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Relaxation of Local Perturbations in the Groundstate of the Heisenberg Ferromagnet

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Abstract. The relation between the relaxation of local perturbations in the groundstate of the ferromagnet and the scattering of spin waves is studied, and it is shown that certain local perturbations return to the groundstate as $t \to \infty$. The effects of the formation of boundstates of magnons are discussed.

Introduction

The problem of the return to equilibrium of a local perturbation of an infinitely extended quantum system has been formulated in the algebraic framework of statistical mechanics in Refs. [1] and [2]. A property which plays a central role in this description is the asymptotic abelianness of the kinematic observables under the time evolution. This condition is known to be true for the free Fermi gas and the X–Y spin model [1, 3, 4]. For these quasi-free systems this is a purely dynamical fact because it holds in the norm of the algebra independently of their actual state. Its origin has to be sought in the dispersive nature of the one-particle wave-packets.

As another illustration of the theory, we study in this note the relaxation of local perturbations in the ground state of the Heisenberg Ferromagnet. For this more complex model one should expect asymptotic abelianness properties only in the weak sense with respect to a given state. Since almost nothing is known on the dynamics of the ferromagnet in states which differ globally from the ground state, we have to limit our investigation to this very special case. It is, nevertheless, useful to examine it in some detail because it allows us to exhibit in a very precise way the mechanism responsible for the relaxation. This mechanism is the scattering of spin waves. We may think that the return to equilibrium in low temperature states can be attributed to the same physical ground.

In the first section, we give an equivalent formulation for the relaxation of local perturbations in the spin wave (or magnon) language. A local perturbation returns to the ground state if and only if the quasi-particles associated with the perturbed state leave asymptotically any finite space region. With the help of the time-dependent scattering theory of magnons [5], we show in Section II that many of them have indeed this behaviour. The effects of the formation of bound states of two magnons are also discussed.

In Section III we consider some dynamical perturbations of the Heisenberg evolution. We may conjecture weak asymptotic abelianness in the ground state with the Heisenberg evolution, but this property is not likely to hold any more with the perturbed evolution. The perturbation might isolate a finite subsystem and thus prevent dissipation, as has been noticed in Ref. [1].

A more physical reason for the lack of asymptotic abelianness is, in our case, the existence of bound states of the quasi-particles in external fields or with magnetic impurities.

I. Local Evanescence and Relaxation of Local Perturbations

We briefly recall the mathematical structure of the ground-state representation of the ferromagnet [6]. Let $\mathscr{A}$ be the C^*-algebra of a spin-$\frac{1}{2}$ lattice Z^ν and $\omega_0 = \prod_{j\in Z^\nu} \otimes |\uparrow\rangle_j$ be the infinite tensor product state in which all spins are in the direction 3 at all lattice points.

The Hilbert space of the irreducible representation of $\mathscr{A}$ defined by ω_0 is the incomplete tensor product $\mathscr{H} = \prod_{j\in Z^\nu} \otimes \mathbb{C}_j^2$ with respect to $\prod_{j\in Z^\nu} \otimes |\uparrow\rangle_j$. We shall still denote by $\mathscr{A}$ and σ_j^1, σ_j^2, σ_j^3 the quasi-local algebra and the spin operators in the representation associated with ω_0.

The group of automorphisms of rotations around the axis 3 is unitary implemented in this representation. Its generator can be written as $N = \sum_{j\in Z^\nu} \frac{1}{2}(1-\sigma_j^3)$, whose spectrum is the set of all non-negative integers $\{n = 0, 1, 2, \ldots\}$. Accordingly, $\mathscr{H}$ can be decomposed in the direct sum $\sum_{n=0}^{\infty} \oplus \mathscr{H}^n$ which diagonalizes N. An orthonormal basis in $\mathscr{H}^n$ is given by the vectors $\sigma_{j_1}^- \sigma_{j_2}^- \ldots \sigma_{j_n}^- \phi_0, j_1, j_2, \ldots, j_n \in Z^\nu$ having exactly n-spin deviations. $\sigma_j^\pm = \sigma_j^1 \pm i\sigma_j^2$, and $\phi_0 \in \mathscr{H}^0$ is the unique vector in $\mathscr{H}$ which is invariant under the translations Z^ν.

The time evolution automorphism τ_t of the quasi-local algebra is also implemented in the representation by a unitary group V_t with generator H. Since the Heisenberg Hamiltonian is invariant under the rotations, V_t commutes with N and is therefore reduced by the subspaces $\mathscr{H}^n$. We denote by $V_t^n(H^n)$ the restriction of $V_t(H)$ to $\mathscr{H}^n$.

$\mathscr{H}^1$ is isomorphic with the Hilbert space of functions $\phi(j)$ on Z^ν with $\sum_{j\in Z^\nu} |\phi(j)|^2 < \infty$. The space $\mathscr{H}^1$ of one spin deviation can be viewed as a one-particle space (one magnon) whose free motion is given by V_t^1. More precisely, in the momentum representation, V_t^1 is an operator of multiplication by the phase $\exp(-i\epsilon(\kappa)t)$ where $\epsilon(\kappa)$ is the kinetic energy of the magnon. For a nearest-neighbour interaction in three dimensions one has simply $\epsilon(\kappa) = J(3 - \cos\kappa^1 - \cos\kappa^2 - \cos\kappa^3)$, $\kappa = (\kappa^1, \kappa^2, \kappa^3)$. In the general case we shall always assume that the spectrum of H^1 is absolutely continuous[1]).

In a similar way one can consider $\mathscr{H}^n$ as a n-magnon space whose interacting motion is governed by V_t^n. The configuration representation of $\mathscr{H}^n$ is given by the set of square summable functions $\phi(j_1, j_2, \ldots, j_n)$ on $\prod^n \times Z^\nu$ which are symmetric in all their arguments and vanish whenever two of them are equal. This latter restriction expresses the kinematical constraints due to the fact that we cannot have more than one spin deviation at a lattice point.

We introduce the operator P_l^n on $\mathscr{H}^n$ which projects on the lattice point l:

$$(P_l^n \phi^n)(j_1 \ldots j_n) = \chi_l^n(j_1 \ldots j_n)\, \phi^n(j_1 \ldots j_n) \tag{1}$$

with

$$\chi_l^n(j_1 \ldots j_n) = \begin{cases} 0 & \text{if } j_\kappa \neq l,\ \kappa = 1 \ldots n \\ 1 & \text{otherwise} \end{cases}$$

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[1]) $\epsilon(\kappa)$ and its derivatives are smooth functions such that the transformation $d\kappa^i = (\partial\epsilon(\kappa)/\partial\kappa^i)^{-1} d\epsilon$ is possible almost everywhere.

It is clear that $(\phi^n, P_l^n \phi^n) = \|P_l^n \phi^n\|^2$ gives the probability of finding a magnon at the lattice point l in the state ϕ^n.

We say that a n-magnon state ϕ^n in $\mathscr{H}^n$ has the property of local evanescence as $t \to +\infty$ ($t \to -\infty$) under V_t^n if the probability of finding a magnon in a finite region goes to zero as $t \to +\infty$ ($t \to -\infty$), that is, if

$$s - \lim_{\substack{t \to +\infty \\ (-\infty)}} P_l^n V_t^n \phi^n = 0$$

for all lattice points $l \in Z^\nu$. This property characterizes typically a scattering state of a multiparticle system.

More generally, we say that a vector $\phi = \sum_{n=0}^{\infty} \oplus \phi^n$ in $\mathscr{H}$ has the property of local evanescence if each component $\phi^n \in \mathscr{H}^n$ has the same property. This is equivalent with

$$s - \lim_{\substack{t \to +\infty \\ (-\infty)}} P_l V_t \phi$$

for all l, where P_l is the direct sum of the P_l^n.

We establish now a relation between the relaxation of a local perturbation of the ground state ω_0 and the local evanescence of magnon states in $\mathscr{H}^n$. We call a local perturbation of ω_0 any state on $\mathscr{A}$ normal with respect to ω_0. We consider more specifically local perturbations $\omega_\phi(A) = (\phi, A\phi)$, $\phi \in \mathscr{H}$, which are vector states.

Proposition 1: Let ω_ϕ be a local perturbation of ω_0 (a vector state). Then

$$\lim_{t \to +\infty} \omega_\phi(\tau_t(A)) = \omega_0(A) \quad \text{for all } A \in \mathscr{A} \tag{2}$$

if and only if ϕ has the property of local evanescence as $t \to +\infty$. An analogous statement is true for the limit $t \to -\infty$.

Proof: It is sufficient to establish (2) for the local elements in $\mathscr{A}$. Since the Pauli matrices $\sigma^0 = 1$, σ^+, σ^- and σ^3 form a linearly independent set on $\mathscr{B}(\mathbb{C}^2)$, each local $A_\mathcal{O}$ in $\prod_{j \in \mathcal{O}} \otimes \mathscr{B}(\mathbb{C}^2)$, $\mathcal{O} c Z^\nu$, can be written in a unique way as a finite linear combination of products $\Gamma_\mathcal{O}^\alpha = \prod_{j \in \mathcal{O}} \sigma_j^{\alpha(j)}$ with $\alpha(j) = 0, +, -$ or 3 and $\alpha = \{\alpha(j), j \in \mathcal{O}\}$. Therefore a state on $\mathscr{A}$ is determined by the value that it takes on such products. In particular:

$$\omega_0(\Gamma_\mathcal{O}^\alpha) = \begin{cases} 1 & \text{if } \alpha(j) = 0 \text{ or } 3 \text{ for all } j \in \mathcal{O} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and we only need to verify (2) on these elements.

We first notice that σ_l^3 is reduced by the subspaces $\mathscr{H}^n$ and is identical with $1 - 2P_l^n$ on them. If ϕ has the property of local evanescence, we obtain

$$s - \lim V_t^* \sigma_l^3 V_t \phi = \phi - 2s - \lim V_t^* P_l V_t \phi = \phi \quad \text{for all } l.$$

More generally we conclude that

$$s - \lim V_t^* \Gamma_\mathcal{O}^\alpha V_t \phi = \phi \quad \text{when } \alpha(j) = 0 \text{ or } 3 \text{ for all } j \in \mathcal{O}. \tag{4}$$

Secondly, in virtue of the identity

$$\|\sigma_l^+ \psi\|^2 = (\psi, \sigma_l^- \sigma_l^+ \psi) = \|P_l \psi\|^2, \quad \psi \in \mathscr{H},$$

we get

$$s - \lim V_t^* \sigma_l^+ V_t \phi = 0 \quad \text{for all } l \in Z^\nu.$$

If in a product $\Gamma_{\mathcal{O}}^\alpha$, $\alpha(j) = +$ for at least one j we have obviously also

$$s - \lim V_t^* \Gamma_{\mathcal{O}}^\alpha V_t \phi = 0. \tag{5}$$

If $\alpha(j) \neq +$ for all j, but $\alpha(j) = -$ for at least one $j \in \mathcal{O}$, $(\Gamma_{\mathcal{O}}^\alpha)^*$ is again a product for which $\alpha(j) = +$ for one j, and

$$s - \lim V_t^* (\Gamma_{\mathcal{O}}^\alpha)^* V_t \phi = 0. \tag{6}$$

From (4), (5) and (6) we conclude that in all cases

$$\lim \omega_\phi(\tau_t(\Gamma_{\mathcal{O}}^\alpha)) = \lim (\phi, V_t^* \Gamma_{\mathcal{O}}^\alpha V_t \phi)$$
$$= \lim (V_t^* (\Gamma_{\mathcal{O}}^\alpha)^* V_t \phi, \phi) = \omega_0(\Gamma_{\mathcal{O}}^\alpha).$$

The converse is immediate. If the locally perturbed state ω_ϕ returns to ω_0, one must have in particular

$$\lim \omega_\phi(\tau_t(P_l)) = \tfrac{1}{2}(1 - \lim \omega_\phi(\tau_t(\sigma_l^3))) = 0$$

for all $l \in Z^\nu$ which is equivalent with $s\text{-}\lim P_l V_t \phi = 0$.

The argument can immediately be extended to a general local perturbation ω of ω_0. Such a state is of the form

$$\omega = \sum_i \lambda_i \omega_{\phi_i}, \quad \lambda_i \geqslant 0, \quad \sum_i \lambda_i = 1 \quad \text{and} \quad \lim \omega(\tau_t(A)) = \omega_0(A)$$

if and only if each ϕ_i, $i = 1, 2, \ldots$ has the property of local evanescence.

As a corollary of Proposition 1, we deduce that $(\mathscr{A}, \tau_t, \omega_0)$ is weakly asymptotically abelian if and only if all vectors in $\mathscr{H}$ are locally evanescent.

In this case (4), (5) and (6) hold on all of $\mathscr{H}$. Then the commutators $[V_t^* A V_t, B]$ converge weakly to zero on $\mathscr{H}$ and we have weak asymptotic abelianness:

$$\lim \omega_0(C[\tau_t(A), B] D) = 0, \quad A, B, C, D \in \mathscr{A} \tag{7}$$

Conversely if (7) is true we choose $A = P_l$, $D = I$, $C = B^*$ in (7) and get

$$\lim \|P_l V_t B\phi_0\| = 0 \quad \text{for all } B \in \mathscr{A}.$$

The result follows from the fact that the set of vectors $B\phi_0$, $B \in \mathscr{A}$, is dense in $\mathscr{H}$.

II. Locally Evanescent Magnon States

a) Scattering states

We have mentioned that a n-magnon state ϕ^n is locally evanescent if it behaves as a scattering state, that is if the probability of finding a magnon in a finite space region vanishes as $t \to +\infty$.

There are many states of this type. To show this, we use a result of Hepp [5] on the time-dependent scattering theory of magnons which was obtained on the basis of Dyson's analysis [7] of the ferromagnet. Let us first describe this result.

We introduce the symmetrized n-particle space

$$\mathscr{F}^n = \prod_{\text{sym}}^{n} \otimes \mathscr{L}^2(Z^\nu).$$

It can be viewed as an ideal n-magnon space with no kinematical constraints.

$\mathscr{H}^n$ is then identified with a subspace of $\mathscr{F}^n$ with the help of the projection $T^n \mathscr{F}^n = \mathscr{H}^n$ defined by:

$$(T^n \phi^n)(j_1, j_2 \ldots j_n) = \begin{cases} \phi^n(j_1, j_2 \ldots j_n) & \text{when } j_1 \neq j_2 \neq \ldots \neq j_n \\ 0 & \text{otherwise} \end{cases}$$

T^n projects out states with more than a single spin deviation at a given lattice point.

$\mathscr{F}^n$ is isomorphic to the symmetrized product of n one-magnon spaces and we can define on it a free evolution operator of ideal independent magnons by $U_t^n = \prod^n \otimes V_t^1$.

An asymptotic condition holds for each n in the following sense [5]

$$s - \lim_{t \to \pm\infty} (V_t^n)^* T^n U_t^n = \Omega_\pm^n \tag{8}$$

and $\Omega_\pm^n$ are isometric from $\mathscr{F}^n$ to $\mathscr{H}^n$ with

$$(\Omega_\pm^n)^* \Omega_\pm^n = I, \quad \Omega_\pm^n (\Omega_\pm^n)^* = F_\pm^n$$

$F_\pm^n$ are projection operators on $\mathscr{H}^n$ characterizing the states in $\mathscr{H}^n$ which evolve asymptotically as n independent ideal magnons.

We expect local evanescence for such states and this is the content of the next proposition.

Proposition 2: The vectors ϕ^n belonging to the subspace $F_+^n \mathscr{H}^n (F_-^h \mathscr{H}^n)$ have the property of local evanescence as $t \to +\infty$ ($t \to -\infty$).

Proof: Since the proof holds for each fixed n-magnon space, we drop the index n in the following. If ϕ belongs to $F_+ \mathscr{H}^n$ ϕ is of the form $\Omega_+ g$ for some $g \in \mathscr{F}^n$ and

$$\lim_{t \to +\infty} \|P_l V_t \phi - P_l T U_t g\| = 0$$

because of the asymptotic condition (8). Therefore it suffices to show that

$$s \lim_{t \to +\infty} P_l T U_t g = 0.$$

It is clear from its definition (1) that P_l can be written as the sum $P_l = \sum_{\kappa=1}^n \chi_l^\kappa$ of projections χ_l^κ with

$$(\chi_l^\kappa \phi)(j_1, j_2 \ldots j_n) = \chi_l(j_\kappa) \phi(j_1, j_2 \ldots j_n)$$

$$\chi_l(j) = \begin{cases} 1 & j = l \\ 0 & j \neq l \end{cases}$$

χ_l^κ commutes with T for $\kappa = 1, 2, \ldots, n$ and in a one-magnon space χ_l^κ is the one-dimensional projection on the lattice point l.

We have

$$\|P_l\, TU_t g\| \leqslant \sum_{\kappa=1}^{n} \|\chi_l^\kappa\, TU_t g\| = \sum_{\kappa=1}^{n} \|T\chi_l^\kappa\, U_t g\| \leqslant \sum_{\kappa=1}^{n} \|\chi_l^\kappa\, U_t g\|.$$

Now $\chi_l^\kappa U_t$ converges strongly to zero on all of $\prod^n \otimes \mathscr{L}^2(Z^\nu)$, and hence on its totally symmetric subspace $\mathscr{F}^n$. It is sufficient to verify this on the dense subset generated by the tensor product states

$$\prod_{i=1}^{n} \otimes g_i, \quad g_i \in \mathscr{H}^1 = \mathscr{L}^2(Z^\nu)$$

$$\|\chi_l^\kappa U_t \prod_{i=1}^{n} \otimes g_i\|^2 = \|\chi_l^\kappa \prod_{i=1}^{n} \otimes V_t^1 g_i\|^2 = \|\chi_l V_t^1 g_\kappa\|^2.$$

This last quantity converges to zero as $t \to +\infty$ since the one-magnon evolution V_t^1 has an absolutely continuous spectrum and χ_l is a finite dimensional projection on $\mathscr{H}^1$. This implies

$$s - \lim_{t \to \infty} P_l\, TU_t g = 0$$

and concludes the proof.

We summarize what we obtain from the combination of Propositions 1 and 2 by the following statement.

For any local perturbation ω of ω_0 normal on $\mathscr{B}(F_+\mathscr{H})$,

$$\lim_{t \to +\infty} \omega(\tau_t(A)) = \omega_0(A), \quad A \in \mathscr{A}.$$

An analogous statement holds for the limit $t \to -\infty$.

We add three remarks:

i) One has to consider separately the limits $t \to +\infty$ and $t \to -\infty$. We cannot infer without further information that a perturbation which relaxes as $t \to +\infty$ has the same behaviour as $t \to -\infty$.

ii) A vector ϕ in $F_\pm\mathscr{H}$ is in general not of the form $A\phi_0$ with some A in $\mathscr{A}$. It only belongs to the closure in $\mathscr{H}$ of the set $\{A\phi_0, A \in \mathscr{A}\}$.

iii) The relaxation is coupled with the phenomena of wave dispersion as in the Fermi gas and the X–Y model. The relaxation times are long and the decay is not exponential. For a next-neighbour interaction, the third component of a reversed spin at the lattice point l reaches its asymptotic value as

$$|(l|V_t^1|l)|^2 \sim (\mathscr{T}_0(Jt))^{2\nu} \sim ct^{-\nu}, \quad t \to \infty.$$

($\mathscr{T}_0(x)$ is the Bessel function of order zero.)

b) Two-magnon boundstates

We have shown that there are many local perturbations of the groundstate ω_0 which return to ω_0. The question remains open to know whether it is true for all of them. All local perturbations performed at a single lattice point, namely perturbations in $\mathscr{H}^0 \oplus \mathscr{H}^1$, have this property since the wave operators $\Omega_\pm$ leave the groundstate and the one-magnon states invariant. But the existence of boundstates of two magnons [8] implies $F_\pm^2 \mathscr{H}^2 \neq \mathscr{H}^2$. A local perturbation which affects simultaneously two lattice points may have a component in the subspace generated by the two-magnon boundstates, that is in the orthogonal complement of $F_\pm \mathscr{H}^2$. Nevertheless we can also expect such a state to be locally evanescent because of the propagation of the 'centre of mass' of the two-magnon boundstate. This is most easily seen for the one-dimensional Heisenberg chain with nearest-neighbour interaction.

Since the total momentum is a constant of the motion, it is convenient to use 'centre of mass' and relative coordinates $R = (j_1 + j_2)/2$, $r = j_1 - j_2$ and to represent $\mathscr{H}^2$ as the direct integral $\int_B^\oplus \mathscr{H}_p^2 dp$ which diagonalizes the total momentum $p = \kappa_1 + \kappa_2$. B is the first Brillouin zone. Each $\mathscr{H}_p^2$ is isomorphic with the set of square summable functions $\psi(r)$ on Z^ν with $\psi(r) = \psi(-r)$ and $\psi(0) = 0$. The evolution $V_t^2 = \int_B^\oplus V_{pt}^2 dp$ is also reduced by this direct integral decomposition and for each p it can have one or several boundstates ψ_p in $\mathscr{H}_p^2$ satisfying

$$V_{pt}^2 \psi_p = \exp(-i\epsilon(p)\, t)\, \psi_p, \quad \|\psi_p\| = 1.$$

For the one-dimensional chain it is known [8] that there is exactly one boundstate for each p with eigenvalue

$$\epsilon(p) = J\tfrac{1}{2}(1 - \cos p). \tag{9}$$

Proposition 3: The direct integral $\psi = \int_B^\oplus \psi_p dp$ of the two-magnon boundstates ψ_p for the Heisenberg chain with nearest-neighbour interaction is locally evanescent at $t = +\infty$ and $t = -\infty$.

Proof: In this case $P_l^2 = \chi_l^1 + \chi_l^2$ and we have to verify that

$$\lim_{t \to \pm\infty} \|\chi_l^1 V_t \psi\| = \lim_{t \to \pm\infty} \|\chi_l^2 V_t \psi\| = 0.$$

We have

$$\|\chi_l^1 V_t \psi\|^2 = \sum_{j>l}^{\infty} |(V_t \psi)(l,j)|^2. \tag{10}$$

Written in terms of relative coordinates and total momentum $(V_t\psi)(l,j)$ takes the form

$$(V_t \psi)(l,j) = \int_B dp \exp(-i\epsilon(p)\, t) \exp\left(ip\left(\frac{l+j}{2}\right)\right) \psi_p(j-l).$$

For fixed j, the function $p \to \psi_p(j-l)$ is square integrable over B:

$$\int_B |\psi_p(j-l)|^2 dp \leqslant \sum_{j>l}^{\infty} \int_B |\psi_p(j-l)|^2 dp = \int_B \sum_{j>l}^{\infty} |\psi_p(j-l)|^2 dp$$

$$= \int_B \|\psi_p\|^2 dp = \|\psi\|^2. \tag{11}$$

Hence, with the form (9) of the energy, $(V_t\psi)(l,j)$ converges to zero as $t \to \pm\infty$.

Moreover, one has by Schwartz's inequality

$$|(V_t \psi)(l,j)|^2 \leqslant B^2 \int_B |\psi_p(j-l)|^2 \, dp.$$

This shows with (11), that each term of the series (10) is majorized uniformly in t by the term of a converging series. Hence $\|\chi_l^1 V_t \psi\|^2$ converges to zero by the dominated convergence theorem.

Proposition 3 indicates that the formation of boundstates of magnons does not prevent, in principle, the return to equilibrium. In fact, if we know that the wave operators $\Omega_\pm^2$ are complete (i.e. if $(F_+^2 \mathscr{H}^2)^\perp = (F_-^2 \mathscr{H}^2)^\perp$ is identical with the subspace generated by the boundstates), we can conclude that all local perturbations at two lattice points will return to ω_0 at $t = +\infty$ and $t = -\infty$. In order to treat perturbations at three and more lattice points, we would have to control the multichannel scattering theory, the formation of boundstates and the completeness of the wave operators in a all n-magnon space.

We have seen that asymptotic abelianness in the groundstate is equivalent with local evanescence of all magnon states. Although this property is plausible, one could presumably not escape to deal with this complex problem if one wishes to prove it.

III. Local Dynamical Perturbations

It is also of interest to study the properties of local dynamical perturbations. A local dynamical perturbation is introduced by adding to the Heisenberg Hamiltonian an element of the quasi-local algebra. The resulting perturbed automorphism of $\mathscr{A}$ has been precisely defined in Ref. [1]. It has been also noted that asymptotic abelianness of $\mathscr{A}$ is not stable, in general, with respect to local dynamical perturbations. We show that we find the same situation in the groundstate of the ferromagnet. Physically, typical local dynamical perturbations are magnetic impurities or locally applied magnetic fields. They allow the possibility of forming boundstates of magnons in the external potential whose occurrence destroys asymptotic abelianness.

We illustrate this point with the simplest possible perturbation, which consists of applying a magnetic field in the 3-direction at a single lattice point, say at $j = 0$. The perturbation is therefore the local element $\lambda\sigma_0^3 = -2\lambda P_0 + \lambda I$. It is equivalent to choose the element μP_0 $(\mu = -2\lambda)$, since a perturbation which is a multiple of the identity does not modify the automorphism. The perturbed automorphism τ_t^μ is implemented in the representation defined by ω_0 with generator (see Ref. [1]):

$$H_\mu = H + \mu P_0.$$

H_μ still commutes with N and the dynamics can be treated separately in each n-magnon space $\mathscr{H}^n$. In $\mathscr{H}^1$, H_μ^1 is simply the kinetic energy of the magnon perturbed by a local potential which is the one-dimensional projection on the lattice point $j = 0$. It is easy to see (see Appendix) that H_μ^1 has exactly one boundstate ϕ_μ, $\|\phi_\mu\| = 1$, for each $\mu \neq 0$ and is absolutely continuous on the orthogonal complement of ϕ_μ.

We then have the

Proposition 4: Let ω_ϕ be a local perturbation of ω_0 with $\phi \in \mathscr{H}^1$, $\|\phi\| = 1$, then for $\mu \neq 0$

$$\lim_{t \to \pm\infty} \omega_\phi(\tau_t^\mu(A)) = |a|^2 \omega_{\phi_\mu}(A) + (1 - |a|^2)\, \omega_0(A), \quad A \in \mathscr{A} \tag{12}$$

with $a = (\phi_\mu, \phi)$.

Proof: If we decompose ϕ in

$$\phi = (\phi_\mu, \phi)\, \phi_\mu + \psi \tag{13}$$

with $(\phi_\mu, \psi) = 0$ we obtain

$$\omega_\phi(\tau_t^\mu(A)) = a(\phi_\mu, (V_{\mu t}^1)^* A V_{\mu t}^1 \psi) + a^*(\psi, (V_{\mu t}^1)^* A V_{\mu t}^1 \phi_\mu)$$

$$+ (\psi, (V_{\mu t}^1)^* A V_{\mu t}^1 \psi) + |a|^2 \omega_{\phi_\mu}(A).$$

As in Proposition 1, we only need to verify (12) on the products $\Gamma_{\mathcal{O}}^\alpha$. We remark first that on the $\Gamma_{\mathcal{O}}^\alpha$ for which the number of indices $\alpha(j) = +$ and $\alpha(j) = -$ are different, all involved states vanish. The other types of products leave $\mathscr{H}^1$ invariant. There are two cases. If $\alpha(j) = 0$ or 3 for all j, $\Gamma_{\mathcal{O}}^\alpha$ is of the form $I - R$ with R of finite rank on $\mathscr{H}^1$ and $\omega_0(\Gamma_{\mathcal{O}}^\alpha) = 1$. If there is a pair of indices $(+, -)$ among the $\alpha(j)$, $\Gamma_{\mathcal{O}}^\alpha$ is itself of finite rank on $\mathscr{H}^1$ and $\omega_0(\Gamma_{\mathcal{O}}^\alpha) = 0$.

Since ψ is in the absolutely continuous subspace of H_μ^1, we use in (13) the fact that

$$\lim_{t \to \pm\infty} \|A V_{\mu t}^1 \psi\| = 0$$

when A is a finite rank operator on $\mathscr{H}^1$. From that and the above description of various $\Gamma_{\mathcal{O}}^\alpha$, we obtain the result of Proposition 4.

Proposition 4 shows that there are local perturbations of ω_0 which do not return to ω_0 with the automorphism τ_t^μ for $\mu \neq 0$. For instance, with an initial perturbation in $\mathscr{H}^1$ not orthogonal to ϕ_μ, the magnetization $m_j(t) = \omega_\phi(\tau_t^\mu(\sigma^3))$ does not relax to its groundstate value 1, but to the stationary distribution $1 - 2|a|^2|\phi_\mu(j)|^2$ where $\phi_\mu(j)$ is the amplitude of the magnon boundstate at lattice point j.

The treatment of local dynamical perturbations which are invariant under the rotations around the 3 axis can always be reduced to a scattering problem of magnons in external fields. When they are able to produce several boundstates, we have the following situations. The evolution of a perturbation of ω_0 reaches a limit which can be ω_0 or a different state as in Proposition 4. It can also have no limit because of the possibility of oscillating interference terms between the various boundstates. Dynamical perturbations of other types (applied fields in the 1 or 2 direction) lead to mathematical problems of a different nature and they must be treated with the methods of field theory.

APPENDIX

The Hamiltonian in the one-magnon space is

$$H_\mu^1 = \epsilon(\kappa) + \mu\chi_0(j), \quad \chi_0(j) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}$$

It leads to the eigenvalue equation

$$\epsilon(\kappa)\, \phi(\kappa) + \mu \int_B \phi(\kappa)\, d\kappa = E\phi(\kappa).$$

The eigenvalue E is given by the solution of

$$f(E) = \int_B \frac{1}{E - \epsilon(\kappa)}\, d\kappa = \frac{1}{\mu}$$

$f(E)$ is a monotonous function of E with range $(-\infty, 0)$ for $E < \inf_{\kappa \in B} \epsilon(\kappa)$ and range $(0, \infty)$ for $E > \sup_{\kappa \in B} \epsilon(\kappa)$. Therefore, there is exactly one solution E for each $\mu \neq 0$[2]. Since the perturbation is of rank one, the wave operators for (H_μ^1, H^1) exist and are complete [9]. This means that H_μ^1 is equivalent to H^1 on its subspace of absolute continuity. In order to conclude that this subspace coincides with the orthogonal complement of the boundstate, we have to know that H_μ^1 has no singularly continuous spectrum. For a rank-one perturbation, the spectral family of H_μ^1 can be computed explicitly and we can check that it is indeed the case owing to the smoothness of the function $\epsilon(\kappa)$. We shall not give the details here.

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[2]) $\epsilon(\kappa)$, $\kappa \in B$, is different from zero almost everywhere.