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Autor: Straumann, N.

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Minimal Assumptions Leading to a Robertson–Walker Model of the Universe

by **N. Straumann**

University of Zürich and SIN

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The homogeneous and isotropic cosmological models of the Robertson–Walker (or Friedmann) type assume the following structure:

i) A time-like vector field $\mathbf{V}$ with $\mathbf{g}(\mathbf{V},\mathbf{V}) = -1$ (which represents the average 4-velocity field of matter) whose integral curves are geodesics.
ii) The vector field $\mathbf{V}$ is hyper-surface orthogonal, i.e. the space-time manifold can be decomposed into a family of space-like surfaces, which are orthogonal to $\mathbf{V}$.
iii) The orthogonal hyper-surfaces are maximally symmetric and hence Riemannian manifolds of constant curvature.
iv) The mapping $\varphi_{12}: O_1 \to O_2$, which is induced by the flow ϕ_t of $\mathbf{V}$ between any two orthogonal hyper-surfaces O_1 and O_2, with their respectively induced metrics $\mathbf{g}_1$ and $\mathbf{g}_2$, satisfies[1]) $\varphi_{12}^*(\mathbf{g}_2) = \lambda_{12}\mathbf{g}_1$, where λ_{12} is a constant. (In co-moving coordinates this means that the metric tensors of all orthogonal hyper-surfaces are equal up to a scaling factor.)

In this note, we show that this structure is already implied by the assumption that the space-time manifold is isotropic at every point relative to the vector field $\mathbf{V}$ in the sense of the definition given below. This fact is certainly known to many people (see for instance Ref. [2]), but we did not find a mathematical proof of it in the literature.

Let there be given a time-like vector field $\mathbf{V}$ with $\mathbf{g}(\mathbf{V},\mathbf{V}) = -1$ on the space-time manifold $(M,\mathbf{g})$ with Lorentz metric $\mathbf{g}$. H_p denotes the subspace of the tangent space $T_p(M)$ orthogonal to $\mathbf{V}(p)$ and $\mathrm{Iso}_p(M)$ denotes the group of all isometries of M with fixed point p. Furthermore, let $SO_3(V_p)$ be the group of all linear transformations of $T_p(M)$ which leave $\mathbf{V}(p)$ invariant and induce special orthogonal transformations in H_p.

Definition: The space-time manifold $(M,\mathbf{g})$ is isotropic at $p \in M$ relative to $\mathbf{V}$ if the following condition is satisfied

$$\{T_p\varphi : \varphi \in \mathrm{Iso}_p(M), \varphi_* \mathbf{V} = \mathbf{V}\} \supseteq SO_3(V_p).$$

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[1]) Unexplained notations are standard, as for instance laid down in Ref. [1].

We assume now that $(M, \mathbf{g})$ is isotropic relative to $\mathbf{V}$ at every space-time point and prove that the properties (i)–(iv) can be deduced from this assumption.

First we show that $\mathbf{V}$ is hyper-surface orthogonal. This means that the differential system $D\colon p \mapsto H_p$ is completely integrable [3]. For this to be true, it is necessary and sufficient – as has been shown by Frobenius [3] – that the following equation holds

$$\boldsymbol{\omega} \wedge \mathbf{d}\boldsymbol{\omega} = 0, \tag{1}$$

where $\boldsymbol{\omega}$ denotes the differential form corresponding to $\mathbf{V}[\boldsymbol{\omega}(\mathbf{X}) = \mathbf{g}(\mathbf{V}, \mathbf{X})$ for all vector fields $\mathbf{X}]$.

In order to prove (1) at an arbitrary point $m \in M$, we consider the exponential mapping Exp_m at the point m. Any $\varphi \in \mathrm{Iso}_m(M)$ with $T_m(\varphi) \in SO_3(V_m)$ can be represented as follows

$$\varphi\colon \mathrm{Exp}_m(X) \mapsto \mathrm{Exp}_m(RX); \quad X \in T_m(M), \quad R \in SO_3(V_m). \tag{2}$$

We introduce the following basis $\mathbf{E}_\mu$ in $T_m(M)\colon \mathbf{E}_i$, $i = 1, 2, 3$ are orthonormal vectors in H_p and $\mathbf{E}_4 = \mathbf{V}(m)$. The normal coordinates in the neighbourhood of m, with respect to this basis, will be denoted by x^μ. In these coordinates, the mapping (2) is given by

$$\varphi\colon x^\mu \mapsto R^\mu{}_\nu x^\nu, \tag{3}$$

where $R^\mu{}_\nu$ is the matrix corresponding to R with respect to the basis $\mathbf{E}_\mu$. Hence $R^i{}_j$, $i, j = 1, 2, 3$, is a special orthogonal matrix, $R^i{}_4 = R^4{}_i = 0$ and $R^4{}_4 = 1$. The Killing fields belonging to the local isometries (3) are of the form

$$K^\mu = L^\mu{}_\nu x^\nu,$$

where

$$(L^\mu{}_\nu) = \left(\begin{array}{c|c} L^i{}_k & 0 \\ \hline 0 & 0 \end{array}\right), \qquad L^i{}_k + L^k{}_i = 0. \tag{4}$$

The Lie derivative of $\boldsymbol{\omega}$ with respect to every Killing field of the form (4) vanishes

$$L_{\mathbf{K}}\, \boldsymbol{\omega} = 0. \tag{5}$$

In components this reads

$$\frac{\partial V_\mu}{\partial x^\nu} L^\nu{}_\lambda x^\lambda + V_\nu L^\nu{}_\mu = 0. \tag{6}$$

Especially for $L^j{}_k = \epsilon_{jks}$, $s = 1, 2, 3$, we obtain

$$\frac{\partial V_i}{\partial x^j} \epsilon_{jks} x^k + \epsilon_{jis} V_j = 0.$$

Taking the derivative of this equation with respect to x^l and putting $x^\mu = 0$ gives

$$\epsilon_{jls} \frac{\partial V_i}{\partial x^j} + \epsilon_{jis} \frac{\partial V_j}{\partial x^l} = 0.$$

For $i = s$ we obtain

$$\epsilon_{ijk} \frac{\partial V_k}{\partial x j} = 0.$$

This implies at the point m ($V_0 = 1$, $V_i = 0$)

$$\epsilon^{\mu\nu\lambda\sigma} V_\nu \partial_\lambda V_\sigma = 0,$$

i.e. equation (1). Hence, we have shown that locally[2]) the space-time manifold can be decomposed into a family of space-like hyper-surfaces orthogonal to $\mathbf{V}$.

Next, we show that the integral curves of $\mathbf{V}$ are geodesics. An isometry φ of M leaves the affine connection $\boldsymbol{\nabla}$ invariant. If in addition V is invariant under φ this means that

$$\varphi^*(\boldsymbol{\nabla}_\mathbf{V}\mathbf{V}) = \boldsymbol{\nabla}_\mathbf{V}\mathbf{V}.$$

Hence we have

$$L_\mathbf{K}(\boldsymbol{\nabla}_\mathbf{V}\mathbf{V}) = 0 \tag{7}$$

for the Killing vector fields (4). Let $\mathbf{W} := \boldsymbol{\nabla}_\mathbf{V}\mathbf{V}$; then (7) reads in components

$$L^\mu{}_\lambda x^\lambda \frac{\partial W^\nu}{\partial x^\mu} - L^\nu{}_\lambda W^\lambda = 0.$$

At the point m ($x^\lambda = 0$) we obtain

$$L^i{}_l W^l = 0.$$

Hence $W^l = 0$ for $l = 1, 2, 3$. This means that $\boldsymbol{\nabla}_\mathbf{V}\mathbf{V}$ is parallel to $\mathbf{V}$. But from $\mathbf{g}(\mathbf{V}, \mathbf{V}) = -1$ we obtain $\mathbf{g}(\mathbf{V}, \boldsymbol{\nabla}_\mathbf{V}\mathbf{V}) = 0$ and consequently $\boldsymbol{\nabla}_\mathbf{V}\mathbf{V} = 0$.

A well-known variational formula for families of geodesics [4] then tells us that the geodesic distance along the integral curves of $\mathbf{V}$ between any two orthogonal hyper-surfaces is constant. This enables us to introduce a 'cosmic time'.

Obviously the sectional curvature of any orthogonal hyper-surface is isotropic. A well-known theorem of Schur implies then that the hyper-surface is a Riemannian manifold of constant curvature. This proves property (iii).

Finally the property (iv) of the flow of $\mathbf{V}$ is a consequence of the following Lemma, which is a sharpening of a statement proved in Ref. [5].

Lemma: Let $(M, \mathbf{g})$ be a Riemannian space of constant curvature with dimension $n \geqslant 3$ and let $\mathbf{A}$ be a covariant tensor field of second rank on M which is invariant under every $\varphi \in \mathrm{Iso}_m(M)$ for any point $m \in M$. Then $\mathbf{A} = \text{const.}\ \mathbf{g}$.

Proof: We introduce normal coordinates around $m \in M$ with respect to an orthonormal basis of $T_m(M)$. Then we have by assumption

$$L_\mathbf{K}\mathbf{A} = 0, \tag{8}$$

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[2]) The considerations in this note are only of a local nature.

for all Killing fields $\mathbf{K}$ with components of the form

$$K^i = \Omega^i{}_j x^j, \quad \Omega^i{}_j + \Omega^i{}_j = 0.$$

In components equation (8) reads

$$K^l \partial_l A_{ij} + A_{lj} \partial_i K^l + A_{il} \partial_j K^l = 0.$$

From this we obtain at the point $m(x^i = 0)$

$$A_{lj} \Omega^l{}_i + A_{il} \Omega^l{}_j = 0$$

or

$$\Omega^l{}_k (\delta^k{}_i A_{lj} + \delta^k{}_j A_{il}) = 0. \tag{9}$$

Hence the bracket in (9) must be symmetric in l and k:

$$\delta_{ik} A_{lj} + \delta_{kj} A_{il} = \delta_{il} A_{kl} + \delta_{lj} A_{ik}. \tag{10}$$

Contraction of k and i gives

$$(n-1) A_{lj} + A_{jl} = \delta_{lj} A^k{}_k. \tag{11}$$

If we interchange in this equation j and l and subtract the two, we get

$$(n-2)(A_{jl} - A_{lj}) = 0.$$

Hence, for $n > 2$, $\mathbf{A}$ must be a symmetric tensor. If we use this in equation (11), we obtain

$$n A_{lj} = \delta_{lj} A^k{}_k.$$

Consequently $\mathbf{A} = \lambda \mathbf{g}$. Since $\mathbf{A}$ and $\mathbf{g}$ are invariant under every $\varphi \in \mathrm{Iso}_m(M)$, $m \in M$, the function λ has to be invariant under φ too: $\varphi^*(\lambda) = \lambda$. Now the set $\{\varphi \in \mathrm{Iso}_m(M) : m \in M\}$ acts transitively on a space with constant curvature. This is geometrically obvious. Alternatively it is easy to show that one can construct out of the Killing fields arising from isometries with fixed point, a Killing field $\mathbf{K}$, which has any prescribed value at a given point $m \in M$ and for which $L_{\mathbf{K}}\lambda = 0$. This also shows that λ has to be a constant.

The property (iv) follows from the Lemma, because the flow ϕ_t of $\mathbf{V}$ commutes with a sufficiently large group of isometries from $\mathrm{Iso}_m(M)$, $m \in M$, which in turn implies that $\varphi_{12}^*(\mathbf{g}_2)$ in property (iv) satisfies on O_1 the assumptions made for the tensor field $\mathbf{A}$ in the Lemma.

Observationally the average large-scale properties of the universe seem to be approximately isotropic around us. In particular, it has been shown that the extragalactic radio sources are distributed approximately isotropically and that the 3°K microwave background radiation is isotropic to a remarkably high degree. If one assumes that the universe looks isotropic for any observer moving with the 'average cosmological fluid' (Copernican principle) then the cosmological model is, as we have shown, necessarily of the Robertson–Walker (Friedmann) type.

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