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# On the Approach to Equilibrium in Fluid Mechanics

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*Abstract.* The method which has been developed for the study of solutions of the family of the kinetic equations of the Enskog–Vlasov type [J. Math. Phys. 15, 35 (1974)] is formulated generally and applied in fluid dynamics. The theory of solving the equations of fluid dynamics is in this way put into the relationship with the theory of solving other dynamical equations of non-equilibrium statistical mechanics. Some basic results in fluid dynamics are reproduced and a new contribution to the problem of finding a natural (both from the physical and mathematical point of view) structure for the space on which the equations of fluid mechanics are defined is made.

## 1. Introduction

A possibility of extracting a general theory from the discussion of the kinetic theory dynamics [1] has been already discussed in Ref. [2]. Since there are now several types of dynamics which can be discussed in a unified way, namely the Enskog–Vlasov dynamics [1], fluid dynamics (Sections 3 and 4 of this paper), non-equilibrium and equilibrium thermodynamics [2] and the Liouville dynamics [2], it is convenient to start this paper with the general theory. The concepts introduced and the relations obtained in the general theory will then simplify the discussion of fluid dynamics. The theory of solving the equations of fluid dynamics will also appear automatically as an organic part of the theory of solving other dynamical equations of non-equilibrium statistical mechanics. Relatively well-developed (both from the physical and mathematical point of view) fluid dynamics might be very useful for the further development of the general theory and in this way for development in other dynamical theories of non-equilibrium statistical mechanics. A detailed study of the relationship between kinetic theory and fluid dynamics, as it appears from the point of view of the general theory, is in Ref. [3]. We also believe that our analysis of the equations of fluid dynamics might bring closer the rather mathematically oriented [4] and the rather physically oriented [5] fluid mechanics.

## 2. A General Theory

This section is an extract and compilation, expressed in terms of the general dynamical system theory, of the results obtained in Refs. [1], [2] and Sections 3 and 4 of this paper. The reader should consult these references for the motivation and concrete illustration of the concepts introduced.

Let us have a class  $\mathcal{S}$  of one-component physical systems and a given set of measurements and observations  $\mathcal{O}$ . The experience obtained from measuring and observing (the observations and measurements are always assumed to be elements of

$\mathcal{O}$ ) the physical systems in  $\mathcal{S}$  is summed up in a phenomenological theory. Let  $f$  characterize completely (with respect to  $\mathcal{O}$ ) the state of the physical systems in  $\mathcal{S}$ . All possible states form a set  $\mathcal{H}$ . We shall assume that  $\mathcal{H}$  is a manifold, in general infinite dimensional. A more detailed specification of  $\mathcal{H}$  has to come as a result of the comparison of the properties of solutions of the time evolution equation of the phenomenological theory

$$\frac{\partial f}{\partial t} = Rf \quad (1)$$

and the results of observations and measurements. The operator  $R$  in (1) is defined through a set of phenomenological quantities  $\mathcal{P}$  whose values can be obtained only from observations and measurements.

The operator  $I: \mathcal{H} \rightarrow \mathcal{H}$  satisfying  $I^2 = 1$  is introduced. The operator  $R$  can be written as

$$R = R^{(+)} + R^{(-)} \quad (2)$$

where

$$R^{(+)} = \frac{1}{2}(R + IRI),$$

$$R^{(-)} = \frac{1}{2}(R - IRI).$$

*Definition.* An element  $f$  of  $\mathcal{H}$  is called an equilibrium state (denoted  $m$ ) iff  $Rf = 0$  and  $If = f$ . Equivalently, the equilibrium state is defined by two equations

$$R^{(+)}f = 0 \quad (3)$$

$$R^{(-)}f = 0. \quad (4)$$

The equilibrium state  $m$  will be physically interpreted as the state characterizing the thermodynamical equilibrium state. A justification of this interpretation will be given in the discussion following the requirements A1, A2 below.

We assume that (1) represents our knowledge of the class  $\mathcal{S}$  of real physical systems provided  $\mathcal{O}$  is given. This assumption defines (from the physical point of view) the family of the operators  $R$  that we want to consider. There is a natural question of defining such a class of  $R$  clearly in mathematical terms. Our study of some concrete dynamical theories of non-equilibrium statistical mechanics [1, 2] allows us to formulate three mathematical requirements A1, A2, A3 that will define the class  $\mathcal{C}$  of the operators  $R$ .

*A1.* Equation (3) is solved by a one parameter (denoted  $\pi_1$ ) of submanifolds  $F_{\pi_1}$  of  $\mathcal{H}$ .  
*A2.* Equation (4) restricted to  $F_{\pi_1}$  for all admissible  $\pi_1$  is equivalent to the equation

$$\text{grad } G(\pi_1, \pi_2) = 0 \quad (4.1)$$

where  $G(\pi_1, \pi_2): F_{\pi_1} \rightarrow \mathbb{R}$  is of class at least  $C^2$ ,  $\pi_2$  is another parameter entering the analysis. It is assumed that the admissible values of  $\pi_1, \pi_2$  form an open subset of  $\mathbb{R}^2$ .

The set of all  $g \in F_\pi$ , satisfying (4.1) is called the thermodynamical manifold and will be denoted by  $\mathcal{M}$ . The elements of  $\mathcal{M}$  are the equilibrium states  $m$ . We shall define

$$\pi_3 = G(\pi_1, \pi_2)(m). \quad (5)$$

Since the function  $G$  is defined by (4.1) up to a constant, we shall introduce the following procedure for determining the sign of  $G$  and the thermodynamic interpretation of  $\pi_1, \pi_2, \pi_3$ . Let our physical system be in the thermodynamical equilibrium state  $m$  corresponding, from the thermodynamical point of view, to a positive energy. Then the sign of  $G$  is chosen as such that  $(\partial/\partial\pi_2)G(\pi_1, \pi_2)(m)$  is positive. If  $(\partial/\partial\pi_1)G(\pi_1, \pi_2)(m)$  is also positive then  $\pi_1 = \beta, \pi_2 = \gamma, \pi_3 = \alpha$ , where  $\beta = T^{-1}$ ,  $T$  is the temperature,  $\gamma = \beta p$ ,  $p$  is the pressure,  $\alpha = \beta\mu$ ,  $\mu$  is the chemical potential. If  $(\partial/\partial\pi_1)G(\pi_1, \pi_2)(m)$  is negative, then  $\pi_1 = \beta, \pi_1 = \alpha, \pi_3 = \gamma$ . The function  $G$  is still determined only up to a positive constant that has to be determined in accordance with the units of measurements used in thermodynamics. We shall call  $m \in \mathcal{M}$  a thermodynamically stable equilibrium state denoted  $m_s$  iff the Hessian of  $G(\pi_1, \pi_2)$  evaluated at  $m_s$  is negative definite in the case  $\pi_1 = \beta, \pi_2 = \gamma, \pi_3 = \alpha$  and positive definite in the case  $\pi_1 = \beta, \pi_2 = \alpha, \pi_3 = \gamma$ . An element  $m \in \mathcal{M}$  is called a critical state iff the Hessian of  $G(\pi_1, \pi_2)$  evaluated at  $m_c$  is not a one-to-one linear operator. Geometric properties of  $\mathcal{M}$  and the function  $\pi_3$  completely determine thermodynamics of the systems considered (see more details in Ref. [2]).

Before discussing trajectories in the vicinity of  $\mathcal{M}$  we need a mathematical preparation.

Let

$$\frac{\partial \varphi}{\partial t} = Q\varphi \quad (6)$$

be the time evolution equation,  $\varphi \in H$ , where  $H$  is a real Hilbert space,  $Q$  is a linear operator. Let us have moreover an operator  $\mathcal{J}: H \rightarrow H$  with  $\mathcal{J}^2 = 1$ . The operators  $Q^{(+)}$  and  $Q^{(-)}$  are defined analogically as  $R^{(+)}$  and  $R^{(-)}$  in (2).

*Definition.* A linear operator  $Q$  is called an Onsager operator iff

- i) the operator  $Q$  is closed and densely defined in  $H$ ,
- ii) the operator  $Q^{(+)}$  is a self-adjoint operator and non-positive,  $Q^{(-)}$  is a skew-adjoint operator.

The Onsager operators have the following properties.

*Property 1.* Onsager operator  $Q$  generates uniquely a semigroup  $U_t, t \geq 0$  of the time evolution operators. The operators  $U_t (t \geq 0)$  are bounded,  $U_0 = id_H$  (the identity operator in  $H$ ) and

$$\lim_{t \rightarrow t_0} U_t \varphi = U_{t_0} \varphi \quad \text{for all } t > t_0 \geq 0 \text{ and all } \varphi \in H.$$

This property is a direct consequence of the Stone–Hille–Yoshida–Phillips theory of semigroup (Chapter IX, Section 8 of Ref. [6]).

*Property 2.* No point of the spectral set of an Onsager operator  $Q$  can lie in  $\text{Re}\lambda > 0$  and the residual spectrum of  $Q$  is empty.

The first part follows from the non-positiveness of  $Q$ , the second part is the consequence of two facts: a)  $Q^\dagger = \mathcal{J}Q\mathcal{J}$ , where the cross denotes adjoint, b) if  $\lambda$  is an

element of the residual spectrum of  $Q$  then the complex conjugate of  $\lambda$  is an element of the point spectrum of  $Q^\dagger$ .

*Property 3.* Let us assume that  $\varphi$  is a real valued function of  $q, p$ ,  $\mathcal{J}\varphi(q, p) = \varphi(q, -p)$  and  $U_t$  ( $t \geq 0$ ) can be realized as a semigroup of integral operators

$$\varphi(q, p, t) = U_t \varphi(q, p, 0) = \int dq_0 dp_0 K(q, p, t; q_0, p_0) \varphi(q_0, p_0, 0). \quad (7)$$

Moreover the scalar product  $\langle \cdot, \cdot \rangle$  in  $H$  is realized by

$$\langle \varphi, \psi \rangle = \int dq_1 dp_1 \int dq dp A(q_1, p_1; q, p) \varphi(q_1, p_1) \psi(q, p). \quad (8)$$

Then the following identity holds

$$\begin{aligned} & \int dq_1 dp_1 A(q_1, p_1; q, p) K(q_1, p_1, t; q_0, p_0) \\ &= \int dq_1 dp_1 A(q_1, p_1; q_0, p_0) K(q_1, -p_1; q, -p, -t). \end{aligned} \quad (9)$$

The proof is straightforward, based on  $Q^\dagger = \mathcal{J}Q\mathcal{J}$  and the known relation between  $K$  corresponding to  $Q$  and  $K^\dagger$  corresponding to  $Q^\dagger$ .

*Property 4.* If  $Q$  is an Onsager operator then  $\mathcal{J}Q$  is a self-adjoint operator. The well-known connection between self-adjoint operators and the variational principle can be used. Equivalently, we can say that  $Q$  is a self-adjoint operator in the pseudo-Hilbert space  $\hat{H}$  that is equipped with the pseudo-scalar product  $\langle \cdot, \mathcal{J} \cdot \rangle$ .

We shall now return to equation (1) and to the study of trajectories in the vicinity of  $m_s \in \mathcal{M}$ . First we construct the Hessian  $Q$  of  $R$  evaluated at  $m_s$  (i.e.  $Q$  depends on  $\pi_1, \pi_2$ ). Let the tangent space  $T_{m_s} \mathcal{H}$  at  $m_s$  be a Hilbert space whose scalar product is not specified. Let  $H$  be the closed complement of  $T_{m_s} \mathcal{M}$  in  $T_{m_s} \mathcal{H}$ .

*A3(a).* There is a scalar product in  $H$  such that  $Q$  is an Onsager operator.

An equivalent formulation of the requirement A3(a) is the following. Let  $H$  be the  $L_2$  space. There exists a linear operator  $A: H \rightarrow H$  such that a)  $A$  is bounded, b)  $\mathcal{J}A\mathcal{J} = A$ , c)  $A$  is self-adjoint and positive definite, d)  $AQ$  is an Onsager operator in  $H$  (in the  $L_2$  space).

*A3(b).* There exists a function  $\mathbb{V}: \mathcal{N} \rightarrow R$ , where  $\mathcal{N} \subset \mathcal{M}$  is a neighborhood of  $m_s$ , such that a)  $\mathbb{V}$  restricted to  $F_{\pi_1} \cap \mathcal{N}$  ( $m_s \in F_{\pi_1}$ ) is identical with  $G$  defined in A2 and if  $G$  reaches at  $m_s$  its maximum or minimum then also  $\mathbb{V}$  reaches at  $m_s$  its maximum or minimum respectively, b) the operator  $A$  introduced in A3(a) equals to the Hessian of  $\mathbb{V}$  at  $m_s$  that is equipped with the corresponding sign to make it positive definite.

The function  $\mathbb{V}$  is physically interpreted as the non-equilibrium extension of the thermodynamic potential  $G$ . The identity (9) is physically interpreted as the Onsager reciprocity relations.

### 3. Fluid Mechanics

The experience obtained from the fluid mechanics type of observations and measurements of a certain family of physical systems (called fluids) has been sum-

marized in the phenomenological theory called fluid mechanics. Fluids cannot be defined in theoretical terms inside fluid mechanics. A phenomenological theory based on more detailed observations and measurements like, for example, kinetic theory [3], has to be used to define fluids theoretically. The state of the system in fluid mechanics is completely described by the triple  $f \equiv (N, E, \mathbf{U})$ , where  $N: \Omega \rightarrow \mathbb{R}$ ,  $E: \Omega \rightarrow \mathbb{R}$ ,  $\mathbf{U}: \Omega \rightarrow T\Omega \equiv \Omega \times \mathbb{R}^3$ ,  $\mathbf{r} \in \Omega$  is a position vector,  $\Omega$  is a bounded open subset of  $\mathbb{R}^3$  with smooth boundary  $\partial\Omega$  and volume one,  $T\Omega$  denotes the tangent bundle of  $\Omega$ . The set of all  $f$  is denoted by  $\mathcal{H}$ . The quantities  $N, E, \mathbf{U}$  are called local density, local energy and local velocity respectively. Dynamics in  $\mathcal{H}$  – fluid dynamics – is given by

$$\frac{\partial f}{\partial t} = R^{(F)} f, \quad (1.a)$$

or in components

$$\begin{aligned} \frac{\partial N}{\partial t} &= -U_\alpha \frac{\partial N}{\partial r_\alpha} - N \frac{\partial U_\alpha}{\partial r_\alpha}, \\ N \frac{\partial U_\alpha}{\partial t} &= -\frac{\partial p}{\partial r_\alpha} - N U_\beta \frac{\partial U_\alpha}{\partial r_\beta} + \eta \frac{\partial^2 U_\alpha}{\partial r_\beta \partial r_\beta} + (\frac{1}{3} \eta + \eta_v) \frac{\partial}{\partial r_\alpha} \frac{\partial U_\beta}{\partial r_\beta}, \\ N \frac{\partial E}{\partial t} &= -N U_\alpha \frac{\partial E}{\partial r_\alpha} + \lambda \frac{\partial^2 \tau}{\partial r_\alpha \partial r_\alpha} - p \frac{\partial U_\alpha}{\partial r_\alpha} + \eta_v \left( \frac{\partial U_\alpha}{\partial r_\alpha} \right)^2 + 2\eta(D_{\alpha\beta} - \frac{1}{3} D_{\gamma\gamma} \delta_{\alpha\beta}) D_{\alpha\beta}, \end{aligned}$$

where

$$D_{\alpha\beta} = \frac{1}{2} \left( \frac{\partial U_\alpha}{\partial r_\beta} + \frac{\partial U_\beta}{\partial r_\alpha} \right).$$

The phenomenological quantities  $\mathcal{P}^{(F)}$  in fluid dynamics (1) are the real-valued twice-differentiable functions  $p, \tau$  of  $E(\mathbf{r}), N(\mathbf{r})$ , called the local pressure and the local temperature, respectively, and the real-valued functions  $\lambda, \eta, \eta_v$  of  $E(\mathbf{r}), N(\mathbf{r})$ , called the coefficient of thermal conductivity, shear viscosity and bulk viscosity respectively. The individuality of each particular fluid is expressed just in the concrete specification of  $\{p, \tau, \lambda, \eta, \eta_v\}$  which can be done – inside fluid mechanics – only through the fluid mechanics observations and measurements. We shall assume that  $\lambda \neq 0$  and  $\tau|_{\partial\Omega} = T$ , where  $T$  is a positive constant.

#### 4. Application of the General Theory in Fluid Dynamics

We shall follow closely Section 2. Equation (1) has now the concrete form (1.a). The operator  $I$  is defined by  $(N, E, \mathbf{U}) \xrightarrow{I} (N, E, -\mathbf{U})$ . The operators  $R^{(F)(+)}_+$  and  $R^{(F)(-)}_-$  are found easily. It follows from the definition of the equilibrium states that  $\mathbf{U}_{\text{eq}} \equiv 0$  (we use the notation  $m \equiv (N_{\text{eq}}, E_{\text{eq}}, \mathbf{U}_{\text{eq}})$ ), and  $N_{\text{eq}}$  and  $E_{\text{eq}}$  are determined by

$$\frac{\partial^2 \tau}{\partial r_\alpha \partial r_\alpha} = 0 \quad (3.a)$$

$$\frac{\partial p}{\partial r_\alpha} = 0. \quad (4.a)$$

Since we have assumed  $\tau|_{\delta\Omega} = T$ , equation (3.a) is solved by  $\tau(N, E) = T$ . Equation (4.a) is clearly equivalent to  $\phi(N, E) = P$ , where  $P$  is a constant. We introduce new quantities  $\beta = (1/T)$ ,  $\gamma = (P/T)$ ,  $V = (1/N)$ . Following the general theory we want to write (4.a) restricted to the manifolds  $\tau(E, N) = T$  in the variational form. This can be done if  $\phi$  and  $\tau$  are not independent but related by

$$\frac{\partial \phi}{\partial E} - \frac{\phi}{\tau} \frac{\partial \tau}{\partial E} = - \frac{1}{\tau} \frac{\partial \tau}{\partial V} \quad (10)$$

The function  $G$  is defined by

$$G = \int_{\Omega} d^3 \mathbf{r} (-S(E, V) + \gamma V + \beta E),$$

where

$$\frac{\partial S}{\partial V} = \frac{\phi}{\tau}, \quad \frac{\partial S}{\partial E} = \frac{1}{\tau}.$$

Equation (10) is equivalent to

$$\frac{\partial^2 S}{\partial E \partial V} = \frac{\partial^2 S}{\partial V \partial E}. \quad (10.a)$$

The thermodynamics of the system is completely determined by

$$\alpha = G|_{\mathcal{M}}$$

and geometric properties (see Ref. [2]) of  $\mathcal{M}$ . The thermodynamically stable equilibrium state  $\mathbf{m}_s \equiv (N_0, E_0, 0)$  is defined as the equilibrium state at which  $G$  reaches its minimum. It is easy to see, just by writing the necessary condition of stationarity of  $G$  explicitly, that the thermodynamical energy  $E_{th}$ , defined in thermodynamics by  $E_{th} = \partial \alpha / \partial \beta$ , equals  $\int_{\Omega} d^3 \mathbf{r} E_{eq}(\mathbf{r})$ , and the thermodynamical volume  $V_{th}$ , defined in thermodynamics by  $V_{th} = \partial \alpha / \partial \gamma$ , equals  $\int_{\Omega} d^3 \mathbf{r} V_{eq}(\mathbf{r})$ .

We shall now study the trajectories in the vicinity of a thermodynamically stable equilibrium state  $\mathbf{m}_s$ . Straightforward calculations give the Hessian  $Q^{(F)}$  of  $R^{(F)}$  evaluated at  $\mathbf{m}_s \equiv (N_0, E_0, 0)$ .

$$\frac{\partial \varphi}{\partial t} = Q^{(F)} \varphi, \quad (6.a)$$

where  $\varphi \equiv (n(\mathbf{r}), e(\mathbf{r}), \mathbf{u}(\mathbf{r}))$  and its first derivatives vanish on  $\partial\Omega$ ,  $\varphi \in H$ ,  $H$  is a Hilbert space and

$$Q^{(F)} \equiv \begin{pmatrix} 0 & 0 & -\frac{1}{V_0} \frac{\partial}{\partial r_1} & -\frac{1}{V_0} \frac{\partial}{\partial r_2} & -\frac{1}{V_0} \frac{\partial}{\partial r_3} \\ V_0 \lambda_1 \tau_n \frac{\partial^2}{\partial r_\alpha \partial r_\alpha} & V_0 \lambda_1 \tau_e \frac{\partial^2}{\partial r_\alpha \partial r_\alpha} & -V_0 P \frac{\partial}{\partial r_1} & -V_0 P \frac{\partial}{\partial r_2} & -V_0 P \frac{\partial}{\partial r_3} \\ -V_0 \dot{p}_n \frac{\partial}{\partial r_1} & -V_0 \dot{p}_e \frac{\partial}{\partial r_1} & V_0 \left( \eta_1 \frac{\partial^2}{\partial r_\alpha \partial r_\alpha} + \eta_2 \frac{\partial^2}{\partial r_1^2} \right) & V_0 \eta_2 \frac{\partial^2}{\partial r_1 \partial r_2} & V_0 \eta_2 \frac{\partial^2}{\partial r_1 \partial r_3} \\ -V_0 \dot{p}_n \frac{\partial}{\partial r_2} & -V_0 \dot{p}_e \frac{\partial}{\partial r_2} & V_0 \eta_2 \frac{\partial^2}{\partial r_1 \partial r_2} & V_0 \left( \eta_1 \frac{\partial^2}{\partial r_\alpha \partial r_\alpha} + \eta_2 \frac{\partial^2}{\partial r_2^2} \right) & V_0 \eta_2 \frac{\partial^2}{\partial r_2 \partial r_3} \\ -V_0 \dot{p}_n \frac{\partial}{\partial r_3} & -V_0 \dot{p}_e \frac{\partial}{\partial r_3} & V_0 \eta_2 \frac{\partial^2}{\partial r_1 \partial r_3} & V_0 \eta_2 \frac{\partial^2}{\partial r_3 \partial r_2} & V_0 \left( \eta_1 \frac{\partial^2}{\partial r_\alpha \partial r_\alpha} + \eta_2 \frac{\partial^2}{\partial r_3^2} \right) \end{pmatrix}$$

where

$$\eta_1 = \eta(V_0, E_0), \quad \eta_2 = \frac{1}{3} \eta(V_0, E_0) + \eta_v(V_0, E_0), \quad \lambda_1 = \lambda(V_0, E_0),$$

$$\tau_n = \frac{\partial \tau}{\partial N} \Bigg|_{m_s}, \quad \tau_e = \frac{\partial \tau}{\partial E} \Bigg|_{m_s}, \quad \dot{p}_n = \frac{\partial \dot{p}}{\partial N} \Bigg|_{m_s}, \quad \dot{p}_e = \frac{\partial \dot{p}}{\partial E} \Bigg|_{m_s},$$

we assume that  $\eta_1$ ,  $\eta_2$  and  $\lambda_1$  are positive. The operator  $Q^{(F)(+)}$  is obtained by putting  $Q_{ij}^{(F)} = 0$  for  $i = 3, 4, 5$ ,  $j = 1, 2$  and  $i = 1, 2$ ,  $j = 3, 4, 5$ . The operator  $Q^{(F)(-)}$  is obtained by putting  $Q_{ij}^{(F)} = 0$  for  $i = 1, 2$ ,  $j = 1, 2$  and  $i = 3, 4, 5$ ,  $j = 3, 4, 5$ .

Following the general theory we shall assume that  $H$  is the  $L_2$  space, i.e.

$$(\varphi_1, \varphi_2) = \int_{\Omega} d^3 \mathbf{r} n_1(\mathbf{r}) n_2(\mathbf{r}) + \int_{\Omega} d^3 \mathbf{r} e_1(\mathbf{r}) e_2(\mathbf{r}) + \int_{\Omega} d^3 \mathbf{r} u_\alpha(\mathbf{r}) u_\alpha(\mathbf{r}).$$

The problem is to find the operator  $A^{(F)}: H \rightarrow H$  satisfying all the properties listed in A3(a) of Section 2. It is clear that the operator  $A^{(F)}$  will have the form

$$A^{(F)} \equiv \begin{pmatrix} A_{11} & A_{12} & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (11)$$

From the requirement that  $A^{(F)} Q^{(F)(+)}$  is self-adjoint, we have

$$A_{22} \tau_n = A_{12} \tau_e, \quad (12)$$

from the requirement that  $A^{(F)}Q^{(F)(-)}$  is skew-adjoint, we have

$$p_n = \frac{A_{11}}{V_0^2} + PA_{12} \quad (13)$$

$$p_e = PA_{22} + \frac{A_{12}}{V_0^2}$$

The solution of (12), (13) is

$$\begin{aligned} A_{12} &= \frac{\tau_n p_e}{\frac{\tau_n}{V_0^2} + P\tau_e} \\ A_{11} &= V_0^2 \left( p_n - \frac{P\tau_n p_e}{\frac{\tau_n}{V_0^2} + P\tau_e} \right) \\ A_{22} &= \frac{\tau_e p_e}{\frac{\tau_n}{V_0^2} + P\tau_e}. \end{aligned} \quad (14)$$

Since  $p$  and  $\tau$  are related by (10), which reads now

$$\tau_n = TV_0^2 p_e - PV_0^2 \tau_e, \quad (10.a)$$

we have

$$\begin{aligned} A_{12} &= \frac{1}{T} \tau_n \\ A_{22} &= \frac{1}{T} \tau_e \\ A_{11} &= V_0^2 \left( p_n - \frac{P}{T} \tau_n \right). \end{aligned} \quad (15)$$

It remains to show that the operator  $A^{(F)}$ , defined in (11), (15), is positive definite. Let us define in a neighborhood  $\mathcal{N}$  of  $m_s$  the function  $\mathbb{V}$  by

$$\mathbb{V} = \int_{\Omega} d^3 \mathbf{r} (-S(E, V) + \frac{1}{2} \beta U_{\alpha} U_{\alpha} + \gamma V + \beta E). \quad (16)$$

One easily recognizes that  $\mathbb{V}$  satisfies the property a), in A3(b) of Section 2 and that

$$A^{(F)} = \frac{1}{\beta} \begin{pmatrix} \frac{\partial^2 \mathbb{V}}{\partial N^2} & \frac{\partial^2 \mathbb{V}}{\partial E \partial N} & 0 & 0 & 0 \\ \frac{\partial^2 \mathbb{V}}{\partial E \partial N} & \frac{\partial^2 \mathbb{V}}{\partial E^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial^2 \mathbb{V}}{\partial U_1^2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial^2 \mathbb{V}}{\partial U_2^2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial^2 \mathbb{V}}{\partial U_3^2} \end{pmatrix} \quad (17)$$

The derivatives in (17) should be understood of course as the functional derivatives (i.e., if using another notation,

$$\frac{\partial^2 \mathbb{V}}{\partial E^2} = \frac{\delta^2 \mathbb{V}}{\delta E(\mathbf{r}) \delta E(\mathbf{r}')} \delta(\mathbf{r} - \mathbf{r}'),$$

etc.). All derivatives in (17) are evaluated at  $m_s$ . It follows from the definition of  $m_s$  that  $A^{(F)}$  is positive definite. Standard arguments can be used to show that  $A^{(F)}Q^{(F)}$  defined on  $L_2$  space is densely defined and closed and  $A^{(F)}Q^{(F)(+)}$  is non-positive. It means, according to the definition in Section 2, that  $A^{(F)}Q^{(F)}$  on  $L_2$  space is an Onsager operator.

## 5. Discussion

We have shown that  $A^{(F)}Q^{(F)}$  defined on  $L_2$  space or  $Q^{(F)}$  defined on the Hilbert space  $H$  equipped with the scalar product  $(\cdot, A^{(F)})$  is an Onsager operator if  $m_s$  is a thermodynamically stable equilibrium state. We can use four properties of the Onsager operators derived in Section 2 and in this way obtain four properties of the solutions of the equations of fluid dynamics. These, or similar results, are already known in fluid dynamics. The advantage of our derivation is, beside the simplicity, the relation – through the general theory – to other dynamical theories of non-equilibrium statistical mechanics.

In view of the critique of the Onsager reciprocity relations in fluid dynamics [7], we would like to point out the application of Property 3 in fluid dynamics. The kernel  $K^{(F)}$  in the integral representation of  $U_t^{(F)}$  is now a  $5 \times 5$  matrix

$$K^{(F)}(\mathbf{r}, t; \mathbf{r}') = (K_{ij}^{(F)}(\mathbf{r}, t; \mathbf{r}')). \quad (7.a)$$

Equation (9) then reads

$$A^{(F)} K^{(F)}(\mathbf{r}, t; \mathbf{r}_0) = \tilde{K}^{(F)}(\mathbf{r}_0; \mathbf{r}, -t) A^{(F)}, \quad (9.a)$$

where the tilde means transposition and the reversion of signs on places  $i = 3, 4, 5$ ;  $j = 1, 2$  and  $j = 3, 4, 5$ ;  $i = 1, 2$ .

The scalar product  $(\cdot, A^{(F)})$  is clearly very convenient from the mathematical point of view (it makes  $Q^{(F)}$  an Onsager operator). At least Properties 1 and 3 have also direct relation to measurements and observations in fluid mechanics. The semi-flow is observed so that the mathematical proof of its existence is a check of the consistency of the mathematical description. For the experimental basis of the Onsager reciprocity relations see Ref. [4]. Another argument for the naturality of  $(\cdot, A^{(F)})$  is based on the relation of  $A^{(F)}$  to  $G$  (and  $V$ ), which is the fundamental quantity in other phenomenological theory, namely thermodynamics. The norm based on the scalar product  $(\cdot, A^{(F)})$  is basically the rate of the entropy production.

Finally we shall point out the difference between the approach to fluid dynamics used here and the approach used in Ref. [4]. We start our discussion with dynamic equations that are accepted as a summary of the experience obtained from fluid mechanical types of observations and measurements. Everything that follows (including the thermodynamical concepts) is a consequence of dynamics and appears naturally in the dynamical system theory type discussions of the solutions of the dynamical equations. On the other hand the fluid dynamics presented in Ref. [4] starts with thermodynamics, generalizes it to the local thermodynamics and then from balance equations, by postulating the linear relationship between fluxes and forces and the Onsager reciprocity relation, the dynamical equations of fluid dynamics are obtained. The thermodynamics and its generalization that stays at the beginning in Ref. [4] and the thermodynamics and its generalization that is a consequence of the postulated fluid dynamics are identical. The advantage of our approach appears clearly in more microscopic phenomenological theories (based on a more detailed set of measurements and observations) like, for example, in kinetic theory [1] where the corresponding thermodynamics and its appropriate non-equilibrium generalizations are not intuitively evident.

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