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On the Absolutely Continuous Subspace of a Self-Adjoint Operator

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Abstract[1]). The absolutely continuous subspace $\mathrm{H}_{ac}(T)$, an object of interest in the scattering theory of quantum mechanics and in other applications involving the spectral analysis of a self-adjoint operator T, is characterized exactly as the closure, in the given Hilbert space H, of the subspace of vectors φ satisfying the resolvent growth condition on individual vectors: $\|R_z(T)\varphi\| = O(y^{-1/2})$, $y = \mathrm{Im}\, z \to O^+$.

One of the underlying mathematical problems occurring in the scattering theory of quantum mechanics (e.g., see Kato [1]) is to determine the closed subspace $H_{ac}(T)$ of H, where T is a self-adjoint operator with spectral family $\{E(\lambda)\}$, H is a complex Hilbert space, and $H_{ac}(T) = \{\varphi \in H \,|\, (E(\lambda)\varphi, \varphi)$ is absolutely continuous, $-\infty < \lambda < \infty\}$. For this purpose, and also for other applications, the following characterization of $H_{ac}(T)$ is perhaps of interest, inasmuch as it involves only estimates on the resolvent $R_z(T) = (T - z)^{-1}$ near the real axis.

Theorem:

$$H_{ac}(T) = \overline{\{\varphi \in H \,|\, \|R_z(T)\varphi\| = O(y^{-1/2}),\ y = \mathrm{Im}\, z \to O^+\}}.$$

That is, the theorem states that a sufficient condition for φ to be in $H_{ac}(T)$ is that there exists a constant M_φ, which may depend on φ, such that $\|R_z(T)\varphi\| \leqslant M_\varphi/\sqrt{y}$ for all $z = x + iy, y > 0$; and that if φ is in $H_{ac}(T)$, then necessarily φ is the limit of such functions. An equivalent characterization holds with $y \to O^-$. Thus in showing an operator T, or a part of T, to be absolutely continuous, one cannot avoid establishing, either directly or indirectly, the existence of a dense subspace of vectors φ satisfying the individual growth rates specified in the theorem.

We will make use of the following lemma on L^1 boundary-values of positive harmonic functions; since the results in this lemma are all known or knowable from the existing lore, we omit the proof. In the following, let $v_y(x)$ denote $v(x,y)$ for y fixed,

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$z = x + iy$, $L^1 \equiv L^1(R)$ the real space $L^1(-\infty,\infty)$, $d\mu$ a positive measure for which $\int_{-\infty}^{\infty} (\lambda^2+1)^{-1} d\mu(\lambda) < \infty$.

Lemma: Let $v(x,y)$ be a positive harmonic function on the half-plane $y = \operatorname{Im} z > 0$. Then:

a) v has a Poisson representation

$$v(x,y) = \int_{-\infty}^{\infty} P_y(\lambda - x)\, d\mu(\lambda), \quad P_y(\lambda - x) = \pi^{-1}\left[\frac{y}{(\lambda - x)^2 + y^2}\right]$$

with $\int_{-\infty}^{\infty} d\mu(\lambda) < \infty$ if and only if $v_y(x) \in L^1(R)$ for some $y > 0$, and then $\int_{-\infty}^{\infty} d\mu(\lambda) = \|v_y\|_{L^1}$ for all $y > 0$;

b) v has such a representation with $d\mu(\lambda) = g(\lambda)\, d\lambda$ with $g \in L^1(R)$ and $g \geqslant 0$ if and only if v_y converges in $L^1(R)$ as $y \to 0^+$, and then $v_y \to g$ in L^1 as $y \to 0^+$;

c) v has representation $v(x,y) = \alpha y + \int_{-\infty}^{\infty} P_y(\lambda - x)\, d\mu(\lambda)$ with $\mu(\lambda)$ absolutely continuous on the open interval (a,b), $-\infty \leqslant a < b \leqslant \infty$, if and only if the functions v_y restricted to (a,b) converge in $L^1_{loc}(a,b)$ as $y \to 0^+$, and then (letting g denote the limit function, $v_y \to g$) one has $\mu(\lambda_2) - \mu(\lambda_1) = \int_{\lambda_1}^{\lambda_2} g(\lambda)\, d\lambda$ for $a < \lambda_1 < \lambda_2 < b$. If $\int_{-\infty}^{\infty} d\mu(\lambda) < \infty$, then in the case $a = -\infty$, one has $\mu(\lambda) = \int_{-\infty}^{\lambda} g(t)\, dt$ for $\lambda < b$, and in the case $b = \infty$, $\mu(\lambda) = \int_{-\infty}^{\infty} d\mu(t) - \int_{\lambda}^{\infty} g(t)\, dt$.

Proof of the theorem: For $z = x + iy$, $y > 0$, let $v = v(x,y) = y\pi^{-1}\|R_z(T)\varphi\|^2 = \pi^{-1}\operatorname{Im}(R_z(T)\varphi,\varphi) = \int_{-\infty}^{\infty} P_y(\lambda - x)\, d(E(\lambda)\varphi,\varphi)$, using the resolvent equation $R_z - R_{z'} = (z - z')\, R_z R_{z'}$ and the spectral representation of $R_z(T)$; v is then a positive harmonic function on the upper half-plane.

If $\|R_z(T)\varphi\|$ is $O(y^{-1/2})$ as $y \to 0^+$, then $0 < v(z) < \pi^{-1} M_\varphi^2$. By the Fatou boundary limit theorem $v_y(x) = v(x,y)$ converges as $y \to 0^+$ for a.e. real x. Thus by the dominated convergence theorem v_y converges in $L_1[\lambda_1,\lambda_2]$ on each finite interval $[\lambda_1,\lambda_2]$. Hence by the lemma (c) applied to the interval $(-\infty,\infty)$, the function $\mu(\lambda) = (E(\lambda)\varphi,\varphi)$ in the representation above is absolutely continuous.

Conversely, suppose $\varphi \in H_{ac}$; then, since the measure is finite, $(E(\lambda)\varphi,\varphi) = \int_{-\infty}^{\lambda} g(x)\, dx$ for some $g \in L_1(R)$, and $v(x,y) = \int_{-\infty}^{\infty} P_y(\lambda - x) g(\lambda)\, d\lambda$. By the lemma (b) $v_y \to g$ in $L_1(R)$ as $y \to 0^+$. On the other hand, letting $H = \bigoplus_{a\in A} H_a$ be the spectral representation (see Dunford and Schwartz [2, p. 1209]) of H with respect to the operator T, U the unitary mapping $U: H \to \bigoplus_{a\in A} L_2(\mu_a)$ given by $x = \bigoplus_{a\in A} \xi_a(T)\, a \to \xi = \bigoplus_{a\in A} \xi_a$, and V the operator UTU^{-1} in $\bigoplus_{a\in A} L_2(\mu_a)$, where $\mu_a(\lambda) = (E(\lambda)\, a, a)$, we have by the functional calclulus

$$\begin{aligned} v(x,y) &= \pi^{-1}\operatorname{Im}(R_z(T)\,\varphi,\,\varphi) = \pi^{-1}(UR_z(T)\,U^{-1}\,\xi,\,\xi) \\ &= \pi^{-1}\operatorname{Im}\,(R_z(V)\,\xi,\,\xi) = \pi^{-1}(\operatorname{Im}\,R_z(V)\,\xi,\,\xi) \\ &= \pi^{-1}\sum_{a\in A}([\operatorname{Im}\,R_z(V)\,\xi]_a,\,\xi_a)_a \\ &= \pi^{-1}\sum_{a\in A}(\operatorname{Im}\,(\lambda - z)^{-1}\,\xi_a(\lambda),\,\xi_a(\lambda))_a \\ &= \sum_{a\in A}\int_{-\infty}^{\infty} P_y(\lambda - x)\,|\xi_a(\lambda)|^2\, d\mu_a(\lambda). \end{aligned}$$

Each function $v^a(x,y) = \int_{-\infty}^{\infty} P_y(\lambda - x)|\xi_a(\lambda)|^2 d\mu_a(\lambda)$ is positive and harmonic. Also $v_y^a(x)$ converges as $y \to O^+$ for a.e. $x \in R$. Moreover, by the Vitali convergence theorem (e.g., see [2, p. 150]), the $L^1(R)$ convergence of v_y, and the inequalities $0 \leqslant v_y^a < v_y$, v_y^a converges in $L^1(R)$ as $y \to O^+$, so that by the lemma (a), (c), $v^a(x,y)$ is the Poisson integral of a finite absolutely continuous measure. Thus by the lemma (b), $v_y^a \to g_a \in L^1(R)$, and $|\xi_a(\lambda)|^2 d\mu_a(\lambda) = g_a(\lambda)\, d\lambda$.

From the $g_a(\lambda)$ we now construct an approximating sequence $\varphi_n \in H_{ac}(T)$ for φ; that is, a sequence φ_n satisfying the resolvent growth condition of the theorem. Since the non-zero ξ_a form a countable set we may index them ξ_{a_i}, $i = 1, 2, 3, \ldots$. Let $E_{an} = \{\lambda \mid |g_a(\lambda)| \leqslant n\}$, and define

$$\xi^n = \bigoplus_{i=1}^{\infty} \xi_{a_i}^n,$$

where $\xi_{a_i}^n = \chi_{E a_i n} \xi_{a_i}$ if $i \leqslant n$, $\xi_{a_i}^n = 0$ if $n < i$. Letting $v_n(x,y) \equiv \pi^{-1} y \|R_z(V)\,\xi_n\|^2$, we have, as above, for $\operatorname{Im} z > 0$,

$$v_n(x,y) = \pi^{-1} \operatorname{Im}(R_z(V)\,\xi^n, \xi^n) = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} P_y(\lambda - x)|\xi_{a_i}^n(\lambda)|^2 d\mu_{a_i}(\lambda)$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} P_y(\lambda - x)|\chi_{E_{a_i n}}(\lambda)\,\xi_{a_i}(\lambda)|^2 d\mu_{a_i}(\lambda)$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} P_y(\lambda - x)\,\chi_{E_{a_i n}}(\lambda) g_{a_i}(\lambda)\, d\lambda \leqslant n^2.$$

Thus, for $\varphi^n = U^{-1}\xi^n$, we have on $\operatorname{Im} z > 0$

$$\|R_z(T)\,\varphi^n\| = \|R_z(V)\,\xi^n\| = \pi^{1/2} y^{-1/2} v_n(z) \leqslant M_{\varphi^n}/\sqrt{y},$$

where $M_{\varphi^n} = \pi^{1/2} n^2$. Hence the φ^n are in $H_{ac}(T)$ and satisfy the characterizing order estimate of the theorem.

It remains to show that the φ^n approximate the given function φ. By Parseval's relation

$$\|\varphi - \varphi^n\|^2 = \|\xi^n - \xi\|^2 = \sum_{i=1}^{\infty} \int_{-\infty}^{\infty} |\xi_{a_i}^n(\lambda) - \xi_{a_i}(\lambda)|^2 d\mu_{a_i}(\lambda)$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{\infty} |\chi_{E_{a_i n}}(\lambda) - 1|^2 |\xi_{a_i}(\lambda)|^2 d\mu_{a_i}(\lambda) + \sum_{i=n+1}^{\infty} \int_{-\infty}^{\infty} |\xi_{a_i}(\lambda)|^2 d\mu_{a_i}(\lambda)$$

$$= \sum_{i=1}^{n} \int_{g_{a_i}(\lambda) > n} g_{a_i}(\lambda)\, d\lambda + \sum_{i=n+1}^{\infty} \|\xi_{a_i}\|_{a_i}^2.$$

Noting that $0 \leqslant v_y^{a_1} + \ldots + v_y^{a_n} \leqslant v_y$ on $\operatorname{Im} z > 0$, and recalling that it was concluded above that $v_y \to g$ and $v_y^a \to g_a$ in $L^1(R)$ as $y \to O^+$, we obtain (a.e. in x) $0 \leqslant g_{a_1} + \ldots + g_{a_n} \leqslant g$, so that $\{\lambda \mid g_{a_i}(\lambda) > n\} \subset \{\lambda \mid g(\lambda) > n\}$ for all i and n. Thus the estimate above becomes

$$\|\varphi^n - \varphi\|^2 \leqslant \int_{g(\lambda) > n} g(\lambda)\, d\lambda + \sum_{i=n+1}^{\infty} \|\xi_{a_i}\|_{a_i}^2;$$

hence by the Fourier representation and the fact that $g \in L^1(R)$, the right-hand side tends to zero as $n \to \infty$, and hence $\varphi_n \to \varphi$ in H.

Remark 1: Some elements of the characterization above, in less explicit form, have been observed previously at various points in the mathematical and physical literatures of scattering theory and of boundary values of analytic functions, but without precise statement and proof of the required resolvent growth rate on individual vectors. Additionally, the specific emphasis placed herein on L^1 boundary values, and, in particular, for just the imaginary part of the matrix elements, is apparently new; it may be formalized in context as follows.

Corollary: $H_{ac}(T)$ consists exactly of those φ for which $\mathrm{Im}\langle R_z\varphi, \varphi\rangle$ attains $L^1(-\infty, \infty)$ boundary values.

Remark 2: One may similarly characterize the absolutely continuous subspace $H_{ac}(T, \Sigma)$ corresponding to an open subset Σ of R; here $H_{ac}(T, \Sigma)$ is the intersection of the subspaces $H_{ac}(T, (\alpha, \beta))$ over all sub-intervals (α, β) in Σ, where

$$H_{ac}(T, (\alpha, \beta)) = \{\varphi \in H \mid (E(\lambda)\,\varphi, \varphi) \text{ is absolutely continuous on } (\alpha, \beta)\}$$

or, equivalently,

$$H_{ac}(T, (\alpha, \beta)) = H_{ac}(T|_{E(\alpha, \beta)H}).$$

Let

$$\Omega(\lambda_1, \lambda_2) = \{z = x + iy \mid \lambda_1 \leqslant x \leqslant \lambda_2, y > 0\};$$

then $H_{ac}(T(\alpha, \beta))$ is the closure of the subspace

$$\{\varphi \in H \mid \|R_z(T)\,\varphi\| \leqslant M(\varphi, \Omega(\lambda_1, \lambda_2))/\sqrt{y}, \quad \text{for all } \alpha < \lambda_1 < \lambda_2 < \beta\}.$$

Remark 3: For an alternate proof of the necessity of the theorem as related to the Kato–Rosenbloom Lemma, and in terms of the spectral family $\{E(\lambda)\}$ rather than the resolvent R_z, for other scattering subspaces related to H_{ac}, and for further references, see [3].

Remark 4: Inasmuch as the above characterization of $H_{ac}(T)$ is independent of both the spectral family $\{E(\lambda)\}$ and the group e^{itT}, it defines an 'absolutely continuous' subspace for certain types of non-self-adjoint operators T (e.g., spectral operators with spectrum on an arc, or dissipative operators with spectrum on a half-plane).

REFERENCES

[1] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, New York 1966).
[2] N. Dunford and J. T. Schwartz, *Linear Operators I and II* (Interscience, New York 1963).
[3] K. Gustafson, *Candidates for σ_{ac} and H_{ac}*, Proc. NATO Adv. Study Inst. on Scattering Theory in Mathematics and Physics (1973), J. LaVita and J. P. Marchand, Reidel, Doortrecht (to appear).