4.6).

The same trick also works in the case of a Dirac field. We write the Euclidean Dirac spinors U^α and W^α as linear combinations of two Euclidean Majorana spinors, see (3.14). Hence, to eliminate the δ-functions from the two point functions of

$$\tilde{U}^{\dot\alpha} = -m^{-1} P^{\beta\dot\alpha} \epsilon_{\beta\gamma} W^\gamma$$

and

$$\tilde{W}^{\dot\alpha} = -m^{-1} P^{\beta\dot\alpha} \epsilon_{\beta\gamma} U^\gamma$$

we have to introduce two contact fields $\eta_1^{\dot\alpha}$ and $\eta_2^{\dot\alpha}$ (anticommuting, mixed two point functions equal to zero, otherwise according to (4.2)). Then we define

$$\eta^{\dot\alpha}(x) = \frac{1}{\sqrt{2}}(\eta_1^{\dot\alpha}(x) + i\eta_2^{\dot\alpha}(x)),$$

$$\xi^{\dot\alpha}(x) = \frac{1}{\sqrt{2}}(\eta_1^{\dot\alpha}(x) - i\eta_2^{\dot\alpha}(x)) \tag{4.7}$$

and

$$\hat{U}^{\dot\alpha}(x) = \tilde{U}^{\dot\alpha}(x) - m^{-1}\eta^{\dot\alpha}(x)$$
$$\hat{W}^{\dot\alpha}(x) = \tilde{W}^{\dot\alpha}(x) - m^{-1}\xi^{\dot\alpha}(x). \tag{4.8}$$

It is straightforward to check that

$$\langle \hat{U}^{\dot\alpha}(x)\, \hat{U}^{\dot\beta}(y)\rangle = \langle \hat{W}^{\dot\alpha}(x)\, \hat{W}^{\dot\beta}(y)\rangle = 0$$

$$\langle \hat{U}^{\dot\alpha}(x)\, \hat{W}^{\dot\beta}(y)\rangle = -\epsilon^{\dot\alpha\dot\beta} S(x-y).$$

IV.2. Relation to earlier work

The Schwinger function corresponding to the Wightman distribution

$$\langle u^{*\dot\alpha_1}(x_1) \ldots u^{*\dot\alpha_n}(x_n)\, u^{\beta_1}(y_1) \ldots u^{\beta_m}(y_m)\rangle$$

can be written as

$$\mathfrak{S}_M^{\dot\alpha_1\ldots\beta_m} = \langle \hat{U}^{\dot\alpha_1}(x_1) \ldots \hat{U}^{\dot\alpha_n}(x_n)\, U^{\beta_1}(y_1) \ldots U^{\beta_m}(y_m)\rangle$$

(Majorana case). In that sense we can say that the Euclidean field U^β corresponds to the relativistic field u^β and $\hat{U}^{\dot\alpha}$ corresponds to $u^{*\dot\alpha}$.

Writing the Dirac spinors as linear combinations of two independent Majorana spinors, (2.8), (3.14) and (4.7), we find the following correspondence for Dirac spinors

$$U^\alpha = 2^{-1/2}(U_1^\alpha + iU_2^\alpha) \leftrightarrow 2^{-1/2}(u_1^\alpha + iu_2^\alpha) = u^\alpha$$

$$W_\alpha = 2^{-1/2}\,\epsilon_{\alpha\beta}(U_1^\beta - iU_2^\beta) \leftrightarrow 2^{-1/2}\epsilon_{\alpha\beta}(u_1^\beta - iu_2^\beta) = v_\alpha^*$$

$$\hat{W}_{\dot\alpha} = 2^{-1/2}\,\epsilon_{\dot\alpha\dot\beta}(\hat{U}_1^{\dot\beta} + i\hat{U}_2^{\dot\beta}) \leftrightarrow 2^{-1/2}\,\epsilon_{\dot\alpha\dot\beta}(u_1^{*\dot\beta} + iu_2^{*\dot\beta}) = v_{\dot\alpha}$$

$$\hat{U}^{\dot\alpha} = 2^{-1/2}(\hat{U}_1^{\dot\alpha} - i\hat{U}_2^{\dot\alpha}) \leftrightarrow 2^{-1/2}(u_1^{*\dot\alpha} - iu_2^{*\dot\alpha}) = u^{*\dot\alpha}. \tag{4.9}$$

To establish the connection with the work in Ref. [OS 4] we write the Dirac spinor as

$$\psi = m^{1/2}\begin{pmatrix} u \\ v \end{pmatrix} \text{ and } \psi^* = m^{1/2}(u^*, v^*),$$

where $u = u^\alpha$, $v = v_{\dot\alpha}$. With γ-matrices defined by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \vec{\gamma} = 0\begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},$$

the Dirac equations (2.1/2) take the usual form $(\gamma^\mu p_\mu + m)\psi = 0$, see [LL 1], §21. According to (4.9) the Euclidean counterparts Ψ^1 and Ψ^2 of ψ and $\psi^+ = \psi^*\gamma^0$ are

$$\Psi^1 = m^{1/2}\begin{pmatrix} U \\ \hat{W} \end{pmatrix} \text{ and } \Psi^2 = m^{1/2}\begin{pmatrix} W \\ \hat{U} \end{pmatrix}, \tag{4.10}$$

where $U = U^\alpha$, $W = W_\alpha$, $\hat{W} = \hat{W}_{\dot\alpha}$, $\hat{U} = \hat{U}^{\dot\alpha}$. For the two point function we find

$$\langle \Psi_\alpha^1(x)\Psi_\beta^2(y)\rangle = (m + ip_0\gamma^0 + \vec{p}\vec{\gamma})_{\alpha\beta}S(x-y)$$

in agreement with the Schwinger function for Ψ^1 and Ψ^2 constructed in [OS 4]. Notice however that the other two point functions such as $\langle\Psi_\alpha^1(x)\Psi_\alpha^{1*}(y)\rangle$ are *not* the same here and in [OS 4]. We now understand where the need for doubling the number of degrees of freedom comes from: If we do not care for behaviour of the Schwinger functions at points of coinciding arguments, then *no* additional degrees of freedom are necessary. However, if we want the Schwinger functions to behave as nicely as possible if two arguments coincide – and we must require that if we want to consider interactions – then we have to introduce contact fields to eliminate unwanted δ-functions. This amounts to a doubling of the number of degrees of freedom.

IV.3. Physical positivity

It was shown in [OS 1], chapter 6, that Schwinger functions of a Wightman quantum field theory satisfy a certain positivity property (axiom (E2)), which we call here 'physical positivity', because it is an immediate consequence of the fact that the Hilbert space of physical states has a positive metric. (Schwinger functions sometimes have other, additional positivity properties, see e.g. [Sy 1], [Ne 2] and Chapter V of this paper.) Because physical positivity will be crucial in the next chapter when we introduce the currents as building stones for theories with local interactions, we briefly explain how it is formulated best in the context of our Euclidean field operators.

Let the field algebra $\hat{\mathfrak{J}}$ be the algebra generated by the operators

$$\{U^{\alpha}(f_{\alpha}), W_{\alpha}(f^{\alpha}), \hat{W}_{\dot{\alpha}}(f^{\dot{\alpha}}), \hat{U}^{\dot{\alpha}}(f_{\dot{\alpha}});\, f^{\cdots}_{\cdots} \in \mathscr{S}(\mathbb{R}^4)\}. \tag{4.11}$$

Let $\hat{\mathfrak{J}}_+$ be the subalgebra of $\hat{\mathfrak{J}}$ generated by

$$\{U^{\alpha}(f_{\alpha}), W_{\alpha}(f^{\alpha}), \hat{W}_{\alpha}(f^{\alpha}), \hat{U}^{\alpha}(f_{\alpha});\, f^{\cdots}_{\cdots} \in \mathscr{S}(\mathbb{R}^4),$$
$$\operatorname{supp} f^{\cdots}_{\cdots} \subset \mathbb{R}^3 x[0, \infty)\}. \tag{4.12}$$

On $\hat{\mathfrak{J}}$ we define an antiautomorphism by

$$\vartheta(U^{\alpha}(f_{\alpha})) = \hat{U}^{\dot{\alpha}}(f_{\vartheta\dot{\alpha}});\quad \vartheta(W_{\alpha}(f^{\alpha})) = \hat{W}_{\dot{\alpha}}(f^{\dot{\alpha}}_{\vartheta})$$
$$\vartheta(MN) = \vartheta(N)\,\vartheta(M);\, \vartheta(\vartheta(M)) = M \tag{4.13}$$

for all $M, N \in \hat{\mathfrak{J}}, f \in \mathscr{S}(\mathbb{R}^4), f_{\vartheta\dot{\alpha}}(x^0, \vec{x}) = \bar{f}_{\alpha}(-x^0, \vec{x}); f^{\dot{\alpha}}_{\vartheta}(x^0, \vec{x}) = \bar{f}^{\alpha}(-x^0, \vec{x})$.

A linear funcational $L(\cdot)$ on $\hat{\mathfrak{J}}$ is said to have the physical positivity property, if

$$L(\vartheta(M)\, M) \geqslant 0, \text{ for all } M \in \hat{\mathfrak{J}}_+. \tag{4.14}$$

The vacuum functional e.g. $\langle\cdot\rangle$ has this property, see [OS 4]

$$\langle\vartheta(M)\, M\rangle \geqslant 0, \text{ for all } M \in \hat{\mathfrak{J}}_+.$$

As in Ref. [OS 4], it is easy to see that there is a unitary involution θ on $\mathscr{E}$ such that for any $M \in \hat{\mathfrak{J}}$

$$\vartheta(M) = \theta M^*\, \theta^{-1}. \tag{4.15}$$

θ is defined by

$$\theta\Omega = \Omega$$
$$\theta A^*_{\beta}(x)\, \theta^{-1} = A^{*\dot{\beta}}_{\vartheta}(x)$$
$$= -m^{-1}[P^{\gamma\dot{\beta}}\, A^*_{\gamma}(-x^0, \vec{x}) + \epsilon^{\dot{\beta}\dot{\gamma}}\, B^*_{\dot{\gamma}}(-x^0, \vec{x})]$$
$$\theta B^*_{\dot{\alpha}}(x)\, \theta^{-1} = B^{*\alpha}_{\vartheta}(x)$$
$$= m^{-1}[-(p^2 + m^2)\, \epsilon^{\alpha\beta}\, A^*_{\beta}(-x^0, \vec{x}) + P^{\alpha\dot{\beta}}\, B^*_{\dot{\beta}}(-x^0, \vec{x})]. \tag{4.16}$$

We leave it to the reader to verify that (4.16) defines a unitary involution θ on $\mathscr{E}$ and that θ satisfies (4.15). Notice that from the definitions (4.10) and (4.13) we obtain

$$\vartheta(\Psi^1_{\alpha}(f^{\alpha})) = \Psi^2_{\dot{\alpha}}((\gamma^0 f)^{\dot{\alpha}}_{\vartheta}), \tag{4.17}$$

where $f = (f^1, \ldots, f^4)$, or formally

$$\vartheta(\Psi^1_{\alpha}(x^0, \vec{x})) = (\Psi^2\, \gamma_0)_{\alpha}\, (-x^0, \vec{x}).$$

V. Euclidean Field Theory for Fermions?

In this chapter we study different possibilities of associating a Euclidean field theory with the Schwinger functions of a free relativistic spin $\frac{1}{2}$ field. Of particular interest to us is the question of stability of a Euclidean field structure when we introduce interactions.

As we have seen in the previous section, the construction of the Euclidean Fermi fields is not unique, because we have some freedom in choosing those two point functions which are not given by any Schwinger function. The freedom of extending the Schwinger functions to points of coinciding arguments however should never be a source of non-uniqueness; if we require that the conventional Euclidean power counting laws for Feynman diagrams be preserved. If one tries to work with extensions of the Schwinger functions which violate this rule (see e.g. [Wi 1, 2]) then it is impossible to introduce an interaction via a Euclidean action such that physical positivity is preserved.

In the following we work with the fields constructed in Chapters III and IV, but our main conclusions will not depend on the special choice of Euclidean fields.

We study three different structures. In Section V.1 we introduce and discuss a gage space $(\mathscr{E}, \mathfrak{N}, m)$ for Euclidean fermions and argue that we cannot pass to interacting theories by simply changing the central normal state m on $\mathfrak{N}$, a procedure which is suggested by the successful use of multiplicative functionals in the bose case[3]).

In Section V.2 we investigate properties of the Grassmann algebra generated by the fields U and W, in particular we establish a Euclidean positivity property and a Markov property.

In Section V.3 we look at Euclidean Fermi currents and briefly discuss the problem of whether there exists *commutative* functional integration for these Euclidean currents. Our conclusion is *negative*.

V.1. The gage space and conditional expectations

In this section we construct a finite regular gage space in the sense of Segal [Se 1, 2] from the Euclidean Majorana fields of Section III.1. Our construction of the gage space can easily be extended to the case of the Euclidean Dirac fields of Section III.2. For most of our statements we refrain from giving the proofs, because they are immediate consequences of results in [Se 1], [Gr 1], [Ne 1, 2], [Wl 1].

We define the gage field D by

$$D(f) = A^{\alpha}(\bar{f}_{\alpha}) + A^{*}_{\alpha}(f^{\alpha})$$

where $\bar{f}_{\alpha}$ is the complex conjugate of f^{α} (notice that $A^{\alpha}(\cdot)$ is antilinear, while $A^{*}_{\alpha}(\cdot)$ is linear), and we let $\mathfrak{N}$ be the von Neumann algebra generated by $\{D(f) \mid f \in \mathscr{E}^{(1)}\}$. It follows from Theorem 5 in [Gr 1], that $(\mathscr{E}, \mathfrak{N}, \langle\cdot\rangle)$ is a strongly finite regular gage space, where $\langle\cdot\rangle$ is the trace on $\mathfrak{N}$ defined by

$$\langle A \rangle = (\Omega, A\Omega)$$

for $A \in \mathfrak{N}$. As usual, for $1 \leqslant p \leqslant \infty$, let the Banach spaces $L^{p}(\mathfrak{N})$ be defined to be the completion of $\mathfrak{N}$ with respect to the norm $\|A\|_{p} \equiv \langle |A|^{p}\rangle^{1/p}$ for $1 \leqslant p < \infty$ and $\|A\|_{\infty} = \|A\|$ (the operator norm), respectively. The elements of $L_{p}(\mathfrak{N})$ may be identified with operators on $\mathscr{E}$. Furthermore the map $A \mapsto A\Omega$, $A \in \mathfrak{N}$, extends to a unitary map I

[3]) This remark is not original. Some people argue, that one should give up centrality of m for interacting theories.

from $L^2(\mathfrak{N})$ onto $\mathscr{E}$. A vector $X \in \mathscr{E}$ is called *positive* if $I^{-1}X \in L^2(\mathfrak{N})$ is a positive operator on $\mathscr{E}$, and a bounded operator on $\mathscr{E}$ is said to be *positivity preserving* if it maps positive vectors onto positive vectors.

We now introduce a local structure on $\mathfrak{N}$. If Λ is an open set in $\mathbb{R}^4$ we let $\mathfrak{N}(\Lambda)$ be the von Neumann algebra generated by

$$\{D(f) \,|\, f = (f^1, f^2) \in \mathscr{E}^{(1)}, \operatorname{supp} f^\alpha \subset \Lambda, \alpha = 1, 2\}.$$

If Λ is an arbitrary subset of $\mathbb{R}^4$ we let

$$\mathfrak{N}(\Lambda) = \bigcap_{\Lambda' \supset \Lambda} \mathfrak{N}(\Lambda')$$

where the intersection is over all open sets Λ' containing Λ. $L^2(\mathfrak{N}(\Lambda))$ is a subspace of $L^2(\mathfrak{N})$, and we denote its image $IL^2(\mathfrak{N}(\Lambda))$ by $\mathscr{E}(\Lambda)$. Let P_Λ be the projection onto $\mathscr{E}(\Lambda)$ and for $A \in L^2(\mathfrak{N}(\Lambda))$ set $p_\Lambda(A) = I^{-1} P_\Lambda LA$. $p_\Lambda(\cdot)$ is a map from $L^2(\mathfrak{N})$ to $L^2(\mathfrak{N}(\Lambda))$ and is called a conditional expectation.

Theorem 5.1 [Wl 1]: For any set Λ,

a) P_Λ is positivity preserving

b) $\|p_\Lambda(A)\|_p \leqslant \|A\|_p$ for all $1 \leqslant p \leqslant \infty$, $A \in L^2(\mathfrak{N}) \cap L^p(\mathfrak{N})$ i.e. p_Λ extends to a contraction on $L^p(\mathfrak{N})$.

Proof: For $A \in L^\infty(\mathfrak{N}) = \mathfrak{N}$ we let $\hat{p}_\Lambda(A)$ be the restriction to $\mathscr{E}(\Lambda)$ of the operator $p_\Lambda(A)$. For R and S in $\mathfrak{N}(\Lambda)$ we find

$$\begin{aligned}(R\Omega, \hat{p}_\Lambda(A)\, S\Omega) &= (RS^*\,\Omega, \hat{p}_\Lambda(A)\,\Omega) \\ &= (RS^*\,\Omega, A\Omega) \\ &= (R\Omega, AS\Omega),\end{aligned}$$

where we have used the cyclicity of the trace $\langle\cdot\rangle = (\Omega, \cdot\,\Omega)$. As $\mathfrak{N}(\Lambda)\,\Omega$ is dense in $\mathscr{E}(\Lambda)$, it follows that for all X, Y in $\mathscr{E}(\Lambda)$

$$(X, \hat{p}_\Lambda(A)\, Y) = (X, AY). \tag{5.1}$$

Now we use that $\hat{p}_\Lambda(A)$ and $p_\Lambda(A)$ must have the same spectrum to conclude from (5.1)

$$p_\Lambda(A) \geqslant 0 \text{ if } A \geqslant 0 \tag{5.2}$$

and

$$\|p_\Lambda(A)\| \leqslant \|A\|. \tag{5.3}$$

Part (a) of Theorem 5.1 follows from (5.2), while (b) is a consequence of (5.3) and results in [Ku 1].

Theorem 5.2: Let Λ be an open set in $\mathbb{R}^4$ with complement $\sim\Lambda$ and boundary $\partial\Lambda$. Then

$$P_\Lambda P_{\sim\Lambda} = P_{\sim\Lambda} P_\Lambda = P_{\partial\Lambda} \text{ (Markov property)}. \tag{5.4}$$

The proof is as in [Ne 3], Theorem 5.

We now study the sharp time subspaces $\mathscr{E}_\tau$ of $\mathscr{E}$ and the sharp time fields. Let Λ_τ be the set $\{x = (x^0, \vec{x}) | x^0 = \tau, \vec{x} \in \mathbb{R}^3\}$ and set $\mathfrak{N}(\Lambda_\tau) = \mathfrak{N}_\tau$ and $\mathscr{E}(\Lambda_\tau) = \mathscr{E}_\tau$. It is easy to see that $\mathfrak{N}_\tau$ is the von Neumann algebra generated by the 'sharp time' fields $D(f_\tau)$, $f_\tau^\alpha(x) = (\delta_\tau \otimes g^\alpha)(x) \equiv \delta(\tau - x^0) \cdot g^\alpha(\vec{x})$, where $g^\alpha_\cdot \in \mathscr{H}_{-1/2}(\mathbb{R}^3)$.

The time-zero subspace $\mathscr{E}_0$ can be naturally identified with the relativistic Fock space $\mathfrak{F}_{\text{rel}}$ of the free spin $\frac{1}{2}$ Majorana field, if we set the Euclidean vacuum Ω equal to the relativistic Fock vacuum Ω_{rel} and define relativistic creation (resp. annihilation) operators by

$$a^*_{\text{rel}.\alpha}(\vec{p}) = \frac{\sqrt{2\omega(\vec{p})}}{(2\pi)^{3/2}} \int A^*_\alpha(x)\, e^{i\vec{p}\vec{x}}\, \delta(x^0)\, d^4x \tag{5.5}$$

(this equation has to be understood in the distributional sense[4])) where

$$\omega(\vec{p}) = \sqrt{m^2 + \vec{p}^2}.$$

In other words, $(\mathscr{E}_0, \mathfrak{N}_0, \langle\cdot\rangle)$ is the gage space for a free relativistic, time-zero Majorana field.

For $A \in L^2(\mathfrak{N}_0)$, we denote the vector $A\Omega$, when interpreted as vector in $\mathfrak{F}_{\text{rel}}$ by ψ_A. The scalar product on $\mathfrak{F}_{\text{rel}}$ is given by $(\psi_A, \psi_B)_{\text{rel}} = \langle A^* B\rangle$ for A, $B \in L^2(\mathfrak{N}_0)$. The free Hamiltonian on $\mathfrak{F}_{\text{rel}}$ is denoted by H_0:

$$H_0 = \int a^*_{\text{rel}.\alpha}(\vec{p})\, a^\alpha_{\text{rel}}(\vec{p})\, \omega(\vec{p})\, d^3p$$

and we write $T_t \equiv \mathbb{U}(1, 1, (t, \vec{0}))$ for the Euclidean 'time' translation group.

Theorem 5.3: a) Let A, $B \in L^2(\mathfrak{N}_0)$. Then

$$\langle A^* T_t B\rangle = (\psi_A, e^{-|t|H_0}\psi_B)_{\text{rel}}$$

(Feynman–Kac formula)

b) There exists a continuous increasing function $\beta(t)$ on $\mathbb{R}^+$ with $\beta(0) = 0$, $\beta(\infty) = 1$, such that for all A, $B \in L^2(\mathfrak{N}_0)$

$$|\langle A^* T_t B\rangle| \leqslant \|A\|_p \|B\|_q$$

for

$$\frac{1}{p} + \frac{1}{q} = 1 + \beta(|t|),\ t \in \mathbb{R}.$$

c) The trace $\langle\cdot\rangle$ has the reflection property

$$\langle A^* T_t B\rangle = \langle A^* T_{-t} B\rangle$$

for all A, $B \in L^2(\mathfrak{N}_0)$, $t \in \mathbb{R}$.

A gage space $(\mathscr{E}, \mathfrak{N}, \langle\cdot\rangle)$ with a local structure as described above and with a Markoff property (5.4) is a natural noncommutative generalization of Markoff processes. The gage fields D are therefore the natural analog to Nelson's free Markoff (Bose)

4) See also [SU 1].

fields [Ne 3]. Introducing interaction however causes problems. In the commutative case the idea was to use multiplicative functionals [Ne 1]. The natural generalization of that concept to the gage space formalism would be to use normal central states ω_κ on $\mathfrak{N}$ to describe the system with interaction (κ denotes some cutoff). The main obstacle to this idea is the Radon–Nikodym theorem (see [Se 1], Theorem 15), which says that if $(\mathscr{E}, \mathfrak{N}, \langle\cdot\rangle)$ is a regular gage space and $\omega(\cdot)$ a normal central state on $\mathfrak{N}$, then there exists a unique positive self-adjoint operator Z affiliated with the center of $\mathfrak{N}$ such that $\omega(A) = \langle AZ\rangle$ for all $A \in \mathfrak{N}$. But in the case discussed above, $\mathfrak{N}$ is a factor and hence there are no normal central states on $\mathfrak{N}$ other than $\langle\cdot\rangle$. The gage space structure is therefore *not stable* under 'turning on interactions'. A further problem is connected with Euclidean covariance. The von Neumann algebra $\mathfrak{N}$ defined above is *not* invariant under Euclidean rotations. We show now that there is no choice of Majorana gage fields such that $\mathfrak{N}$ is Euclidean invariant. In the case of Dirac fields, however, it is possible to construct Euclidean covariant gage fields.

We begin our argument in a general context. Let $\Psi^\alpha(h_\alpha)$, $h = (h_\alpha) \in \mathscr{S}(\mathbb{R}^4)^{\times m}$, be an m component, free Euclidean Fermi field, acting as a bounded operator on the Fock space $\mathscr{E} = \oplus_{n=0}^\infty \mathscr{H}^{@n}$, where $\mathscr{H}$ is some complex Hilbert space containing $\mathscr{S}(\mathbb{R}^4)^{\times\bar{m}}$ with scalar product $(\cdot,\cdot)$. We assume that Ψ transforms covariantly under some representation of $SU(2) \times SU(2)$. We let $A^\alpha(f_\alpha)$ and $A_\alpha^*(f_\alpha)$ $(\alpha = 1,\ldots,\bar{m} \leqslant m)$ be the annihilation and creation operators associated with Ψ; $A^\alpha(f_\alpha)\Omega = \vec{0}$, for all $(f_\alpha) \in \mathscr{H}$, where Ω is the vacuum in $\mathscr{E}$. With $\mathbb{U}(u_1, u_2)$ being the unitary representation of $SU(2) \times SU(2)$ on $\mathscr{E}$ we assume that

$$\mathbb{U}(u_1,u_2)\, A^\alpha(f_\alpha)\, \mathbb{U}^{-1}(u_1,u_2) = A^\alpha((Vf_R)_\alpha), \tag{5.6}$$

where $V = V(u_1, u_2)$ is some finite dimensional representation of $SU(2) \times SU(2)$ and $f_R(x) = f(R^{-1}x)$, $R = R(u_1, u_2)$ being the element in SO_4 corresponding to (u_1, u_2). Also

$$\langle A^\alpha(f_\alpha)\, A_\beta^*(g^\beta)\rangle \equiv (f,g). \tag{5.7}$$

We now call D^α a gage field associated with Ψ^α if D^α is a densely defined operator-valued function on $\mathscr{H}$ of the form

$$D(f) = D_\alpha(f^\alpha) = A^\alpha(\overline{f_\alpha}) + A_\alpha^*((Cf)^\alpha)$$

for some (anti-)linear mapping C on $\mathscr{H}$ such that

$$\langle D(f)\, D(g)\rangle = \langle D(g)\, D(f)\rangle \tag{5.8}$$

and

$$\mathrm{D}(f)^* = D(h)$$

for some $h = h(f) \in \mathscr{H}$, for all $f, g \in \mathscr{H}$ see [Gr].

Lemma 5.4: Let D be a gage field associated with Ψ. Then

1) C is linear and $C = C^T$
2) If D transforms covariantly under the representation $\mathbb{U}(u_1, u_2)$ of $SU(2) \times SU(2)$ then

$$V(u_1,u_2)\, C = C\overline{V}(u_1,u_2)$$

for all $(u_1, u_2) \in SU(2) \times SU(2)$, i.e. V has to be a potentially real representation.

Proof: By (5.7) and (5.8), for all $f, g \in \mathscr{H}$

$$\langle D(f)D(g)\rangle = (\bar{f}, Cg)$$
$$= \langle D(g)\,D(f)\rangle = (\bar{g}, Cf)$$
$$= (\overline{Cf}, g)$$
$$= (\bar{f}, C^T g).$$

From the first and the third line we conclude that C is linear and from the first and the last line that $C = C^T$.

For (2) we compute

$$\mathbb{U}(u_1, u_2)\,D(f)\,\mathbb{U}^{-1}(u_1, u_2) = A^\alpha((V\bar{f}_R)_\alpha) + A^*_\alpha((VCf_R)^\alpha)$$

and if D^α should transform covariantly then we must have

$$C\overline{V\bar{f}} = VCf, \text{ i.e. } C\bar{V} = VC.$$

Q.E.D.

It is well known [Wg 1, 2], [Ma 1] that

a) Even (odd) dimensional irreducible representations of $SU(2)$ are pseudoreal (potentially real).
b) The direct product of two pseudoreal (potentially real) representations contains only potentially real irreducible components.
c) The direct product of a pseudoreal and a potentially real representation contains only pseudoreal representations.

According to (3.11), for Majorana fields we have $V(u_1, u_2) = u_1$, which is a pseudoreal representation. By Lemma 5.4 this *excludes* the possibility of Euclidean covariant gage fields.

However, if we take the direct sum of two spin $\frac{1}{2}$ representations of $SU(2) \times SU(2)$ then we can obtain a potentially real representation:

$V(u_1, u_2) = \begin{pmatrix} \bar{u}_1 & 0 \\ 0 & u_1 \end{pmatrix}$ is potentially real; for, with $C = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, we have $C\bar{V} = VC$. Hence we have proven the following:

Theorem 5.5: There are Euclidean covariant gage fields associated with a *Dirac* field.

For the sake of concreteness let us exemplify Theorem 5.5. Let A_1^α and A_2^α be the annihilation operators belonging to the two Majorana fields U_1 and U_2, resp. which constitute the Dirac field U; see (4.9). We let $A^*_{j\alpha}(\cdot)$ be *linear*, $j = 1, 2$; $\alpha = 1, 2$. We define the (four component) gage field D by

$$D(f) = A_1^\alpha(\bar{f}_\alpha) + A^*_{2\alpha}(\epsilon^{\alpha\beta} f_\beta) + A_2^\alpha(\epsilon_{\alpha\beta}\bar{f}^{2+\beta}) + A^*_{1\alpha}(f^{2+\alpha}), \tag{5.9}$$

$\alpha, \beta = 1, 2$; $f = (f_1, f_2, f^3, f^4)$.

Then using

$$\langle A_i^\alpha(\bar{f}_\alpha)\,A^*_{j\beta}(g^\beta)\rangle = \delta_{ij}(f_\alpha, Sg^\alpha) \equiv \delta_{ij} \sum_\alpha \int f_\alpha(x)\,S(x-y)\,g^\alpha(y)\,dxdy$$

we find

$$\begin{aligned}\langle D(f)\,D(g)\rangle &= (f_\alpha, Sg^{2+\alpha}) + (\epsilon_{\alpha\beta} f^{2+\beta}, S\epsilon^{\alpha\gamma} g_\gamma)\\ &= (f_\alpha, Sg^{2+\alpha}) + (f^{2+\alpha}, Sg_\alpha)\\ &= \langle D(g)\,D(f)\rangle.\end{aligned}$$

Furthermore with $\mathbb{U} = \mathbb{U}(u_1, u_2)$ and $R = R(u_1, u_2)$ we infer from (3.11) that

$$\begin{aligned}\mathbb{U} D(f)\,\mathbb{U}^{-1} &= A_1^\alpha(\overline{\bar{u}_{1\alpha}^\beta f_{R\beta}}) + A_{2\alpha}^*(u_{1\beta}^\alpha \epsilon^{\beta\gamma} f_{R\gamma})\\ &\quad + A_2^\alpha(\bar{u}_{1\alpha}^\beta \epsilon_{\beta\gamma} f_R^{2+\gamma}) + A_{1\alpha}^*(u_{1\beta}^\alpha f^{2+\beta})\\ &= D(\hat{f}),\end{aligned}$$

where $\hat{f} = (\hat{f}_1, \hat{f}_2, \hat{f}^3, \hat{f}^4)$ is defined by

$$\begin{aligned}\hat{f}_\alpha(x) &= \bar{u}_{1\alpha}^\beta f_{R\beta}(x) = \bar{u}_{1\alpha}^\beta f_\beta(R^{-1}x)\\ \hat{f}^{2+\alpha}(x) &= u_{1\beta}^\alpha f_R^{2+\beta}(x) = u_{1\beta}^\alpha f^{2+\beta}(R^{-1}x).\end{aligned}$$

This establishes the Euclidean covariance of the field $D(f)$ and proves Theorem 5.5. Notice that $D(f)^* = D(g)$, $g_\alpha = f^{2+\alpha}$, $g^{2+\alpha} = f_\alpha$.

It is easy to see that the n-point functions of the gage fields D defined in (5.9) can be analytically continued in the time variables to real times (i.e. to the Minkowski-space region). The distributions obtained in this way are *Poincaré covariant.* They are well suited for the description of free, relativistic Fermi (gage-) fields. The (time 0-) relativistic gage field can be identified with the (time 0-) Euclidean field $D(\delta_0 \otimes \cdot)$.

In spite of its success in the free field case the gage space approach seems to us to be *unnatural* as a general framework for Fermi field theories:

a) As noted above there are *no* non-trivial 'multiplicative functionals' on a gage space $(\mathscr{E}, \mathfrak{N}, \langle\cdot\rangle)$. Thus, it is impossible to describe cutoff interactions within the framework of the free field gage space.
b) In *general* it seems to be impossible to formulate the principle of locality in terms of (anti-) commutation properties of the gage fields.
c) As the reader may easily convince himself none of the *local* Euclidean Fermi currents (such as the one used to describe the Yukawa interaction) can be expressed in terms of the Euclidean gage field D. (Similar negative experiences on the level of the Hamiltonian formalism were earlier made in [Gr 1]). Remarks (b) and (c) are of course connected with each other.

In conclusion we think that although the gage space approach (i.e. a probabilistic approach) is attractive for the description of free fields it yields at best some technical tools (such as 'conditioning' in the sense of [GRS 1] or Hölder inequalities) for the construction of *local,* interacting fields.

V.2. The field algebra

Let U be the Euclidean Majorana field constructed in Section III.1, and let $\mathfrak{J}$ be the Grassmann algebra generated (algebraically) by

$$\left\{U(f) = \sum_\alpha U^\alpha(f_\alpha) \,|\, f \in \mathscr{E}^{(1)}\right\}.$$

Then $\langle \cdot \rangle \equiv (\Omega, \cdot \Omega)$ defines a linear functional on $\mathfrak{J}$. Our first goal is to construct an *antilinear* operation σ mapping $\mathfrak{J}$ into itself, such that

$$\langle \sigma(M)\, M \rangle \geqslant 0, \text{ for all } M \in \mathfrak{J}. \tag{5.10}$$

Notice that vacuum expectation values of elements in $\mathfrak{J}$ can always be expressed in terms of Schwinger functions. In particular (5.10) is a positivity property of the Schwinger functions and we may think of it as the Fermi field analog of the Nelson–Symanzik positivity property for Boson Schwinger functions. In order to construct σ we let X be an arbitrary vector in $\mathscr{E}$. Then, with : : denoting Wick ordering,

$$\begin{aligned}\left(:\prod_{i=1}^{n} U^{\alpha_i}(f_i):\Omega, X\right) &= \left(\prod_{i=1}^{n} (-\epsilon^{\alpha_i\beta_i} A^{*}_{\beta_i}(f_i))\,\Omega, X\right)\\ &= \left(\Omega, \prod_{i=n}^{1} (-\epsilon_{\alpha_i\beta_i} A^{\beta_i}(f_i))\, X\right)\\ &= \left(\Omega, :\prod_{i=n}^{1} (-\epsilon_{\alpha_i\beta_i} U^{\beta_i}(\overline{f_i})):X\right).\end{aligned}$$

We conclude that σ, defined by linear extension of

$$\sigma\left(:\prod_{i=1}^{n} U^{\alpha_i}(f_i):\right) = :\prod_{i=n}^{1} (-\epsilon_{\alpha_i\beta_i} U^{\beta_i}(\overline{f_i})): \tag{5.11}$$

satisfies inequality (5.10). Notice that σ is *not* an antiautomorphism (i.e. a * operation) on $\mathfrak{J}$.

Because Ω is cyclic and separating for $\mathfrak{J}$ we can reconstruct $\mathscr{E}$ just from $\mathfrak{J}$, $\langle \cdot \rangle$ and the mapping σ: $\mathfrak{J}$ with inner product $(M,N) = \langle \sigma(M) N \rangle$ is a pre-Hilbert space whose closure we denote by $\mathscr{E}_{\mathfrak{J}}$. Then the map $I: M \mapsto M\Omega$, $M \in \mathfrak{J}$ extends to a unitary from $\mathscr{E}_{\mathfrak{J}}$ and $\mathscr{E}$. As in the previous section we can unitarily embed the relativistic Fock space $\mathfrak{J}_{\mathrm{rel}}$ in $\mathscr{E}$.

A local structure on $\mathfrak{J}$ can be introduced as before. For $\Lambda \subset \mathbb{R}^4$ an open set, we let $\mathfrak{J}(\Lambda)$ be the Grassmann algebra generated by

$$\{U(f) \mid f \in \mathscr{E}^{(1)}, \operatorname{supp} f^{\alpha} \subset \Lambda, \alpha = 1, 2\}$$

and for a general set $\Lambda \subset \mathbb{R}^4$ we define

$$\mathfrak{J}(\Lambda) = \bigcap_{\{\Lambda' \mid \Lambda \subset \Lambda' \text{open}\}} \mathfrak{J}(\Lambda').$$

By P_{Λ} we denote again the orthogonal projection onto $\mathfrak{J}(\Lambda)\Omega$. Let $T_t = (\mathbb{1}, \mathbb{1}, (t, \vec{0}))$ as in V.1.

Theorem 5.6:

1) For all open sets Λ,

$$P_{\Lambda} P_{\sim\Lambda} = P_{\sim\Lambda} P_{\Lambda} = P_{\partial\Lambda} \quad \text{(Markoff property)}$$

2) For all $M, N \in \mathfrak{J}_0 = \mathfrak{J}(\{t = 0\})$

$$\langle \sigma(M)\, T_t N \rangle = (\psi_M, e^{-|t| H_0} \psi_N)_{\mathrm{rel}} \quad \text{(Feynman–Kac)}$$

where $\psi_M = M\Omega \in \mathfrak{J}_{\mathrm{rel}}$ etc.

3) For all $M, N \in \mathfrak{J}_0$

$$\langle MT_t N \rangle = \langle MT_{-t} N \rangle \quad \text{(Reflection property)}$$

Is this structure, when abstracted from the special case of *free* fields a useful non-commutative version of functional integration, if we want to study interacting models involving fermions? The answer is probably no. The problem is again that it is not stable under turning on interactions. (There is of course no covariance problem in this set up.) If we replace the functional $\langle \cdot \rangle$ by $\langle \cdot e^{-V_\kappa} \rangle / \langle e^{-V_\kappa} \rangle \equiv \langle \cdot \rangle_\kappa$, where V_κ is some cutoff Euclidean action – see section V.3 – then in general $\langle \sigma(M) M \rangle_\kappa$ will not be positive anymore and we would have to find a new σ-operation. A possible way out of this difficulty would be to try to choose σ such that it defines a * *operation* on $\mathfrak{J}$, i.e. such that it be antilinear on $\mathfrak{J}$ and $\sigma(M \cdot N) = \sigma(N)\,\sigma(M)$. However, this is impossible as shown in the following:

Theorem 5.7: Let $\mathfrak{J}$ be the Grassmann algebra generated by some vector space W. Let $\langle \cdot \rangle$ be a linear functional on $\mathfrak{J}$ such that $\langle P \rangle = 0$ for all odd monomials P (P is an odd monomial if it is equal to $f_1 f_2 \ldots f_k, f_i \in W$, k odd). Then there is *no * operation* on $\mathfrak{J}$, such that $\langle P^* P \rangle > 0$ for all $P \in \mathfrak{J}$, $P \neq 0$.

Proof: Let $f \in W$ and suppose there is a * operation on $\mathfrak{J}$. Then $f^* \in \mathfrak{J}$ and we can write it as

$$f^* = P + Q$$

where $P(Q)$ is a sum of even (odd) monomials in elements of W. Using the assumptions of our theorem we find

$$\langle f^* f \rangle = \langle Qf \rangle = -\langle fQ \rangle = -\langle ff^* \rangle. \tag{5.12}$$

The left-hand side of (5.12) must be nonnegative, the right-hand side nonpositive and hence $\langle f^* f \rangle = 0$. If we assume that this implies: $f = 0$, then the existence of a * operation on $\mathfrak{J}$ necessarily implies that $W = \{\vec{0}\}$. This proves the theorem.

We could also try to make $\mathfrak{J}$ a * algebra and $\langle \cdot \rangle$ a state on $\mathfrak{J}$ by giving up anticommutativity of the Euclidean fields U^α, by changing the Schwinger functions in points of coinciding arguments. But such a procedure always conflicts with conventional Euclidean power counting. Furthermore the fields obtained in this manner are so singular that it is impossible to introduce cutoff Euclidean actions wich are well defined and satisfy physical positivity.

V.3. Euclidean currents

An attractive idea seems to be to work with the commutative algebra generated by Euclidean Fermi currents only, since all physical information can in principle be obtained from vacuum expectation values of products of currents. The interesting question of course is whether the current algebra can be connected with commutative functional integration; one might be hopeful for an affirmative answer, because in Schwinger's model for one (space) dimensional Quantum Electrodynamics [Sc 1, SL 1] the vector currents can be interpreted as derivatives of a Bose field, for which commutative integration is well known to apply. We will show, however, that this result is misleading. In the case of interest to us it is not possible to imbed the current algebra in a commutative *-algebra such that the currents are formally self-adjoint.

This rules out the possibility that the current functional is the Laplace transform of some probability measure. We do however not exclude more general forms of functional integration, where the measures are signed or even complex and where the currents do not correspond to real functions.

For the Dirac fields constructed in Chapter IV the unrenormalized Euclidean currents are given (formally) by

$$\mathfrak{J}_s(x) = :\Psi^2(x)\Psi^1(x): \qquad \text{(scalar current)}$$

$$\mathfrak{J}_V^\mu(x) = :\Psi^2(x)\,\gamma^\mu\Psi^1(x): \quad \text{(vector current)}$$

or using (4.10)

$$\mathfrak{J}_s(x) = m(:W_\alpha(x)\,U^\alpha(x): + :\hat{U}^{\dot{\alpha}}\,\hat{W}_{\dot{\alpha}}:)$$

and similarly for $\mathfrak{J}_V^\mu$. For simplicity our following discussion will mostly be limited to scalar currents.

By (4.17), $\mathfrak{J}_s(x)$ satisfies

$$\vartheta(\mathfrak{J}_s(x^0,\vec{x})) = \mathfrak{J}_s(-x^0,\vec{x})$$

and thus the formal expression

$$\mathfrak{J}_s(f) = \int dx\,\mathfrak{J}_s(x)\,f(x)$$

is ϑ-invariant if only the real function f is invariant under 'time reflection': $f(x^0,\vec{x}) = f(-x^0,\vec{x})$.

To make things rigorous we introduce cutoffs, but we want to do it without destroying the ϑ-invariance of $\mathfrak{J}_s$. There are two possibilities: (a) replacing the space time *continuum* by a space time *lattice*, [GRS 1] or (b) replacing the fields $\Psi^i(x)$ by

$$\Psi_n^i(x) \equiv (\chi_n * \Psi^i)\,(x), \tag{5.13}$$

where $\chi_n(x) = \delta(x^0) \otimes \rho_n(\vec{x})$ is a cutoff function (for the spatial momentum components). We will choose $\rho_n(\vec{x})$ to be the Fourier transform of the characteristic function $\tilde{\rho}_n(\vec{k})$ of the set $S_n = \{\vec{k}\,|\,|\vec{k}| \leqslant n\}$. As in (5.13) we also define (momentum) cutoff versions of the fields U^α, W^α, etc. Now we define the cutoff Euclidean current to be

$$\mathfrak{J}_n(f) = \int dx\, :\Psi_n^2(x)\,\Psi_n^1(x):\,f(x),$$

where f is some test function.

Theorem 5.8 [OS 4]: Let $f \in L^1(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$, and let A be an arbitrary polynomial in $\Psi^i(g)$, $g \in \mathscr{S}(\mathbb{R}^4)$. Then for all $n \in \mathbb{Z}_+$

a) $\langle A\,e^{\rho J_n(f)}\rangle$ is an entire analytic function of ρ, and
b) If f is real and ϑ-invariant, i.e. if

$$f(x^0,\vec{x}) = \overline{f(x^0,\vec{x})} = f(-x^0,\vec{x}),$$

then the functional $\langle A\,e^{J_n(f)}\rangle$ satisfies *physical positivity*.

The significance of this theorem is of course that the physical positivity property is shown to be *stable under turning on interactions*. We now prove that there is no way of embedding the field algebra $\mathfrak{J}$ in a * algebra $\mathfrak{J}_*$ such that a Euclidean Fermi current $\mathfrak{J}$ is self-adjoint with respect to the * operation of $\mathfrak{J}_*$.

Theorem 5.9: Let * be an arbitrary * operation and $\mathfrak{J}_*$ the * algebra generated algebraically by the operators

$$\{\Psi^1(f_1), \Psi^{1*}(f_2), \Psi^2(f_3), \Psi^{2*}(f_4) \mid f_1, \ldots, f_4 \text{ in } \mathscr{S}(\mathbb{R}^4)\},$$

where Ψ^1 and Ψ^2 are arbitrary Euclidean Fermi fields. Let $\mathfrak{J}_*$ be represented as a subalgebra of the algebra $B(\mathscr{H})$ of all bounded operators on a complex Hilbert space $\mathscr{H}$. Suppose J is an *even*, self-adjoint, operator-valued functional on the field algebra $\mathfrak{J}$ generated by Ψ^1, Ψ^2, (e.g. a current). Then J commutes with all operators in the weak closure of $\mathfrak{J}_*$ on $\mathscr{H}$. If Ψ^1 and Ψ^2 are *free*, Euclidean Fermi fields, J is a multiple of the identity.

Proof: Since J is even, it commutes with all operators in $\mathfrak{J}$. Since $J = J^*$, J commutes with all operators in $\mathfrak{J}^* = \{A \mid A^* \in \mathfrak{J}\}$. Since $\mathfrak{J}_*$ is generated by $\{\mathfrak{J}, \mathfrak{J}^*\}$, J commutes with all operators in $\mathfrak{J}_*$: here we say that J commutes with the bounded operator A if $(J^*\psi, A\varphi)_{\mathscr{H}} = (A^*\psi, J\varphi)_{\mathscr{H}}$, for all φ and ψ in $D(J) = D(J^*)$. Therefore, since the weak and the strong closure of $\mathfrak{J}_*$ coincide, J commutes with all operators in the weak closure $\overline{\mathfrak{J}_*}$ of $\mathfrak{J}_*$.

If Ψ^1 and Ψ^2 are *free* Euclidean Fermi fields then

$$\langle A T_x B T_x^* \rangle \to \langle A \rangle \langle B \rangle, \text{ as } |x| \to \infty, \tag{5.14}$$

for all A and B in $\mathfrak{J}_*$. Here $x = \langle t, \vec{x} \rangle \in \mathbb{R}^4$ and $|x|$ denotes the Euclidean length of x; $\{T_x\}$ denotes space-time translations on $\mathscr{H} = \mathscr{E}$. In the case of *free* fields $\{T_x\}$ is contained in $\overline{\mathfrak{J}_*}$. Thus $T_x J T_x^* = J$ is *independent* of x! By (5.14) it follows that

$$J = \langle J \rangle \cdot I$$

This completes the proof of the theorem.

We can now argue that Theorem 5.9 no longer applies if we work with the Euclidean Fermi *currents* only and never introduce Euclidean Fermi *fields*. Then the Euclidean Fermi *currents* might turn out to be self-adjoint and there would exist a satisfactory commutative functional integration theory. Let us therefore consider the current functional

$$Z_n(f) = \langle e^{J_n(f)} \rangle, f \in \mathscr{S}_{\text{real}}(\mathbb{R}^4),$$

where J_n is the scalar current built from the *cutoff* free Euclidean fields Ψ_n^1, Ψ_n^2. The existence of Z_n is guaranteed by Theorem 5.8. Taking the structure of boson theories as a guide we might hope that $Z_n(f)$ is the *Laplace* transform of a probability measure μ on the σ-algebra generated by the Borel cylinder sets of $\mathscr{S}'_{\text{real}}(\mathbb{R}^4)$, i.e.

$$Z_n(f) = \int_{\mathscr{S}_1} e^{\Phi(f)} d\mu_n(\Phi). \tag{5.15}$$

Such a structure would lead to a satisfactory functional integration theory for Fermions. However it can be shown that $Z_n(f)$ does *not* have the necessary positivity property consistent with the integral representation (5.15):

(5.15) is equivalent to:

$$\sum_{i,j=1}^{N} \bar{c}_i c_j Z_n(f_i + f_j) \geqslant 0, \tag{5.16}$$

for arbitrary N, arbitrary complex numbers $c_1, \ldots, c_N$ and *real* Schwartz space functions $f_1, \ldots, f_N$. This inequality can be disproved by explicit counter-examples.

The situation remains the same if we consider the renormalized limit functional $Z_{\text{ren}}(f)$, $f \in \mathcal{S}_{\text{real}}(\mathbb{R}^d)$, (constructed for the case of two space-time dimensions in [Si 1]; also it is known from covariant perturbation theory that there exists a Schwartz space norm $|\cdot|_{\mathcal{S}}$ such that for $|f|_{\mathcal{S}}$ small enough

$$Z_{\text{ren}}(f) = \exp \sum_{n=2}^{\infty} L^{(n)}(f, \ldots, f),$$

where $L^{(n)}(\xi_1, \ldots, \xi_n)$ is the renormalized nth order loop and $\sum_{n=2}^{\infty} L^{(n)}(f, \ldots, f)$ converges absolutely; the techniques of [Si 1] permit one to define $Z_{\text{ren}}(f)$ without restrictions of the size of $|f|_{\mathcal{S}}$).

It is possible to find test functions such that the inequality (5.16) is false for Z_{ren}. The simplest counter-example to (5.16) can be constructed by using the 'mean field' calculations of Ref. [CW 1].

(The experienced reader will conclude from the fact that (5.15), (5.16) are *not* valid for the scalar current that it is hard (if not impossible) to prove correlation inequalities in models like the Yukawa model in two dimensions.)

Note added in proof. We would like to point out again that the fact that the functionals $Z_n(f)$ and $Z_{\text{ren}}(f)$ violate inequality (5.16) does *not* depend on the existence of Euclidean Fermi fields but is a direct consequence of the relativistic, free field theory and the Bargmann–Hall–Wightman theorem. This result does not apply to the regularized currents associated with a free, massless Dirac field in two space-time dimensions for which inequality (5.16) is true.

Acknowledgments

We are grateful to Profs. S. Coleman and G. Mackey for a helpful conversation on the material presented in Sections V.3, V.1, and to Profs. O. Lanford and A. Yildiz for convincing us that a paper which attempts to clarify the present situation concerning Euclidean Fermi field theory deserves to be made public.

REFERENCES

[CW 1] S. Coleman, E. Weinberg, *Radiative corrections as the origin of spontaneous symmetry breaking*, Phys. Rev. *D7*, 1888 (1973).

[E 1] G. Velo and A. S. Wightman (eds.), *Constructive Quantum Field Theory*, Springer Lecture Notes in Physics, Vol. 25, 1973.

[Gr 1] L. Gross, *Existence and uniqueness of physical ground states*, Jour. Funct. Anal. *10*, 52 (1972).

[GRS 1] F. Guerra, L. Rosen and B. Simon, *The $P(\varphi)_2$ Euclidean Quantum Field theory as classical statistical mechanics*, preprint.

[Ku 1] R. A. Kunze, *L_p Fourier transforms on locally compact unimodular groups*, Trans. Amer. Math. Soc. *89*, 519 (1958).

[LL 1] L. D. Landau and E. M. Lifschitz, *Relativistische Quantentheorie* (Akademie Verlag, Berlin 1971).

[Ma 1] G. W. Mackey, *On induced representations of groups*, Am. Jour. Math. *73*, 576 (1951).

[Ne 1] E. Nelson, *Quantum fields and Markoff fields*, Amer. Math. Soc. Summer Institute on PDE, held at Berkeley, 1971.

[Ne 2] E. Nelson, *Construction of quantum fields from Markoff fields*, J. Funct. Anal. *12*, 97 (1973).

[Ne 3] E. Nelson, *The free Markoff field*, J. Funct. Anal. *12*, 211 (1973).

[OS 1] K. OSTERWALDER and R. SCHRADER, *Axioms for Euclidean Green's functions*, Commun. Math. Phys. *31*, 83 (1973).
[OS 2] K. OSTERWALDER and R. SCHRADER, *Axioms for Euclidean Green's functions, II*, to appear in Comm. Math. Phys.
[OS 3] K. OSTERWALDER, *Euclidean Green's functions and Wightman distributions*, in *Constructive Quantum Field Theory*, G. Velo and A. S. WIGHTMAN (eds.), Springer Lecture Notes in Physics, Vol. 25, 1973.
[OS 4] K. OSTERWALDER and R. SCHRADER, *Euclidean Fermi fields and a Feynman–Kac formula for Boson–Fermion models*, Helv. Phys. Acta *46*, 277 (1973).
[SU 1] R. SCHRADER and D. UHLENBROCK, *Markov structures on Clifford algebras*, preprint.
[Sc 1] J. SCHWINGER, *Gauge invariance and mass*, Phys. Rev. *128*, 2425 (1962).
[Se 1] I. SEGAL, *A non-commutative extension of abstract integration*, Ann. of Math. *57*, 401 (1953) and Ann. of Math. *58*, 595 (1953).
[Si 1] E. SEILER, *Schwinger functions for the Yukawa model in two dimensions with space-time cutoff*, preprint.
[SL 1] J. H. LÖWENSTEIN and J. A. SWIECA, *Quantum electrodynamics in two dimensions*, Ann. Physics *68*, 172 (1971).
[Sy 1] K. SYMANZIK, *Euclidean quantum field theory*, in Proc. Int. School of Phys. 'Enrico Fermi', Varenna Course XLV, R. JOST (ed.) (Academic Press, New York 1969).
[v.W. 1] B. L. VAN DER WAERDEN, *Die Gruppentheoretische Methode in der Quantenmechamk* (Springer, Berlin 1932).
[We 1] H. WEYL, *Elektron und Gravitation I*, Zeits für Physik *56*, 330 (1929).
[Wg 1] E. WIGNER, *Group Theory* (Academic Press, New York and London 1959).
[Wg 2] E. WIGNER, *On representations of certain finite groups*, Am. Jour. Math. *63*, 57 (1941).
[Wl 1] I. F. WILDE, *The free Fermion as a Markov field*, Jour. Funct. Anal. *15*, 12 (1974).
[Wi 1] D. N. WILLIAMS, *Euclidean Fermi fields with a Hermitean Feynman–Kac–Nelson formula*, to appear in Commun. Math. Phys.
[Wi 2] D. N. WILLIAMS, *Euclidean Fermi fields with a Hermitean Feynman–Kac–Nelson formula II*, preprint.