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# Schwinger Functions and their Generating Functionals, I

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*Abstract.* (Euclidean) Markoff fields and in particular the fields of the $P(\varphi)_2$ models in two-dimensional space-time are studied. It is shown that the states on the Markoff fields, i.e. the generating functionals for the Schwinger functions of the $P(\varphi)_2$ models with different boundary conditions, converge as the interaction region tends to $\mathbb{R}^2$. The generating functionals in the infinite volume limit are in 1–1 correspondence with Euclidean invariant measures on $\mathscr{S}'$; (here $\mathscr{S} = \mathscr{S}_{\text{real}}(\mathbb{R}^2)$). Existence of coincident Schwinger functions and continuity in the time arguments are proven. The Wightman axioms are verified for the quantum fields in the infinite volume limit. Some results on the general structure of Markoff field theory are presented.

## 1. Introduction

This is the first part of two papers on the Schwinger functions, the generating functionals and the infinite volume limit interacting measures of the well-known $P(\varphi)_2$ models in the Euclidean formulation. The material of these papers is organized as follows:

PART I

*Section 2.* Review of estimates for $\varphi$-perturbations of the $P(\varphi)_2$ quantum field Hamiltonian.

*Section 3.* Analysis of Markov field theory, the generating functional for the Schwinger functions, applications to the $P(\varphi)_2$ models with space cutoff.

*Section 4.* The infinite volume limit for the generating functionals of the $P(\varphi)_2$ models, verification of the Wightman axioms, sharp-time Euclidean fields.

PART II (to be published)

*Section 5.* $\mathscr{S}$-quasi-invariance of the interacting measure for the $P(\varphi)_2$ models in the infinite volume limit. Local Markov property and DLR equations. Canonical structure of the $P(\varphi)_2$ quantum field theory.

*Section 6.* Asymptotic perturbation expansion for Araki functionals and Euclidean local number operators.

REMARK.

The results of Section 2 and a certain number of results proven in Sections 3 and 4 are basic for the analysis presented in Section 5 and Section 6.

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### *1.1. Main results of Part I*

We prove that the generating functionals for the Schwinger functions (Euclidean Green's functions [1, 2, 3] etc.) exist in the infinite volume limit and are in a 1–1 correspondence with Euclidean invariant measures on $\mathscr{S}'$ for the $P(\varphi)_2$ models in the Euclidean formulation of quantum field theory. We prove the existence of coincident Schwinger functions [4], continuity of sharp-time Schwinger functions, and we verify a special form of the Osterwalder–Schrader axioms [3, 5] yielding Wightman's axioms [6, 7]. We show that the physical vacuum is an analytic vector for the (time 0–) quantum fields $\varphi(f)$, $f \in \mathscr{S}(\mathbb{R})$. All the essential results of this paper and Part II and their proofs are formulated within the framework of Markov and Euclidean field theory in the sense of Symanzik and Nelson [1, 2, 4, 5, 8–10].

*Acknowledgments.* I thank Prof. Lascoux for financial support of my participation at a meeting on Euclidean field theory in Paris which was a source of inspiration for this paper. This meeting was organized by F. Guerra and B. Simon. I also thank Prof. B. Simon for numerous helpful conversations and very valuable criticism. Finally I thank the Département de Physique Théorique de l'Université de Genève for the kind hospitality at a time when most of this work was done.

REMARKS.

These two papers extend the results of an unpublished manuscript written in June 1973. I am grateful to B. Simon and O. Bratteli for pointing out some errors and suggesting changes in the first and second version of these papers.

After the completion of this work I learned that Prof. G. Hegerfeldt has also studied generating functionals for the Schwinger functions from an abstract point of view (preprints, University of Göttingen, 1973). He describes Nelson–Symanzik and Osterwalder–Schrader positivity in the language of generating functionals.

## 2. Review of Estimates for $\varphi$-Perturbations of the $P(\varphi)_2$ Quantum Field Hamiltonian

In this section we summarize some basic results on perturbations of the Hamiltonian $H_l$ with space cutoff $l$ by functions of the (time 0–) quantum field $\varphi$. These results were first proven by Glimm and Jaffe [11] and later with some modifications by Guerra, Rosen and Simon [12–14]. They are one of the starting points for this paper.

Let us begin Section 2 by summarizing some fundamental field theoretic terminology, such as Fock space, quantum field (boundary conditions) etc. We also collect some results on the spatially cutoff $P(\varphi)_2$ quantum field Hamiltonian $H_l$. As general references we recommend Glimm and Jaffe [15, 16]; see also Ref. [4].

### *2.1. The Fock space for the free quantum field, spatially cutoff $P(\varphi)_2$ Hamiltonians with different boundary conditions*

Let $\mathscr{F}$ be the symmetric Fock space over $L^2(\mathbb{R})$ [4, 10, 15, 16] associated with the creation and annihilation operators $a^*(f)$, $a(f)$, $(f \in L^2(\mathbb{R}))$ and the vacuum $\Omega_0$, which has the property that $a(f)\Omega_0 = \vec{0}$, for all $f$ in $L^2(\mathbb{R})$. We define the *free one-particle*

*Hamiltonian* $\mu_{l,s}$ by

$$\mu_{l,s} := \sqrt{-\frac{d^2}{dx^2} + m_0^2 + s^2(1-\chi_l)} \tag{2.1}$$

where $\chi_l(x)$ is the characteristic function of the interval $[-l/2, l/2]$. Note that $\mu_{l,s}$ is to be understood as the square root of the operator

$$-\frac{d^2}{dx^2} + m_0^2 + s^2(1-\chi_l)$$

which is positive and self adjoint (s.a.) on $L^2(\mathbb{R})$. Thus $\mu_{l,s}$ is well defined. The positive real number $s$ is called *mass parameter* [4].

*Remark:* We say that an operator $A$ is well defined (s.a., positive...) on a Hilbert space $\mathscr{H}$ iff its domain of definition $D(A)$ is dense in $\mathscr{H}$ ($A$ is s.a. on $D(A)$; the quadratic form determined by $A$ is positive...).

The *free Hamiltonian* $H_{l,s}^0$ on $\mathscr{F}$ is defined to be the *biquantization* of $\mu_{l,s}$, i.e.

$$H_{l,s}^0 := d\Gamma(\mu_{l,s}) = \int dk\, dl a^*(k)\, \tilde{\mu}_{l,s}(k,l)\, a(l), \tag{2.2}$$

where $\tilde{\mu}_{l,s}(k,l)$ is the kernel of $\mu_{l,s}$ in the momentum space representation. If $s=0$

$$\tilde{\mu}_{l,s=0}(k,l) = \sqrt{k^2+m_0^2}\,\delta(k-l). \tag{2.3}$$

The operator $H_{l,s}^0$ is *positive and s.a.* on $\mathscr{F}$.

We define the (time 0–) *Newton–Wigner field* $\varphi_0$:

$$\varphi_0(x) := \frac{1}{\sqrt{4\pi}} \int dk\, e^{ikx}\{a^*(k) + a(-k)\} \tag{2.4}$$

and the (time 0–) quantum field by

$$\varphi_{l,s}(x) := (\mu_{l,s}^{-1/2} * \varphi_0)(x). \tag{2.5}$$

The canonically conjugate momentum is given by

$$\pi_{l,s}(x) := i[H_{l,s}^0, \varphi_{l,s}(x)]. \tag{2.6}$$

*Notations:* If $s=0$ we write $\mu$ for $\mu_{l,s=0}$, $H_0$ for $H_{l,0}^0$, $\varphi$ for $\varphi_{l,0}$ and $\pi$ for $\pi_{l,0}$. (The objects $\mu_{l,0}$, $H_{l,0}^0, \ldots$ are *independent* of $l$). We call these boundary conditions *free* boundary conditions.

We now define Wick powers of the fields $\varphi_{l,s}$.

$$:\varphi_{l,s}(f)^n: := \sum_{m=0}^{[n/2]} \frac{(-1)^m n!}{2^m m!(n-2m)!} \varphi_{l,s}(f)^{n-2m} \|\mu^{-1/2} f\|_2^{2m}.$$

Note that $\|\mu^{-1/2} f\|_2$ is independent of $s$ and $l$. Therefore this definition of Wick powers of the field agrees with the usual definition of Wick powers *only if* $s=0$.

Let $P$ be a *real polynomial* with *even, positive* leading coefficient. We define the *interaction Hamiltonian* with space cutoff $l$ by

$$V_{l,s} \equiv V_{l,s}(P) := \int_{-l/2}^{l/2} dx : P(\varphi_{l,s}) : (x). \tag{2.7}$$

The *spatially cutoff* $P(\varphi)_2$ *Hamiltonian* is given by

$$\hat{H}_{l,s} \equiv \hat{H}_{l,s}(P) := H^0_{l,s} + V_{l,s}. \tag{2.8}$$

*Theorem 2.1: Let $0 \leqslant s < \infty$ and $l < \infty$. Then the operator $V_{l,s}$ is s.a. on $\mathscr{F}$, $\hat{H}_{l,s}$ is essentially s.a. on $D(H^0_{l,s}) \cap D(V_{l,s})$ and*

$$\hat{H}_{l,s} \geqslant -O(1) \cdot l > -\infty \tag{2.9}$$

*uniformly in $l$ and $s \geqslant 0$.*

*We set*

$$E_{l,s}(P) := \inf \operatorname{spec} \hat{H}_{l,s}(P)$$

*and*

$$H_{l,s} := \hat{H}_{l,s} - E_{l,s}(P). \tag{2.10}$$

*The operator $H_{l,s}$ has a unique ground state $\Omega_{l,s} \in \mathscr{F}$ corresponding to the eigenvalue 0 and*

$$(\Omega_0, \Omega_{l,s}) \neq 0. \tag{2.11}$$

*Proof:* Precise formulations of this theorem and proofs follows from Refs. [4, 15–19].

*Notations:* If $s = 0$ we write $V_l$ for $V_{l,0}$, $\hat{H}_l$ for $\hat{H}_{l,0}$, $E_l(P)$ for $E_{l,0}(P)$.

We now consider the case where $s = \infty$ (see Ref. [4]): The free one-particle Hamiltonian $\mu_{l,\infty}$ is defined to be the square root of $-(d^2/dx^2)_{l,D} + m_0^2$ on the space $L^2_{l,D}$ of square integrable functions with support in $[-l/2, l/2]$. Here $(d^2/dx^2)_{l,D}$ is the Laplacian on $L^2_{l,D}$ with *Dirichlet boundary conditions* at $x = \pm l/2$.

We define $\mathscr{F}_{l,D}$ to be the symmetric Fock space over $L^2_{l,D}$.

*Notations:* We write $H^0_{l,D}$ for $H^0_{l,\infty}$, $V_{l,D}$ for $V_{l,\infty}$, $H_{l,D}$ for $H_{l,\infty}$, etc. We call $H_{l,D}$ the Hamiltonian with half Dirichlet boundary conditions [4].

*Theorem 2.2: If $l < \infty$, then the operator $V_{l,D}$ is s.a. on $\mathscr{F}_{l,D}$, $H_{l,D}$ is essentially s.a. on $D(H^0_{l,D}) \cap D(V_{l,D})$ and has a unique vacuum $\Omega_{l,D}$ corresponding to the eigenvalue 0 and*

$$(\Omega_0, \Omega_{l,D}) \neq 0.$$

*Furthermore*

$$E_{l,D} = \inf \operatorname{spec} \hat{H}_{l,D} = \lim_{s \to \infty} E_{l,s} = \sup_s E_{l,s}. \tag{2.12}$$

*Proof:* The theorem follows from Refs. [15, 16] by essentially the same proofs that work for Theorem 2.1. See Ref. [4] for a careful analysis of the Dirichlet boundary conditions.

### 2.2. $\varphi$-Perturbations of the $P(\varphi)_2$ Hamiltonians

All the results of Sub-section 2.2 (and most of the results in Section 3) concern the spatially cutoff $P(\varphi)_2$ theory with *free boundary conditions*, i.e. these results are formulated in terms of $\varphi$, $\pi$, $H_l$, ... These results extend to the case of Dirichlet boundary conditions (see Sub-section 3.5).

Let $P$ be the polynomial defined in Sub-section 2.1 (2.7), and let $Q$ be some real polynomial with $\deg Q < \deg P$. Let $h$ be a real function with $\|h\|_\infty \leqslant 1$ and $\operatorname{supp} h \subset [-l/2, l/2]$. We set

$$\delta H(h) := \int :Q(\varphi):(x)\, h(x)\, dx, \tag{2.13}$$

and

$$\delta E(l,h) := \inf \operatorname{spec}(H_l + \delta H(h)). \tag{2.14}$$

*Estimate I: Under these assumptions on $P$, $Q$, and $h$*

$$\pm\, \delta H(h) \leqslant H_l + \delta E(l, \mp h)$$

***and***

$$|\delta E(l,h)| \leqslant a\|h\|_1 + b_l \cdot \operatorname{diam}(\operatorname{supp} h), \tag{2.15}$$

***where** a depends only on $P$ and $Q$ but not on $h$, and $b_l$ depends only on $P$ and $|b_l| = O(1/l)$ (**as** $l \to \infty$).*

*Corollary: On the physical Hilbert space we have under the same assumptions and for arbitrary $\epsilon > 0$*

$$\pm\, \delta H(h) < \epsilon H_{\mathrm{ren.}} + c(\epsilon)\, \|h\|_1 \tag{2.16}$$

*where $c(\epsilon)$ only depends on $P$, $Q$, and $\epsilon$ but not on $h$.*

*Proof:* See Ref. [11]. The form in which we have stated (2.15) and (2.16) is proven in Ref. [13]. See also Simon [14].

*Estimate II: Let $\operatorname{supp} h \subset [-l/2, l/2]$. Then*

$$\pm\, \varphi(h) \leqslant K\|h\|_1 (H_l + I) \tag{2.17}$$

*where the constant $K$ depends only on $P$ but neither on $l$ nor on $h$. If $K\|h\|_1 < 1$ we get*

$$\pm\, \varphi(h) < H_l + K\|h\|_1. \tag{2.18}$$

*Proof:* Inequality (2.17) was first proven in Ref. [11]; another proof is given in Simon [14]. Inequality (2.18) is a trivial consequence of (2.17) and the positivity of $H_l$.

Q.E.D.

As an illustration of the methods developed in Refs. [12, 13, 20] we prove the following:

*Estimate III:*

$$\pm \lambda\varphi(h) \leqslant H_l + K_2\lambda^2 \|h\|_2^2 + K_3 \cdot \mathrm{diam}(\mathrm{supp}\, h). \tag{2.19}$$

*Proof:* We use Nelson's symmetry [9, 20] for the proof of (2.19). Let supp $h \subseteq [-a, a]$. We set

$$\hat{H}_T(t) := \hat{H}_T + \lambda h(t) \int_{-T/2}^{T/2} dx\, \varphi(x),$$

$$U_T(h, a, -a) := s - \lim_{N\to\infty} \prod_{m=0}^{N} \exp\left[-\frac{2a}{N}\hat{H}_T\left(-a + \frac{2ma}{N}\right)\right]$$

and the existence of the limit follows from the results of Sub-section 3.2.

Nelson's symmetry yields:

$$\begin{aligned}(\Omega_0, \exp[-T(\hat{H}_l + \lambda\varphi(h))]\,\Omega_0) &= \left(\exp\left[-\left(\frac{l}{2} - a\right)\hat{H}_T\right]\Omega_0,\ U_T(h, a, -a)\right.\\ &\qquad \left.\times \exp\left[-\left(\frac{l}{2} - a\right)\hat{H}_T\right]\Omega_0\right)\\ &\leqslant (\Omega_0, \exp[-(l-2a)\,\hat{H}_T]\,\Omega_0)\|U_T(h, a, -a)\|\\ &= (\Omega_0, \exp[-T\hat{H}_{l-2a}]\,\Omega_0)\|U_T(h, a, -a)\|.\end{aligned}$$

Now

$$\|U_T(h, a, -a)\| \leqslant \exp\left[-\int_{-a}^{a} dt |\inf \mathrm{spec}\ \hat{H}_T(t)|\right] \tag{2.20}$$

and

$$\begin{aligned}\inf \mathrm{spec}\ \hat{H}_T(t) \geqslant \inf \mathrm{spec}&\left(\epsilon H_0 + \lambda h(t) \int_{-T/2}^{T/2} dx\, \varphi(x)\right)\\ &+ (1-\epsilon) \inf \mathrm{spec}\left(H_0 + \frac{1}{1-\epsilon} V_T(P)\right),\end{aligned} \tag{2.21}$$

for all $0 < \epsilon < 1$. Clearly

$$\epsilon H_0 + \lambda h(t) \int_{-T/2}^{T/2} dx\, \varphi(x) \geqslant -\frac{K_2}{2\epsilon}\lambda^2 |h(t)|^2\, T \tag{2.22}$$

for some $K_2 < \infty$. From the linear lower bound (2.9) (Guerra [20]), and from Refs. [12] and [13] we know that

$$H_0 + \frac{1}{1-\epsilon} V_T \geqslant (-\alpha_\infty(P) + O(\epsilon))\, T + O(1), \quad \text{for } 0 \leqslant \epsilon \leqslant \tfrac{1}{2} \tag{2.23}$$

where

$$\alpha_\infty(P) = \lim_{l\to\infty} - \frac{E_l(P)}{l},$$

(Guerra [20]).

Choosing now $\epsilon = \frac{1}{2}$ we easily complete the proof of Estimate III by standard arguments [11–13].

Q.E.D.

But choosing $\epsilon = \epsilon(t) = 1/(2+t^2)$ in the above proof and using (2.20) to (2.23) and

$$\left| - \frac{E_l(P)}{l} - \alpha_\infty(P) \right| = 0\left(\frac{1}{l}\right)$$

we get

*Estimate IV:*

$$\pm \lambda\varphi(h) \leqslant H_l + K_4 \lambda^2 \|(2+x^2)^{1/2} h\|_2^2 + K_5 \tag{2.24}$$

*where $K_4$ is independent of $P$, $l$, $\lambda$ and $h$, and $K_5$ depends only on $P$.*

The techniques used for the proofs of Estimates III and IV, namely twice Nelson's symmetry, convergence of $-[E_l(P)]/l$ to $\alpha_\infty(P)$, as $l \to \infty$, concavity of $E_l(P)$, etc. are typical for the GRS proof of Estimate I such as presented in Refs. [12] and [13].

*Remark:* Let $P(x) = \sum_{m=0}^n a_m x^{2m} + \mu x$, $a_n > 0$ be a real polynomial. Let $H_{l,s}(P)$ be the Hamiltonian with mass parameter $s$, $0 < s \leqslant \infty$, associated with the polynomial $P$. Then Estimate I with $Q(x) = x$ or $x^2$ and Estimates II–IV hold for the quantum field $\varphi_{l,s}$, and the Hamiltonian $H_{l,s}(P)$ with constants that are *independent* of $s$; this is a consequence of correlation inequalities (Sub-section 3.5.)

Estimates I–IV also hold for the Hamiltonians $H_{l,D}$ with half Dirichlet boundary conditions, uniformly in $l$ [11, 14].

## 3. Analysis of the Generating Functionals for the Schwinger Functions with Space-Time Cutoff

### *3.1. Introduction*

Throughout Section 3 we work with free boundary conditions, i.e. the mass parameter $s = 0$. The results of this section extend to arbitrary mass parameter and Dirichlet boundary conditions by a monotonicity argument ('Griffiths inequalities' [4, 21]). This is explained in Sub-section 3.5.

Let $\Lambda = \Lambda_l^T := [-l/2, l/2] \times [-T/2, T/2]$ be a space-time region and let $h_1 \ldots h_m$ be test functions in $\mathscr{S}_r(\mathbb{R}^2)$ with time-ordered supports. We define the $m$-point Schwinger functions with space-time cutoff $\Lambda$

$$\mathfrak{S}_m^{\Lambda}(h_1, \ldots, h_m) := \frac{1}{Z_\Lambda} \int \prod_{i=1}^{m} dt_i \left( \Omega_0, \exp\left[ -\left(\frac{T}{2} - t_1\right) \hat{H}_l \right] \varphi(h_1(\cdot, t_1)) \right.$$
$$\left. \times \exp[-(t_1 - t_2) \hat{H}_l] \ldots \varphi(h_m(\cdot, t_m)) \exp\left[ -\left(\frac{T}{2} + t_m\right) \hat{H}_l \right] \Omega_0 \right) \tag{3.1}$$

where

$$Z_\Lambda := (\Omega_0, \exp(-T \hat{H}_l) \Omega_0);$$

see Refs. [4, 22–24].

In this section we want to show that for all $l < \infty$ and $T \leqslant \infty$ there exists a generating functional $J_{\Lambda_l^T}(\cdot)$ defined on $\mathscr{S}_r(\mathbb{R}^2)$ such that

$$\mathfrak{S}_m^{\Lambda_l^T}(h_1, \ldots, h_m) = i^m \frac{\partial^m}{\partial \lambda_1 \ldots \partial \lambda_m} J_{\Lambda_l^T}\left( \sum_{j=1}^{m} \lambda_j h_j \right) \Bigg|_{\lambda_1 = \cdots = \lambda_m = 0}. \tag{3.2}$$

We show that the function $J_{\Lambda_l^T}(\lambda h)$ is the boundary value of an entire analytic function $J_{\Lambda_l^T}(\zeta \cdot h)$, as $\operatorname{Im} \zeta \to 0$, for arbitrary $l < \infty$, $T \leqslant \infty$.

We derive uniform bounds on $|J_{\Lambda_l^\infty}(\zeta h)|$ and we show that the family $\{J_{\Lambda_l^\infty}(\zeta h) \,|\, 0 \leqslant l < \infty\}$ is equicontinuous in the test function $h$ in some norm on $\mathscr{S}_r(\mathbb{R}^2)$, which does not depend on $\zeta$. Our results imply that the functionals $J_{\Lambda_l^\infty}(h)$ are the Fourier transforms of measures $\nu_{\Lambda_l^\infty}$ on $\mathscr{S}' \equiv \mathscr{S}_r'(\mathbb{R}^2)$ which are closely related to the path space measures determined by the s.a. semigroups $e^{-tH_l}$ (extensively used by Nelson [25], Glimm and Jaffe [11, 26] and Rosen [18]). The results of Section **3** are basic for the proofs of our main results in Section **4**, where we construct the limit measures $\nu_{\Lambda = \mathbb{R}^2}$ and discuss its properties, e.g. Euclidean invariance.

### *3.2. Markoff and Euclidean fields, the Euclidean Fock space*

We summarize here some Markoff field terminology in the sense of Refs. [1, 2, 4, 9] for *Bose* field theories in a two-dimensional space-time. We motivate the use of this terminology in the study of the $P(\varphi)_2$ models, and we mention the connections between Bose quantum field theory models, conservative Markoff processes on compact spaces and Markoff field theory.

We start with a probabilistic definition of fields and Markoff fields [2, 4, 9, 27].

Let $J$ be a non-linear functional on the space $\mathscr{S} = \mathscr{S}_r(\mathbb{R}^2)$ with the properties:

i) $J$ is *normalized*, i.e. $J(0) = 1$.
ii) $J$ is *continuous* on $\mathscr{S}$ in the topology of $\mathscr{S}(\mathbb{R}^2)$.
iii) $J$ is of *positive type*, i.e. given arbitrary complex numbers $c_1, \ldots, c_n$ and test functions $f_1, \ldots, f_n$ then

$$\sum_{i,j=1}^{n} \bar{c}_i c_j J(f_j - f_i) \geqslant 0.$$

iv) $$J(f) = \overline{J(-f)}.$$

A theorem of Minlos [28] (generalization of a well-known theorem of Bochner), tells us that $J$ is the Fourier transform of a unique measure $\nu$ defined on the $\sigma$-ring generated by the Borel cylinder subsets of $\mathscr{S}'$, i.e.

$$J(f) = \int_{\mathscr{S}'} e^{iq(f)}\, d\nu(q). \tag{3.3}$$

We define the Hilbert space

$$\mathscr{H}_\nu := L^2(\mathscr{S}', d\nu) \tag{3.4}$$

and the unitary groups

$$\{e^{is\phi(f)} | s \in \mathbb{R}\},$$
$$[e^{is\phi(f)}](q) := e^{isq(f)}, \quad q \text{ in } \mathscr{S}'. \tag{3.5}$$

Because of 3.2, ii) these groups are strongly continuous on $\mathscr{H}_\nu$ and hence have an infinitesimal generator $\Phi(f)$, $[\Phi(f)](q) = q(f)$, which is called the *field* associated with $\nu$, $J$.

Let $s$, $t$ be real numbers and $s < t$. The subspace $\mathscr{H}_\nu(s,t)$ of $\mathscr{H}_\nu$ is defined to be the closure of the linear hull of all vectors of the form $\{e^{i\phi(f)} I / f \in \mathscr{S}, \operatorname{supp} f \subseteq \mathbb{R}_x \times [s,t]\}$ where $I$ is the function identically 1 on $\mathscr{S}'$ and is also denoted by $\Omega$. The orthogonal projection onto $\mathscr{H}_\nu(s,t)$ is denoted by $E_\nu(s,t)$. If 'sharp-time fields' $\Phi(g \otimes \delta_t)$ exist (as limits of the fields $\Phi(f)$ with $f$ in $\mathscr{S}$) we define the subspaces $\mathscr{H}_\nu(t)$ generated by $\{\exp[i\phi(g \otimes \delta_t)] / g \in \mathscr{S}_r(\mathbb{R})\}$ and the corresponding s.a. projections $E_\nu(t)$.

If the functional $J$ is space-time translation invariant we define 'space-time translations' by

$$T_{\langle x,t\rangle} I = I, \quad T_{\langle x,t\rangle} \exp[i\phi(f)]\, T_{\langle -x,-t\rangle} = \exp[i\phi(f_{\langle x,t\rangle})] \tag{3.6}$$

for all $\langle x,t\rangle$ in $\mathbb{R}^2$, where $f_{\langle x,t\rangle}(y,s) := f(y-x, s-t)$. The group $\{T_{\langle x,t\rangle} | \langle x,t\rangle \in \mathbb{R}^2\}$ is a continuous unitary group on $\mathscr{H}_\nu$;

$$T_t := T_{\langle 0,t\rangle}.$$

*Definition 3.1:* The field $\Phi$ is called a *Markoff* (Bose) field if $\Phi$ is a field and the *Markoff property*

$$E_\nu(t) = E_\nu(-\infty, t)\, E_\nu(t, \infty) = E_\nu(t, \infty)\, E_\nu(-\infty, t) \tag{3.7}$$

holds for all $t$ in $\mathbb{R}$.

Let $F$ be a measurable, $\nu$-integrable function on $\mathscr{S}'$. We define the expectation value of $F$ with respect to $\nu$

$$\langle F\rangle_\nu := \int_{\mathscr{S}'} F(q)\, d\nu(q) = (\Omega, F\Omega). \tag{3.8}$$

*Definition 3.2:* The field $\Phi$ is called a Euclidean Bose field if the functional $J$ is *Euclidean invariant* and all the moments of the measure $\nu$, i.e. the Schwinger functions, $\{\langle\phi(f_1)\ldots\phi(f_n)\rangle_\nu\}_{n=0}^\infty$ exist, are tempered, and have a unique analytic continuation in the time variable to the Wightman distributions [6, 7] smeared with the test functions $f_1, \ldots, f_n$. (See Sub-section 3.3.)

The connection between this definition of Schwinger functions and definition (3.1) is explained in Sub-section 3.3. See also Refs. [4, 22, 23].

One of our goals is to construct a Euclidean invariant functional $J$ with the properties 3.2, i)–iv) whose Schwinger functions are the Schwinger functions of the $P(\varphi)_2$ model and obey the Osterwalder–Schrader axioms [3, 5, 29].

We now explain the connection between quantum field theory, Markoff field theory, and Markoff processes and the relevance of the latter in the study of the $P(\varphi)_2$ models [27, 30, 31]. We assume that we are given *quantum fields* on a Hilbert space $\mathscr{H}_W$, a Hamiltonian $H$ on $\mathscr{H}_W$ with a *groundstate* $\Omega$ and a maximal abelian von Neumann algebra $\mathscr{M}(0)$ generated by $\{e^{i\varphi(f)} | f \in \mathscr{S}_r(\mathbb{R})\}$, where $\varphi(f)$ is the (time 0–) quantum field. We assume that $\Omega$ is *cyclic* for $\mathscr{M}(0)$. It is shown in Ref. [16] (and references given there) that we can pass to the *Schrödinger representation* of $\mathscr{H}_W$ and $\mathscr{M}(0)$; Let $X$ be the *spectrum* of $\mathscr{M}(0)$ (which can be replaced by $\mathscr{S}_r'(\mathbb{R})$ with some topology that makes it a compact Hausdorff space). Then

$\mathscr{M}(0) = C(X)$ (algebra of continuous functions on $X$);

$\mathscr{H}_W = L^2(X, d\mu)$ for some regular Borel probability measure $\mu$ on $X$;

$\Omega = I.$

One can show

*Theorem 3.1: The following are equivalent*

(*I*) *The s.a. semigroup* $\{e^{-tH} | t \geqslant 0\}$ *preserves positivity of functions in* $L^2(X, d\mu)$ *and* $\Omega \equiv I$ *is invariant under* $\{e^{-tH} | t \geqslant 0\}$.

(*II*) *There exists a Markoff field* $\Phi$ *on a Hilbert space* $\mathscr{H}_v$ *such that for all* $\Psi$ *and* $\theta$ *in* $\mathscr{H}_v(0)$

$(\Psi, T_t\, \theta) = (\Psi, T_{-t}\theta)$ (*reflection principle* [2])

*with the following identifications:* (*a*) $\mathscr{H}_W = \mathscr{H}_v(0)$, (*b*) $\Omega = I$, (*c*) $\varphi(f) = \phi(f \otimes \delta_0)$, *for all* $f$ *in* $\mathscr{S}_r(\mathbb{R})$, (*d*) $e^{-|t|H}\theta = E_v(0)\, T_t \theta$, *for all* $\theta$ *in* $\mathscr{H}_W$. *Let* $f$ *be in* $\mathscr{S}$ *and* $\operatorname{supp} f \subseteq \mathbb{R}_x \times [0, \infty)$. *Then* (*e*)

$$\lim_{T\to\infty}\left(\lim_{N\to\infty}\left(\Omega, \prod_{m=0}^{N}\left(\exp\left[-\frac{T}{N}H\right]\exp\left[i\frac{T}{N}\varphi\left(f\left(\cdot, \frac{m}{N}T\right)\right)\right]\right)\Omega\right)\right) = J(f).$$

The functional $J$ is also called the *characteristic functional* (of the Markoff process on $X$ determined by $\{e^{-tH} | t \geqslant 0\}$; see Ref. [27]).

The part (II) $\Rightarrow$ (I) of this theorem is due to Nelson [2].

The part (I) $\Rightarrow$ (II) is due to Simon [30]. (See also Ref. [27].)

*Remarks:* The part (I) $\Rightarrow$ (II) follows from the fact that $\{e^{-tH} | t \geqslant 0\}$ determines a so-called *self adjoint, conservative Markoff process* [27, 31]. This Markoff process determines a measure $\rho$ on the *path space* $Q_+$ (e.g. $= \mathscr{S}_r'(\mathbb{R} \times \mathbb{R}_+)$) [31, 32]. Since $e^{-tH} I = I$, for all $t \geqslant 0$, or equivalently the Markoff process is *conservative*, there is an extension $Q$(e.g. $= \mathscr{S}'$) of the path space $Q_+$, and the path space measure $\rho$ uniquely determines a measure $\nu$ on $Q$. The Fourier transform of $\nu$ is the functional $J$. Equation (3.7) *is* the Markoff property of the Markoff process determined by $\{e^{-tH} | t \geqslant 0\}$. Identifications (d) and (e) represent the *Feynman–Kac* (*–Nelson*) formula [4, 9].

*The free, Euclidean Markoff field* [10]

Conditions (I) of Theorem 3.1 hold for $\mathscr{H}_W = \mathscr{F}$, $H = H_l$, where $H_l$ is defined in (2.8), (2.10) and $l \in [0, \infty)$, and for the groundstate $\Omega_l$ of $H_l$; see Refs. [15, 16]. Therefore there exists a corresponding Markoff process and a Markoff field theory [27, 30].

For $l = 0$, $H = H_0$ we have

$$J_0(f) = \exp[-\tfrac{1}{2}\langle f,f\rangle] \tag{3.9}$$

where

$$\langle f,g\rangle := \int d^2\xi d^2\eta \overline{f(\xi)} S_2^0(\xi-\eta) g(\eta)$$

and $S_2^0(\xi-\eta)$ is the kernel of the operator $(-\Delta + m_0^2)^{-1}$ (which is the free two-point Schwinger function).

The Fourier transform of $J_0$ is the *Gaussian measure* $\nu_0$ on $\mathscr{S}'$ with *covariance* (*operator*)

$$(-\Delta + m_0^2)^{-1}. \tag{3.10}$$

The Hilbert space $\mathscr{H}_{\nu_0} = \mathscr{K}$ associated with $J_0$, $\nu_0$ is called the *Euclidean Fock space*. The subspaces $\mathscr{K}(s,t)$, $\mathscr{K}(t)$ and the projections $E(s,t)$, $E(t)$ are defined in the same way as $\mathscr{H}_\nu(s,t)$, $\mathscr{H}_\nu(t)$, $E_\nu(s,t)$, $E_\nu(t)$, respectively.

*Identifications:*

$$\mathscr{K}(0) = \mathscr{F}, \quad \phi(f\otimes\delta_0) = \varphi(f)$$

$$E(0)\, T_t\theta = e^{-|t|H_0}\theta, \quad \text{for all } \theta \text{ in } \mathscr{F}. \tag{3.11}$$

The function $I$ in $L^2(\mathscr{S}', d\nu_0) \equiv \mathscr{K}$ is denoted by $\Omega_0$ (and is not distinguished from the vacuum in $\mathscr{F}$).

### 3.3. *General properties of the characteristic functional or the generating functional for the Schwinger functions*

In this sub-section we study some general properties of the generating functional for the Schwinger functions or the characteristic functional, defined in 3.2, i)–iv); Theorem 3.1, Identification (e). Since we are mainly interested in the application of the results of this sub-section to the study of the $P(\varphi)_2$ models we do not try to prove the most general results (but see Ref. [27]). The assumptions for this sub-section are as follows:

i) The semigroup $\{e^{-tH}/t \geqslant 0\}$ with the invariant state $\Omega$ obeys conditions (I) of Theorem 3.1.

ii) Let $\varphi(f)$ be the (time 0–) quantum field on the Hilbert space $\mathscr{H}_W = L^2(X, d\mu)$. We assume that there is a norm $|||\cdot|||$ which is continuous on $\mathscr{S}(\mathbb{R})$ such that

a) $$|\varphi(f)| \leqslant |||f|||(H+I). \tag{3.12}$$

b) The vector $\Omega$ is in the domain of $\varphi(f)$ if $|||f||| < \infty$ and

$$\|\varphi(f)\Omega\|_{\mathscr{H}_W} \leqslant O(1)|||f|||. \tag{3.13}$$

Note that in the case of the $P(\varphi)_2$ models these assumptions can be verified for $H = H_l$, $\mathscr{H}_W = \mathscr{F}$ and some norm $|||\cdot|||$ determined for example by Estimate IV which is *independent* of $l$. For the $P(\varphi)_2$ models we can show more:

$$|\varphi(f)|^2 \leqslant |||f|||^2(H_l' + I)^2, \tag{3.12'}$$

uniformly in $l < \infty$, which yields **3.3**, ii), (a) and (b). (See Refs. [**11, 12**] and sub-section **3.4**.)

*I. Continuity properties of functionals obeying 3.2, i)–iv)*

Let $J(\cdot)$ be a functional obeying **3.2**, i)–iv), i.e.

$$J(f) = \int_{\mathscr{S}'} e^{iq(f)}\, d\nu(q), \quad f \in \mathscr{S},$$

for some probability measures $\nu$ on $\mathscr{S}'$. Clearly,

$$|J(f+f') - J(f)|^2 \leqslant \int_{\mathscr{S}'} d\nu(q)\, q(f')^2. \tag{3.14}$$

We now want to study the class of functionals such that

$$J(\zeta f) := \int_{\mathscr{S}'} e^{i\zeta q(f)}\, d\nu(q)$$

is holomorphic in $\zeta$ in a certain domain in $\mathbb{C}$ (containing $\mathbb{R}$) for all $f$ in a certain class of test functions and we then derive bounds on the right-hand side of (3.14). Since

$$|J(\zeta f)| \leqslant J(i\lambda f), \quad \text{where } \lambda := +\operatorname{Im}\zeta, \tag{3.15}$$

it is obvious that the domain of analyticity of $J(\zeta f)$ is a strip around the real axis.
Let $\|\cdot\|_s$ be some norm which is continuous on $\mathscr{S}(\mathbb{R}^2)$. We define the test function spaces:

$$\mathscr{V} := \text{completion of } \mathscr{S}(\mathbb{R}^2) \text{ with respect to the norm } \|\cdot\|_s$$

$$\mathscr{V}_r := \{f/f \in \mathscr{V}, f \text{ is real valued}\}. \tag{3.16}$$

*Lemma 3.2: Assume that*

$$\sup_{\substack{|\operatorname{Im}\zeta| \leqslant \alpha \\ \|f\|_s = 1}} |J(\zeta f)| < \tfrac{1}{8} K_\alpha < \infty. \tag{3.17}$$

*Then the absolute value of*

$$\left\langle \prod_{i=1}^m \phi(f_i) \right\rangle_\nu = \int_{\mathscr{S}'} d\nu(q) \prod_{i=1}^m q(f_i)$$

*is bounded by*

$$O(1)^m\, m! \prod_{i=1}^m \|f_i\|_s. \tag{3.18}$$

*In particular*

$$\langle\phi(f)^2\rangle_\nu \leqslant \frac{K_\alpha}{\alpha^2}\|f\|_s^2 \tag{3.18'}$$

*and for*

$$2|\operatorname{Im}\zeta|(\|f\|_s + \|f'\|_s) \leqslant \alpha$$

$$|J(\zeta(f+f')) - J(\zeta f)| \leqslant \left[\int_0^1 ds\, J(2i\lambda(f+sf'))\right]\langle\phi(f')^2\rangle_\nu \leqslant \frac{K_\alpha^2}{\alpha^2}\|f'\|_s^2. \tag{3.19}$$

*Proof:* Let $h := f/\|f\|_s$. Then $J(\zeta h)$ is holomorphic inside and on the circle

$$\Gamma = \{\zeta/|\zeta| = \alpha/2\} \quad \text{and} \quad |J(\zeta h)|_{\zeta\in\Gamma}| < \tfrac{1}{8}K_\alpha$$

Hence

$$\langle\phi(f)^m\rangle_\nu = \|f\|_s^m\langle\phi(h)^m\rangle_\nu = (-i)^m\|f\|_s^m \frac{\partial^m}{\partial\zeta^m} J(\zeta h)|_{\zeta=0}$$

exists for all $m < \infty$. Using Cauchy's integral formula we get

$$\frac{\partial^m}{\partial\zeta^m} J(\zeta h)|_{\zeta=0} = \frac{m!}{2\pi i}\int_\Gamma dz \frac{J(zh)}{z^{m+1}}.$$

Hence

$$\left|\frac{\partial^2}{\partial\zeta^2} J(\zeta h)|_{\zeta=0}\right| \leqslant \frac{K_\alpha}{\alpha^2}$$

This proves (3.18′).

For the proof of (3.18) we replace $J(\zeta h)$ by $J(\sum_{i=1}^m \zeta_i h_i)$ and use the same analyticity arguments.

Using Duhamel's formula, we get

$$|J(\zeta(f+f')) - J(\zeta f)|^2 = \left|\int_0^1 ds \int_{\mathscr{S}'} d\nu(q)\, \exp[i\zeta(q(f) + sq(f'))]\, q(f')\right|^2$$

$$\leqslant \left[\int_0^1 ds\, J(2i\lambda(f+sf'))^{1/2}\right]^2 \langle\phi(f')^2\rangle_\nu$$

$$\leqslant \left[\int_0^1 ds\, J(2i\lambda(f+sf'))\right]\langle\phi(f')^2\rangle_\nu \leqslant \frac{1}{8}\frac{K_\alpha^2}{\alpha^2}\|f'\|_s^2$$

(by applying the Schwartz inequality first to $\int_{\mathscr{S}'} d\nu(q) \ldots$ and then to $\int_0^1 ds \ldots$).

Q.E.D.

*Remark:* Let $\Phi$ denote the field associated with $J$ and let $:\Phi^m:(\cdot)$ denote the $m$th Wick power of $\Phi$ (where Wick products are defined in Segal's probabilistic way [4]). Let

$$Q(\boldsymbol{\zeta}\cdot\mathbf{h}) := \sum_{m=0}^{n} \zeta_m :\phi^m:(h_m)$$

and

$$G(Q,\boldsymbol{\zeta}\cdot\mathbf{h}) := \int_{\mathscr{S}'} d\nu(q)\exp[iQ(\boldsymbol{\zeta}\cdot\mathbf{h})(q)].$$

Then Lemma 3.2 can easily be generalized to apply to the functional $G(Q,\boldsymbol{\zeta}\cdot\mathbf{h})$, provided this functional exists and has suitable analyticity properties in $\zeta_1, \ldots \zeta_n$. We will eventually use such generalizations (Part II).

*II. Analyticity properties of the characteristic functional associated with* $\{e^{-tH}|t \geqslant 0\}$

We assume here that **3.3**, i) and ii) hold. We first prove a result on the Schwinger functions associated with the Hamiltonian H. We define the following test function spaces over $\mathbb{R}$:

$$\mathscr{W} := \{f/|||f||| < \infty\} \quad \text{and} \quad \mathscr{W}_r := \{f/f \in \mathscr{W}, f \text{ is real-valued}\} \tag{3.20}$$

(more precisely, $\mathscr{W}$ is the completion of $\mathscr{S}(\mathbb{R})$ in the norm $|||\cdot|||$).

*Lemma 3.3:* i) *Let* $f_1, \ldots, f_m$ *be in* $\mathscr{W}$. *Then on the set*

$$\{t_1, \ldots, t_m| -\infty < t_1 < t_2 < \ldots < t_m < \infty\}$$

*the expectation value* (*e.v.*)

$$(\Omega, \varphi(f_1)\exp[-(t_2 - t_1)H]\varphi(f_2) \ldots \varphi(f_{m-1})\exp[-(t_m - t_{m-1})H]\varphi(f_m)\Omega)$$

*exists and is* $C^\infty$ *in* $t_1, \ldots t_m$. *It is called 'm point Schwinger function'.*

ii) *Let f and g be in* $\mathscr{W}$. *Then the e.v.* $(\Omega, \varphi(f)e^{-\tau H}\varphi(g)\Omega)$ *exists and is bounded on* $\{\tau \geqslant 0\}$. *It is* $C^\infty$ *in* $\tau$ *on* $\{\tau > 0\}$ *and continuous at* $\tau = 0$. *The e.v.* $(\Omega, \varphi(\bar{f})e^{-\tau H}\varphi(f)\Omega)$ *is a convex function of* $\tau$ *on* $\{\tau \geqslant 0\}$.

*Proof:* Let $R = (H + I)^{-1}$. From **3.3**(ii) we know that

$$\|R^{1/2}\varphi(f)R^{1/2}\| \leqslant |||f|||, \quad \|\varphi(f)\Omega\| \leqslant O(1)|||f|||.$$

Hence

$$\|e^{-\tau_1 H}H^m\varphi(f)H^n e^{-\tau_2 H}\| \leqslant |||f|||\,\|(H + I)^{1/2}H^m e^{-\tau_1 H}\| \cdot \|(H + I)^{1/2}H^n e^{-\tau_2 H}\|.$$

But

$$\|(H + I)^{1/2}H^m e^{-\tau H}\| \leqslant \sup_{\lambda \geqslant 0} \sqrt{\lambda + 1}\,\lambda^m e^{-\tau\lambda} \leqslant O(1)^m \frac{m!}{\tau^{m+1/2}}.$$

These bounds prove the existence of the e.v.'s defined in (i) and imply that they are $C^\infty$ on $\{t_1, \ldots, t_m|t_1 < t_2 < \ldots < t_m\}$.

*Remark:* The e.v.'s defined in (i) are of course holomorphic in $t_1, \ldots t_m$ on $\{t_1, \ldots, t_m / \operatorname{Re} t_1 < \operatorname{Re} t_2 < \ldots < \operatorname{Re} t_m\}$.

*Proof of (ii):* Since by 3.3(ii)(b) $\Omega$ is in the domain of definition of $\varphi(f)$, for arbitrary $f \in \mathscr{W}$, $\varphi(f)\Omega$ is in $\mathscr{H}_W$. The continuity of $(\Omega, \varphi(f)\, e^{-\tau H} \varphi(g)\, \Omega)$ on $\{\tau \geqslant 0\}$ follows now from the strong continuity of $e^{-\tau H}$ on $\mathscr{H}_W$ on $\{\tau \geqslant 0\}$. Finally, for $j = 1,2$ and $\tau > 0$

$$\frac{d^j}{d\tau^j} (\Omega, \varphi(\bar{f})\, e^{-\tau H} \varphi(f)\, \Omega) = (-1)^j\, (\Omega, \varphi(\bar{f})\, H^{j/2}\, e^{-\tau H}\, H^{j/2}\, \varphi(f)\, \Omega).$$

Thus

$(\Omega, \varphi(\bar{f})\, e^{-\tau H} \varphi(f)\, \Omega)$ *is convex on* $\{\tau \geqslant 0\}$.

Q.E.D.

*Definitions:* Let $f$ be a function on $\mathbb{R}^2$. We set

$$\|f\|_s := \int |||f(\cdot, t)|||\, dt + \sup_t |||f(\cdot, t)|||. \tag{3.21}$$

Let

$$\underline{t}(f) := \inf\{t / \exists\, x \text{ such that } \langle x, t \rangle \in \operatorname{supp} f\}$$

$$\bar{t}(f) := \sup\{t / \exists\, x \text{ such that } \langle x, t \rangle \in \operatorname{supp} f\}. \tag{3.22}$$

Let $\mathscr{V}$ denote the completion of $\mathscr{S}(\mathbb{R}^2)$ in the norm $\|\cdot\|_s$,

$$\mathscr{V}_0 := \{f / f \in \mathscr{V}, -\infty < \underline{t}(f) < \bar{t}(f) < \infty\} \tag{3.23}$$

and let $\mathscr{V}_r$, $\mathscr{V}_{0r}$ denote the real parts of $\mathscr{V}$, $\mathscr{V}_0$, respectively. Using Theorem 3.1(II), Identifications (c) and (d) we conclude that

$$\begin{aligned}(\Omega, \phi(h \otimes \delta_t)\, \phi(h' \otimes \delta_{t'})\, \Omega) &= (\Omega, \phi(h \otimes \delta_0)\, T_{t'-t}\, \phi(h' \otimes \delta_0)\, \Omega) \\ &= (\Omega, \varphi(h)\, e^{-|t'-t|H}\, \varphi(h')\, \Omega).\end{aligned} \tag{3.24}$$

Thus

$$\begin{aligned}|(\Omega, \phi(f)\, \phi(g)\, \Omega)| &= \left| \int dt\, dt' (\Omega, \varphi(f(\cdot, t))\, e^{-|t'-t|H}\, \varphi(g(\cdot, t'))\, \Omega) \right| \\ &\leqslant \int dt\, dt' \|\varphi(\overline{f(\cdot, t)})\, \Omega\| \cdot \|\varphi(g(\cdot, t'))\, \Omega\| \\ &\leqslant \int dt\, dt' |||f(\cdot, t)||| \cdot |||g(\cdot, t')||| < \|f\|_s \cdot \|g\|_s,\end{aligned} \tag{3.25}$$

i.e. the two-point Schwinger function is continuous in each argument in the norm $\|\cdot\|_s$.

*Lemma 3.4: Suppose that $f$ is in $\mathscr{V}_{0r}$ and* $\operatorname{supp} f \subseteq [-T/2, T/2]$. *Then*

$$\lim_{N \to \infty} \left( \Omega, \prod_{n=0}^{N-1} \left( \exp\left[ -\frac{T}{N} H \right] \exp\left[ i \frac{T}{N} \varphi\left( f\left( \cdot, \left( \frac{n}{N} - \tfrac{1}{2} \right) T \right) \right) \right] \right) \Omega \right) \equiv J(f)$$

*exists, and the characteristic functional $J(f)$ has a unique extension from $\mathscr{V}_{0r}$ to $\mathscr{V}_r$.*

*Proof:* Let

$$J_N(f) := \left(\Omega, \prod_{n=0}^{N-1} \left(\exp\left[-\frac{T}{N}H\right] \exp\left[i\frac{T}{N}\varphi(f(\cdot,\ldots))\right]\right)\Omega\right).$$

From Theorem 3.1(II) we know that

$$J_N(f) = \left(\Omega, \exp\left[i\sum_{n=0}^{N-1}\frac{T}{N}\phi(f_n^N \otimes \delta_{((n/N)-1/2)T})\right]\Omega\right),$$

where

$$f_n^N(x) := f\left(x, \left(\frac{n}{N} - \tfrac{1}{2}\right)T\right).$$

Thus, by Duhamel's formula,

$$|J_N(f) - J_M(f)|^2 \leqslant \left(\Omega, \left[\frac{T}{N}\sum_{n=0}^{N-1}\phi(f_n^N \otimes \delta_{((n/N)-1/2)T}) - \frac{T}{M}\sum_{m=0}^{M-1}\phi(f_m^M \otimes \delta_{((m/M)-1/2)T})\right]^2 \Omega\right).$$

We now use (3.24) and get

$$\begin{aligned}|J_N(f) - J_M(f)|^2 \leqslant \sum_{n,n'=0}^{N-1}\left(\frac{T}{N}\right)^2\left(\Omega, \varphi(f_n^N)\exp\left[-\left|\frac{n-n'}{N}\right|TH\right]\varphi(f_{n'}^N)\Omega\right)\\ -2\sum_{\substack{n=0\\ m=0,\ldots,M-1}}^{N-1}\left(\frac{T}{N}\right)\left(\frac{T}{M}\right)\left(\Omega, \varphi(f_n^N)\exp\left[-\left|\frac{n}{N}-\frac{m}{M}\right|TH\right]\right.\\ \left.\times\varphi(f_m^M)\Omega\right) + \sum_{m,m'=0}^{M-1}\left(\frac{T}{M}\right)^2\left(\Omega, \varphi(f_m^M)\exp\left[-\left|\frac{m-m'}{M}\right|TH\right]\right.\\ \left.\times\varphi(f_{m'}^M)\Omega\right).\end{aligned} \tag{3.26}$$

Since $f$ is in $\mathscr{V}$ and $\mathscr{V}$ is the completion of $\mathscr{S}(\mathbb{R}^2)$ in the norm $\|\cdot\|_s$, $|||f(\cdot,t)|||$ is continuous in $t$. We now use Lemma 3.3(ii) (continuity of two-point Schwinger function) and conclude that the first and the last term on the right-hand side of (3.26) are *Riemann sums* which tend to

$$\int dt\, dt'(\Omega, \varphi(f(\cdot,t)) \exp[-|t-t'|H]\,\varphi(f(\cdot,t'))\,\Omega) = (\Omega, \phi(f)^2\,\Omega),$$

as $N \to \infty$, $M \to \infty$.

The second term on the right-hand side of (3.26) is a Riemann sum which tends to

$$-2\int dt\,dt'(\Omega,\varphi(f(\cdot,t))\exp[-|t-t'|H]\,\varphi(f(\cdot,t'))\,\Omega).$$

Hence

$$|J_N(f)-J_M(f)|^2\to 0,\quad \text{as } N\to\infty,\ M\to\infty.$$

Therefore the limit

$$\lim_{N\to\infty} J_N(f)\equiv J(f)$$

exists.

Obviously

$$|J_N(f+f')-J_N(f)|\leqslant \sum_{n,n'=0}^{N-1}\left(\frac{T}{N}\right)^2\left(\Omega,\varphi(f_n'^N)\exp\left[-\left|\frac{n-n'}{N}\right|TH\right]\varphi(f_{n'}'^N)\,\Omega\right). \tag{3.27}$$

The right-hand side of (3.27) tends to $(\Omega,\phi(f')^2\Omega)$ as $N\to\infty$. Hence

$$|J(f+f')-J(f)|\leqslant(\Omega,\phi(f')^2\Omega)\leqslant\|f'\|_s^2$$

by (3.25). Thus $J(\cdot)$ has a unique extension to $\mathscr{V}$.

Q.E.D.

*Corollary 3.5: Let $\mathscr{M}_\nu$ be the von Neumann algebra generated by operators of the form*

$$\left\{\exp\left[\sum_{i=1}^{M}\phi(f_i\otimes\delta_{t_i})\right]\Big/ f_i\in\mathscr{W}_r,\quad i=1,\ldots,M,\ M<\infty\right\}.$$

*Then*

i) *$\mathscr{M}_\nu$ is maximal abelian on $\mathscr{H}_\nu$ and the vector $\Omega$ (the function $I$ in $L^2(\mathscr{S}',d\nu)$) is cyclic and separating for $\mathscr{M}_\nu$.*

ii) *If $\{F_i\}_{i\in I}$ is a bounded net in $\mathscr{M}_\nu$ then $F_i\to F$, strongly, iff*

$$\|(F_i-F)\,\Omega\|\to 0.$$

*Proof:* Because of Lemma 3.4 $\mathscr{M}_\nu$ contains the von Neumann algebra $\tilde{\mathscr{M}}$ generated by $\{e^{i\phi(f)}|f\in\mathscr{S}\}$. The vector $\Omega$ is cyclic for $\tilde{\mathscr{M}}$ and therefore $\tilde{\mathscr{M}}$ is maximal abelian (see Ref. [15]). Since $\mathscr{M}_\nu$ is abelian, we have $\mathscr{M}_\nu=\tilde{\mathscr{M}}$. Therefore $\Omega$ is cyclic for $\mathscr{M}_\nu$ and, since $\mathscr{M}_\nu$ is maximal abelian, $\Omega$ is separating for $\mathscr{M}_\nu$. Let $\mathscr{D}:=\{G\Omega/G\in\mathscr{M}_\nu\}$; $\mathscr{D}$ is dense in $\mathscr{H}_\nu$. Let $\theta\in\mathscr{D}$, i.e., $\theta=G\Omega$, for some $G\in\mathscr{M}_\nu$. Then

$$\|(F_i-F)\,\theta\|=\|(F_i-F)\,G\Omega\|=\|G(F_i-F)\,\Omega\|\leqslant\|G\|\cdot\|(F_i-F)\,\Omega\|$$

which tends to 0 as $i\to\infty$. Thus $\{F_i\}_{i\in I}$ converges on $\mathscr{D}$ and, since it is bounded, on $\mathscr{H}_\nu$.

Q.E.D.

Let $f$ be a test function in $\mathscr{W}_r$ and let $\zeta$ be a complex number.

On the domain $D(H^{1/2})\times D(H^{1/2})$ we define the sesquilinear form

$$H_{\zeta f}:=H+i\zeta\varphi(f). \tag{3.28}$$

If

$$|\zeta|\ |||f|||<1 \tag{3.29}$$

then by **3.3**, ii) (a) $H_{\zeta f}$ is *sectorial.*

$$\operatorname{Re} H_{\zeta f} \geqslant -|\operatorname{Im}\zeta|\ |||f|||>1$$

$$|\operatorname{Im} H_{\zeta f}| < O(1)[\operatorname{Re} H_{\zeta f} + |\operatorname{Im}\zeta|\ |||f|||]. \tag{3.30}$$

(Hence for $\zeta$ purely imaginary $H_{\zeta f}$ determines a unique s.a. operator, also denoted by $H_{\zeta f}$, which is bounded below by $-|\operatorname{Im}\zeta|\cdot|||f|||$.)

*Lemma 3.6: If inequality (3.29) holds then $H_{\zeta f}+|\zeta|\cdot|||f|||$ is the generator of a contraction semigroup denoted by*

$$\{\exp[-t(H_{\zeta f}+|\zeta|\ |||f|||)]|t\geqslant 0\}$$

*and*

$$\exp[-tH_{\zeta f}]\,\theta = E_{\nu}(0)\exp[-i\zeta\phi(f\otimes\chi_{[0,t]})]\,T_t\,\theta \tag{3.31}$$

*for all $\theta$ in $\mathscr{H}_W$, $t\geqslant 0$; where $\chi_{[0,t]}$ is the characteristic function of $[0,t]$.*

*Proof:*

$1^0$. By assumption (**3.3**, i)) $\varphi(g)$ is s.a. on $\mathscr{H}_W$ for all $g$ in $\mathscr{S}_r(\mathbb{R})$ and $\varphi(g)$ represents a real-valued function in $L^2(X,d\mu)$, since $\|\varphi(g)\,\Omega\|<\infty$, by **3.3**, ii) (b). Since $\mathscr{W}_r$ is the completion of $\mathscr{S}_r(\mathbb{R})$ in $|||\cdot|||$, there is a sequence $\{g_n\}_{n=0}^{\infty}\subset\mathscr{S}_r(\mathbb{R})$ converging to $f$ in $|||\cdot|||$. Therefore $\varphi(g_n)\,\Omega\to\varphi(f)\,\Omega$, strongly, as $n\to\infty$. Thus $\varphi(f)$ represents a real-valued function in $L^2(X,d\mu)$ and hence is s.a.

$2^0$. Let

$$F_N(x) := \begin{cases} N, & x\geqslant N \\ x, & -N\leqslant x\leqslant N \\ -N, & x\leqslant -N. \end{cases}$$

Then $F_N(\varphi(f))$ is well-defined (via spectral decomposition of $\varphi(f)$) and is in $\mathscr{M}(0)$ $(=C(X)=L^\infty(X))$, for all $N<\infty$. By the Feynman–Kac–Nelson formula (see Refs. [**4, 9, 22, 23**]).

$$\exp\{-t[H+i\zeta F_N(\varphi(f))]\}\,\theta = E_\nu(0)\exp[-i\zeta\int_0^t dt'\,F_N(\phi(f\otimes\delta_{t'}))]\,T_t\,\theta,$$

for any $\theta$ in $\mathscr{H}_W$; and for all $N<\infty$ $\exp\{-t[H+i\zeta F_N(\varphi(f))]\}$ is an operator valued holomorphic function of $\zeta$, for $|\operatorname{Im}\zeta|<1/|||f|||$ with the bound

$$\|\exp\{-t[H+i\zeta F_N(\varphi(f))]\}\|\leqslant\exp(t|\operatorname{Im}\zeta|\ |||f|||) \tag{3.32}$$

(which is *uniform* in $N$).

We now show that $\exp\{-t[H+i\zeta F_N(\varphi(f))]\}$ converges strongly to the right-hand side of (**3.31**), as $N\to\infty$. Because of (**3.32**) it suffices to prove convergence on a dense set and, for example, all real $\zeta$. By Corollary **3.5** this follows from the convergence of

$$\exp\left[-i\zeta\int_0^t dt'\,F_N(\phi(f\otimes\delta_{t'}))\right]\Omega \quad\text{to}\quad \exp[-i\zeta\phi(f\otimes\chi_{[0,t]})]\,\Omega.$$

But for real $\zeta$

$$\left\| \left( \exp\left[ -i\zeta \int_0^t dt'\, F_N(\phi(f \otimes \delta_{t'})) \right] - \exp[-i\zeta\phi(f \otimes \chi_{[0,t]})] \right) \Omega \right\|^2$$

$$\leqslant \zeta^2 \left( \Omega, \left[ \int_0^t dt' \{ F_N(\phi(f \otimes \delta_{t'}) - \phi(f \otimes \delta_{t'}) \} \right]^2 \Omega \right)$$

$$= \zeta^2 \int_0^t dt' \int_0^t dt'' (\Omega, (F_N(\varphi(f)) - \varphi(f)) \exp(-|t' - t''|H) (F_N(\varphi(f)) - \varphi(f)) \Omega)$$

$$\leqslant (\zeta t)^2 \| (F_N(\varphi(f)) - \varphi(f)) \Omega \|^2$$

which tends to 0, as $N \to \infty$, since $\varphi(f)$ is in $L^2(X, d\mu)$.

Q.E.D.

*Corollary 3.7: Let* $H_{\zeta f}(t) := H + i\zeta\varphi(f(\cdot, t))$ *and let* $|\mathrm{Im}\,\zeta|\ \|f\|_s < 1$ *where*

$$\|f\|_s = \int |||f(\cdot, t)|||\, dt + \sup_t |||f(\cdot, t)|||.$$

*Then for all* $\theta$ *in* $\mathscr{H}_W$

$$U(\zeta f, t, s)\, \theta = s - \lim_{N \to \infty} \prod_{n=0}^{N-1} \exp\left[ -\frac{t-s}{N} H_{\zeta f}\left( \frac{n}{N} t + \left( 1 - \frac{n}{N} \right) s \right) \right] \theta$$

*exists, is holomorphic in* $\zeta$ *for* $|\mathrm{Im}\,\zeta| < 1/\|f\|_s$ *and bounded by*

$$\| U(\zeta f, t, s) \| < \exp(|\mathrm{Im}\,\zeta|\, \|f\|_s). \tag{3.33}$$

*Finally*

$$U(\zeta f, t, s)\, \theta = E_v(0) T_{-s} \exp\left[ -i\zeta \int_s^t dt' \int dx\, \Phi(x, t') f(x, t') \right] T_t\, \theta. \tag{3.34}$$

*Proof:* Let

$$U_N(\zeta f, t, s) = \prod_{n=0}^{N-1} \exp\left[ -\frac{t-s}{N} H_{\zeta f}\left( \frac{n}{N} t + \left( 1 - \frac{n}{N} \right) s \right) \right].$$

From Lemma 3.6 and its proof we know that for all $N < \infty$ $U_N(\zeta f, t, s)$ is holomorphic in $\zeta$ for $|\mathrm{Im}\,\zeta| < 1/[\sup_t |||f(\cdot, t)|||]$ hence for $|\mathrm{Im}\,\zeta| < 1/\|f\|_s$ and bounded:

$$\| U_N(\zeta f, t, s) \| \leqslant \exp \sum_{n=0}^{N-1} |\mathrm{Im}\,\zeta|\ |||f(\cdot, \tau_n^N)|||\ \frac{(t-s)}{N}$$

where

$$\tau_n^N := \frac{n}{N} t + \left( 1 - \frac{n}{N} \right) s.$$

The right-hand side of this inequality tends to

$$\exp\left\{|\mathrm{Im}\,\zeta| \int_s^t dt' \,|||f(\cdot,t')|||\right\} < \exp\{|\mathrm{Im}\zeta|\cdot\|f\|_s\} \tag{3.35}$$

as $N\to\infty$, since $f_t(\cdot) := f(\cdot,t)$ is continuous in $t$ in the norm $|||\cdot|||$.

By Vitali's theorem it suffices to show that $U_N(\zeta f,t,s)$ converges to the right-hand side of (3.34) for *real* $\zeta$. By Lemma 3.6

$$U_N(\zeta f,t,s)\,\theta = E_\nu(O)\,T_{-s}\exp\left[i\zeta \sum_{n=0}^{N-1} \int_{\tau_n^N}^{\tau_{n+1}^N} dt' \int dx\, \phi(x,t')f(x,\tau_n^N)\right]T_{-t}\,\theta.$$

Hence, by Corollary 3.5 and the bound (3.35), $U_N(\zeta f,t,s)\theta$ converges to the right-hand side of (3.34), as $N\to\infty$, iff

$$\exp\left[i\zeta \sum_{n=0}^{N-1} \int_{\tau_n^N}^{\tau_{n+1}^N} dt' \int dx\, \phi(x,t')f(x,\tau_n^N)\right]\Omega$$

tends to

$$\exp\left[i\zeta \int_s^t dt' \int dx\, \phi(x,t')f(x,t')\right]\Omega$$

as $N\to\infty$. Since $f_t(\cdot)$ is continuous in $t$ in the norm $|||\cdot|||$ this follows almost in the same way as Lemma 3.4, (3.26). But this and (3.35) complete the proof of the corollary.

Q.E.D.

*Theorem 3.8: Let* $J(\zeta f) := (\Omega, \exp[i\zeta\phi(f)]\Omega)$ *(see Lemma 3.4 for real* $\zeta$*). Then*

a) $J(\zeta f)$ *is holomorphic in* $\zeta$ *on the domain* $\{z \,|\, |\mathrm{Im}\,z| < 1/\|f\|_s\}$ *and* $|J(\zeta f)| < \exp(|\mathrm{Im}\,\zeta| \times \|f\|_s)$.

b) $J(\zeta f)$ *is continuous in* $f$ *in the norm* $\|\cdot\|_s$*, for* $\|f\|_s < 1/|\mathrm{Im}\,\zeta|$.

c) *Let* $f_1,\ldots,f_m$ *be test functions in* $\mathscr{V}_{0r}$ *with strictly time-ordered supports, i.e.*

$\underline{t}(f_1) < \bar{t}(f_1) < \underline{t}(f_2) < \cdots < \underline{t}(f_m) < \bar{t}(f_m)$.

*Then*

$$(-i)^m \frac{\partial^m}{\partial\lambda_1\ldots\partial\lambda_m} J\left(\sum_{i=1}^m \lambda_i f_i\right)\Bigg|_{\lambda_1=\cdots=\lambda_m=0}$$

$$= \int \prod_{i=1}^m dt_i (\Omega, \varphi(f_1(\cdot,t_1)) \exp[-(t_2-t_1)H]\,\varphi(f_2(\cdot,t_2))\ldots\varphi(f_{m-1}(\cdot,t_{m-1}))$$

$$\times \exp[-(t_m-t_{m-1})H]\,\varphi(f_m(\cdot,t_m))\,\Omega). \tag{3.36}$$

*(Compare (3.36) with (3.1) and (3.2).)*

d) *Let* $\|\cdot\|_0$ *be some norm which is continuous on* $\mathscr{S}(\mathbb{R}^2)$. *Suppose that* $|J(\zeta f)| \leqslant K_\alpha < \infty$ *if* $|\mathrm{Im}\,\zeta|\,\|f\|_0 \leqslant \alpha$ *for some positive* $\alpha$. *Then the absolute value of the so-called coincident* *m-point Schwinger function* [4]

$$\mathfrak{S}_m(f_1,\ldots,f_m) := \int\limits_{\mathscr{S}'} d\nu(q)\, q(f_1)\ldots q(f_m)$$

$$= (-i)^m \frac{\partial_m}{\partial\lambda_1\ldots\partial\lambda_m} J\left(\sum_{i=1}^m \lambda_i f_i\right)\bigg|_{\lambda_1=\cdots=\lambda_m=0} \tag{3.37}$$

*is bounded by*

$$O(1)^m \cdot m! \prod_{i=1}^m \|f_i\|_0.$$

*In particular, this holds for* $\|\cdot\|_0 = \|\cdot\|_s$.

e) *The distributions* $\mathfrak{S}_m(x_1,t_1,\ldots,x_m,t_m)$ *are tempered and positive in the sense of Osterwalder and Schrader* [3].

*Proof:* (a) is an immediate consequence of Corollary 3.7, inequalities (3.14) and (3.25). The bound follows from (3.33). (b) follows from Lemma 3.2.

*Proof of (c):* Because of (a) it is clear that

$$(-i)^m \frac{\partial^m}{\partial\lambda_1\ldots\partial\lambda_m} J\left(\sum_{i=1}^m \lambda_i f_i\right)\bigg|_{\lambda_1=\cdots=\lambda_m=0}$$

exists and is equal to

$$\int d\nu(q) \prod_{i=1}^m q(f_i) \equiv \left\langle \prod_{i=1}^m \phi(f_i)\right\rangle_\nu. \tag{3.38}$$

By construction of the path space measure $\nu$ [27, 30, 32], it is clear that (3.38) is identical with the right-hand side of (3.36). See also Refs. [4, 14, 22–24].

(d) follows directly from Lemma 3.2 (3.18), and yields obviously the temperedness of the distributions $\mathfrak{S}_m(x_1,t_1,\ldots,x_m,t_m)$ (by the nuclear theorem). The Osterwalder–Schrader positivity follow directly from the existence and positivity of the Hamiltonian and the positive definiteness of the metric on $\mathscr{H}_W$, whence (e).

*Remark:* Obviously the right-hand side of (3.36) is the *time-smeared m-point Schwinger function* as defined in Lemma 3.3. Theorem 3.8 justifies that we call $J$ *generating functional* for the Schwinger functions and the moments of the measure $\nu$ the *Schwinger functions.*

### *3.4. Applications of the results of Sub-sections 3.2 and 3.3 to the $P(\varphi)_2$ quantum field models*

Let $H_l$ be the $P(\varphi)_2$ quantum field Hamiltonian with space cutoff $l$ and ground-state $\Omega_l$ such as defined in 2.2 (2.10), (2.11) with mass parameter $s = 0$.

We have mentioned in Sub-section 3.2 that conditions (I) of Theorem 3.1 hold for $\mathscr{H}_W = \mathscr{F}$, $H = H_l$, $\Omega = \Omega_l$ and $0 \leqslant l < \infty$. We now verify Hypotheses 3.3, ii) (a) and (b) of Sub-section 3.3. All estimates that we use in this sub-section are *uniform* in $l$ and henceforth will eventually hold in the limit $l = \infty$.

Let

$$|||f|||_0 := \sqrt{\int dx(2+x^2)|f(x)|^2} \geqslant \|f\|_2,$$

then

$$\pm\,\lambda\varphi(f) \leqslant H_l + \lambda^2 K_4 |||f|||_0^2 + K_5,$$

*uniformly* in $l$ (with constants $K_4$ and $K_5$ independent of $\lambda, f$, and $l$) which follows from Estimate IV, Sub-section 2.2. Hence

$$\pm\,\varphi\left(\frac{f}{|||f|||_0}\right) \leqslant H_l + K_4 + K_5$$

and therefore

$$\pm\,\varphi(f) \leqslant |||f|||_1(H_l + I), \tag{3.40}$$

for some norm $|||\cdot|||_1$ with $\|f\|_2 \leqslant |||f|||_0 \leqslant |||f|||_1 \leqslant O(1)|||f|||_0$ and *uniformly* in $l$.

Since the interaction $V_l(P)$ (defined in (2.7)) is a function of the field we have

*Lemma 3.9:*

$$\pi(f) := i[H_0, \varphi(f)] = i[H_l, \varphi(f)]$$

*and*

$$0 \leqslant e^{\pm i\varphi f} H_l e^{\mp i\varphi(f)} = H_l \mp \pi(f) + \|f\|_2^2,$$

*hence*

$$\pm\,\pi(f) \leqslant \|f\|_2(H_l + I) \leqslant |||f|||_1(H_l + I);$$

*see Ref.* [*17*].

From Lemma 1.1 of Ref. [11] we get

$$\varphi(f)^2 \leqslant |||f|||^2(H_l + I)^2, \tag{3.41}$$

*uniformly* in $l$, where

$$\|f\|_2 \leqslant |||f|||_1 \leqslant |||f||| \leqslant O(1)\,|||f|||_1.$$

Thus

$$\|\varphi(f)\,\Omega_l\|_{\mathscr{F}} \leqslant |||f|||\cdot\|(H_l + I)\,\Omega_l\|_{\mathscr{F}} = |||f|||. \tag{3.42}$$

We have now proven

*Theorem 3.10: For $\mathscr{H}_W = \mathscr{F}$, $H = H_l$, $\Omega = \Omega_l$ and all $l$, $0 \leqslant l < \infty$ the assumptions 3.3, i) and ii), (3.12) and (3.13) hold. Thus all the results of Sub-section 3.3 apply to the $P(\varphi)_2$ quantum field theory models with the following identifications:*

$$\mathfrak{S}_m(x_1, t_1, \ldots) \to \mathfrak{S}_m^l(x_1, t_1, \ldots) = (\Omega_l, \varphi(x_1)\exp(-|t_2 - t_1|H_l)\,\varphi(x_2)\ldots\varphi(x_m)\,\Omega) \tag{3.43}$$

$$U(\zeta f, t, s) \rightarrow$$

$$U_l(\zeta f, t, s) = s - \lim_{N\to\infty} \prod_{n=0}^{N} \exp\left\{-\frac{t-s}{N}\left[H_l + i\zeta\varphi\left(f\left(\cdot, \frac{n}{N}t + \left(1-\frac{n}{N}\right)s\right)\right)\right]\right\} \quad (3.44)$$

$$J(\zeta f) \rightarrow J_l(\zeta f) = \lim_{\substack{t\to\infty \\ s\to-\infty}} (\Omega_l, U_l(\zeta f, t, s)\, \Omega_l) \quad (3.45)$$

*and* $\nu \rightarrow \nu_l =$ *probability measure on* $\mathscr{S}'$ *with Fourier transform* $J_l$. (3.46)

*Definition:* Let $\nu$ be a probability measure on $\mathscr{S}'$. A *$\nu$-multiplicative functional* $\hat{J}$ [9, 33] is the Fourier transform of a probability measure $\hat{\nu}$ on $\mathscr{S}'$ which is *absolutely continuous* with respect to $\nu$. By the Radon–Nikodyin theorem there is a *non-negative function* $F$ in $L^1(\mathscr{S}', d\nu)$ such that

$$d\hat{\nu}(q) = F(q)\, d\nu(q).$$

(The function $\sqrt{F(q)}$ is a vector in $L^2(\mathscr{S}', d\nu) = \mathscr{H}_\nu$.)

Let $\nu_0$ be the Gaussian measure on $\mathscr{S}'$ with covariance operator $(-\Delta + m_0^2)^{-1}$ defined in Sub-section 3.2 (3.10). We want to discuss a class of $\nu_0$-multiplicative functionals. In particular we want to show that

$$J_l(\zeta f) = \lim_{T\to\infty} J_{\Lambda_l^T}(\zeta f), \quad \|f\|_s < \infty \quad (3.47)$$

for a sequence $\{J_{\Lambda_l^T}\}_{T<\infty}$ of $\nu_0$-multiplicative functionals. We prove (3.47) by means of the *Feynman–Kac–Nelson formula.*

Let $\{H(t) = H_0 + W(t)\}$ be a family of s.a. operators on the Fock space $\mathscr{F}$ which are bounded below. Here $H_0$ is the free Hamiltonian and $W(t)$ is a s.a. function of the (time 0–) fields, (i.e. $W(t)$ is affiliated with $\mathscr{M}(0) = C(X)$ in the notation of Sub-section 3.2). Let $f$ be in $\mathscr{S}_r(\mathbb{R}^2)$ and $\zeta \in \mathbb{C}$. We want to solve the differential equation

$$\frac{d}{dt} U(\zeta f, t, s) = -\,(H(t) + i\zeta\varphi(f(\cdot, t)))\; U(\zeta f, t, s),$$

with $U(\zeta f, s, s) = 1$.

Since the semigroup $\{e^{-tH_0} | t \geqslant 0\}$ determines a Markoff process on $X$ and $W(t) + i\zeta\varphi(f(\cdot, t))$ is affiliated with $C(X)$, this equation can be solved by use of the Feynman–Kac formula, i.e. by means of a path space integration (under suitable assumptions on $W(t)$). We now specify the class of interactions $W(t)$ of interest: Let

$$Q(x, t | \xi) := \sum_{m=0}^{2n} a_m(x, t)\, \xi^m; \; a_m(x, t) \quad (3.48)$$

continuous and

a) $\sup_{t\in\mathbb{R}} [\int |a_m(x, t)|^\alpha dx] < \infty$, for $\alpha = 1, 2$ and all $m = 1, \ldots, 2n$,

b) $Q(x, t | \xi) \geqslant K > -\infty$, for all $x$, $t$ and $\xi$, where $K$ is independent of $x$, $t$ and $\xi$.

We set

$$W_Q(t) := \sum_{m=0}^{2n} \int a_m(x, t) : \varphi(x)^m : dx. \quad (3.49)$$

Properties of the s.a. operator $W_Q(t)$ have been discussed in Refs. [15, 16, 26] and references given there. It is shown in Refs. [4, 17, 18] etc. that $H_Q(t) := H_0 + W_Q(t)$ is essentially s.a. on $D(H_0) \cap D(W_Q(t))$ and bounded from below for all $t \in \mathbb{R}$. Let

$$|||f(\cdot,t)||| := O(1)\sqrt{\int dx(2+x^2)|f(x,t)|^2},$$

$$\|f\|_s := \sup_t |||f(\cdot,t)||| + \int dt |||f(\cdot,t)||| < \infty. \tag{3.50}$$

Then $|\varphi(f(\cdot,t))|^2 \leqslant \epsilon H_Q(t)^2 + c(\epsilon,f)$ for all $t$ and for arbitrary $\epsilon > 0$ and some $c(\epsilon,f)$ which is finite if, for example, $\|f\|_s < \infty$. (The inequality follows from results in Refs. [17] and [18] and (2.22).)

Hence the following definition makes sense:

$$U_Q^N(\zeta f,t,s) := \prod_{n=0}^{N-1} \exp\left[-\frac{t-s}{N}(H_Q(\tau_n^N) + i\zeta\varphi(f(\cdot,\tau_n^N)))\right],$$

where

$$\tau_n^N = \frac{n}{N}t + \left(1-\frac{n}{N}\right)s.$$

Obviously $\|U_Q^N(\zeta f,t,s)\| < K(Q,f,t,s)$, uniformly in $N$, for some $K(Q,f,t,s) < \infty$ $(-\infty < s < t < \infty)$.

*Theorem 3.11: Under conditions (3.48a, b), and for* $\|f\|_s < \infty$

$$U_Q(\zeta f,t,s) = s - \lim_{N\to\infty} U_Q^N(\zeta f,t,s)$$

*exists, is a bounded operator and*

$$U_Q(\zeta f,t,s)\,\theta = E(O)\,T_{-s}\exp\left[-\sum_{m=0}^{2n}\int_s^t dt'\int dx\, a_m(x,t'):\phi^m:(x,t') + i\zeta\int_s^t dt'\int dx\, f(x,t')\,\phi(x,t')\right]T_t\,\theta \tag{3.51}$$

*for arbitrary* $\theta$ *in* $\mathscr{F}$.

*Remarks:* Formula (3.51) is called the generalized Feynman–Kac–Nelson formula. The path space integration is hidden in the (probabilistic) definition of the free Euclidean Markoff field $\Phi$ (see Refs. [14, 27 and 30] for an elaboration of this point). The proof of Theorem 3.11 is implicit in Refs. [4, 14, 22, 23], and therefore not given here. Note that Theorem 3.11 is a generalization of Lemma 3.6 and Corollary 3.7 for $H = H_0$, $\Omega = \Omega_0$ (and it can be proven in quite a similar way). The operator $\exp[-\sum_{m=0}^{2n}\int_s^t dt'\int dx\ldots]$ $(\zeta = 0)$, on the right-hand side of (3.51) is the Euclidean analogue of the Bogoliubov $S$-matrix [34] and is Euclidean covariant ('Nelson's symmetry').

*Application of Theorem 3.11:* Let $\Lambda$ be a Borel set in $\mathbb{R}^2$ and let $\chi_\Lambda$ be the characteristic function of $\Lambda$. We define

$$\Lambda_l^T := [-l/2, l/2] \times [-T/2, T/2]. \tag{3.52}$$

Let $P$ be the polynomial determining the interaction Hamiltonians $V_l(P)$ (Sub-section 2.1 (2.7)). We set

$$P_\Lambda(x,t|\xi) := \chi_\Lambda(x,t)\, P(\xi). \tag{3.53}$$

Suppose now that $\Lambda$ is a bounded region. We set

$$H_{P_\Lambda}(t) := H_0 + \int dx\, \chi_\Lambda(x,t) : P(\varphi) : (x). \tag{3.54}$$

We now define the 'partition function'

$$\begin{aligned} Z_\Lambda &:= (\Omega_0, U_{P_\Lambda}(0,+\infty,-\infty)\,\Omega_0) \\ &= (\Omega_0, e^{-V_\Lambda}\Omega_0) > 0 \end{aligned} \tag{3.55}$$

where

$$V_\Lambda := \int_\Lambda dx\, dt : P(\phi) : (x,t)$$

and the functional

$$\begin{aligned} J_\Lambda(\zeta f) &:= \frac{1}{Z_\Lambda} \lim_{\substack{t\to+\infty \\ s\to-\infty}} (\Omega_0, U_{P_\Lambda}(\zeta f, t, s)\,\Omega_0) \\ &= \frac{1}{Z_\Lambda} (\Omega_0, e^{i\zeta\phi(f)} e^{-V_\Lambda}\Omega_0). \end{aligned} \tag{3.56}$$

Obviously $J_\Lambda$ is the Fourier transform of the measure

$$d\nu_\Lambda(q) = \frac{1}{Z_\Lambda} e^{-V_\Lambda}\, d\nu_0(q) \tag{3.57}$$

and is a $\nu_0$-*multiplicative* functional.

The following estimates are immediate consequences of the results of Sections 2 and 3

$$Z_{\Lambda_l^T} \geq e^{-TE_l(P)} |(\Omega_0, \Omega_l)|^2 > 0$$

(see Theorem 2.1). We set

$$|||f(\cdot,t)||| = \sqrt{\int dx (2+x^2) |f(x,t)|^2}. \tag{3.58}$$

From Estimate IV, Sub-section 2.2, we infer

$$\pm \lambda \|f\|_s \varphi\left(\frac{f(\cdot,t)}{|||f(\cdot,t)|||}\right) \leq H_l + \lambda^2 \|f\|_s^2 K_4 + K_5,$$

hence by linearity of $\varphi$ and since $|||f(\cdot,t)|||/\|f\|_s < 1$

$$\pm \lambda\varphi(f(\cdot,t)) \leq H_l + \left(\lambda^2 \|f\|_s K_4 + \frac{K_5}{\|f\|_s}\right) |||f(\cdot,t)|||. \tag{3.59}$$

Using Theorem **3.11** and equation (3.56) we now get

$$|(\Omega_0, \exp[i\zeta\phi(f)]\exp[-V_{\Lambda_l^T}]\,\Omega_0)| \leqslant \exp[-TE_l(P) + (|\mathrm{Im}\,\zeta|^2\|f\|_s^2 K_4 + K_5)].$$

Therefore

$$|J_{\Lambda_l^T}(\zeta f)| \leqslant \frac{1}{|(\Omega_0, \Omega_l)|^2}\exp(|\mathrm{Im}\,\zeta|^2\|f\|_s^2 K_4 + K_5). \tag{3.60}$$

By Lemma **3.2** $J_{\Lambda_l^T}$ is continuous in the norm $\|\cdot\|_s$, uniformly in $T$, for each fixed $l < \infty$.

It is easy to show by means of the FKN formula (3.51) that for test functions $h_1, \ldots, h_m$ with time-ordered supports and $\|h_i\|_s < \infty$ for all $i = 1, \ldots, m$

$$(-i)^m \frac{\partial^m}{\partial\lambda_1\ldots\partial\lambda_m} J_{\Lambda_l^T}\left(\sum_{i=1}^m \lambda_i h_i\right)\Bigg|_{\lambda_1=\cdots=\lambda_m=0} = \mathfrak{S}_m^{\Lambda_l^T}(h_1, \ldots, h_m) \tag{3.61}$$

where $\mathfrak{S}_m^{\Lambda_l^T}(\cdots)$ is the space-time cutoff $m$-point Schwinger function defined in (3.1).

Since the eigenvalue 0 of the Hamiltonian $H_l$ is *simple* and *isolated* (see Refs. [16, 17]) and since $(\Omega_0, \Omega_l) \neq 0$ we have

$$\Omega_l = \lim_{T\to\infty}\frac{\exp[-(T/2)\,H_l]\,\Omega_0}{\|\exp[-(T/2)\,H_l]\,\Omega_0\|} = \lim_{T\to\infty}\frac{\exp[-(T/2)\,H_l]\,\Omega_0}{Z_{\Lambda_l^T}^{1/2}} \tag{3.62}$$

If $f$ is a test function with $\|f\|_s < \infty$, $\mathrm{supp}f \subseteq \mathbb{R}_x \times [-t/2, t/2]$. Then

$$J_{\Lambda_l^T}(\zeta f) = \frac{(\exp[-(T-t/2)\,H_l]\,\Omega_0, U_{P_{\Lambda_l^T}}(\zeta f, t/2, -t/2)\exp[-(T-t/2)\,H_l]\,\Omega_0)}{(\exp[-(T-t/2)\,H_l]\,\Omega_0, \exp(-tH_l)\exp[-(T-t/2)\,H_l]\,\Omega_0)}.$$

Thus by (3.60) and (3.62)

$$\lim_{T\to\infty} J_{\Lambda_l^T}(\zeta f) = \left(\Omega_l, U_{P_{\Lambda_l^\infty}}\left(\zeta f, \frac{t}{2}, -\frac{t}{2}\right)\Omega_l\right) = J_l(\zeta f).$$

By Estimate IV or (3.59)

$$|J_l(\zeta f)| \leqslant \exp(|\mathrm{Im}\,\zeta|^2\|f\|_s^2 K_4 + K_5) \tag{3.63}$$

*uniformly* in $l$.

Because of the bounds (3.60) and (3.63) and Lemma **3.2**

$$J_l(\zeta f) = \lim_{T\to\infty} J_{\Lambda_l^T}(\zeta f) \quad \text{for all } f \text{ with } \|f\|_s < \infty. \tag{3.64}$$

For later use we want to improve estimate (3.60).

*Assumption:*

$$\inf(\mathrm{spec}\,H_l\backslash\{0\}) \geqslant m > 0, \tag{3.65}$$

uniformly in $l$, i.e. there exists a *uniform mass gap* [24]. Thus

$$(\exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0, \exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0) \leqslant |(\Omega_l, \Omega_0)|^2 + \exp[-(T-\tau)\,m]$$

$$\leqslant e^{\tau m}(|(\Omega_l, \Omega_0)|^2 + e^{-Tm}).$$

It is shown in Ref. [12] that $|(\Omega_l, \Omega_0)| \geqslant e^{-O(1)\cdot l}$. Therefore for $T \geqslant \alpha \cdot l$, for some positive $\alpha$ depending on $m$

$$(\exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0,\ \exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0) \leqslant 2e^{\tau m}|(\Omega_l, \Omega_0)|^2.$$

Hence

$$\begin{aligned}&(\exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0,\ \exp[-\tfrac{1}{2}(T-\tau)\,H_l\,\Omega_0)\\&\quad \geqslant |(\Omega_l, \Omega_0)|^2 \geqslant \tfrac{1}{2}\exp(-\tau m)(\exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0,\ \exp[-\tfrac{1}{2}(T-\tau)\,H_l]\,\Omega_0)\end{aligned} \tag{3.66}$$

provided $T \geqslant \alpha l$.

Let $\mathscr{V}$ be the completion of $\mathscr{S}(\mathbb{R}^2)$ in the norm $\|\cdot\|_s$ (see (3.23)), and let

$$\mathscr{V}_\tau := \{f \mid f \in \mathscr{V},\ \operatorname{supp} f \subseteq \Lambda_\infty^\tau\}. \tag{3.67}$$

Then

$$\begin{aligned}|J_{\Lambda_l^T}(\zeta f)| &\leqslant \frac{\exp(|\mathrm{Im}\,\zeta|^2\|f\|_s^2 K_4 + K_5)\|\exp[-\tfrac{1}{2}(T-t)\,H_l]\,\Omega_0\|^2}{|(\Omega_l, \Omega_0)|^2}\\&\leqslant 2\exp(\tau m)\exp(|\mathrm{Im}\,\zeta|^2\|f\|_s^2 K_4 + K_5).\end{aligned} \tag{3.68}$$

We summarize our results in

*Theorem 3.12: For each $\zeta$ in $\mathbb{C}$ the family $\{J_l(\zeta f)\}_{l=0}^{\infty}$ is bounded in absolute value by* $\exp(|\mathrm{Im}\,\zeta|^2\|f\|_s^2 K_4 + K_5)$ *and (by Lemma 3.2) continuous in the norm $\|\cdot\|_s$, uniformly in $l < \infty$.*

*For $f$ in $\mathscr{V}_\tau$ the family $\{J_{\Lambda_l^T}(\zeta f) \mid T \geqslant \alpha \cdot l, l < \infty\}$ is bounded in absolute value by* $2\exp(\tau m)\exp(|\mathrm{lim}\,\zeta|^2\|f\|_s^2 K_4 + K_5)$ *and (by Lemma 3.2) continuous in the norm $\|\cdot\|_s$ on $\mathscr{V}_\tau$, uniformly in $T \geqslant \alpha l$ and $l \leqslant \infty$.*

### *3.5. Schwinger functions with half-Dirichlet boundary conditions*

In this sub-section we review the results of Sub-sections **3.3** and **3.4** for arbitrary mass parameter $s \geqslant 0$.

Let $V_{l,s}(P)$ be the interaction and $H_{l,s}(P)$ the Hamiltonian with mass parameter $s \geqslant 0$ corresponding to the polynomial $P$ (see Sub-section 2.1). It is obvious that for $0 \leqslant s \leqslant \infty$ and for arbitrary, *fixed* $l < \infty$, *all* the results of Sub-sections **3.3** and **3.4** hold (by the same proofs that work for $s = 0$). In particular the coincident Schwinger functions $\mathfrak{S}_m^{l,s}(x_1, t_1, \ldots, x_m, t_m)$ exist and have the properties stated in Theorems **3.8** and **3.9** (with estimates that possibly depend on $l$ and $s$).

We now consider a particular class of interactions. We choose the polynomial $P$ to have the form

$$P(\zeta) = \sum_{m=0}^{n} \alpha_m\,\xi^{2m} + \mu\xi \tag{3.69}$$

and we take $\mu \geqslant 0$. (Of course $a_n > 0$.)

*Remark:* The restriction to positive $\mu$ is no loss of generality, since there is a symmetry between the cases $\mu > 0$ and $\mu < 0$; see Ref. [4].

*Theorem 3.13 (essentially due to Refs. [4, 21]): Let P be as in (3.69). Then*

$$0 \leqslant \mathfrak{S}_m^{l,s}(x_1, t_1, \ldots, x_m, t_m) \leqslant \mathfrak{S}_m^{l,0}(x_1, t_1, \ldots, x_m, t_m) \tag{3.70}$$

*for all $l$, $s \geqslant 0$.*

*The sequence $\{\mathfrak{S}_m^{l,s}(x_1, t_1, \ldots, x_m, t_m)\}_{s=0}^{\infty}$ is decreasing as s increases and*

$$\lim_{s\to\infty} \mathfrak{S}_m^{l,s}(x_1, t_1, \ldots, x_m, t_m) \equiv \mathfrak{S}_m^{l,D}(x_1, t_1, \ldots, x_m, t_m) \geqslant 0 \tag{3.71}$$

*exists and is called the m-point Schwinger function with half-Dirichlet boundary conditions.*

*Proof:* The theorem is a consequence of the GRS (Griffiths) inequalities for the $P(\varphi)_2$ models, where $P$ is as in (3.69). The proof of (3.70) and (3.71) is given in Refs. [4, 29]. Because of (3.70) and Theorems 3.9 and 3.12 the distributions $\mathfrak{S}_m^{l,D}(x_1, t_1, \ldots, x_m, t_m)$ are tempered with an order that does *not* depend on $l$.

Q.E.D.

*Corollary 3.14: (a)*

$$J_{l,D}(\zeta f) = \sum_{m=0}^{\infty} \frac{(i\zeta)^m}{m!} \mathfrak{S}_m^{l,D}(f, \ldots, f) \tag{3.72}$$

*exists is entire analytic in $\zeta$, for each $f \in \mathscr{V}$, and*

$$|J_{l,D}(\zeta f)| \leqslant J_{l,D}(i \operatorname{Im} \zeta f) \leqslant J_{l,D}(-i|\operatorname{Im} \zeta||f|) \leqslant J_l(-i|\operatorname{Im} \zeta||f|) \tag{3.73}$$

*where $|f|$ denotes the absolute value of $f$.*

*(b) Estimates I–IV of Sub-section 2.2 hold (with constants that are independent of s and l).*

*(c) All the results of Sub-sections 3.3 and 3.4 hold (with estimates that are uniform in l).*

*Proof:* Using (3.70) and (3.71) we get

$$\left| \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \mathfrak{S}_m^{l,D}(f, \ldots, f) \right| \leqslant \sum_{m=0}^{\infty} \frac{|\zeta|^m}{m!} \mathfrak{S}_m^{l,D}(|f|, \ldots, |f|)$$

$$\leqslant \sum_{m=0}^{\infty} \frac{|\zeta|^m}{m!} \mathfrak{S}_m^{l}(|f|, \ldots, |f|).$$

By Theorems 3.9 and 3.11

$$\sum_{m=0}^{\infty} \frac{|\zeta|^m}{m!} \mathfrak{S}_m^{l}(|f|, \ldots, |f|) = J_l(-i|\zeta||f|) \tag{3.74}$$

which is finite for all $|\zeta| < \infty$ and all $f \in \mathscr{V}$. This proves (3.72). The first inequality in (3.73) is obvious (see (3.15)), the second follows from (3.71) (positivity of the distributions $\mathfrak{S}_m^{l,D}(\cdots)$!), and the third follows from (3.72) and (3.70). This proves (3.73).

*Proof of (b):* It is easy tc show that for each fixed $l < \infty$ and all $s \in [0, \infty]$ $H_{l,s} + \lambda\varphi_{l,s}(f)$ has a ground state $\Omega_{l,s}(\lambda f)$ with

$$(\Omega_{l,s}(\lambda f), \Omega_{l,s}) \neq 0. \tag{3.75}$$

Let

$$\delta E_{l,s}(\lambda f) := \inf \operatorname{spec}(H_{l,s} + \lambda \varphi_{l,s}(f)).$$

The eigenvalue

$$\delta E_{l,s}(\lambda f) \tag{3.76}$$

is *simple* and *isolated.* (For a proof of (3.75) and (3.76) see Refs. [4, 15, 16].) Hence

$$\lim_{T\to\infty} \frac{\exp[-T/2(H_{l,s} + \lambda\varphi_{l,s}(f) - \delta E_{l,s}(\lambda f))]\,\Omega_{l,s}}{\|\exp[-T/2(\cdots)]\,\Omega_{l,s}\|} = \Omega_{l,s}(\lambda f). \tag{3.77}$$

Now let $h_T := f \otimes \chi_{[-T/2,T/2]}$. Then by (3.75)–(3.77)

$$\begin{aligned}\delta E_{l,s}(\lambda f) &= -\lim_{T\to\infty} \frac{1}{T} \log\,(\Omega_{l,s}, \exp[-T(H_{l,s} + \lambda\varphi_{l,s}(f))]\,\Omega_{l,s})\\ &= -\lim_{T\to\infty} \frac{1}{T} \log J_{l,s}(i\lambda h_T)\\ &\geqslant -\lim_{T\to\infty} \frac{1}{T} \log J_l(-i|\lambda||h_T|) = \delta E_l(|\lambda||f|).\end{aligned} \tag{3.78}$$

But (3.78) yields Estimates II–IV for arbitrary $s \leqslant \infty$, by some standard arguments [11]. The same arguments apply to $\varphi$-perturbations of the type $:\varphi^2:(h)$, since here the Wick-ordering just means subtraction of a (infinite) constant from $\varphi^2(h)$. (The general form of Estimate I follows from methods of Glimm and Jaffe [11]). Finally (c) follows from (a) and (b) and Theorem 3.1(I) for $H = H_{l,s}$, $\Omega = \Omega_{l,s}$.

Q.E.D.

*Remarks on the $P(\varphi)_2$ Schwinger functions with half-Dirichlet boundary conditions [4] on a compact, regular set $\Lambda \subset \mathbb{R}^2$*

We summarize briefly some results on the $P(\varphi)_2$ Schwinger functions with half-Dirichlet boundary conditions on a compact, regular set $\Lambda$ in $\mathbb{R}^2$ which are studied in more detail in Refs. [4, 14, 29]. We call a set $\Lambda \subset \mathbb{R}^2$ *regular* if it has continuous, piecewise smooth boundaries and $\Lambda$ is the same as the closure $\overline{\Lambda_{\text{int.}}}$ of its interior $\Lambda_{\text{int.}}$.

Let $P$ be a polynomial satisfying condition (3.69). Let $\Delta_\Lambda$ be the Laplacian on the space $L^2(\Lambda)$ with Dirichlet boundary conditions at $\partial\Lambda$ and let $S^{\Lambda}_{m_0,D}(\xi,\eta)$ be the kernel of the s.a. operator $(-\Delta_\Lambda + m_0^2)^{-1}$. We define the generating functional for the free Dirichlet 'Euclidean' field:

$$J^0_{\Lambda,D}(f) := \exp[-\tfrac{1}{2}(f\chi_\Lambda, S^\Lambda_{m_0,D} * (f\chi_\Lambda))_{L^2(\Lambda)}], \tag{3.79}$$

where $f$ is in $\mathscr{S}$ and $\chi_\Lambda$ is the characteristic function of the set $\Lambda$.

It follows from Section 3.2 that the functional $J^0_{\Lambda,D}$ is the Fourier transform of a Gaussian measure $\nu^\Lambda_D$ on $\mathscr{S}'$ with mean 0 and covariance $S_{m_0,D}$. We set

$$V_\Lambda \equiv V_\Lambda(P) := \int_\Lambda d^2\xi : P(\Phi) : (\xi), \tag{3.80}$$

where the Wick ordering : : is the one determined by the free, Gaussian measure $\nu_0$ (Sections 2.1, 3.2, Ref. [4]). It is shown in Refs. [4, 14] that the function $e^{-V_\Lambda}$ on $\mathscr{S}'$ is positive, $\nu_D^\Lambda$-measurable and $\nu_D^\Lambda$-integrable. Hence we can define the measure

$$d\nu_D^{\Lambda,P} := \frac{1}{Z_D^{\Lambda,P}} e^{-V_\Lambda(P)} d\nu_D^\Lambda, \tag{3.81}$$

where $Z_D^{\Lambda,P} := \int_{\mathscr{S}'} d\nu_D^\Lambda e^{-V_\Lambda(P)}$. It is shown in Refs. [4, 14] that the moments

$$\mathfrak{S}_m^{\Lambda,D}(x_1, t_1, \ldots, x_m, t_m) := \int_{\mathscr{S}'} d\nu_D^{\Lambda,P}(q) \prod_{l=1}^m q(x_l, t_l)$$

of the measure $\nu_D^{\Lambda,P}$ exist and are tempered distributions. As a consequence of Theorem 3.13′ and Corollary 3.14′ stated below this is true even for $\Lambda = \Lambda_l^\infty$.

It is obvious that the moments (or Schwinger functions)

$$\{\mathfrak{S}_m^{\Lambda_l^\infty,D}(x_1, t_1, \ldots, x_m, t_m)\}_{m=0}^\infty$$

are 'time'-translation invariant. (See Ref. [4].) We define

$$l_\Lambda := \sup\{|x| \,\|\, \exists t \text{ such that } \langle x, t\rangle \in \Lambda\}.$$

*Theorem 3.13′: (Proven in Refs. [4, 14, 21].) Let $\Lambda' \subset \Lambda \subseteq \mathbb{R}^2$ be regular sets. Then*

$$0 \leqslant \mathfrak{S}_m^{\Lambda',D}(x_1, t_1, \ldots, x_m, t_m) \leqslant \mathfrak{S}_m^{\Lambda,D}(x_1, t_1, \ldots, x_m, t_m)$$

$$\leqslant \mathfrak{S}_m^{l_\Lambda,D}(x_1, t_1, \ldots, x_m, t_m) \leqslant \mathfrak{S}_m^{l_\Lambda,0}(x_1, t_1, \ldots, x_m, t_m).$$

From this theorem, our results in Section 3.4 and results proven in Ref. [29] we now get

*Corollary 3.14′: Let $\Lambda$ be some regular set in $\mathbb{R}^2$ with $l_\Lambda < \infty$. Then (a)*

$$J_{\Lambda,D}^P(\zeta f) := \int_{\mathscr{S}'} d\nu_D^{\Lambda,P}(q)\, e^{i\zeta q(f)} = \sum_{m=0}^\infty \frac{(i\zeta)^m}{m!} \mathfrak{S}_m^{\Lambda,D}(f_1, \ldots, f)$$

*exists, is entire analytic in $\zeta$, for each $f$ in $\mathscr{V}$ and*

$$|J_{\Lambda,D}^P(\zeta f)| \leqslant J_{\Lambda,D}^P(i|\operatorname{Im}\zeta| f) \leqslant J_{l_\Lambda,D}(-i|\operatorname{Im}\zeta\| f|) \leqslant J_{l_\Lambda}(-i|\operatorname{Im}\zeta\| f|)$$

*All results of Sub-section 3.4 remain true (with bounds that are uniform in $\Lambda$).*

*(b) The Schwinger function $\{\mathfrak{S}_m^{\Lambda_l^\infty,D}\}_{m=0}^\infty$ are positive in the sense of Osterwalder and Schrader [3, 5, 29].*

*Remark:* Part (a) is obvious. Part (b) is proven in Ref. [29].

From Theorem 3.13′, Corollary 3.14′, and the estimates of Sub-sections 3.4 and 2.2 we derive the following useful bounds:

*Proposition 3.15: (a) Let $h$, $h_1$, $h_2$ be real functions on $\mathbb{R}^1$ such that the norms $|||h|||$, $|||h_1|||$ and $|||h_2|||$ are finite, where $|||\cdot|||$ is given by (3.58). Let $\chi_T$ be the characteristic function of the interval $[-T, T]$. Then*

$$0 \leqslant J_{\Lambda,D}^P(\pm ih \otimes \chi_T) \leqslant \exp[2T(K_4|||h|||^2 + K_5)]$$

*and*

$$|\mathfrak{S}_2^{\Lambda,D}(h_1 \otimes \delta_{t_1}, h_2 \otimes \delta_{t_2})| \leqslant \mathfrak{S}_2^{l_\Lambda,0}(|h_1| \otimes \delta_{t_1}, |h_2| \otimes \delta_{t_2}) \leqslant O(1)|||h_1|||\cdot|||h_2|||,$$

*uniformly in* $\Lambda$ *(and* $t_1, t_2$*).*

*(b) Let B be a compact, regular set in* $\mathbb{R}^2$. *We set* $K(B) := \max_t \|\chi_B(\cdot,t)\|_1$ *and assume that* $|\mathrm{Im}\,\zeta|\cdot K\cdot K(B) < 1$, *where K is the constant occurring in Estimate II, Sub-section 2.2. Then*

$$|J^P_{\Lambda,D}(\zeta\chi_B)| \leqslant J_{l_\Lambda}(-i|\mathrm{Im}\,\zeta|\chi_B) \leqslant \exp(K|\mathrm{Im}\,\zeta|\,\|\chi_B\|_1).$$

*Proof:* The estimates of Part (a) follow directly from Theorem **3.13′**, Corollary **3.14′**, Lemma **3.9** and (**3.42**). The estimate of Part (b) follows from Theorem **3.13′**, Corollary **3.14′**, Theorem **3.10**, and from Estimate II of Sub-section 2.2.

Q.E.D.

*Remarks:* The results summarized in Theorem **3.13′**, Corollary **3.14′**, and Proposition **3.15** tell us that all the results proven in Section **4** for the Schwinger functions $\{\mathfrak{S}_m^{l,D}\}_{m=0}^\infty$ and the generating functional $J_{l,D}$ remain true under the substitution:

$$\mathfrak{S}_m^{l,D} \mapsto \mathfrak{S}_m^{\Lambda_l^\infty,D}, \quad J_{l,D} \mapsto J^P_{\Lambda_l^\infty,D}.$$

Those results are basic for our verification of the Osterwalder–Schrader axioms (Axioms (E0′)–(E3) of Ref. [5]) presented in Sub-section **4.3**. They have applications in Ref. [29].

## 4. The Infinite Volume (Thermodynamic) Limit

### *4.1. The infinite volume limit for the generating functionals*

In this sub-section we show that for all $f$ in $\mathscr{V}$ and all $\zeta$ in $\mathbb{C}$

$$J(\zeta f) := \lim_{l\to\infty} J_l(\zeta f) \tag{4.1}$$

exists is continuous in the norm $\|\cdot\|_s$ and Euclidean invariant, under the condition that

(C1) the polynomial

$$P_\lambda(\xi) = \lambda \sum_{m=0}^{2n} a_m \xi^m \tag{4.2}$$

determining the interaction $V_l(P_\lambda)$ and the Hamiltonian $H_l(P_\lambda)$ is *positive* and the quotient $\lambda/m_0^2$ is sufficiently *small*, such that the Glimm–Jaffe–Spencer cluster expansion [24] *converges*.

We also show that for $f$ in $\mathscr{V}_\tau$, where $\tau < \infty$, and all $\zeta$ in $\mathbb{C}$

$$J(\zeta f) = \lim_{\substack{l\to\infty \\ T\to\infty \\ T\geqslant \alpha l}} J_{\Lambda_T^l}(\zeta f) = \lim_{l\to\infty} J_l(\zeta f)$$

exists.

We then show that for all $f$ in $\mathscr{V}$ and $\zeta \in \mathbb{C}$

$$J_D(\zeta f) := \lim_{l\to\infty} J_{l,D}(\zeta f) \tag{4.3}$$

exists, is continuous in the norm $\|\cdot\|_s$ and Euclidean invariant under the following condition:

(C2) The polynomial $P$ is of the form

$$P(\xi) = \sum_{m=0}^{n} a_m \xi^{2m} + \mu\xi \quad (\mu \geqslant 0) \tag{4.4}$$

(such that Nelson's convergence theorem [21] for the Schwinger functions applies).

The proof of (4.1) and (4.3) is based on

*Theorem 4.1: Let $m$ be an arbitrary integer and $f_1, \ldots, f_m$ be functions in $C_0^\infty(\mathbb{R}^2)$. Then:*

*(I) (Glimm–Jaffe–Spencer [24]) Under the condition (C1)*

$$\lim_{\substack{T\to\infty\\ l\to\infty\\ T\geqslant\alpha l}} \mathfrak{S}_m^{\Lambda_l^T}(f_1,\ldots,f_m) = \lim_{l\to\infty} \mathfrak{S}_m^l(f_1,\ldots,f_m)$$
$$=: \mathfrak{S}_m(f_1,\ldots,f_m)$$

*exists and is Euclidean invariant.*

*(II) (GRS [4, 14]) Under the condition (C2)*

$$\lim_{l\to\infty} \mathfrak{S}_m^{l,D}(f_1,\ldots,f_m) =: \mathfrak{S}_m^D(f_1,\ldots,f_m)$$

*exists and is Euclidean invariant.*

$$\mathfrak{S}_m^{l,D}(x_1,t_1,\ldots,x_m,t_m) \uparrow \mathfrak{S}_m^D(x_1,t_1,\ldots,x_m,t_m) \tag{4.5}$$

*in the sense of convergence in $\mathscr{S}'(\mathbb{R}^{2m})$.*

*Proof:* (I) is proven in the ingenious paper of Glimm, Jaffe and Spencer [24]; (II) is proven for the non-coincident Schwinger functions in Ref. [14]. But because of Theorem **3.13** (3.70) and Theorems 3.12, 3.8, (II) holds for the coincident functions as well. The monotonicity (4.5) is the consequence of a beautiful application of the GRS inequalities, Refs. [4, 21]. Another version of (II) is proven in Ref. [29].

*Theorem 4.2 (first main result): Let (C1), ((C2)) be true.*

*(I) For all $f$ in $\mathscr{V}$ and all $\zeta$ in $\mathbb{C}$*

$$J_{(D)}(\zeta f) := \lim_{l\to\infty} J_{l,(D)}(\zeta f)$$

*exists, is an entire analytic function in $\zeta$ of order $\leqslant 2$, for each $f$ in $\mathscr{V}$, is continuous in the norm $\|\cdot\|_s$, for each $\zeta$ in $\mathbb{C}$ and is Euclidean invariant.*

$$|J_{(D)}(\zeta f)| \leqslant O(1)\exp(|\mathrm{Im}\,\zeta|^2 \|f\|_s^2 \cdot K_4). \tag{4.6}$$

*The functional $J_{(D)}(\cdot)$ is the Fourier transform of a unique measure $\nu_{(D)}$ on $\mathscr{S}'$ (which is Euclidean invariant), and $J_{(D)}(\cdot)$ is the generating functional for the Glimm–Jaffe–Spencer–(Nelson)–Schwinger functions obtained in Theorem 4.1.*

*(II) The distributions* $\{\mathfrak{S}_m^{(D)}(x_1,t_1,\ldots,x_m,t_m)\}_{m=0}^{\infty}$ *are continuous in each argument in the norm* $\|\cdot\|_s$ *and they are positive in the sense of Osterwalder and Schrader [3, 5].*

$$|\mathfrak{S}_m^{(D)}(f_1,\ldots,f_m)| \leqslant O(1)^m \sqrt{m!}\, \|f_1\|_s \cdots \|f_m\|_s. \tag{4.7}$$

*(III) For all* $f$ *in* $\mathscr{V}_\tau$, $\tau < \infty$, *and all* $\zeta$ *in* $\mathbb{C}$

$$J(\zeta f) = \lim_{\substack{l\to\infty \\ T\to\infty \\ T\geqslant \alpha l}} J_{\Lambda_l^T}(\zeta f) = \lim_{l\to\infty} J_l(\zeta f)$$

*exists.*

*Proof:* Because of Theorem 3.12 and Corollary 3.14

$$J_{l,(D)}(\zeta f) = \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \mathfrak{S}_m^{l,(D)}(f,\ldots,f), \tag{4.8}$$

where the series on the right-hand side of (4.8) converges absolutely for all $f$ in $\mathscr{V}$, $\zeta$ in $\mathbb{C}$ and *uniformly* in $l$, i.e. for fixed $\zeta$ and $f$ and all $\epsilon > 0$ there is a $n_0(\epsilon)$ independent of $l$ such that

$$\left|\sum_{m=n}^{\infty} \frac{\zeta^m}{m!} \mathfrak{S}_m^{l,(D)}(f,\ldots,f)\right| < \epsilon$$

for all $n \geqslant n_0(\epsilon)$ and all $l < \infty$.

Applying now Theorem 4.1 we conclude that

$$J_{l,(D)}(\zeta f) \to \sum_{m=0}^{\infty} \frac{\zeta^m}{m!} \mathfrak{S}_m^{(D)}(f,\ldots,f) =: J_{(D)}(\zeta f) \tag{4.9}$$

as $l \to \infty$, for fixed $f$ in $C_0^\infty(\mathbb{R}^2)$. But, since for each $\zeta$ in $\mathbb{C}$ the families $\{J_{l,(D)}(\zeta f)\}_{l<\infty}$, $\{\mathfrak{S}_m^{l,(D)}(\zeta f,\ldots,\zeta f)\}_{l<\infty}$ are *equicontinuous* in $f$ in the norm $\|\cdot\|_s$, (4.9) holds for all $f$ in $\mathscr{V}$. Euclidean invariance of $J_{l,(D)}(\zeta f)$ follows directly from (4.9) and the Euclidean invariance of the Schwinger functions in the limit $l = \infty$. The bound (4.6) and the continuity of $J_{(D)}(\zeta f)$ in $\|\cdot\|_s$ follow from Theorem 3.12 and Corollary 3.14. Since $J_{l,(D)}(\zeta f)$ obeys the properties 3.2, i)–iv) of Sub-section 3.2, for all $l < \infty$, we now conclude that $J_{(D)}(f)$ obeys the same properties and hence it is the Fourier transform of a unique measure on $\mathscr{S}'$. This completes the proof of (I).

*Proof of (II):* From Theorems 3.8 and 4.1 we conclude that

$$|\mathfrak{S}_m^{(D)}(f_1,\ldots,f_m)| \leqslant O(1)^m m! \prod_{i=1}^{m} \|f_i\|_s.$$

Since the order of the entire analytic function $J_{(D)}(\zeta f)$ of $\zeta$ at $\infty$ is $\leqslant 2$, it is easy to conclude that

$$|\mathfrak{S}_m^{(D)}(f_1,\ldots,f_m)| = \left|\frac{\partial^m}{\partial\zeta_1\ldots\partial\zeta_m} J_{(D)}\left(\sum_{i=1}^{m} \zeta_i f_i\right)\right|_{\zeta_1=\cdots=\zeta_m=0} \leqslant O(1)^m \sqrt{m!} \prod_{i=1}^{m} \|f_i\|_s.$$

The rest is obvious.

*Proof of (III):* This follows directly from Theorem 3.12 and Theorem 4.1(I).

Q.E.D.

*Corollary 4.3: Under condition (C1) or (C2) the functional $J_{(D)}(f)$ determines a Euclidean field $\Phi$ (Definition 3.2, Sub-section 3.2) on a Hilbert space $\mathscr{H}_{v_{(D)}} = L^2(\mathscr{S}', dv_{(D)})$. The vacuum $\Omega_{(D)} = I$ (= function identically 1 on $\mathscr{S}'$) is an analytic vector for the fields $\Phi(f), f \in \mathscr{V}$. The coincident Schwinger functions exist.*

*Proof:* This is a direct consequence of Theorems 4.1 and 4.2, Estimate IV (Sub-sections 2.2 and 3.5). It is easy to verify that because of Theorem 4.1(I), (II), Estimate IV holds in the infinite volume limit, too. For this, and a Euclidean transcription of Estimate IV, see Ref. [29].

Q.E.D.

We now want to prove the existence of (*time 0–*) *quantum fields* in the infinite volume limit as operators on $\mathscr{H}_{W,(D)}$.

### 4.2. (*Time 0–*) *quantum fields in the infinite volume limit*

We prove the following general result:

*Theorem 4.4: Suppose that*

a) *$J$ is a functional on $\mathscr{S}$ obeying properties 3.2, i)–iv) and that the Schwinger functions $\{\mathfrak{S}_m\}_{m=0}^{\infty}$ associated with $J$ are positive in the sense of Osterwalder and Schrader [3, 5] and are tempered distributions.*
*Suppose moreover that*

b) *the two-point function $\mathfrak{S}_2(f \otimes \delta_t, f \otimes \delta_0)$ is bounded in some neighborhood of $t = 0$ for all $f$ in $\mathscr{S}(\mathbb{R})$.*

*Then the (time 0–) fields $\varphi(f) := \Phi(f \otimes \delta_0)$ exist and are self adjoint on $\mathscr{H}_v$, for all $f$ in $\mathscr{S}_r(\mathbb{R})$.*

*Proof:*

Step 1°. It follows from hypothesis (a) that the function $\mathfrak{S}_2(f \otimes \delta_t, f \otimes \delta_0)$ exists and is $C^\infty$ in $t$ on $\{t \| t| > 0\}$ for $f$ in $\mathscr{S}(\mathbb{R})$. Thus hypothesis (b) is meaningful. It is easy to see that

$$\mathfrak{S}_2(f \otimes \delta_t, f \otimes \delta_0) = \mathfrak{S}_2(f \otimes \delta_{|t|}, f \otimes \delta_0)$$

under hypothesis (a).

It is shown in Refs. [3, 5, 29] that $\mathfrak{S}_2(\bar{f} \otimes \delta_t, f \otimes \delta_0)$ is *decreasing* in $t$ and *convex* on $\{t | t > 0\}$. It follows now from hypothesis (b) that $\mathfrak{S}_2(f \otimes \delta_t, f \otimes \delta_0)$ is bounded and continuous in $t$, *even at $t = 0$*. (For details see Ref. [29].)

Step 2°. Let $\chi_n(t) := \sqrt{n/\pi}\, e^{-nt^2}$, $n = 1, 2 \dots$. Obviously

$$\{\chi_n\}_{n=1}^{\infty} \subset \mathscr{S}_r(\mathbb{R}) \quad \text{and} \quad \chi_n \to \delta_0, \tag{4.10}$$

as $n \to \infty$, weakly on $C(\mathbb{R})$. Let $f$ be in $\mathscr{S}_r(\mathbb{R}_x)$. Then $\{\exp[is\Phi(f \otimes \chi_n)]\}_{n=1}^{\infty}$ is a sequence of unitary groups on $\mathscr{H}_v$. We show that

$$s - \lim_{n \to \infty} \exp[is\Phi(f \otimes \chi_n)] =: \exp[is\Phi(f \otimes \delta_0)] \tag{4.11}$$

exists for all $s$ in $\mathbb{R}$ and thus $\{\exp[is\Phi(f \otimes \delta_0)]\}$ is a unitary group on $\mathscr{H}_v$.

By construction, the vector $I$ is cyclic and separating for the algebra generated by $\{e^{i\Phi(f)}|f\in\mathscr{S}\}$ on $\mathscr{H}_v$. Thus (4.11) is proven if we can show that

$$\|(\exp[is\Phi(f\otimes\chi_n)]-\exp[is\Phi(f\otimes\chi_m)])\,I\|\to 0, \tag{4.12}$$

as $m, n\to\infty$, for all $s$ in $\mathbb{R}$.

Step $3^0$. Proof of (4.12): By Duhamel's formula (3.14)

$$\|(\exp[is\Phi(f\otimes\chi_n)-\exp[is\Phi(f\otimes\chi_m)])\,I\|^2\leqslant s^2\int dt\,dt'\,\overline{(\chi_n(t)-\chi_m(t))}(\chi_n(t')-\chi_m(t'))$$
$$\times\,\mathfrak{S}_2(f\otimes\delta_{t-t'},f\otimes\delta_0). \tag{4.13}$$

By step $1^0$ $\mathfrak{S}_2(f\otimes\delta_{t-t'},f\otimes\delta_0)$ is jointly continuous in $t\in\mathbb{R}$ $t'\in\mathbb{R}$. Hence by (4.10) the right-hand side of (4.13) tends to 0, as $m, n\to\infty$.

Q.E.D.

*Corollary 4.5: Under conditions (C1), (C2), respectively the (time 0–) quantum fields defined by*

$$\varphi(f):=\phi(f\otimes\delta_0),\quad f\text{ in }\mathscr{S}_r(\mathbb{R}), \tag{4.14}$$

*exist (in the infinite volume limit) and are s.a. on* $\mathscr{H}_{v_{(D)}}$. *The functional* $J_{(D)}(f\otimes\delta_0)$ *is continuous in* $f$ *in the norm* $|||\cdot|||$ *(defined in (3.58)), for real valued* $f$. *Therefore*

$$J_{(D)}(f\otimes\delta_0) \tag{4.15}$$

*is the Fourier transform of a measure* $\mu_{(D)}$ *on* $\mathscr{S}_r'(\mathbb{R})$.

*Proof:* By Theorem 3.13

$$\begin{aligned}\mathfrak{S}_2^{l,D}(f\otimes\delta_t,f\otimes\delta_0)&\leqslant\mathfrak{S}_2^{l,D}(|f|\otimes\delta_t,|f|\otimes\delta_0)\\&\leqslant\mathfrak{S}_2^{l}(|f|\otimes\delta_t,|f|\otimes\delta_0)\\&=(\Omega_l,\varphi(|f|)\,e^{-|t|H_l}\,\varphi(|f|)\,\Omega_l)\\&\leqslant|||f|||^2,\quad\text{by (3.42)}\end{aligned} \tag{4.16}$$

From Theorem 4.1, (4.16), Theorem 3.10 and Lemma 3.3, ii) it follows that under the conditions (C1)

$$\mathfrak{S}_2^{l}(f\otimes\delta_t,f\otimes\delta_0)\to\mathfrak{S}_2(f\otimes\delta_t,f\otimes\delta_0)\leqslant|||f|||^2,$$

as $l\to\infty$, and $\mathfrak{S}_2(f\otimes\delta_t,f\otimes\delta_0)$ is bounded, continuous, and convex on $\{t\geqslant 0\}$. From Theorem 4.1, (4.16), Corollary 3.14 and Lemma 3.3(ii) it follows that under the conditions (C2)

$$\mathfrak{S}_2^{l,D}(f\otimes\delta_t,f\otimes\delta_0)\to\mathfrak{S}_2^{D}(f\otimes\delta_t,f\otimes\delta_0)(\leqslant|||f|||^2),$$

as $l\to\infty$, and $\mathfrak{S}_2^{D}(f\otimes\delta_t,f\otimes\delta_0)$ is bounded and continuous on $\{t\geqslant 0\}$. Hence Theorem 4.4 applies and proves the first part of the corollary.

$$|J_{(D)}((f+f')\otimes\chi_n)-J_{(D)}(f\otimes\chi_n)|^2\leqslant\mathfrak{S}_2^{(D)}(f'\otimes\chi_n,f'\otimes\chi_n)\leqslant|||f'|||^2 \tag{4.17}$$

by (4.16), for all $n\leqslant\infty$.

Hence $J_{(D)}(f \otimes \delta_0)$ is normalized, of positive type and continuous on $\mathscr{S}_r(\mathbb{R})$. We apply the theorem of Minlos to conclude that it is the Fourier transform of a unique measure $\mu_{(D)}$ on $\mathscr{S}_r'(\mathbb{R})$.

Q.E.D.

*Remark:* $\mu_{(D)}$ can be obtained from $\nu_{(D)}$ by restriction. We now show that $J_{(D)}(\lambda(f \otimes \delta_0))$ is the boundary value of a function which is *analytic* in a strip *around the real axis*, for each $f$ with $|||f|||_{1,\infty} := \|f\|_\infty + \|f\|_1 < \infty$.

Because of Euclidean invariance of the functional $J_{(D)}$ we can study $J_{(D)}(\lambda(\delta_0 \otimes f))$. We prove the following more general result:

*Lemma 4.6: Let $\zeta_1, \ldots, \zeta_n$ be arbitrary complex numbers in the strip $\Sigma_1 := \{\zeta \| \mathrm{Im}\, \zeta| < 1\}$. Suppose that $K \cdot (\sum_{i=1}^n |||f_i|||_{1,\infty}) \leqslant 1$, where $K$ is the constant defined in Estimate II (2.17), Sub-section 2.2.*

*Then $J_{(D)}(\sum_{i=1}^n \zeta_i (f_i \otimes \delta_{t_i}))$ is analytic in $\zeta_1, \ldots, \zeta_n$ on the domain $\Sigma_1^{\times n}$ and continuous in the variables $t_1, \ldots, t_n$. Moreover*

$$\left| J_{(D)}\left( \sum_{i=1}^n \zeta_i (f_i \otimes \delta_{t_i}) \right) \right| \leqslant \exp\left[ K \left( \sum_{i=1}^n |\mathrm{Im}\, \zeta_i| \, |||f_i|||_{1,\infty} \right) \right] \tag{4.18}$$

on $\sum_1^{\times n}$.

*Proof:* Let $l/2 \geqslant \max_{i=1,\ldots,n} |t_i| + 1$ and let

$$\xi(x) := \begin{cases} 1, & |x| \leqslant \frac{1}{2} \\ 0, & |x| \geqslant 1 \end{cases}, \quad 0 \leqslant \xi(x) \leqslant 1, \ \xi \text{ is } C^\infty; \quad \xi^t(x) := \xi(x - t).$$

Let $\chi_m^t(x) := \chi_m(x - t)$ where $\chi_m$ is defined in (4.10). Obviously

$$\|\chi_m^{t_i} \xi^{t_i}\|_1 < 1, \quad \mathrm{supp}(\chi_m^{t_i} \xi^{t_i}) \subseteq [-l/2, l/2], \quad \chi_m^{t_i} \xi^{t_i} \to \delta_{t_i}$$

on $C(\mathbb{R})$, as $m \to \infty$, for all $i = 1, \ldots, n$.

Suppose that $K \cdot (\sum_{i=1}^n |||f_i|||_{1,\infty}) \leqslant 1$. Then by Estimate II, Sub-section 2.2

$$\pm \left\{ \sum_{i=1}^n f_i(t)\, \varphi(\chi_m^{t_i} \cdot \xi) \right\} < H_l + K \cdot \left( \sum_{i=1}^n |f_i(t)| \right) \tag{4.19}$$

Hence by Corollary 3.7 (and Theorem 3.10, Corollary 3.14)

$$\left| J_{l,(D)}\left( \sum_{i=1}^n \zeta_i (\chi_m^{t_i} \xi^{t_i} \otimes f_i) \right) \right| \leqslant J_{l,(D)}\left( i \sum_{i=1}^n \mathrm{Im}\, \zeta_i (\chi_m^{t_i} \xi^{t_i} \otimes f_i) \right)$$

$$< \exp\left[ K \left( \sum_{i=1}^n |\mathrm{Im}\, \zeta_i| \, |||f_i|||_{1,\infty} \right) \right]. \tag{4.20}$$

Now by Theorem 4.2 and (4.20)

$$J_{(D)}\left( \sum_{i=1}^n \zeta_i (\chi_m^{t_i} \xi^{t_i} \otimes f_i) \right) = \lim_{l \to \infty} J_{l,(D)}\left( \sum_{i=1}^n \zeta_i (\chi_m^{t_i} \cdot \xi^{t_i} \otimes f_i) \right).$$

Therefore $J_{(D)}(\sum_{i=1}^n \zeta_i (\chi_m^{t_i} \xi^{t_i} \otimes f_i))$ is analytic in $\zeta_1, \ldots, \zeta_m$ on the domain $\Sigma_1^{\times n}$ and on this domain

$$\left| J_{(D)}\left( \sum_{i=1}^n \zeta_i (\chi_m^{t_i} \cdot \xi^{t_i} \otimes f_i) \right) \right| < \exp\left[ K \left( \sum_{i=1}^n |\mathrm{Im}\, \zeta_i| \, |||f_i|||_{1,\infty} \right) \right].$$

Now by (4.11), (4.12) and Corollary 4.5

$$J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(\delta_{t_i} \otimes f_i)\right) = \lim_{m\to\infty} J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(\chi_m^{t_i}\xi^{t_i} \otimes f_i)\right) \tag{4.21}$$

*uniformly* in $t_1, \ldots, t_n$ on compact sets (because of uniform continuity of $\mathfrak{S}_2^{(D)}(f \otimes \delta_t, f \otimes \delta_0)$ in $t$ on compact sets). Therefore $J_{(D)}(\sum_{i=1}^{n} \zeta_i(\delta_{t_i} \otimes f_i))$ is analytic on $\Sigma_1^{\times n}$ and on this domain

$$\left|J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(\delta_{t_i} \otimes f_i)\right)\right| < \exp\left[K\left(\sum_{i=1}^{n} |\mathrm{Im}\, \zeta_i|\, |||f_i|||_{1,\infty}\right)\right]. \tag{4.22}$$

We now show continuity in $t_1, \ldots, t_n$.

By Euclidean invariance of the functionals $J, J_D$

$$J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(\delta_{t_i} \otimes f_i)\right) = J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(f_i \otimes \delta_{t_i})\right).$$

In order to establish continuity of the functions $J_{(D)}(\sum_{i=1}^{n} \zeta_i(f_i \otimes \delta_{t_i}))$ in $t_1, \ldots, t_n$ for arbitrary, fixed $\zeta_1, \ldots, \zeta_n$ in their domain of holomorphy $\Sigma_1^{\times n}$ containing $\mathbb{R}^n$ it suffices to show that for arbitrary, *real* $\zeta_1, \ldots, \zeta_n$

$$J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(f_i \otimes \delta_{t_i'})\right) \to J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(f_i \otimes \delta_{t_i})\right)$$

as $t_j' \to t_j$, for $j = 1, \ldots, n$.

We use Duhamel's formula and the unitarity of the operators $\exp[i\zeta_i \Phi(f_i \otimes \delta_{t_i})]$, for real $\zeta_i$, and conclude that

$$\left|J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(f_i \otimes \delta_{t_i'})\right) - J_{(D)}\left(\sum_{i=1}^{n} \zeta_i(f_i \otimes \delta_{t_i})\right)\right|$$

$$\leqslant \sum_{i,j=1}^{n} |\zeta_i||\zeta_j|\, \mathfrak{S}_2^{(D)}(f_i \otimes (\delta_{t_i} - \delta_{t_i'}), f_j \otimes (\delta_{t_j} - \delta_{t_j'})).$$

Estimate (4.22) tells us that the vacuum $\Omega_{(D)} = I$ is in the domain of the (time 0–) fields $\varphi_{(D)}(f_i) = \Phi(f_i \otimes \delta_0)$, $i = 1, \ldots, n$. As in step $1^0$ of the proof of Theorem 4.4 we may show now that the two-point functions $\mathfrak{S}_2^{(D)}(f_i \otimes \delta_{t_i'}, f_j \otimes \delta_{t_j'})$ are jointly continuous in $t_i'$ and $t_j'$, $i, j = 1, \ldots, n$. Thus

$$\mathfrak{S}_2^{(D)}(f_i \otimes (\delta_{t_i} - \delta_{t_i'}), f_j \otimes (\delta_{t_j} - \delta_{t_j'})$$

tends to 0 as $t_m' \to t_m$ for all $m = 1, \ldots, n$. This completes the proof of the lemma.

Q.E.D.

*Remark:* Estimates I and IV of Sub-section 2.2 imply that for $f$ and $g$ in $\mathscr{S}(\mathbb{R})$ the family

$$\{\mathfrak{S}_2^{l,(D)}(f \otimes \delta_t, g \otimes \delta_0) = (\Omega_{l,(D)}, \varphi_{(D)}(f) \exp[-|t|H_{l,(D)}]\, \varphi_{(D)}(g)\, \Omega_{l,(D)})\}_{l<\infty}$$

of two-point functions is equicontinuous in $t \in \mathbb{R}$. Therefore

$$J_{(D)}\left(\sum_{i=1}^{n} f_i \otimes \delta_{t_i}\right) = \lim_{l \to \infty} J_{l,(D)}\left(\sum_{i=1}^{n} f_i \otimes \delta_{t_i}\right)$$

for $f_1, \ldots, f_n$ in $\mathscr{S}_r(\mathbb{R})$.

The proof is straightforward and is left to the reader.

*Theorem 4.7* (*second main result*):

(*I*) *The vacuum $\Omega_{(D)} = I$ in the Hilbert space $\mathscr{H}_{\nu_{(D)}}$ is an analytic vector for the sharp time fields $\Phi(f \otimes \delta_t)$ for arbitrary $t$ and all $f$ with $|||f|||_{1,\infty} < \infty$. In particular $\Omega_{(D)}$ is analytic for $\varphi_{(D)}(f)$ if $|||f|||_{1,\infty} < \infty$.*

(*II*) *Let $f_1, \ldots, f_m$ be test functions with $|||f_i|||_{1,\infty} < \infty$, $i = 1, \ldots, m$. Then the m-point Schwinger function*

$$\mathfrak{S}_m^{(D)}(f_1 \otimes \delta_{t_1}, \ldots, f_m \otimes \delta_{t_m})$$
$$= (\Omega_{(D)}, \varphi_{(D)}(f_1) \exp[-|t_1 - t_2| H_{(D)}] \ldots \exp[-|t_{m-1} - t_m| H_{(D)}] \varphi_{(D)}(f_m) \Omega_{(D)})$$

*exists, is analytic in $s_1 = t_1 - t_2, \ldots, s_{m-1} = t_{m-1} - t_m$ on $\{\langle s_1, \ldots, s_{m-1}\rangle | \mathrm{Re}\, s_i \neq 0, i = 1, \ldots, m\}$ and continuous in $t_1, \ldots, t_m$ on $\mathbb{R}^m$. Furthermore*

$$|\mathfrak{S}_m^{(D)}(f_1 \otimes \delta_{t_1}, \ldots, f_m \otimes \delta_{t_m})| \leqslant O(1)^m\, m! \prod_{i=1}^{m} |||f_i|||_{1,\infty}.$$

*Proof:* Clearly

$$\mathfrak{S}_m^{(D)}(f_1 \otimes \delta_{t_1}, \ldots, f_m \otimes \delta_{t_m}) = \frac{\partial^m}{\partial \zeta_1 \ldots \partial \zeta_m} (-i)^m J_{(D)}\left(\sum_{i=1}^{m} \zeta_i (f_i \otimes \delta_{t_i})\right)\Bigg|_{\zeta_1 = \cdots = \zeta_m = 0}.$$

Applying Lemma 3.2 and using the analyticity of $J_{(D)}(\sum_{i=1}^{m} \zeta_i (f_i \otimes \delta_{t_i}))$ in $\zeta_1 \cdots \zeta_m$ the theorem follows easily.

The analyticity properties of the Schwinger function $\mathfrak{S}_m^{(D)}(f_1 \otimes \delta_{t_1}, \ldots, f_m \otimes \delta_{t_m})$ in $s_1, \ldots, s_{m-1}$ is a standard result, given Estimate II or IV.

Q.E.D.

*Remark:* Assume that the hypotheses of Theorems **3.12** and **4.2**(III) hold, i.e. that the polynomial $P$ is such that the Cluster Expansion of Ref. [24] converges. Let $f_1, \ldots, f_n$ be real functions on $\mathbb{R}^1$ with

$$\operatorname{supp} f_i \subseteq \left[-\frac{\tau}{2}, \frac{\tau}{2}\right], \quad i = 1, \ldots, n \quad \text{and} \quad K\left(\sum_{i=1}^{n} |||f|||_{1,\infty}\right) \leqslant 1.$$

Then the estimates of Lemma **4.6** and Theorem **4.7** hold already on the level of space-time cutoff generating functionals and Schwinger function.

Let $l \geqslant l_0 > \tau/\alpha$ (for some finite $l_0$) and $T \geqslant \alpha l$, where the number $\alpha$ is such as in Theorem **3.12**. Let $t_i \in (-l_0/2, l_0/2)$ and $\zeta_i \in \Sigma_1$, $i = 1, \ldots, n$. Then

$$\left|J_{\Lambda_l^T}\left(\sum_{i=1}^{n} \zeta_i (\delta_{t_i} \otimes f_i)\right)\right| \leqslant O(1)\, e^{\tau_m} \exp\left[K\left(\sum_{i=1}^{n} |\mathrm{Im}\, \zeta_i|\, |||f_i|||_{1,\infty}\right)\right], \tag{4.23}$$

uniformly in $l$ and $T$. Also, bounds on the space-time cutoff, sharp-space Schwinger functions similar to the ones of Theorem 4.7(II) hold and are uniform in $l \geqslant l_0$ and $T \geqslant \alpha \cdot l$.

Straightforward generalizations of these bounds based on Estimates I and II of Sub-section 2.2 and results of Dimock [35] now imply that the sharp-time Schwinger functions

$$\mathfrak{S}_m^{(D)}(f_1 \otimes \delta_{t_1}, \ldots, f_m \otimes \delta_{t_m})$$

have a perturbation expansion in the coupling constant $\lambda$ defined in (4.2) which is asymptotic to the exact solution at $\lambda = 0$. (See Ref. [35] and Part II of this paper, and Ref. [29] for applications of this result.)

### 4.3. *Verification of the Osterwalder–Schrader axioms for $P(\varphi)_2$ models under condition (C2)*

In this sub-section we verify the axioms proposed by Osterwalder and Schrader [3, 5] in the form of axioms (E0′)–(E3) of Ref. [5] and hence, *a fortiori*, Ref. [5], the Wightman axioms (up to the uniqueness of the physical vacuum) for the $P(\varphi)_2$ models with half-Dirichlet boundary conditions under condition (C2). Those axioms were verified for the $P(\varphi)_2$ models under condition (C1) in Ref. [5].

First we briefly outline what the Osterwalder–Schrader axioms for one neutral, scalar Bose field in two space-time dimensions (axioms (E0′)–(E3) of Ref. [5]) are. These axioms are formulated in terms of the Schwinger functions $\{\mathfrak{S}_m\}_{m=0}^{\infty}$:

Axiom (E0′) is a distribution property for the Schwinger functions formulated below.

Axiom (E1) says that the Schwinger functions ought to be Euclidean invariant.

Axiom (E2) is the *Osterwalder–Schrader positivity condition:* For all finite sequences $f_0, \ldots, f_N$ of test-functions $f_n \in \mathscr{S}(\mathbb{R}^{2n}_{<})$ [3, 5]

$$\sum_{n,m=0}^{N} \mathfrak{S}_{n+m}(\overline{\theta f_n} \otimes f_m) \geqslant 0 \tag{4.24}$$

where $\overline{\theta f_n}(x_1, t_1, \ldots, x_n, t_n) = \overline{f(x_1, -t_1, \ldots, x_n, -t_n)}$.

Finally, axiom (E3) says that the Schwinger functions are symmetric in the space-time arguments.

From our results in Sub-section 3.5 (Corollaries 3.14, 3.14′) and from Theorem 4.2 and Corollary 4.3 we know that axioms (E1), (E2) and (E3) hold for the $P(\varphi)_2$ models with half-Dirichlet boundary conditions under condition (C2) in the infinite volume limit. Among the axioms (E1)–(E3) the only axiom which is not quite obvious is (E2). This axiom holds for the Schwinger functions $\{\mathfrak{S}_m^{l,D}\}_{m=0}^{\infty}$ and $\{\mathfrak{S}_m^{\Lambda_l^{\infty}, D}\}_{m=0}^{\infty}$ for all $l < \infty$ and hence it is true in the limit $l = \infty$.

It follows from the existence of a positive, s.a. quantum field Hamiltonian ($H_{l,D}$, $H_{\Lambda_l^{\infty}, D}$, respectively). More details concerning the verification of axiom (E2) are given in Ref. [29].

We can now formulate axiom (E0′). Since the Schwinger functions $\mathfrak{S}_m^{(D)}(\xi_1, \ldots, \xi_m)$ are Euclidean- (in particular translation-) invariant, there are distributions $S_{m-1}^{(D)}(q_1, \ldots, q_{m-1})$, $q_i = \xi_{i+1} - \xi_i$, such that

$$S_{m-1}^{(D)}(q_1, \ldots, q_{m-1}) = \mathfrak{S}_m^{(D)}(\xi_1, \ldots, \xi_m), \quad S_0^{(D)} = \mathfrak{S}_1^{(D)}(\xi), \quad S_{-1}^{(D)} = 1. \tag{4.25}$$

Axiom (E0′) in the form of Ref. [5] requires that

$$|S_{m-1}^{(D)}(h_1, \ldots, h_{m-1})| \leqslant K_0^m (m!)^L \prod_{i=1}^{m-1} |h_i|_{\mathscr{S}}, \tag{4.26}$$

for some constants $K_0$, $L$ and a Schwartz space norm $|\cdot|_{\mathscr{S}}$ which are *independent* of $m$.

*Theorem 4.8 (third main result): Let*

$$|h|_{\mathscr{S}} := \sup_{q \in \mathbb{R}^2} |(1 + |q|^2)^{1+\epsilon} h(q)|, \tag{4.27}$$

*for some positive $\epsilon$. With this Schwartz space norm there exists a positive, finite constant $K_0$ such that*

$$|S_{m-1}^D(h_1, \ldots, h_{m-1})| \leqslant K_0^m (m!)^3 \prod_{i=1}^{m-1} |h_i|_{\mathscr{S}}, \tag{4.28}$$

*where $\{S_{m-1}^D\}_{m=-1}^\infty$ are the Schwinger functions of a $P(\varphi)_2$ model under condition (C2) in the infinite volume limit, i.e. axiom (E0′) (4.26) holds with $L = 3$ and $|\cdot|_{\mathscr{S}}$ as in (4.27).*

*Proof:* Let $\{\Delta\}$ be a covering of $\mathbb{R}^2$ by unit squares centered at the points $\langle i, j\rangle \in \mathbb{R}^2$, $(\{\chi_\Delta | \Delta \in \{\Delta\}\}$ is a partition of I). Given two squares $\Delta_a$, $\Delta_b$ we define

$$\Delta_a + \Delta_b = \{q | q = q_a + q_b, q_a \in \Delta_a, q_b \in \Delta_b\}.$$

Let $\chi_{\Delta_0 + \cdots + \Delta_j}$ be the characteristic function of the set $\Delta_0 + \cdots + \Delta_j$. Then

$$\|\chi_{\Delta_0 + \cdots + \Delta_j}\|_1 = (j+1)^2, \quad \text{where } \Delta_l \in \{\Delta\}. \tag{4.29}$$

From Proposition 3.15, Sub-section 3.5 and from Theorem 4.2, Sub-section 4.1 we know that

$$\left| J_D\left(\sum_{i=0}^{m-1} \zeta_i \chi_{\sum_{l=0}^i \Delta_l}\right)\right| \leqslant \exp\left[K\left(\sum_{i=0}^{m-1} |\mathrm{Im}\,\zeta_i|(i+1)^2\right)\right],$$

provided $K[\sum_{i=0}^{m-1} |\mathrm{Im}\,\zeta_i|(i+1)] \leqslant 1$, where $K$ is the constant determined by Estimate II of Sub-section 2.2.

Hence, by the Cauchy integral formula

$$|\mathfrak{S}_m^D(\chi_{\Delta_0}, \ldots, \chi_{\sum_{l=0}^{m-1}\Delta_l})| \leqslant (K')^m m! \prod_{i=0}^{m-1} \|\chi_{\sum_{l=0}^i \Delta_l}\|_1. \tag{4.30}$$

Now

$$h = \sum_{\Delta \in \{\Delta\}} h \cdot \chi_\Delta. \tag{4.31}$$

Let $\Delta_0$ be the unit square centered at $\langle 0, 0\rangle$. Clearly

$$\int d^2 \xi \chi_{\Delta_0}(\xi) = 1.$$

We now use (4.29), (4.30) and (4.31) and the first Griffiths inequality $\mathfrak{S}_m^D(\xi_1,\ldots,\xi_m) \geqslant 0$ (which is true under condition (C2)) to estimate $S_{m-1}^D(h_1,\ldots,h_{m-1})$.

$$S_{m-1}^D(h_1,\ldots,h_{m-1}) =$$

$$\sum_{\substack{\Delta_i \in \{\Delta\},\\ i=1,\ldots,m}} \int \prod_{j=0}^{m-1} d^2\xi_j\, \mathfrak{S}_m^D(\xi_0,\ldots,\xi_{m-1})\,\chi_{\Delta_0}(\xi_0) \prod_{i=1}^{m-1} h_i(\xi_i-\xi_{i-1})\chi_{\Delta_i}(\xi_i-\xi_{i-1}).$$

But

$$\left|\chi_{\Delta_0}(\xi_0) \prod_{i=1}^{m-1} h_i(\xi_i-\xi_{i-1})\,\chi_{\Delta_i}(\xi_i-\xi_{i-1})\right| \leqslant \chi_{\Delta_0}(\xi_0) \prod_{j=1}^{m-1} \chi_{\sum_{l=0}^{j}\Delta_l}(\xi_j)\|h_j\chi_{\Delta_j}\|_\infty$$

and therefore

$$\left|\int \prod_{j=0}^{m-1} d^2\xi_j\, \mathfrak{S}_m^D(\xi_0,\ldots,\xi_{m-1})\,\chi_{\Delta_0}(\xi_0) \prod_{i=1}^{m-1} h_i(\xi_i-\xi_{i-1})\,\chi_{\Delta_i}(\xi_i-\xi_{i-1})\right|$$

$$\leqslant \mathfrak{S}_m^D(\chi_{\Delta_0},\ldots,\chi_{\sum_{l=0}^{j}\Delta_l},\ldots,\chi_{\sum_{l=0}^{m-1}\Delta_l}) \prod_{i=1}^{m-1} \|h_i\chi_{\Delta_i}\|_\infty$$

$$\leqslant (K')^m\, m!\,(m!)^2 \prod_{i=1}^{m-1} \|h_i\chi_{\Delta_i}\|_\infty, \quad \text{by (4.29), (4.30).}$$

With the definition (4.27) of $|\cdot|_{\mathscr{S}}$ we now get

$$|S_{m-1}^D(h_1,\ldots,h_{m-1})| \leqslant (K')^m (m!)^3 \prod_{i=1}^{m-1} |h_i|_{\mathscr{S}}$$

$$\times \left\{ \sum_{\substack{\Delta_i \in \{\Delta\},\\ i=1,\ldots m-1}} \prod_{j=1}^{m-1} (1+[\operatorname{dist}(0,\Delta j)]^2)^{-1-\epsilon} \right\}$$

$$\leqslant K_0^m (m!)^3 \prod_{i=1}^{m-1} |h_i|_{\mathscr{S}},$$

for some $K_0 < \infty$. This completes the proof of Theorem 4.8.

Q.E.D.

We have now completed the verification of the Osterwalder–Schrader axioms [5]. It is shown in Ref. [5] that axioms (E0′)–(E3) are sufficient for the reconstruction of relativistic quantum fields which satisfy the Wightman axioms (up to the uniqueness of the physical vacuum).

I thank Professor K. Osterwalder for suggesting to me the problems solved in this subsection.

For more detailed results concerning the quantum fields of the $P(\varphi)_2$ models under conditions (C1), (C2) and other axioms, see Ref. [29].

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